Rescue Policies for Small Businesses in the COVID-19 Recession*

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May 11, 2023

Abstract

While the COVID-19 pandemic had a large and asymmetric impact on firms, many countries quickly enacted massive business rescue programs which are specifically targeted to smaller firms. Little is known about the effects of such policies on business entry and exit, investment, factor reallocation, and macroeconomic outcomes. This paper builds a general equilibrium model with heterogeneous and financially constrained firms in order to evaluate the short- and long-term consequences of small firm rescue programs in a pandemic recession. We calibrate the stationary equilibrium and the pandemic shock to the U.S. economy, taking into account the factual Paycheck Protection Program (PPP) as a specific policy. We find that the policy has only a modest impact on aggregate output and employment because (i) jobs are saved predominately in the smallest firms that account for a minor share of employment and (ii) the grant reduces the reallocation of resources towards larger and less impacted firms. Much of the reallocation effects occur in the aftermath of the pandemic episode. By preventing inefficient liquidations, the policy dampens the long-term declines of aggregate consumption and of the real wage, thus delivering small welfare gains.

JEL classification: E22, E65, G38, H25

Keywords: COVID-19, Heterogeneous Firms, Business Subsidies, Paycheck Protection Program

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*We are grateful to Ivo Bakota, Basile Grassi, Marek Ignaszak, Matthias Kredler, Alexander Ludwig, Francesco Pappada', Mathias Trabandt and seminar participants at the University of Barcelona, the Munich Center for the Economics of Ageing, Goethe University Frankfurt, FAU/IAB, and conference audiences at IMF-TARC Conference 2021, Econometric Society Winter Meeting 2021, SED Annual Meeting 2022, Oslo Macro Conference 2022, Mannheim Workshop on Heterogeneous Firms and Macroeconomics 2022 for insightful comments. Alessandro Di Nola thanks the German Research Foundation (grant No. SCHO 1442/2) for financial support.

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1 Introduction

The 2020-21 recession induced by the COVID-19 pandemic differs from regular business-cycle downturns in important ways. Government-mandated shutdown policies, individual demand adjustments and disruptions of global production chains had a large and asymmetric impact on private businesses. In particular, the magnitude of output and employment declines were larger for smaller firms (see Bloom et al., 2021; Cajner et al., 2020).[1] Furthermore, business closures in the U.S. have increased sharply, again with much variation by firm size (Crane et al., 2021; Chetty et al., 2020). To stabilize income losses in the short term and to prevent a severe and long-lasting impact on production capacities, governments in many countries swiftly implemented small business rescue programs in the form of grants or conditional loans. For instance, in March 2020 the U.S. enacted the Coronavirus Aid, Relief, and Economic Security (CARES) Act that allocated over $600 billion for the Paycheck Protection Program (PPP). In 2020, over three quarters of U.S. small businesses received the PPP loan and most are eventually forgiven (Autor et al., 2022).

Little is known about the effects of such a large-scale business rescue policy from a macroeconomic perspective. On the one hand, offering liquidity to small businesses can prevent productive firms from selling their capital or even from permanently shutting down, impeding inefficient capital liquidation and facilitating a quicker economic recovery once the pandemic terminates. On the other hand, such rescue plans can inadvertently prolong the lives of unproductive ("zombie") firms, thus hampering efficient capital reallocation. Furthermore, the PPP program was designed to favor timeliness over targeting (Autor et al., 2022), resulting in an unprecedented fiscal cost as firms that are not impacted or at risk of liquidation also received the forgivable loan. Given that targeting financial aid to impacted firms requires a greater administrative burden, it is important to understand the cost-effectiveness of a targeted rescue policy compared to the rather universal PPP program.

The goal of this paper is to quantify the short- and long-run macroeconomic effects of the small business rescue policy enacted in the COVID-19 pandemic, and to evaluate alternative policies. To this end, we build a general equilibrium model with firms that differ in productivity, capital, and financial assets or debt. We interpret these entities as "small firms" in a non-corporate sector that can only borrow against collateral, face financial constraints

[1] Using anonymized administrative data provided by ADP (a large private provider of payroll services), Cajner et al. (2020) show that businesses with fewer than 50 employees reduced employment by more than 25 percent from March to April 2020, whereas larger firms saw declines of 15-20 percent over the same time period.

and capital adjustment frictions due to partial irreversibility of their investments. Small firms incur overhead expenses including fixed operating costs and payroll expenses. Our model also includes a corporate sector where firms are not financially constrained. The pandemic has an asymmetric impact on firms in the two sectors. Although the corporate sector is not the center of our analysis, it is important for factor reallocation and hence for macroeconomic adjustment following the pandemic shock.

We first calibrate our model such that the stationary equilibrium matches relevant aggregate and firm-level moments of the pre-pandemic U.S. economy. We draw data from various sources including semi-aggregate tables of the Statistics of U.S. Businesses (SUSB), the Business Dynamics Survey (BDS), as well as micro data from the Kaufman Firm Survey (KFS) to inform us about the balance sheets of small firms in the U.S. Our model replicates the heavily skewed firm size distribution observed in the SUSB, the observation that the firm exit rate decreases in firm size observed in the BDS, firm-level investment rate patterns, and the share of indebted firms. In our calibrated model, the decisions to enter, exit, and invest depend not only on a firm's productivity and capital, but also its indebtedness.

We model the pandemic shock with six components: A shut-down shock that affects a fraction of small firms, a TFP shock on the corporate sector, a demand shock affecting the marginal utility of consumption, a labor supply shock affecting the marginal utility of leisure, a credit supply shock, and a shock to the number of potential entrants. The shut-down shock captures the impact of government mandated lock-down policies that forced businesses offering "social" goods and services to temporarily close at the beginning of the pandemic. The demand shock is important for explaining the observed sharp drop in consumption, and the labor supply shock captures the drop in employment, possibly due to health risks of in-person working. The credit supply shock reflects changes in credit conditions, and a change in the number of potential entrants stands for altered business opportunities during the COVID-19 recession. We calibrate the pandemic shock to match the changes of U.S. output, consumption, aggregate employment, employment in small firms, entry and exit rates at the onset of the pandemic, while taking into account the PPP policy and their take-up rates by small firms.

Based on our calibrated model, we compare the PPP to the laissez-faire economy, a counterfactual scenario with no government intervention. We find that the PPP prevents 62% of permanent exits of small businesses at the onset of the pandemic. Nonetheless, the PPP has only modest effects on aggregate output and employment. The reason is twofold: *(i)* The PPP dampens the reallocation of resources towards the less impacted corporate sector. *(ii)* The PPP prevents business exits mostly in very small firms that account for a small fraction of total employment. The lack of aggregate impact echoes previous findings in the literature (see Crouzet and Mehrotra, 2020) showing that although smaller firms are more exposed to aggregate volatility, the difference has only modest implications on aggregate fluctuations. Therefore, policies aiming to stabilize smaller firms should not be expected to have large macroeconomic consequences.

Next, we simulate a counterfactual policy that is similar to the PPP but only gives financial aids to impacted firms. Since the share of impacted firms is calibrated to be 12%, the fiscal cost of the targeted policy is only one sixth of the cost of the PPP. The targeted policy leads to a smaller employment improvement compared to the PPP over a 10–year period, but it is more cost-effective. Specifically, we compare the cost of an average employment increase by 1% over a 10-year period under the two policies and find that the cost of the targeted policy is 31% of the cost of the PPP. Further, we find that the targeted grant creates fewer "zombie firms" in comparison to the baseline grant, where a zombie firm is defined as a firm whose survival is caused by the policy but is socially inefficient.

We further compare the policy impact on the extensive and intensive investment margins and over different time horizons. While the PPP mitigates the investment decline in the short run primarily via the exit margin, it also has a positive effect on the intensive investment margin of expanding firms over the long run (years 3–10). Overall, the policy stimulates investment in small firms, and hence dampens the persistent reallocation of capital and labor towards the corporate sector that would occur in the absence of government intervention. By preventing inefficient capital liquidations, the PPP mitigates the long-term declines of aggregate consumption and of the real wage, thus weakening the labor supply increase in the aftermath of the pandemic recession, which ultimately brings about a small welfare gain.

Related literature Our work relates to different strands of the macroeconomic literature. Several studies analyze the impact of health policies in the COVID-19 pandemic by integrating epidemiological dynamics into macroeconomic general-equilibrium models (e.g. Eichenbaum et al., 2020; Glover et al., 2020) or demand and supply spillovers in multi-sector models (e.g. Guerrieri et al., 2020; Baqaee and Farhi, 2020). By focusing on the effects of business rescue policies, our model features a representative household and treats the pandemic shock as an exogenous event which impacts both the productivity of firms and household preferences for consumption and leisure in order to generate the factual employment, consumption and investment responses during the 2020 recession. While simplifying our model analysis, this modeling choice obviously rules out potential feedback effects of business rescue policies on the health sector.

Other recent work evaluates the macroeconomic and distributional impacts of fiscal policy in the COVID-19 recession. Bayer et al. (2020) and Bigio et al. (2020) analyze conditional and unconditional transfers to heterogeneous households. Faria-e-Castro (2021) studies the effectiveness of different types of fiscal policy, including transfers to firms, without considering firm heterogeneity. Complementary to these studies, our work focuses on the macroeconomic and welfare consequences of business rescue policies, while abstracting from distributional implications.

Further contributions examine the role of firms in the pandemic recession. Bilbiie and Melitz (2020) show how price rigidity amplifies the entry and exit dynamics, and Elenev et al. (2020) study the impact of different firm bailout policies, focusing on the linkages between the financial intermediation and production sectors. Gourinchas et al. (2022) calibrate a static model with heterogeneous firms and find that government grants were quite effective in reducing business exits but also costly due to the lack of targeting. These papers do not allow for persistent firm heterogeneity by productivity or financial assets and thus they do not examine the reallocation of production factors across firms and over time.

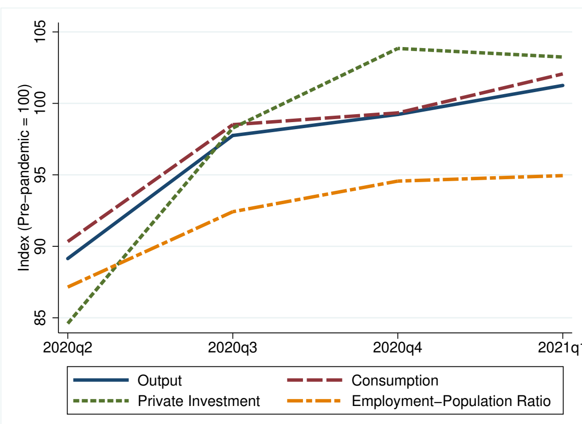
Most closely related to our work are Buera et al. (2021a) and Jo et al. (2021). Buera et al. (2021a) examine the impact of a pandemic shock on heterogeneous firms facing financial frictions, also including occupational choice and labor market frictions. Jo et al. (2021) use a model setting similar to ours which additionally features households with different health status (and hence endogenous pandemic dynamics). Different from our work, both papers do not analyze the role of government grants to small firms for firm selection and macroeconomic dynamics, and their models do not feature the partial irreversibility of capital investments that is central for our study.

Finally, we build on a large literature that incorporates heterogeneous, financially constrained firms into macroeconomic models. Our model is based on Khan and Thomas (2013), where we assume partially irreversible capital investments. Unlike their paper, but as in Khan et al. (2016), entry and exit are endogenous outcomes, yet all debt in our model is secured by collateral so that default does not occur. Instead our model distinguishes between forced and voluntary exit events, where the former depends crucially on the debt position of the firm. The long-term macroeconomic impact of credit-subsidy policies on heterogeneous firms has been studied by Buera et al. (2013), Buera et al. (2021b) and Jo and Senga (2019). While they focus on stationary environments, the reallocation effects via extensive (entry and exit) and intensive (factor intensity) margins are common to our work.

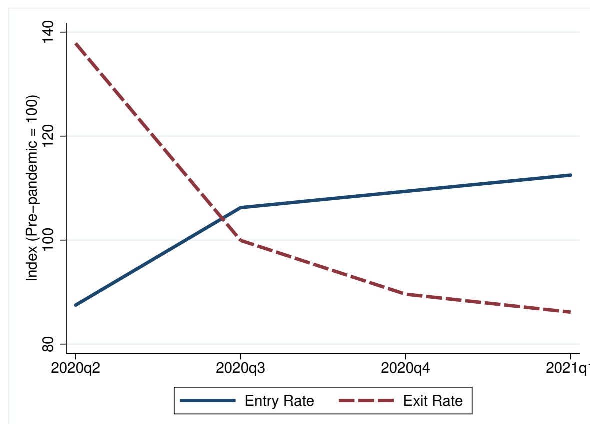
The rest of this paper is organized as follows. Section 2 briefly reviews the response of the U.S. economy in the COVID-19 recession and the PPP policy. Section 3 presents the model and Section 4 the calibration. Section 5 shows findings of the paper, and Section 6 concludes.

2 COVID-19 Pandemic in the U.S. and Rescue Policies

The recession induced by the COVID-19 pandemic differs from past recessions in important ways. The pandemic shock to the macroeconomy is deep but short-lived. Figure 1a shows the macroeconomic impact of the COVID-19 pandemic on the U.S. economy. Compared to the first quarter of 2020, the aggregate economy took a dramatic downturn in the second quarter of 2020: total non-farm output fell by 10.9%, employment by 12.9%, consumption by 9.7% and private domestic investment by 15.4%.



(a) Aggregate Variables



(b) Entry and Exit Rates

Notes for panel (a): Source: FRED Economic data from St. Louis Fed. "Pre-pandemic" refers to 2020 Q1.
Notes for panel (b): Source: Business Employment Dynamics from the U.S. Bureau of Labor Statistics. Entry and exit rates are establishment birth and death rates based on firms with fewer than 500 employees. "Pre-pandemic" refers to the long-run average between 1993 and 2019.

Figure 1: Macroeconomic Impacts of the COVID-19 Pandemic

To mitigate the spread of the coronavirus, many governments imposed strict shutdown and social-distancing policies at the beginning of the pandemic. The economic impact of the pandemic was felt disproportionately by small businesses as they face tighter borrowing constraints and may experience greater difficulties to liquidate their fixed capital. Bartlett and Morse (2020) report that small businesses are more likely to experience closures, temporary or permanent, than larger businesses. Bloom et al. (2021) show that small firms experience a larger drop in sales. Based on data from the ADP Research Institute, Cajner et al. (2020)

show that firms with fewer employees experience greater employment losses compared to larger firms. Based on their data, employment in firms with fewer than 500 employees drops by 23% relative to the pre-pandemic level, whereas employment in larger firms drops by only 18%.

Figure 1b shows the impact of the pandemic on entry and exit rates of small firms which is taken from the Business Economic Dynamics statistics of the Bureau of Labor Statistics (BLS-BED). Relative to the pre-pandemic long-run average, the rate at which small firms permanently exit sharply increased at the onset of the pandemic by 38%, and the entry rate of small firms dropped by 12%. Both rates quickly recovered from the third quarter of 2020 onward.

In response to the potentially devastating impacts of the pandemic on small businesses, the U.S. allocated over $600 billion for the Paycheck Protection Program (PPP) starting in April 2020.[2] The program offers forgivable loans up to 2.5 times the average monthly payroll to small businesses with up to 500 employees. PPP loans feature an attractive interest rate of 1% p.a. and can be forgiven when certain requirements are met. Appendix A provides details on the terms of PPP loans and forgiveness requirements. According to the U.S. Small Business Administration (SBA), as of October 17, 2022, 93% of all PPP loans issued in 2020 have been forgiven, which amounts to 96% of the total loan value.[3]

Despite the high degree of heterogeneity in the pandemic impact on businesses, the distribution of PPP loans is largely untargeted and prioritizes speedy loan disbursement (Autor et al., 2022). According to Autor et al. (2020), initially about 70% of eligible firms applied for PPP loans, and Borawski and Schweitzer (2021) report that by the end of 2020 PPP loans had been taken up by 76% of U.S. small businesses.[4]

A few empirical studies estimate the initial employment effect of the PPP program. Utilizing the PPP eligibility size threshold to differentiate treatment and control groups, Autor et al. (2020, 2022) find that the policy increased employment at eligible firms by about three percent at a high cost of around $200,000 per job-year. Using different data sources, Chetty et al. (2020) and Hubbard and Strain (2020) find somewhat smaller employment effects in the range of 1-2%. Bartlett and Morse (2020) and Hubbard and Strain (2020) also find that the PPP had a significant impact on the survival probability of smaller businesses. Finally, Kurmann et al. (2021) use real-time establishment data and find that pandemic policies, such as the PPP, are effective in mitigating the negative employment effect on small businesses by relaxing financial constraints.

[2]In this paper, we only focus on the small firm rescue program issued in the year 2020.

[3]See "2022 PPP forgiveness platform lender submission metrics reports, version 42" posted on sba.gov.

[4]The total take-up rate of PPP loans in both 2020 and 2021 is around 94% (Autor et al., 2022).

To study the impact of a business rescue policies in the short- and long-run, including spill-over effects and reallocation of production factors in general equilibrium, we consider a model in which financially constrained small firms and unconstrained large firms are differentially impacted by a pandemic shock in the following sections.

3 The Model

We consider a discrete-time general equilibrium model of a closed economy. A single consumption/investment good is produced by firms that belong either to the corporate or to the non-corporate sector.[5] While firms in the corporate sector face no financial frictions, non-corporate firms (termed "small firms" in the following) can only borrow against the collateral value of their capital and they cannot raise equity from outside investors. Firms in the non-corporate sector make entry and exit decisions. In particular, small firms face idiosyncratic productivity risk and may decide to liquidate their firm if they are unable or unwilling to continue operating.

While there is no aggregate risk, we consider the economy's response to a one-time unexpected shock and the transition path back to the unique steady-state equilibrium. To simplify notation, we index several variables by the time index t, indicating dependence on the aggregate state vector which is either constant (in steady state) or converges deterministically after the one-time shock.[6] The one-time shock includes a shut-down shock on a fraction of small firms, a credit supply shock to all small firms, a shock to business creation, a TFP shock on the corporate sector, and shocks to household preferences over consumption and leisure.

3.1 Environment

3.1.1 Small Firms

Small firms use capital k and labor ℓ to produce output $xf(k,\ell)$ where f is a strictly increasing, strictly concave, and decreasing returns to scale production function, and x is exogenous, idiosyncratic productivity which follows a monotone Markov chain on finite set $\mathbb{X}$ with transition probabilities $g(x'|x)$.

[5] To keep our model reasonably simple, we do not differentiate between the goods produced for social and non-social consumption. Given that many social goods have close non-social substitutes (e.g. restaurants vs food at home, health clubs vs home gyms, or movie theaters vs streaming services), this seems a reasonable short-cut.

[6] Further simplifying notation, the time index is dropped from the firms' state and decision variables in which case we use the prime superscript to denote next period values of these variables.

Small firms incur fixed operating cost $c^f(k)$ which capture general and administrative expenses. We assume that $c^f(0) > 0$ and $c^{f'}(k) \geq 0$ for all k. With capital fixed at the beginning of the period, the firm chooses labor input to maximize profits

$$\pi_t(x, k) \equiv \max_{\ell \geq 0} x f(k, \ell) - w_t \ell - c^f(k) \ ,$$

where w_t is the real wage in period t.

Every period, a fraction δ of installed capital depreciates and the firm decides about next period's capital stock k'. Capital investment is partially irreversible: whenever a firm downsizes its capital stock to $k' < (1-\delta)k$, it only receives $\theta[(1-\delta)k - k']$ units of output where $\theta < 1$ is the price of used capital. Likewise, an exiting firm only receives $\theta(1-\delta)k$ units of output. We denote the expenditures for a firm's gross investment (or the negative of the liquidation value when gross investment is negative) by the function

$$\xi[k' - (1-\delta)k] \equiv \begin{cases} k' - (1-\delta)k & \text{if } k' \geq (1-\delta)k \ , \\ \theta[k' - (1-\delta)k] & \text{if } k' < (1-\delta)k \ . \end{cases}$$

Small firms can save and borrow in the capital market at safe gross interest rate $1/q_t$. Borrowing is secured by the collateral value of capital. The newly issued debt is constrained by

$$b' \leq \lambda k' \ , \tag{1}$$

where $\lambda = \theta(1-\delta)$ is the value of collateral capital.[7] Furthermore, small firms cannot raise equity which implies that the dividend payout in period t must be non-negative:

$$\pi_t(x, k) - b + q_t b' - \xi[k' - (1-\delta)k] \geq 0 \ , \tag{2}$$

where b is the debt due in period t (or the negative value of savings when $b < 0$).

Events within a period proceed in the following three stages. First, incumbent firms draw their productivity x and exit exogenously with probability ψ. Those firms that do not exit exogenously may be forced to exit or decide to exit voluntarily, as described in the next paragraph. Further, a mass M of potential entrants draw initial productivity, debt and capital

[7] Different from corporate firms in the U.S., small firms hold mostly secured debt (see Azariadis et al. (2016)). Since all debt is secured by the (deterministic) collateral value, there is no equilibrium default in our model, different from, e.g., Jo et al. (2021). Our modeling of the pandemic shock below includes a tightening of the credit constraint parameter λ, while still precluding default. Abstracting from firm default should not be a major concern, given that bankruptcy filings of small businesses actually declined during 2019-2020 (see Wang et al., 2021), although the exit rate increased.

(x, b, k) from a joint distribution $\Phi(x, b, k)$ and decide to enter whenever the value of the new firm exceeds the installation cost net of debt which is $k - b$.[8] Second, incumbent and entrant firms hire labor and produce. Third, firms make borrowing and investment decisions, b' and k', together with the dividend payout in (2).

A firm that was active in the previous period and does not exit exogenously may exit at the beginning of a period due to two distinct events. First, low-productivity and highly indebted firms may not be able to satisfy the two financial constraints (1) and (2); these *illiquid* firms are then forced to exit. This happens if and only if $\pi_t(x, k) - b + \theta(1 - \delta)k < 0$.[9] Second, the firm may voluntarily decide to exit whenever the value of liquidation is greater than the value of staying in business. Regardless of the cause of exit, the firm's capital is liquidated and the owner retains the liquidation value net of debt repayment (or financial savings when $b < 0$) which is $\theta(1 - \delta)k - b \geq 0$.

All firms are owned by households and maximize their contribution to household utility. We describe and characterize the firms' entry, exit, investment and borrowing policies in Section 3.2.

3.1.2 Corporate Firms

Firms in the corporate sector use capital K^c and labor services L^c to produce output $F(K^c, L^c)$ with strictly increasing, concave, constant returns production function F. In a given period t, capital is predetermined and corporate firms hire labor to maximize profits which are denoted $\Pi_t(K_t^c) = \max_{L_t^c} F(K_t^c, L_t^c) - w_t L_t^c$. Over time, capital in the corporate sector depreciates at rate δ. Like all small firms, corporate firms are owned by households and choose optimal investment in the owners' interest. To simplify the exposition, we incorporate the corporate firms' investment decision into the representative household's optimization problem described in the next subsection.

[8]Our modeling of the entry process follows Clementi and Palazzo (2016) (and differs from Hopenhayn, 1992) in that a bounded measure of potential, heterogeneous entrants first observe their type before deciding to pay the installation cost. For this reason we do not need to separately specify an entry cost to limit entry. If there were a positive entry cost (and hence additional investment irreversibility), the efficiency gains from small-firm rescue policies are likely somewhat larger.

[9]If this inequality holds, it can be easily shown that any investment/borrowing policy (k', b') must either violate the borrowing constraint or the non-negative dividend constraint, given that $q_t \leq 1$ (which is true in steady state and in the transition after the pandemic shock). Conversely when $\pi_t(x, k) - b + \theta(1 - \delta)k$ is positive but small, the firm can only remain liquid if it sells all capital except k' which is constrained by $[\theta - q_t\lambda]k' \leq \pi_t(x, k) - b + \theta(1 - \delta)k$. When $\pi_t(x, k) - b + \theta(1 - \delta)k = 0$, the firm is forced to liquidate all capital ($k' = 0$) and to set $b' = 0$. If this firm stays in business, it will be forced to exit next period because of $\pi_{t+1}(x', 0) = -c^f(0) < 0$.

3.1.3 Households

There is a unit mass of households with unit time endowment who derive period utility $u(C, 1-L)$ which is strictly increasing in consumption C and leisure $1-L$, with L denoting labor supply. Future utility is discounted with factor $\beta < 1$ per period. Households own all firms.

Households can perfectly insure against idiosyncratic business risks. As a result, their consumption, labor supply and investment decisions are described by the utility maximization problem of a representative household. In period t, the representative household takes as given the aggregate state vector $X_t = (K_t^c, A_t, \mu_t^0)$ where K_t^c is the capital stock of the corporate sector, A_t are financial assets, and μ_t^0 is the measure of small firms over idiosyncratic states (x, b, k) prior to entry and exit. The household decides about consumption C_t, labor supply L_t, gross investment I_t^c in the corporate sector, and financial assets for the next period A_{t+1} which are traded at price q_t. Further, the household decides the entry of small firms $d_t^e(x, b, k)$ which equals one when potential firm (x, b, k) enters and zero otherwise, liquidation of small firms $d_t^l(x, b, k)$ which equals one when firm (x, b, k) is liquidated and zero otherwise, and investment and borrowing/savings decisions of active firms, $k_t'(x, b, k)$ and $b_t'(x, b, k)$. In recursive notation, the representative household's problem is

$$V_t(X_t) = \max \ u(C_t, 1-L_t) + \beta V_{t+1}(X_{t+1}) \ , \tag{3}$$

subject to the budget constraint

$$\begin{aligned} C_t + I_t^c + q_t A_{t+1} + M \int [k-b] d_t^e(x,b,k) \ d\Phi(x,b,k) \leq \ & w_t L_t + A_t + \Pi_t(K_t^c) \\ & + \int \pi_t(x,k) - b + q_t b_t'(x,b,k) - \xi[k_t'(x,b,k) - (1-\delta)k] \ d\mu_t(x,b,k) \\ & + \int [\theta(1-\delta)k - b][\psi + (1-\psi) d_t^l(x,b,k)] d\mu_t^0(x,b,k) \ , \end{aligned} \tag{4}$$

where

$$\mu_t = (1-\psi)(1-d_t^l)\mu_t^0 + M d_t^e \Phi \tag{5}$$

is the measure of active firms in period t (i.e., incumbent firms that do not exit plus entrants), the accumulation equation for capital in corporate firms,

$$K_{t+1}^c = \ (1-\delta) K_t^c + I_t^c \ , \tag{6}$$

financial constraints for continuing small firms,

$$b'_t(x,b,k) \leq \lambda k'_t(x,b,k) \ , \qquad \pi_t(x,k) - b + q_t b'_t(x,b,k) - \xi[k'_t(x,b,k) - (1-\delta)k] \geq 0 \ , \quad (7)$$

and the dynamic evolution of the distribution measure of small firms which requires that the measure of small firms at the beginning of the next period satisfies

$$\mu^0_{t+1}(A) = \int \mathbb{I}_{(x', b'_t(x,b,k), k'_t(x,b,k)) \in A} g(x'|x) \ d\mu_t(x,b,k) \ , \quad (8)$$

for all Borel sets $A \subset \mathbb{X} \times \mathbb{R} \times \mathbb{R}_+$.

The budget constraint (4) says that the household's expenditures for consumption, investment in corporate firms, financial assets and the investment expenditures of entrant firms (left-hand side) do not exceed the sum of labor income, the stock of financial assets at the beginning of the period, profit incomes of corporate and small firms, and the liquidation values of exiting small firms (right-hand side).

3.2 Competitive Equilibrium

3.2.1 Definition

Given an initial state (K^c_0, A_0, μ^0_0) with $A_0 = \int b \ d\mu^0_0(x,b,k)$, a *competitive equilibrium* is a sequence of market prices (w_t, q_t), consumption, labor supply and investment decisions $(C_t, L_t, I^c_t, K^c_{t+1}, A_{t+1})$, entry, exit, investment and borrowing decisions $(d^e_t, d^l_t, k'_t, b'_t)$ and distribution measures μ^0_{t+1}, μ_t of small firms, for all $t \geq 0$, such that

(i) The representative household solves the utility maximization problem (3)–(8).

(ii) The markets for labor and financial assets clear, i.e. in all periods t,

$$L_t = \int \ell_t(x,k) \ d\mu_t(x,b,k) + L^c_t \ ,$$
$$A_{t+1} = \int b'_t(x,b,k) \ d\mu_t(x,b,k) \ ,$$

with labor demand $\ell_t(x,k) = \text{argmax}_\ell \, x f(k,\ell) - w_t \ell$ and $L^c_t = \text{argmax}_{L^c} \, F(K^c_t, L^c) - w_t L^c$.

A *stationary competitive equilibrium* is a competitive equilibrium with constant state vector (K^c, A, μ^0) and constant market prices (w, q).

The goods market is in equilibrium because of Walras's law: the binding household budget constraint together with the other market-clearing conditions implies that consumption plus investment equals aggregate output,

$$C_t + I_t = Y_t \equiv F(K_t^c, L_t^c) + \int [xf(k, \ell_t(x,k)) - c^f(k)] \; d\mu_t(x,b,k) \ ,$$

where

$$\begin{aligned} I_t =& I_t^c + M \int k d_t^e(x,b,k) \; d\Phi(x,b,k) - \int \theta(1-\delta)k[\psi + (1-\psi)d_t^l(x,b,k)] \; d\mu_t^0(x,b,k) \\ &+ \int \xi[k_t'(x,b,k) - (1-\delta)k] \; d\mu_t(x,b,k) \end{aligned}$$

is aggregate gross investment in corporate firms and in small firms. The latter includes investment spending of entrant firms net of the liquidation of capital of exiting firms at the beginning of the period (first line), and the gross investment of all active firms at the end of the period (second line). The aggregate capital stock, predetermined at the beginning of period t, is $K_t = K_t^c + \int k \; d\mu_t^0(x,b,k)$.

3.2.2 Equilibrium Characterization

The first-order condition of the household's financial savings problem implies that the asset price is

$$q_t = \beta \frac{u_C(C_{t+1}, 1 - L_{t+1})}{u_C(C_t, 1 - L_t)} \ . \tag{9}$$

Let $v_t^0(x,b,k)$ be the value of a small firm before the exit decision and let $v_t(x,b,k)$ be the value of a continuing firm. These values represent the marginal contributions of the small firm to the household's utility, measured in units of the period-t consumption/investment good. That is, payments accruing in the next period $t+1$ are priced with the financial discount factor q_t. The value of a continuing small firm satisfies the Bellman equation

$$\begin{aligned} v_t(x,b,k) = \max_{b',k'} \ & \pi_t(x,k) - b + q_t b' - \xi[k' - (1-\delta)k] \\ & + q_t \left[\psi(\theta(1-\delta)k' - b') + (1-\psi)\mathbb{E}_{x'|x} v_{t+1}^0(x', b', k') \right] \ , \end{aligned} \tag{10}$$

$$\text{s.t. } b' \leq \lambda k' \quad \text{and} \quad \pi_t(x,k) - b + q_t b' - \xi[k' - (1-\delta)k] \geq 0 \ .$$

At the beginning of the next period, firms exit exogenously with probability ψ. Otherwise, the continuation value is $\mathbb{E}_{x'|x} v_{t+1}^0(x', b', k')$ where the expectation is taken over realizations of next period's productivity x' conditional on current productivity x.

The value of a firm at the beginning of period t and prior to endogenous exit (forced liquidation or voluntary exit), $v_t^0(x,b,k)$, equals $\theta(1-\delta)k-b$ when the firm is illiquid, i.e. if $\pi_t(x,k)-b+\theta(1-\delta)k<0$ (see Section 3.1.1), or when $v_t(x,b,k)<\theta(1-\delta)k-b$ in which case the firm is voluntarily liquidated. Otherwise the firm remains active. This implies that

$$v_t^0(x,b,k)=\begin{cases}\theta(1-\delta)k-b & \text{if } \pi_t(x,k)-b+\theta(1-\delta)k<0 \text{ or } v_t(x,b,k)<\theta(1-\delta)k-b\ ,\\ v_t(x,b,k) & \text{else}\ ,\end{cases} \tag{11}$$

with liquidation policy function

$$d_t^l(x,b,k)=\begin{cases}1 & \text{if } \pi_t(x,k)-b+\theta(1-\delta)k<0 \text{ or } v_t(x,b,k)<\theta(1-\delta)k-b\ ,\\ 0 & \text{else.}\end{cases} \tag{12}$$

Regarding entry decisions, it is optimal to invest into a new firm (x,b,k) in period t if the firm value is greater than the installation cost net of debt, which leads to the entry policy function

$$d_t^e(x,b,k)=\begin{cases}1 & \text{if } v_t(x,b,k)\geq k-b\ ,\\ 0 & \text{else.}\end{cases} \tag{13}$$

Finally, the first-order conditions for labor supply and investment in corporate firms are

$$0= u_C(C_t,1-L_t)w_t-u_{1-L}(C_t,1-L_t)\ , \tag{14}$$

$$1= q_t\left[1-\delta+F_K(K_{t+1}^c,L_{t+1}^c)\right]\ . \tag{15}$$

3.2.3 Firm Policies in Stationary Equilibrium

The borrowing and savings policies of small firms can be characterized as in Khan and Thomas (2013). Firms with sufficiently high savings (low debt) are not threatened by illiquidity in any future state. These *unconstrained firms* are able to pay positive dividends while keeping the buffer stock of financial savings sufficiently high. On the other hand, if financial savings are low (or debt is high), a firm expects to be forced to liquidate in some future state with positive probability. These *constrained firms* value retained earnings higher than dividends; therefore they pay no dividends until they build up enough savings and become unconstrained, unless they exit before.

We describe the investment and borrowing policies and the value functions of both firm types in a stationary equilibrium. The same logic applies for the transitional dynamics and is described in Appendix F. In stationary equilibrium, the financial discount factor is $q = \beta$ and the time index is dropped from all variables. Consider first the Bellman equations of unconstrained firms (sub-index u):

$$
\begin{aligned}
v_u(x,b,k) = \max_{k'\geq 0}\ & \pi(x,k) - b + qb'(x,k) - \xi[k' - (1-\delta)k] && (16)\\
& + q\left[\psi(\theta(1-\delta)k' - b'(x,k)) + (1-\psi)\mathbb{E}_{x'|x} v_u^0(x', b'(x,k), k')\right] \ , \\
v_u^0(x,b,k) = \ & \max[\theta(1-\delta)k - b, v_u(x,b,k)] \ , && (17)
\end{aligned}
$$

where the policy function $b'(x,k)$ satisfies the two financial constraints (7) and ensures that the firm stays unconstrained in the next period. We characterize the minimum savings policy with this property below.[10] These two equations demonstrate that value functions of unconstrained firms take the form $v_u^0(x,b,k) = V^0(x,k) - b$ and $v_u(x,b,k) = V(x,k) - b$ (before and after exit). In words, the marginal value of financial assets held by the firm equals one. This is because a marginal payout today has the same value for the household owner as keeping these funds in the firm and receiving the payout later on.

It further follows from the first Bellman equation that unconstrained firms are indifferent regarding the level of savings $-b'(x,k)$, as long as they ensure that the firm remains unconstrained in the future. Dropping financial savings from the Bellman equation obtains

$$
\begin{aligned}
V(x,k) = \max_{k'\geq 0} \pi(x,k) \ & - \xi[k' - (1-\delta)k] && (18)\\
& + q\left[\psi\theta(1-\delta)k' + (1-\psi)\mathbb{E}_{x'|x} \max[\theta(1-\delta)k', V(x',k')]\right] \ .
\end{aligned}
$$

The unique value function solving this equation is strictly increasing in x (because π is strictly increasing in x and the Markov process for x is monotone). This defines a cutoff productivity level

$$
\tilde{x}\,(k) = \min\{x \in \mathbb{X} | V(x,k) \geq \theta(1-\delta)k\} \qquad (19)
$$

such that all unconstrained firms with capital k and productivity below $\tilde{x}\,(k)$ liquidate the firm, while all others stay. The piecewise linear investment function $\xi[k' - (1-\delta)k]$ further

[10] Equation (17) permits voluntary exit of unconstrained firms. By definition, unconstrained firms never become illiquid and hence they are never forced to exit.

implies that optimal investment follows an (s, S) policy:

$$k'(x,k) = \begin{cases} k^{**}(x) \ , & \text{if } (1-\delta)k > k^{**}(x) \ , \\ (1-\delta)k \ , & \text{if } (1-\delta)k \in [k^{*}(x), k^{**}(x)] \ , \\ k^{*}(x) \ , & \text{if } (1-\delta)k < k^{*}(x) \ , \end{cases} \tag{20}$$

where the downward and upward investment levels are the solutions to the problems

$$\begin{aligned} k^{**}(x) &= \arg\max_{k' \geq 0} -(1 - q\psi(1-\delta))\theta k' + q(1-\psi)\mathbb{E}_{x'|x} \max[\theta(1-\delta)k', V(x',k')] \ , \\ k^{*}(x) &= \arg\max_{k' \geq 0} -(1 - q\psi\theta(1-\delta))k' + q(1-\psi)\mathbb{E}_{x'|x} \max[\theta(1-\delta)k', V(x',k')] \ . \end{aligned}$$

It remains to characterize the savings policies ensuring that the firm remains unconstrained. Again following Khan and Thomas (2013), we assume that the firm keeps savings as low as possible (debt as high as possible) to just ensure that both financial constraints are satisfied and that the firm remains unconstrained in the future. This is achieved if

$$b'(x,k) = \min \left[\lambda k'(x,k), \min_{x' \geq \tilde{x}(k'(x,k))} \hat{B}(x', k'(x,k)) \right] \ , \tag{21}$$

where $\hat{B}(x,k)$ is the highest debt level that is consistent with non-negative dividends, investment policy k' and borrowing policy b', satisfying

$$\hat{B}(x,k) = \pi(x,k) + qb'(x,k) - \xi[k'(x,k) - (1-\delta)k] \ . \tag{22}$$

The minimum savings policy rule characterized by (21) and (22) entails that the firm enters a given period with debt $b \leq \hat{B}(x,k)$ whenever $x \geq \tilde{x}(k)$. In this event, the unconstrained firm remains active and pays non-negative dividend

$$d(x,b,k) = \pi(x,k) - b + qb'(x,k) - \xi[k'(x,k) - (1-\delta)k] \ .$$

When $x < \tilde{x}(k)$, however, the (unconstrained) firm is voluntarily liquidated and pays out $\theta(1-\delta)k - b \geq 0$ to the owner household.

If a firm's debt b is larger than $\hat{B}(x,k)$ (or savings are smaller than $-\hat{B}(x,k)$), the firm cannot follow the unconstrained investment and borrowing policies described above since the firm may face a binding borrowing constraint with positive probability in some future state. For such constrained firms it is optimal to pay zero dividends because the value of retained earnings exceeds the value of an immediate dividend payout.

These considerations imply that the firm's value at the beginning of the period is

$$v^0(x,b,k) = \begin{cases} v_u^0(x,b,k) \ , & \text{if } b \le \hat{B}(x,k) \ , \\ v_c^0(x,b,k) \ , & \text{otherwise} \ , \end{cases} \tag{23}$$

where the value of a constrained firm before exit is

$$v_c^0(x,b,k) = \begin{cases} \theta(1-\delta)k - b & \text{if } \pi(x,k) - b + \theta(1-\delta)k < 0 \text{ or } v_c(x,b,k) < \theta(1-\delta)k - b \ , \\ v_c(x,b,k) & \text{else} \ , \end{cases} \tag{24}$$

with $v_c(x,b,k)$ denoting the value of a continuing constrained firm. This firm chooses investment policy k' and borrowing policy b' such that dividends in the current period are zero. The corresponding Bellman equation is

$$v_c(x,b,k) = \max_{k' \ge 0} q\left[\psi(\theta(1-\delta)k' - b') + (1-\psi)\mathbb{E}_{x'|x} v^0(x',b',k')\right] \ , \tag{25}$$

$$\text{s.t. } b' = \frac{1}{q}\left[b - \pi(x,k) + \xi[k' - (1-\delta)k]\right] \le \lambda k' \ .$$

The borrowing constraint gives rise to an upper bound for next period's capital

$$k' \le \bar{k}'(x,b,k) = \begin{cases} \frac{\pi(x,k) + (1-\delta)k - b}{1-q\lambda} & \text{if } \pi(x,k) + q\lambda(1-\delta)k - b \ge 0 \ , \\ \frac{\pi(x,k) + \theta(1-\delta)k - b}{\theta - q\lambda} & \text{else} \ . \end{cases}$$

In the second case, when the firm has low productivity or high debt, the borrowing constraint necessitates some liquidation of capital, i.e. $\bar{k}'(x,b,k) < (1-\delta)k$. Otherwise, when the firm is productive enough or debt is sufficiently low, investment up to $\bar{k}'(x,b,k) - (1-\delta)k \ge 0$ is permitted.

3.3 Pandemic Shock and Rescue Policies

Suppose that the economy is in a stationary equilibrium and that the pandemic shock hits in period $t = 0$. The shock is a one-time unexpected event that fades out over time.[11] We assume that the pandemic not only affects productivity, but also the credit supply, the flow of business entry, demand for goods and services, and the willingness of households to participate in the labor market.

Specifically, the shock has six components. As we explain in the calibration section below,

[11] Since the shock is completely unforeseen, it has no impact on households' and firms' behavior ex-ante.

these six shocks allow us to match the declines of aggregate output, aggregate consumption, employment in both sectors, as well as the responses of the small firm entry and exit rates at the onset of the pandemic. All six shocks decline with persistence factor $\rho < 1$.

First, there is a TFP shock ν^c on the corporate sector such that the corporate production function becomes $(1 + \nu^c \rho^t) F(K_t^c, L_t^c)$. Second, there is a shut-down shock ν^n on a fraction η_i of impacted small firms such that the firm productivity of impacted firms becomes $(1+\nu^n \rho^t)x$. We interpret the impacted firms as those small firms producing social goods and services that are particularly vulnerable to government-induced lockdown measures.

Third, there is a credit shock to all small firms. The credit shock is modeled as a shock $\nu^\lambda < 0$ to the borrowing constraint of all small firms, possibly reflecting impaired access to bank credit at the beginning of the pandemic. Using the Bank Lending Survey administered by the Federal Reserve Board, Bodovski et al. (2021) document that U.S. banks' lending standards for firms tightened at the beginning of the COVID-19 recession, in contrast to other developed economies that experienced an easing of lending standards in the year 2020. According to the Small Business Credit Survey (Federal Reserve Banks, 2021), the share of small businesses that applied for non-PPP financing went down from 2019 to 2020, both for employer and non-employer businesses, while the application approval rate also dropped by over 10 p.p. Bräuning et al. (2022) find that credit supply to small firms with high leverage was tightened during the pandemic.

In our model, we assume that small firms face the borrowing constraint $b' \leq \lambda_t \, k'$ on the transition path, where $\lambda_t = (1 + \nu^\lambda \rho^t)\lambda$. Note that the credit shock does not affect the resale value of capital (which remains at $\theta < 1$), but rather tightens the credit limit below the collateral value of capital, i.e. $\lambda_t < \theta(1 - \delta)$. Therefore, default is still precluded in the pandemic period and thereafter (cf. footnote 7).

Fourth, there is a shock ν^M to the mass of potential entrants to the small firm sector which is $M_t = (1+\nu^M \rho^t)M$. One explanation for this shock is that the pandemic event both created and destroyed business opportunities of potential entrepreneurs. Data from the Business Formation Statistics (BFS) indicate a sharp increase in business applications beginning in June 2020. As Decker and Haltiwanger (2022) point out, the BFS data captures the creation of new firms and can be viewed as an indicator for entrepreneurial activity.

Fifth, there is a demand shock ν^d such that marginal utility of consumption becomes $(1 + \nu^d \rho^t) u_C(C_t, 1 - L_t)$. Such a preference shock captures the observed drop in aggregate consumption at the onset of the pandemic in response to stay-at-home orders and increased health risks of consuming social goods or services. Finally, there is a labor supply shock ν^ℓ such that the marginal utility of leisure becomes $(1+\nu^\ell \rho^t)u_{1-L}(C_t, 1-L_t)$. This reflects employment adjustments possibly due to increased health risks associated with in-person work.

We also take into account the PPP policy in the baseline calibration. Because of the high forgiveness rate of PPP loans (see Section 2), we model the rescue policy as a grant rather than a loan. We assume that an exogenous $\eta = 0.76$ share of firms receive the grant in period $t = 0$ and that the probability of receiving the grant is independent of whether the firm is impacted, thus capturing the lack of targeting of the PPP policy.[12] Moreover, the grant is unconditional and does not need to be repaid. As a robustness check, we show in Appendix C that our main results hold when the grant is conditional on payroll spending.

Let $b_p(x_0, k)$ be the amount of the grant offered to a firm with capital k and whose absent-of-shock productivity is x_0 in the impact period ($t = 0$). In line with the actual PPP policy, we assume that the grant amount is equal to 10 weeks of payroll as follows:

$$b_p(x_0, k) = X_p w^* \ell^* (x_0, k) \ , \tag{26}$$

where $X_p = 2.5/3$ (our model period is a quarter), w^* is the wage rate in the steady state, and $\ell^*(x_0, k)$ is labor demand in the steady state.

To model the grant, we only need to modify the profit of small firms on the transition path. In $t = 0$, the profit function reads

$$\pi_0(x, k, \iota, s) = \max_{\ell \geq 0} (1 + \nu_0^n \iota) x f(k, \ell) + s b_p(x, k) - w_0 \ell - c^f(k) \ , \tag{27}$$

where ι is a dummy variable indicating whether the firm is impacted by the pandemic and s is a dummy variable for receipt of the grant. In $t \geq 1$, we have

$$\pi_t(x, k, \iota) = \max_{\ell \geq 0} (1 + \nu_t^n \iota) x f(k, \ell) - w_t \ell - c^f(k) \ .$$

Note that it is always optimal for firms to choose to take up the maximum grant because grants prevent forced liquidations (for constrained firms) or raise dividends (for unconstrained firms). We provide further details and describe how we solve the model on the transition path in Appendix F.2.

The government finances the rescue grant by imposing a lump-sum tax on households. Since households own small firms who benefit from the rescue grant and since households are

[12] Using data from the Small Business Administration, Borawski and Schweitzer (2021) document that 76% of U.S. small businesses received the PPP loan in 2020. In addition, PPP loans reached small businesses in all industries without substantial variation.

perfectly insured, the fiscal cost of the grant has no direct bearing on household consumption and labor supply decisions apart from the effects on equilibrium prices. In addition, since the representative household is financially unconstrained and there is no distortionary taxation, the timing of lump-sum taxation is irrelevant as Ricardian equivalence applies in our model.[13]

4 Calibration

4.1 Steady-State Calibration

We calibrate the model in steady state to the U.S. economy prior to the COVID-19 pandemic. Each period in our model corresponds to a quarter of a calendar year. We define small firms as businesses with fewer than 500 employees, which is the threshold for eligibility of the PPP loans.

4.1.1 Functional Form Assumptions

In the corporate sector, the production function is

$$F(K^c, L^c) = A(K^c)^{\alpha}(L^c)^{1-\alpha} \ .$$

In the non-corporate sector, each firm has the production technology

$$xf(k, \ell) = xA(k^{\gamma_1}\ell^{1-\gamma_1})^{\gamma_2} \ ,$$

where $\gamma_1 \in (0, 1)$ is the capital share and $\gamma_2 \in (0, 1)$ is a span-of-control parameter. The production function exhibits decreasing returns to scale, possibly due to diminishing returns of managerial supervision as in Lucas (1978). The log productivity $\ln(x)$ follows an AR(1) process with mean $\ln(\bar{x})$, standard deviation ε_x and autocorrelation ρ_x. Operating costs take the affine-linear form $c^f(k) = c_0 + c_1 k$.

Among potential firm entrants, the distribution of initial productivity is independent of the initial distributions of debt and capital. We assume that the initial productivity is drawn from the stationary distribution of firm productivity x. The initial capital distribution follows a truncated Pareto distribution with shape parameter α_k. The initial debt-to-asset ratios are drawn from a three-point distribution.

[13] Of course, Ricardian equivalence could fail for many reasons, such as distortionary taxation. While studying the timing of taxation and the implications of rising public debt during the COVID-19 pandemic is an interesting issue, its analysis is beyond the scope of this paper.

The utility function of the representative household is

$$U(C, 1-L) = \frac{C^{1-\sigma}}{1-\sigma} + \zeta(1-L) \ .$$

4.1.2 Calibration Strategy and Data

Most data targets are obtained from publicly available tables from the Statistics of U.S. Businesses (SUSB) and the Business Dynamics Statistics (BDS) in 2010-2018. We use the confidential data from the Kauffman Firm Survey (KFS) to calculate statistics involving firms' balance sheet information and firm dynamics. The KFS is a single-cohort longitudinal dataset. The first survey of KFS was conducted in 2004 on a representative sample of new firms. A follow-up survey is conducted every year until 2011. We obtain fixed expenses from an analysis by Sageworks based on Census 2007 data. Appendix G provides information on how some of the moments are calculated.

Table 1 shows the values of parameters that are determined outside of the model. Following Jo and Senga (2019), we assume that $\gamma_1 = 0.3182$ and $\gamma_2 = 0.88$. The bottom panel of the table shows the distribution of initial debt-to-asset ratios which approximate the observed distribution in the Kauffman Firm Survey (KFS). Specifically, we compute the ratio of net financial debts to non-financial assets for small firms in their first year of operation from the KFS. The three values of our discrete distribution are the 25th, the 50th, and the 75th percentiles of the actual distribution in the KFS, and the corresponding probabilities are 0.25, 0.5, and 0.25. All other parameters listed in Table 1 are set to standard values.

Table 2 lists parameters calibrated internally and the main data targets that help identifying each parameter. It is well understood that all these parameters jointly take an impact on various model statistics, but we can nonetheless determine which parameter mostly affects which target. To calibrate the parameters, we compute the model counterparts of these targets and choose parameter values to minimize the sum of squared percentage distances between model and data moments.

The three parameters of the firm-specific productivity process are set to match the distribution of firms over four employment size classes and the autocorrelation of employment in continuing firms.[14] The two parameters of the operating cost function and the exogenous exit rate are set to match the exit rate of all small firms, the exit rate of very small firms (less than ten employees), and the fixed-expense-to-revenue ratio. The shape parameter of the

[14] We also tried an alternative calibration in which we target firm-level employment volatility instead of the firm-size distribution. Our main findings remain intact; see Appendix E for details.

Parameter	Description	Value	Source
β	Subjective discount factor	0.989	Annual interest rate of 4%
σ	CRRA utility coefficient	2.000	Standard
α	Capital Share corporate sector	0.300	Standard
δ	Capital depreciation rate	0.015	Annual depreciation rate of 6%
γ_1	Capital Share small firms	0.318	Jo and Senga (2019)
γ_2	Span of control	0.880	Jo and Senga (2019)
A	TFP shifter	0.250	Normalization

Discrete distribution of debt-asset ratios of entrant firms (Source: KFS)

Value	Probability
-0.094	0.25
0.125	0.75
0.868	0.25

Table 1: External Parameters

initial capital distribution targets the size of entrant firms, and the mass of potential entrants controls the overall size of employment in smaller firms in relation to the corporate sector.

Following Khan and Thomas (2013), we calibrate the resale value of capital θ to match the serial correlation of firms' investment from Cooper and Haltiwanger (2006). Cooper and Haltiwanger (2006) use a 17-year panel data of a sample of mature firms to compute statistics of firms' investment rates. To compute the model counterpart of the investment rate moments, we use a simulated sample of unconstrained firms over 17 years, defining the investment rate of firm i in period t as $\frac{k_{i,t+1}-(1-\delta)k_{i,t}}{k_{i,t}}$. As another statistic (not used as calibration target), we calculate the frequency of lumpy investment, defined as an annual investment rate of over 20%.

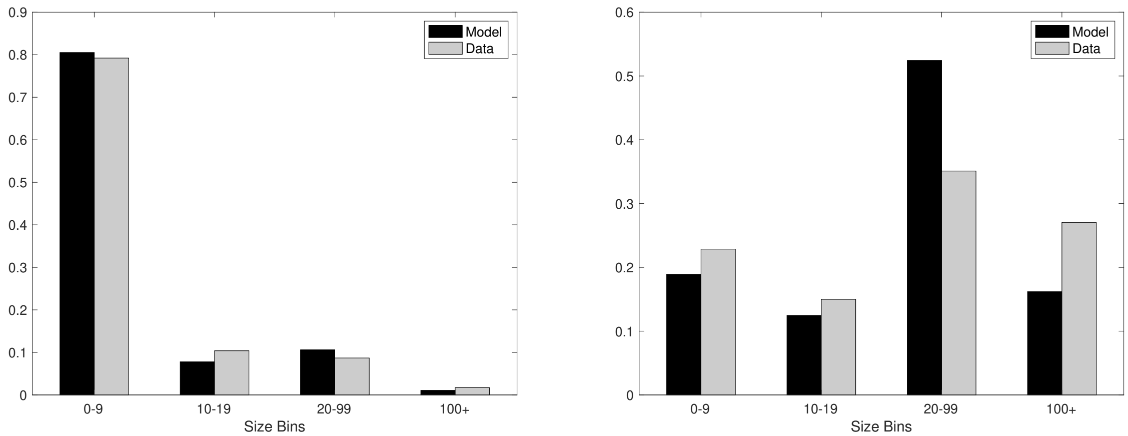
4.1.3 Parameters and Model Fit

Table 2 shows the calibrated parameter values, and Table 3 and Figure 2 show the fit of targeted moments. Our model fits the data well. Regarding the firm-size distribution, we match the fact that the distribution of small businesses is heavily skewed towards firms with fewer employees (see Figure 2.a): Around 80% of small businesses have less than 10 employees and only a small fraction, 1.7%, has more than 100 employees. We also match relatively well the employment shares by firm size bins, which are more evenly distributed across the four bins. In addition, the exit rate of firms with less than 10 employees is substantially higher than the average exit rate, as the data indicates. We further replicate the pattern that entering

Parameter	Description	Value	Main Target	Data Source
Preferences				
ζ	Marginal utility of leisure	23.420	Time spent in market work	Standard
AR(1) of idiosyncratic productivity x				
ε_x	Standard deviation	0.098	Firm size distribution (4 size classes)	BDS
ρ_x	Autocorrelation	0.957	Autocorr. of employment in continuing small firms	KFS
$\bar{x}$	Mean of $\ln(x)$	1.074	Average firm size	BDS
Entry and exit				
M	Mass of potential entrants	0.045	Small firm share of employment	SUSB
α_k	Shape parameter initial capital distrib.	0.446	Average employment of entrants	BDS
Small firm technology				
c_0	Operating cost parameter, intercept	0.165	Exit rate of smallest firms ($<$ 10 workers)	BDS
c_1	Operating cost parameter, slope	0.005	Fixed expense to revenue ratio	Sageworks
ψ	Exogenous exit rate	0.004	Exit rate of small firms	BDS
θ	Resale value of capital	0.909	Serial correlation of investment rate	Cooper and Haltiwanger (2006)

Notes: Data sources include the Business Dynamics Statistics (BDS), Statistics of U.S. Businesses (SUSB), Kauffman Firm Survey (KFS) and data from Sageworks.

Table 2: Internal Parameters and Calibration Targets



(a) Firm Shares

(b) Employment Shares

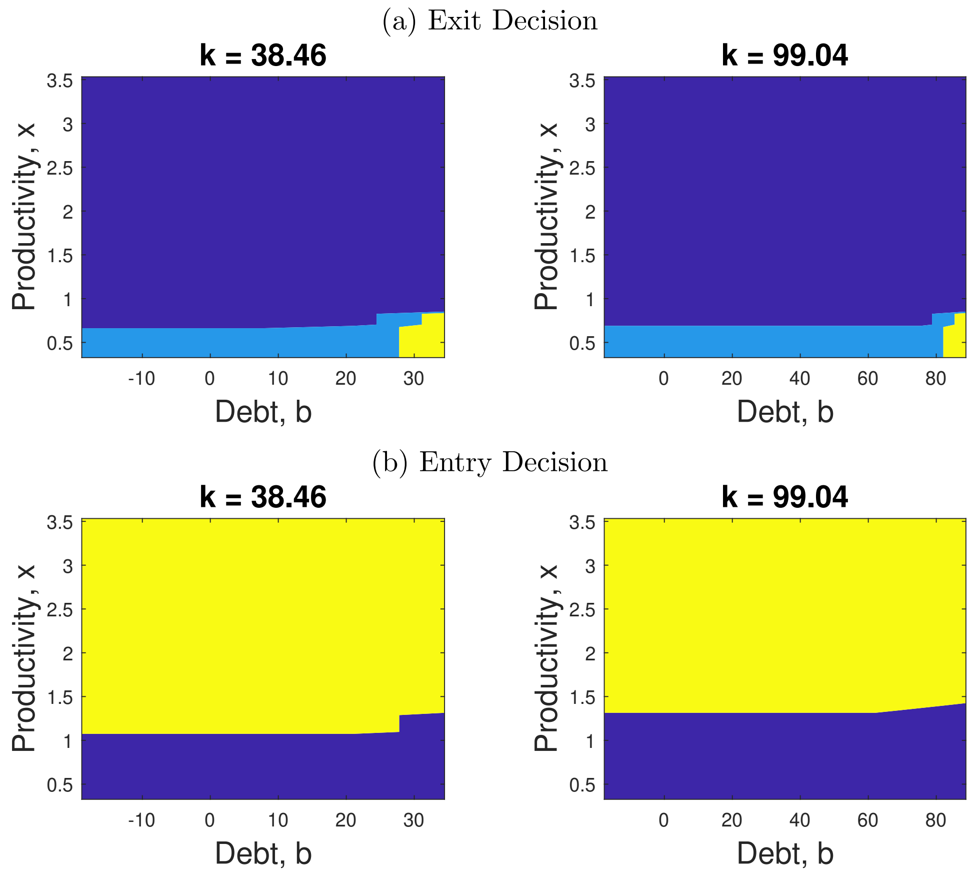
Figure 2: Model Fit. Firm and Employment Distribution by Size Class

firms have fewer employees and the weak positive serial correlation of firms' investment rate. In terms of untargeted moments, our model slightly undershoots the share of indebted firms, and it roughly matches the frequency of lumpy investment.

Moment	Data	Model
Targeted		
Average employment in small firms	9.2516	8.8026
Average employment, age 0	5.2935	5.3071
Small firm share of employment	0.4895	0.4746
Small firm exit rate	0.0198	0.0210
Fixed-expense-to-revenue ratio	0.2448	0.2286
Autocorr. employment	0.9667	0.9527
Time spent in market work	0.3300	0.3175
Exit rate, emp. size 0 to 9	0.0246	0.0251
Serial corr. investment rate	0.0580	0.0591
Firm size distribution	See Fig 2	See Fig 2
Untargeted		
Share of firms with debt	0.3288	0.2190
Freq. positive lumpy investment	0.1860	0.2123
Share forced exit	–	0.5364
Share voluntary exit	–	0.3147
Share exogenous exit	–	0.1489

Notes: The table shows model statistics of the benchmark economy and the empirical counterparts based on data from BDS, SUSB and KFS. Firm and employment shares by employment size are shown in Figure 2.

Table 3: Model Fit



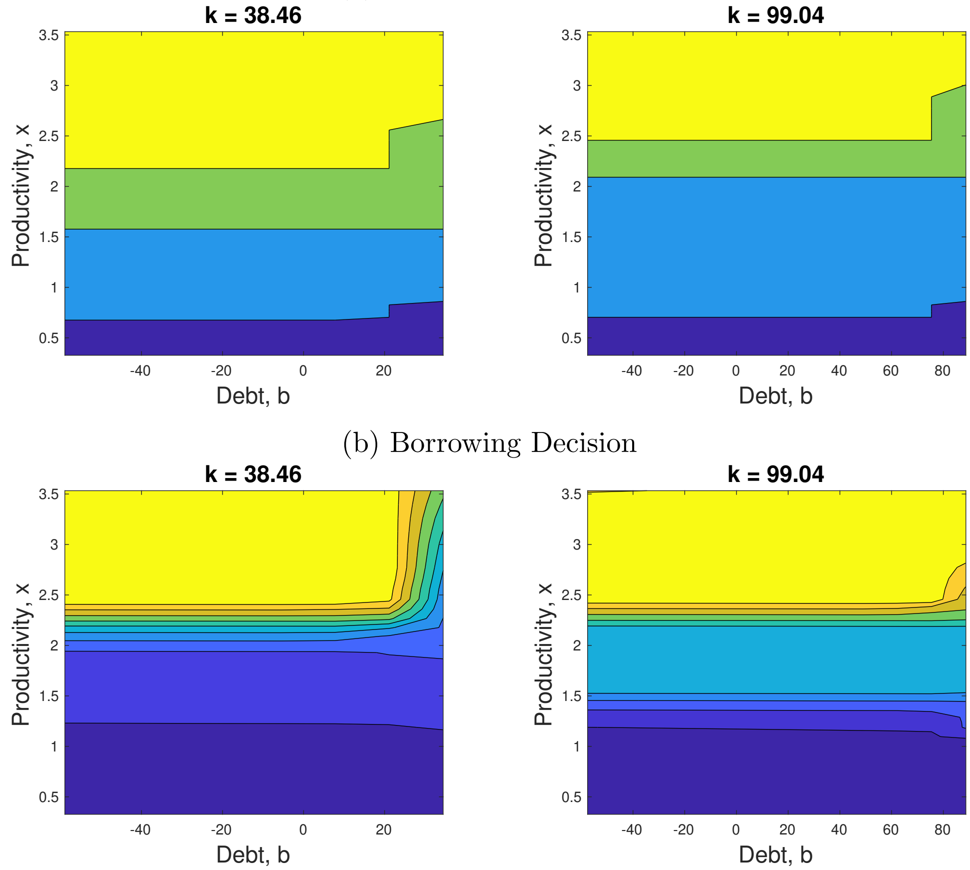
Notes on Panel (a): Dark blue: stay. Light blue: voluntary exit. Yellow: exit.
Notes on Panel (b): Yellow: entry. Dark blue: no entry.

Figure 3: Small Firm Exit and Entry Policy Functions

To illustrate the dependence of exit, entry, investment and borrowing decisions on productivity and debt, we show the optimal policy for small firms with two different levels of capital in Figures 3 and 4. Incumbent firms with high debt and low productivity are forced to liquidate because they cannot raise more debt or equity. The remaining low productivity firms voluntarily liquidate because the value of the firm is below the liquidation value. Overall, over half of all exit events in our model are forced, about a third is voluntary, and about 15 percent are triggered by the exogenous exit shock (see Table 3).[15] Regarding the entry decision, firms with high productivity or low debt choose to enter. Because of the partial irreversibility of capital investment, the productivity threshold for entry is higher compared to the productivity

[15] Only 6.5% of firms in the calibrated model are *unconstrained* in the definition of Section 3.2.3, i.e. they have accumulated enough savings to never face the risk of binding financial constraints in the future (in steady state).

(a) Investment Decision



Notes on Panel (a): Dark blue: exit. Light blue: negative investment (no exit). Green: small positive investment (investment rate less than 100%). Yellow: large positive investment (investment rate greater than 100%).

Notes on Panel (b): Lighter colors indicate more borrowing ($b' - b$).

Figure 4: Small Firm Investment and Borrowing Policy Functions

thresholds for staying.

The investment decision, shown in the top panel of Figure 4, is also influenced by both productivity and indebtedness: High productivity or low-debt firms have a higher investment rate. Similarly, more productive firms choose to borrow more, but only when they are sufficiently far from their borrowing capacity. As the bottom left graph of Figure 4 shows, the collateral constraint especially limits the borrowing policy of firms with low capital but high productivity and debt.

4.2 Calibrating the COVID-19 Shock

The calibration of the pandemic shock is based on the scenario with the PPP grant. As described in Section 3.3, we assume that the shock has six components which decay exponentially with parameter $\rho \in (0, 1)$. Since some small firms had to shut down under lockdown policies at the beginning of the pandemic, we assume that the shock to impacted small firms is $\nu^n = -1$. We calibrate the initial magnitudes of the other pandemic shocks $(\nu^c, \nu^\lambda, \nu^M, \nu^d, \nu^\ell)$ and the share of impacted small firms η_i to match the impacts of the COVID-19 pandemic on total output, consumption, employment, employment in small firms, exit rate and entry rate in the second quarter of 2020 (which corresponds to the first period of the transition). We calibrate the decay parameter ρ to match the change in total output in the third quarter of 2020.

Table 4 shows the values of calibrated shock parameters and Table 5 compares the pandemic impact in the data to the one in the model. Our calibration reveals that 12% of small firms are impacted by the lock-down shock. The TFP shock on the corporate sector is smaller than the average productivity shock in the non-corporate sector. The credit supply to small firms decreases by 15.8% and the mass of potential entrants increases by 24% on impact. The former is required to generate an increase of firm exit in spite of the PPP grant paid out to over three quarters of small firms. A greater number of potential entrants is necessary to match the rather modest decline of the firm entry rate in the second quarter of 2020.

Parameter	Description	Value
η_i	Fraction of impacted small firms	0.1200
ν^λ	Credit shock on small firms	-0.1580
ν^M	Shock to the mass of potential entrants	0.2400
ν^c	Productivity shock the corporate sector	-0.0070
ν^d	Preference shock	-0.1843
ν^l	Labor supply shock	0.2500
ρ	Autocorrelation	0.1100

Table 4: Calibrated Pandemic Shock Parameters

There is a sizable preference shock such that the marginal utility of consumption drops and the marginal disutility of working increases upon impact. Those shocks make sure that the decline of consumption and aggregate employment in the model matches their data counterparts. As seen in Table 5, the baseline grant economy tracks the observed pandemic impact closely, including the change in private investment and output in small firms, which we do not target in the calibration procedure.

Figure 5 shows the paths of aggregate variables in our model. A salient feature of Figure

Description	Data	Model
Targeted		
Output, 2020Q2	-10.857	-10.8976
Output, 2020Q3	-2.246	-2.4738
Consumption, 2020Q2	-9.667	-9.8018
Total employment, 2020Q2	-12.850	-11.7066
Employment small, 2020Q2	-16.021	-16.2132
Exit rate, 2020Q2	37.8400	37.7459
Entry rate, 2020Q2	-12.5000	-12.1711
Untargeted		
Private investment, 2020Q2	-15.398	-16.5231
Small firm output, 2020Q2	-15.650	-15.8452

Notes: The pandemic shocks are calibrated so that the "Grant baseline" model economy matches the data. *Data sources:* GDP, consumption, investment and aggregate employment are taken from FRED (fred.stlouisfed.org). Employment by firm size comes from Cajner et al. (2020). Small firm output is from Bloom et al. (2021) and small firm entry and exit rates come from BLS-BED.

Table 5: Pandemic Impact (% Deviation from Pre-Pandemic Levels)

5a is the speed of the recovery. With a calibrated persistence $\rho = 0.11$, output bounces back to just 2.5% below the initial steady state level in the second period of the pandemic after falling by almost 10.9% on impact. The figure also shows that our model captures the notable feature of the pandemic recession that consumption drops almost as much as output, in line with the factual dynamics shown in Figure 1. Also similar to their data counterparts, the exit rate falls below the pre-pandemic level after its increase in the impact period. The entry rate returns to its steady state level already in the quarter after the shock, while in the data it even exceeds the long-term pre-pandemic level.

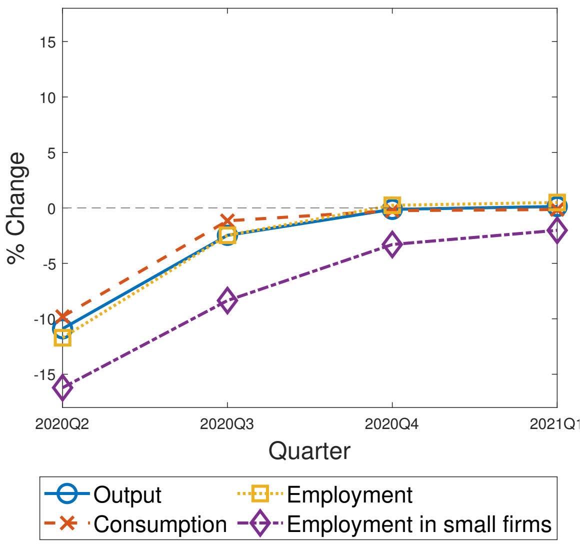
5 Findings

5.1 Impact of the Rescue Grant

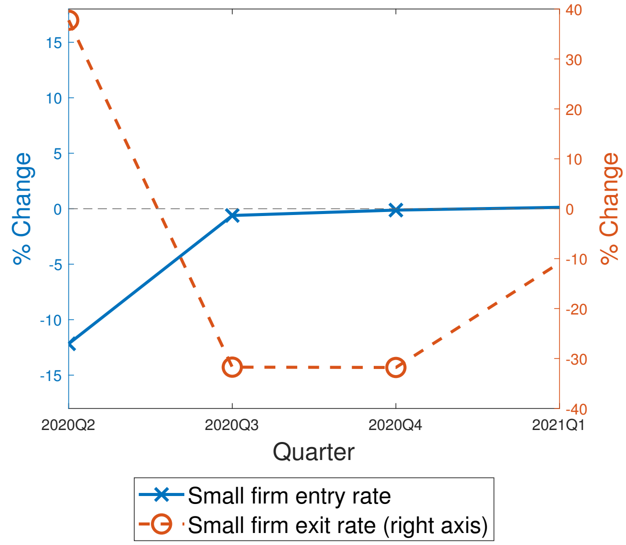
To understand the macroeconomic impact of the rescue grant, we consider a counterfactual laissez-faire economy which is absent of any government intervention. Figure 6 shows the impulse response of the aggregate economy under the baseline policy environment with the rescue grant and under the laissez-faire environment over a 40-quarter horizon.[16]

Based on our calibrated model, in the absence of the rescue grant, the pandemic shock

[16]Impulse responses of other variables not shown in this section are reported in Appendix D.1.

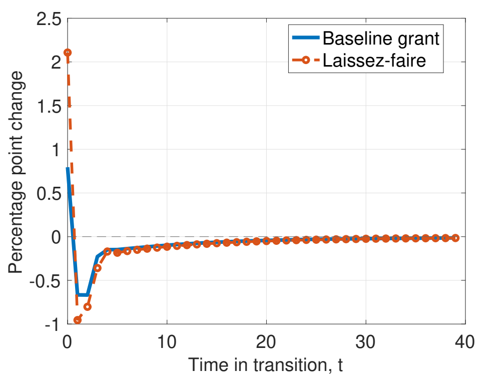


(a) Aggregate Variables

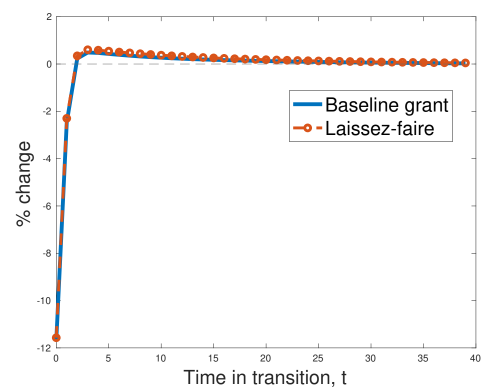


(b) Entry and Exit Rates

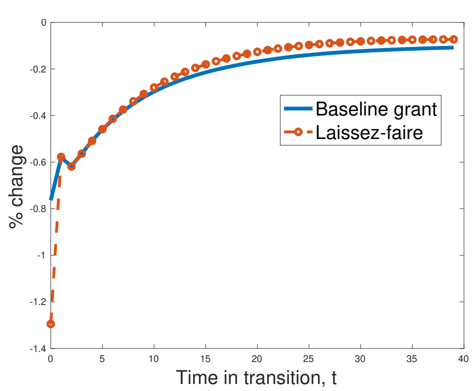
Figure 5: Adjustment to the Pandemic Shock in the Baseline Calibration



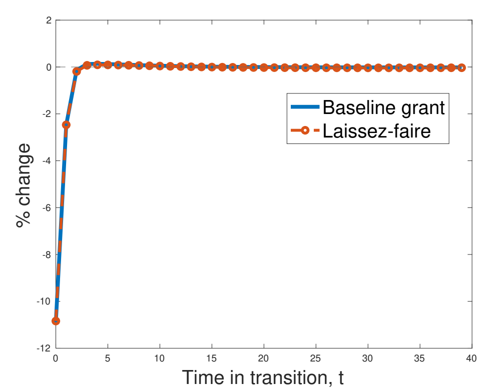
(a) Small Firm Exit Rate



(b) Aggregate Employment



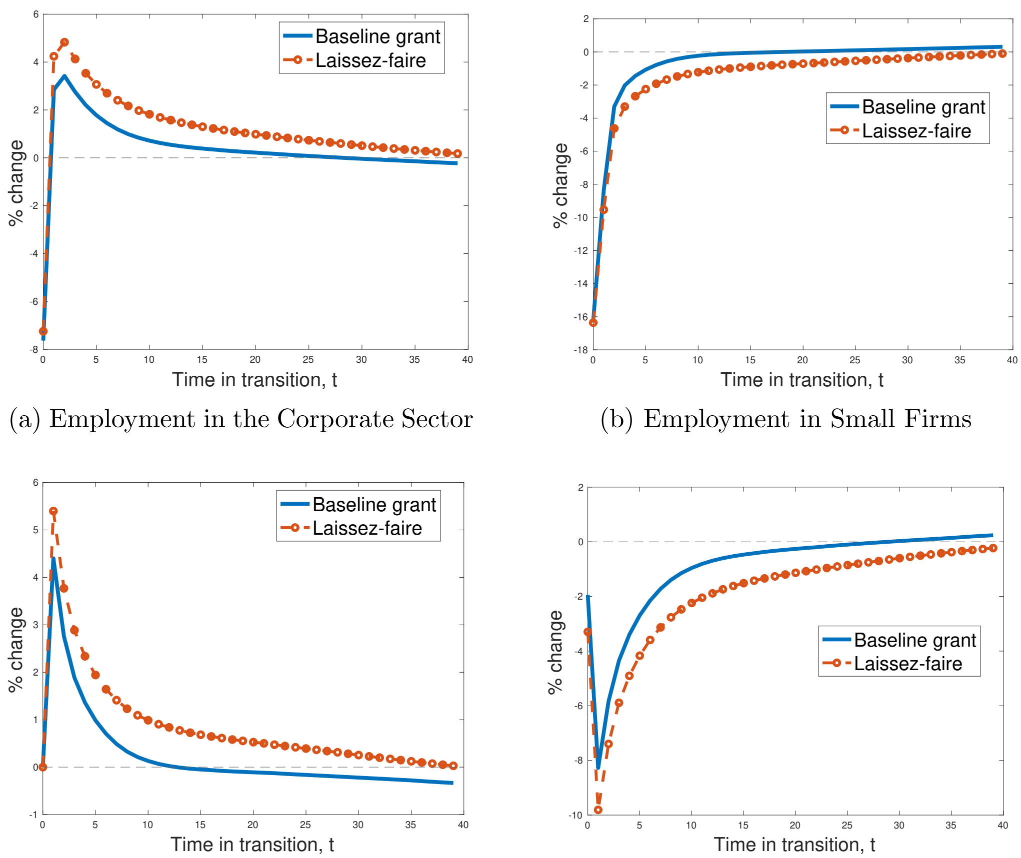
(c) Aggregate Capital



(d) Aggregate Output

Figure 6: Impulse Response to the Pandemic Shock

induces an increase in the firm exit rate by over 2.1 p.p. upon impact (Figure 6a). In contrast, under the baseline grant environment, the exit rate increases by only 0.8 p.p. on impact. Overall, the baseline grant is successful in preventing business exits by 1.3 p.p. The reduction in business exits leads to a smaller drop in the aggregate capital in the small firm sector and in the overall economy on impact (Figures 7d and 6c). Note that the stock of corporate capital and capital in continuing small firms is predetermined and does not respond immediately to the pandemic shock.



(a) Employment in the Corporate Sector

(b) Employment in Small Firms

(c) Capital in the Corporate Sector

(d) Capital in Small Firms

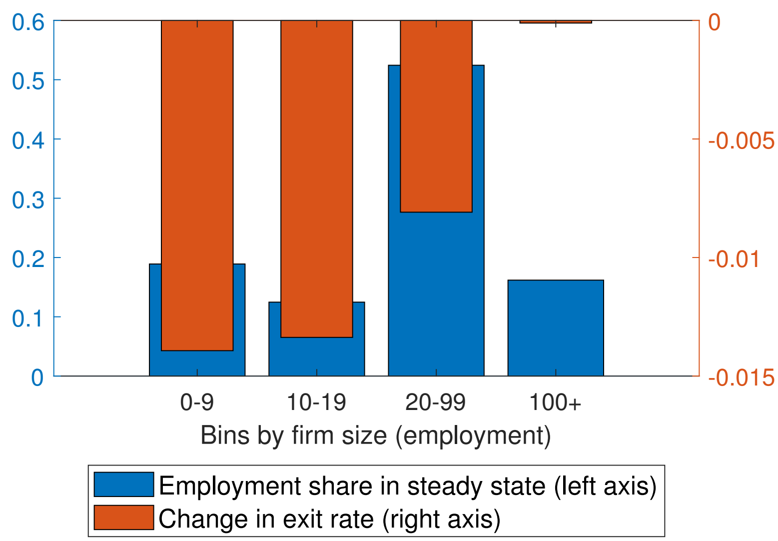
Figure 7: Impulse Response to the Pandemic Shock by Sector

In subsequent quarters, aggregate capital remains persistently below its pre-pandemic level and the recovery is notably slower compared to the laissez-faire scenario. This suggests that the grant has mostly a temporary benefit but has persistent adverse consequences on aggregate capital.

The reduction in business exits does not translate into significant improvements in aggregate output or employment (Figures 6b and 6d) for the following reasons. First, the grant dampens the reallocation of labor and capital towards the corporate sector. Figure 7 shows that, relative to the laissez-faire economy, the rescue grant reduces the increase of capital and employment in the corporate sector and the opposite in the non-corporate sector. Thus, although the grant saves many small firms, it mitigates the reallocation of production factors towards the less impacted corporate sector, which ultimately mutes improvements in aggregate output and employment.[17]

The second reason for the modest responses of aggregate macro variables is that the grant rescues mainly firms employing only few workers. Figure 8 shows the change in the exit rate induced by the grant (relative to laissez-faire) for each bin of firms by employment size and the corresponding share of total small firm employment in steady state. While the rescue grant reduces the exit rate in all firms, the effect is much stronger for firms with fewer than 20 workers and these firms account for around 30% of employment in the small-firm sector.

The third reason is that, while fewer firms exit because of the grant, those that stay choose to disinvest more in the short run (see Section 5.4). As a result, the short-run impact of the grant on employment is negligible.



Notes: Average effect on the exit rate (top bars, inverted scale, right axis) measures the difference in the average quarterly exit rate between the baseline grant environment and the laissez-faire environment in the impact period ($t = 0$). Bins are determined according to the steady state employment distribution of all incumbent firms. Bottom bars show the employment shares in steady state (left axis).

Figure 8: Impact of the Grant on the Exit Rate and Employment Shares by Firm Size Bins

[17]Focusing on rescue policies for small firms, our model abstracts from any financial frictions or adjustment costs faced by firms in the corporate sector. Incorporating such features should attenuate the divergence between the two sectors seen in Figure 7.

	Baseline grant	Targeted grant	Targeted/Baseline
Fiscal cost (fraction of GDP)	4.99%	0.78%	15.6%
Small-firm emp. saved over 10 years	0.76%	0.39%	51.3%
Cost per perc. emp. saved (fraction of GDP)	6.54%	2.02%	30.9%

Notes: The fiscal cost is computed as a fraction of GDP in the steady state. Employment saved is computed as the per-quarter difference in small-firm employment relative to the laissez-faire economy over a 10-year period from the onset of the pandemic. The last column shows the ratio between the first two columns.

Table 6: Cost per Job Saved in Small Firms

In our baseline policy scenario, the grant is available without conditions. This is a reasonably good approximation of the factual PPP policy, whose forgiveness requirements were quite loose from the beginning (see Section 2). As a robustness experiment, we also analyze the case of a conditional grant which imposes a minimum employment requirement in the period the grant is received. Results are presented in Appendix C. In comparison to the baseline grant, the conditional grant mitigates the employment loss in the impact quarter while the initial increase of the firm exit rate is 1 p.p., a bit larger than under the baseline grant. Intuitively, while jobs in impacted firms are saved under the conditional grant, other firms now prefer to exit. Importantly, the differences between the two policies are rather small and the main reallocation effects are also observed in the alternative scenario.

5.2 Targeted Rescue Policy

The PPP aids were dispersed in a highly timely fashion, but there is a lack of targeting such that the take-up of the loans is almost universal (Autor et al., 2022). The lack of targeting has been criticized for incurring an unprecedented fiscal cost and the potential of tying up resources in unproductive firms that could find more productive use elsewhere in the economy. In this subsection, we quantify the economic trade-off of targeting rescue grants.[18] To this end, we consider a counterfactual rescue grant that targets only impacted firms. Other aspects of the policy, including grant amount and timing, remain the same as the baseline rescue grant. Since only a fraction $\eta_i = 12\%$ of small firms is shut down in the first period, the fiscal cost of the grant is now substantially reduced. The first row of Table 6 shows that the baseline grant is over six times more expensive than the targeted grant.

The targeted grant leads to a more limited improvement in small firm employment over

[18] Our analysis does not account for the potential administrative costs associated with granting targeted aids in a timely fashion. In addition, we only focus on employment in small firms here.

the laissez-faire economy: over a 10-year horizon, the average employment improvement under the targeted grant is 0.39%, about half of that under the baseline grant. To compare the cost-effectiveness of the two rescue grants, we compute the cost of saving one percent of job-quarters under the two rescue policies by taking the ratio of the first two rows of Table 6. We find that the cost of saving 1% of jobs (over a 10-year period) under the targeted grant is 30.9% of the cost under the baseline grant.

As an alternative experiment, we vary the size of the targeted grant in order to examine how the number of jobs saved and the cost effectiveness change with the grant volume. While a much larger targeted grant saves more jobs, it becomes considerably less cost effective, even in comparison to the untargeted baseline grant. This is because a larger grant provides windfall gains to those impacted firms which are already saved with a smaller grant while the number of saved firms diminishes at the margin. Conversely, a targeted grant half as large as considered in the benchmark is a bit more cost effective, but it saves fewer jobs (see Appendix B for details).

5.3 Zombie Firms

The rescue policy prevents the liquidation of otherwise viable firms that do not have access to external funding at the time of the shock. On the other hand, it might inadvertently prolong the life of unproductive firms whose capital may be more valuable in other (corporate or non-corporate) firms. Such adverse consequences might be expected, especially because the rescue grant is untargeted. In this section we use our model to quantify the risk of generating such zombie firms.

Specifically, a firm is *saved* by a grant in period $t = 0$ if the grant-receiving firm stays active but would have exited if it were not given the grant. We call a saved firm a *zombie* if the firm's liquidation value is larger than its hypothetical value to the owner household if financial constraints were not present. In other words, if the rescue policy not merely alleviates financial constraints but instead prevents a more efficient reallocation of capital, the firm survives although it is socially inefficient that it does.

Formally, a firm with productivity x, capital k, and impact status $\iota \in \{0, 1\}$ is a zombie firm in $t = 0$ if $V_t(x, k, \iota) < \theta(1 - \delta)k$, where V_t is the value of an unconstrained firm (net of financial assets) in period t. As further outlined in Section 3.2.3 and in Appendix F.2, this value satisfies the recursion

$$V_t(x,\kappa,\iota) = \max_{\ell\geq 0, k'\geq 0} (1+\iota\nu_t^n)xf(k,\ell) - w_t\ell - \xi[k'-(1-\delta)k] - c^f(\kappa)$$
$$+ q_t\left[\psi\theta(1-\delta)k' + (1-\psi)\mathbb{E}_{x'|x}V_{t+1}^0(x',k',\iota)\right] .$$

The zombie condition $V_t(x,k,\iota) < \theta(1-\delta)k$ says that the value of liquidated capital exceeds the expected discounted profit value of the firm, so that the household would be better off if capital were liquidated and invested somewhere else.

Table 7 shows the fraction of saved firms and the fraction of zombies among saved firms on impact under the baseline grant policy and under the counterfactual targeted grant policy. Upon impact, only 1.35% of firms saved by the baseline grant are zombies. The targeted grant leads to an even smaller rate of zombie firms of 0.36%.

	Baseline grant	Targeted grant
Fraction of saved firms	47.1%	68.5%
Fraction of zombie firms among saved firms	1.35%	0.36%

Notes: Fraction of saved firms is the measure of saved firms divided by the measure of grant-receiving firms that would exit without the grant. See the text for definitions of saved firms and zombie firms. All calculations are based on period $t = 0$.

Table 7: Zombie Firms

The reason why the targeted grant policy creates fewer zombie firms is that the targeted policy saves in proportion a larger share of high-productivity firms, namely all those impacted firms which would exit without the grant. While these firms are also saved with the untargeted grant, the latter policy additionally saves many low-productivity firms that are not impacted by the pandemic, a larger fraction of which turn out to be zombie firms. Nonetheless, the number of zombie firms is relatively small even in the baseline grant scenario. This is because the least productive firms choose to exit the economy independent of receiving the grant or not, while the grant helps many viable firms to overcome temporary liquidity shortages emanating from either the tightening of credit conditions or from the shutdown shock.

5.4 Policy Effects on Small Firm Investment

In addition to saving small firms from exiting, the rescue grant also plays a role in firms' investment decisions. The pandemic shock may force small firms to liquidate part of their capital in order to stay in operation. Due to the partial irreversibility of capital, such capital liquidation may be inefficient. The rescue policy can prevent firms from liquidating their capital and improve economic efficiency; it may even lead to an increase in capital investment of firms that are not directly impacted by the pandemic shock. We quantify the impact of the rescue policy on investment by computing the amount of capital liquidation prevented and the amount of additional investment boosted by the policy.

There are in total four types of capital adjustments: upward and downward adjustments by continuing firms, initial capital installation by entrants, and liquidation by exiting firms. We compute the aggregate levels of capital adjustments as follows:

- Upward capital adjustment by continuing firms:

$$\mathcal{A}_{u,t+1} = \int \max\{k_t'(x,b,k) - (1-\delta)k, 0\} d\mu_t(x,b,k) \ .$$

- Downward capital adjustment by continuing firms:

$$\mathcal{A}_{d,t+1} = \int \min\{k_t'(x,b,k) - (1-\delta)k, 0\} d\mu_t(x,b,k) \ .$$

- Initial capital installation by entrants:

$$\mathcal{A}_{ue,t} = M_t \int k d_t^e(x,b,k) d\Phi(x,b,k) \ .$$

- Capital liquidation by exiting firms:

$$\mathcal{A}_{de,t} = -\int k(\psi + (1-\psi)d_t^l(x,b,k)) d\mu_t^0(x,b,k) \ .$$

Note the differing time indices in the above definitions. This is because capital adjustment at entering and exiting firms changes the capital stock in the current period, while capital installations and deinstallations of continuing firms materialize only in the next quarter. As a consequence, in the impact period of the pandemic ($t = 0$) the capital stocks of continuing firms remain at their steady-state values, $\mathcal{A}_{u,t=0} = \mathcal{A}_u$ and $\mathcal{A}_{d,t=0} = \mathcal{A}_d$.

In Table 8, we compute steady-state capital adjustment rates, defined as capital adjustments divided by the steady-state capital in the small firm sector. While the aggregate investment rate equals the depreciation rate of 1.5%, there are sizable capital reallocations across firms, the majority of which occur at the intensive margin.

Up. adj. by continuing firms	0.056
Down. adj. by continuing firms	-0.052
Capital bought by entrants	0.015
Capital sold by exiters	-0.004
Overall small-firm investment rate	0.015

Notes: Capital adjustment rates and the overall investment rate are computed relative to steady state small firm capital.

Table 8: Four Types of Capital Adjustment Rates in the Steady State

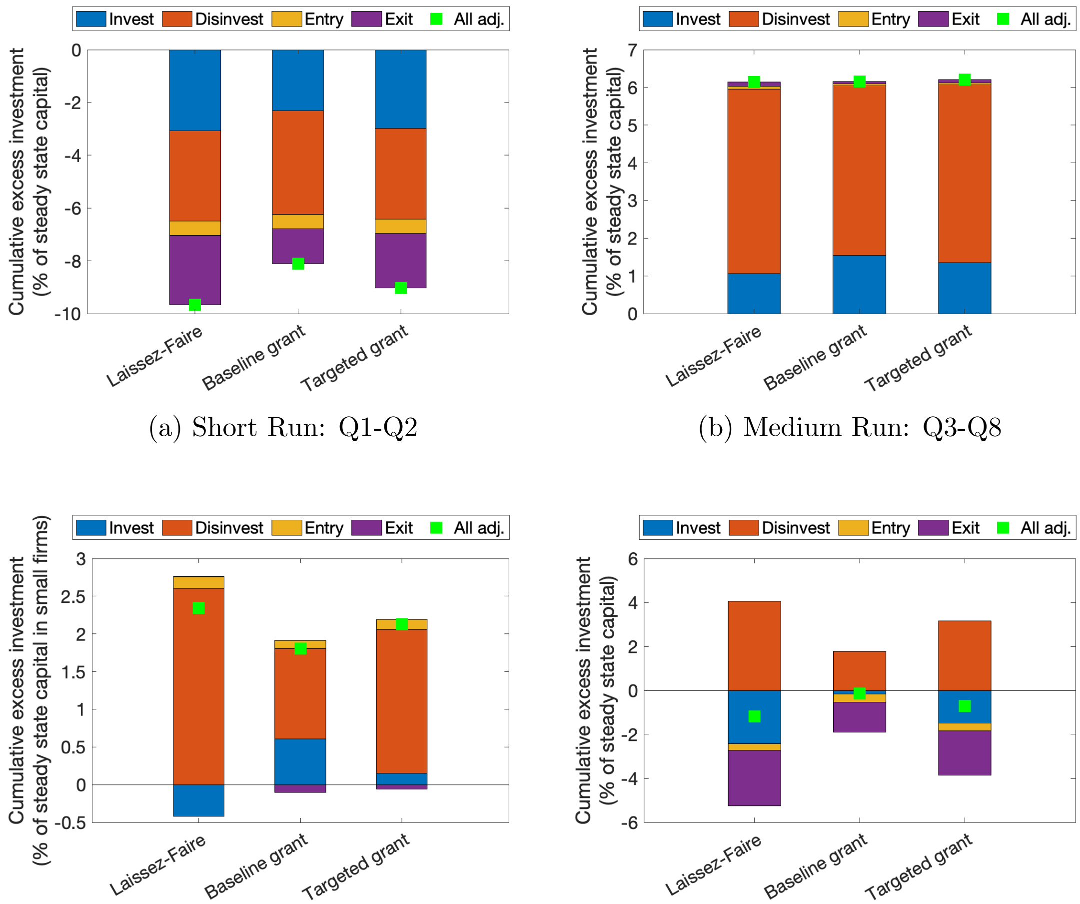
Next, we calculate the cumulative impact of the pandemic on the four types of capital adjustments. We do this separately for four time spans: (a) Short run: change from the pre-pandemic time to the end of Q2; (b) medium run: Q3 to Q8; (c) long run: Q9 to Q40; (d) total impact: Q1 to Q40. We show the results in Figure 9, where the cumulative excess investments are expressed as percentages of the steady-state capital stock in small firms. For example, we plot $\frac{\Delta \mathcal{A}_{u,sr}}{K_{small}}$ in panel (a), where

$$\Delta \mathcal{A}_{u,sr} = \sum_{t=0}^{1} (\mathcal{A}_{u,t} - \mathcal{A}_u)$$

with $\mathcal{A}_u$ being the pre-pandemic (steady state) value.

Relative to the laissez-faire response, the baseline grant reduces capital losses at exiting firms in the short run,[19] and it also helps continuing firms to build capital, throughout all time spans (see blue bars in Figure 9). On the other hand, the baseline grant induces greater disinvestment of capital on impact and smaller capital gains on the disinvestment margin in the recovery (see red bars in Figure 9). This is because low-productivity firms saved by the grant still need to liquidate a portion of their capital on impact, while these firms would exit in the absence of the grant. In the medium and long run, continuing firms disinvest less in the laissez-faire scenario in comparison with the grant scenario because firms that survive the initial impact under laissez-faire happen to be less indebted than those in the grant scenario (see the next section). In line with the findings of Section 5.2, the targeted grant not only saves fewer jobs, but also prevents fewer capital losses in small firms. The differences in comparison to the baseline grant are seen in all capital adjustment margins except firm entry where similarly

[19] As seen in Figure 6a, the exit rate falls persistently below its long run value from the second quarter onward. Yet the impact of firm exit on cumulative investment in the medium- and long-run is negligible. This is because the reduction of firm exit is mostly observed in the smallest firms which have the highest exit rates, hence the effect on the capital stock is small.



(a) Short Run: Q1-Q2

(b) Medium Run: Q3-Q8

(c) Long Run: Q9-Q40

(d) Q1-Q40

Notes: See the text for the definition of cumulative excess investment. Invest. = upward capital adjustment by continuing firms; Disinvest. = downward capital adjustment by continuing firms; Entry = capital bought at entry; Exit = capital sold at exit.

Figure 9: Cumulative Excess Investment by Types of Capital Adjustment

small capital losses are observed in all three scenarios shown in Figure 9.

5.5 Firm Indebtedness

On the transition path, the firm exit rate drops below the steady state level after the initial impact in both the baseline economy and the laissez-faire economy (Figure 6a). To understand why the exit rate drops, it is helpful to look at changes in firm indebtedness on the transition path. Figure 10 shows that firms are on average less indebted in both policy scenarios.[20] One

[20]Note that firm debt in our model only includes non-PPP debt taken up by small businesses.

reason is the credit shock, which tightens the debt limit of firms. While this shock is only short-lived, there are other reasons for the highly persistent reduction in firm indebtedness which are specific to each policy scenario. In the baseline economy, continuing firms remain less indebted because of the grant. In the laissez-faire economy, many highly indebted firms exit on impact while continuing firms make smaller capital investment. The long-lasting reduction in firm indebtedness ultimately leads to fewer downward capital adjustments in the aftermath of the pandemic shock, and this effect is stronger under the laissez-faire scenario (see Figure 9.b-c).

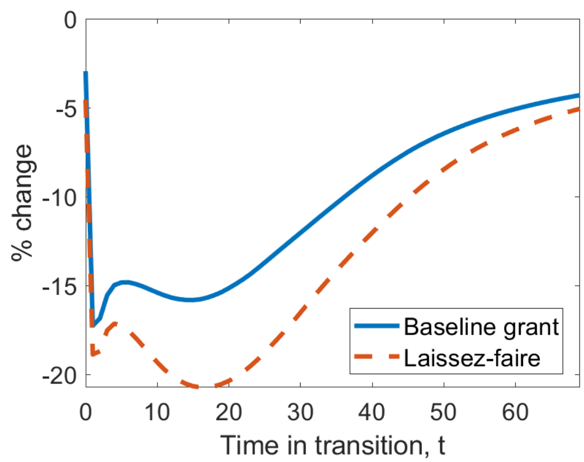


Figure 10: Firm Indebtedness Measured as the Average Debt-Capital Ratio (b/k)

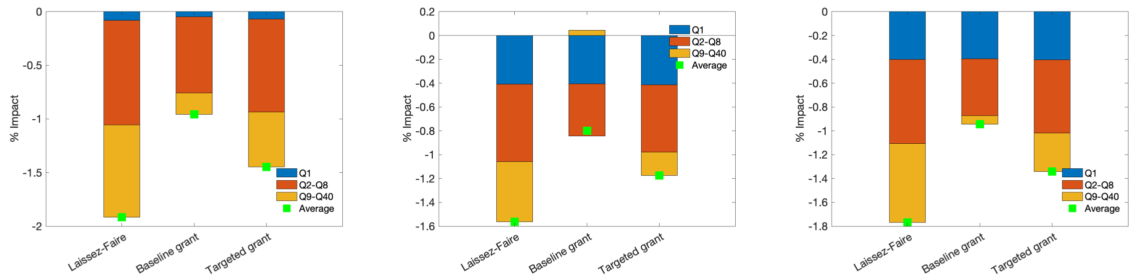
5.6 Short- and Long-Run Effects

The impulse response functions in Section 5.1 highlight the persistent effects of the rescue grant on aggregate capital and on the sectoral allocation of labor and capital. To better understand the impact of the rescue grant over time, we decompose the cumulative effect of the pandemic into short-, medium- and long-run effects and compare the baseline policy environment with the two counterfactual environments: laissez-faire and targeted grant.

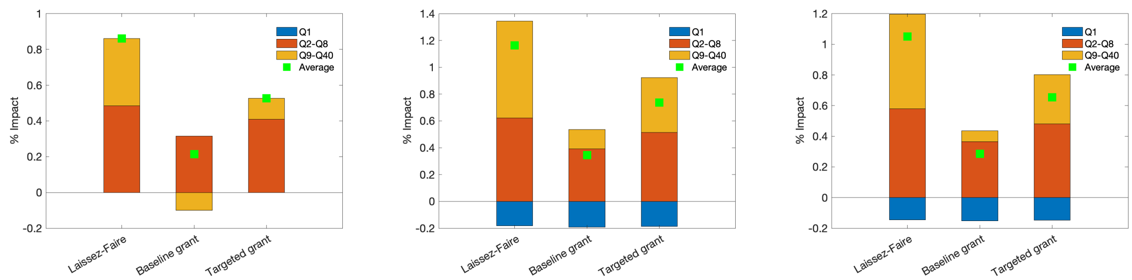
Figure 11 shows the result of the decomposition. For each policy scenario, the net height of the bar (indicated by the green square) represents the overall average quarterly effect of the pandemic over a ten-year period. The three components representing the short-, medium- and long-run contributions display the decomposition of the overall average.[21]

Focusing on output in the two sectors, Figure 11c shows that the short-run output loss in small firms is similar across the three policy environments. This is because the output loss

[21] For details about this decomposition, see Appendix D.2.



(a) Small Firm Capital (b) Small Firm Employment (c) Small Firm Output



(d) Corporate Capital (e) Corporate Employment (f) Corporate Output

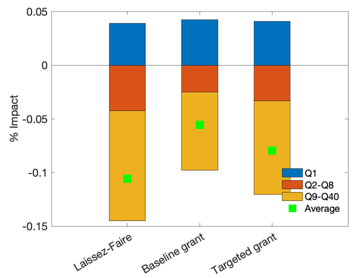
Notes: The net height of the bars represents the average quarterly effect of the pandemic over 10 years. Short-run: the first quarter; medium-run = quarters 2 to 8; long-run = quarters 9 to 40.

Figure 11: Short- and Long-Run Effects

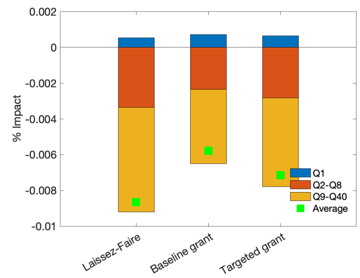
is mainly due to the productivity loss and labor demand adjustments, while firm exits play a less important role (see Section 5.1). Figure 11f shows that the short-run output loss in the corporate sector is also common across the three policy environments. Here the reason is that the short-run response is mostly due to falling TFP and employment, while capital adjustments are negligible in the first two quarters.

In contrast, the medium- and long-run output effects of the pandemic exhibit quite different patterns under the three policy environments. In the laissez-faire scenario, the forced liquidation of small firms and the slow recovery of the non-corporate sector allow the corporate sector to absorb more capital and labor to expand its production (Figure 11.d-e), which explains the long-run positive effect of the pandemic on corporate output. Furthermore, the exit of small firms in the laissez-faire scenario has long-lasting consequences since it takes time for new firms to enter, which slows down the build-up of small firm capital.

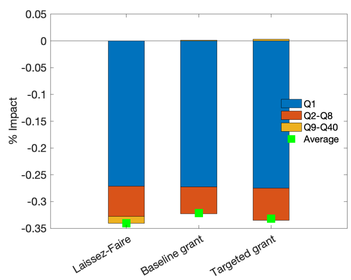
The baseline grant results in markedly smaller medium- and long-run factor reallocations compared to the laissez-faire environment. Specifically, the baseline grant prevents the long-term losses of small firm output and the opposite long-term output gains of the corporate



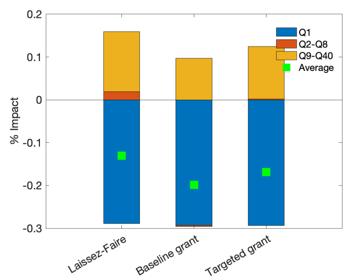
(a) Wage, w_t



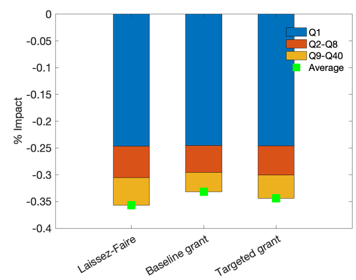
(b) Price of Financial Asset, q_t



(c) Aggregate Output



(d) Aggregate Employment



(e) Aggregate Consumption

Figure 12: Short- and Long-Run Effects (Other Variables)

sector. This is because the grant, due to its improvement on the balance sheets of financially constrained small firms, helps to stimulate investment, thus preserving capital in the non-corporate sector, so that less capital and labor is reallocated to the corporate sector. The targeted grant, which only reaches those small firms hit by the shutdown shock, also dampens the long-run factor reallocations although to a smaller extent than the universal grant.

Turning the attention to aggregate variables, Figure 12 shows that the pandemic shock has a strong short-run impact on output, employment and consumption under all scenarios considered. Most of the drop in consumption happens in the short-run in all three cases, largely due to the sizable but short-lived demand shock. The baseline grant has a small mitigating effect on consumption in the medium and long run. The reason is the following: the baseline grant prevents firm liquidation and therefore the household suffers less from losses of capital liquidation, leaving more resources for consumption and savings. At the same time, the discount factor falls by less (i.e., the increase of the interest rate is dampened), thus inducing the household to save less and consume more in comparison to the laissez-faire (and targeted grant) scenarios.

Finally, persistently lower consumption stimulates labor supply in the long run, which is seen as a reduction of the real wage (the marginal rate of substitution between hours of work and consumption falls) which ultimately raises employment. Due to the smaller decline of consumption under the baseline grant, these adjustments are weaker so that the average aggregate employment decline is larger in the grant scenario.

5.7 Welfare Analysis

Since the baseline grant and the targeted grant bring about smaller declines of average consumption and larger declines of aggregate employment over the ten years following the pandemic shock compared to the laissez-faire scenario, they benefit the representative household who derives utility from consumption and leisure. To properly measure the welfare effect of the rescue grants, we consider a consumption equivalent variation (CEV) that lasts four quarters. That is, we ask how much consumption would have to increase over the first year of the pandemic event in the laissez-faire scenario to make the household indifferent to the respective grant scenario. The one-year CEV defined in this way is given by[22]

$$CEV = \left(\frac{V_0^G - V_0^{LF}}{\tilde{V}^C} + 1 \right)^{\frac{1}{1-\sigma}} - 1 \ ,$$

where V_0^G and V_0^{LF} are, respectively, the values of the representative household under the grant and the laissez-faire economies at $t = 0$ of the pandemic. $\tilde{V}^C$ is the value derived from consumption under the laissez-faire economy in the first four periods of the pandemic, i.e.

$$\tilde{V}^C \equiv \sum_{t=0}^{3} \beta^t \left(1 + \nu^d \rho^t\right) \frac{(C_t^{LF})^{1-\sigma}}{1-\sigma} \ .$$

We find that the baseline grant is equivalent to a 0.79% increase in consumption in the first year of the pandemic, while the targeted grant is equivalent to a 0.44% increase. Both policies help mitigating the medium- and long-term consumption loss which is valued by the household, as are lower average working hours.

6 Conclusions

The COVID-19 pandemic caused a deep but short-lived economic recession, which prompted many governments to enact massive rescue policies targeted at small firms. In order to evaluate

[22]See Appendix H for the derivation.

the macroeconomic impact of such policies, we build a general equilibrium model where firms are subject to financial constraints and can only liquidate their capital at a loss.

Based on our calibrated model, we find that an unconditional grant policy, such as the PPP enacted in the U.S. in the year 2020, is highly effective in preventing the exit of small businesses, but has only modest impact on aggregate outcomes such as consumption, employment and output. This happens for two reasons. First, the grant mostly prevents the smallest firms from exiting. Second, it dampens the reallocation of resources away from smaller, financially constrained firms towards larger and less impacted corporate firms. This result echoes previous findings in the literature (see Crouzet and Mehrotra, 2020) showing that while small firms are typically hit harder during recessions, their volatility has only a small effect on aggregate fluctuations and that policies relaxing financing constraints are unlikely to generate a sizable macroeconomic impact.

Next to the short-run effects, we also find that the grant mitigates the reallocations of capital and labor that would otherwise occur in the aftermath of the pandemic shock. This is because the grant not only saves many small firms, but also has a lasting impact on the financial position of these firms, hence supporting their investment for many years. Furthermore, the grant slightly dampens the declines of aggregate consumption and of the real wage in the years after the pandemic, thus bringing about welfare gains.

When the universal grant is replaced by a targeted grant policy that specifically targets impacted firms, the dampening effect on factor reallocation and the aggregate implications are muted. However, the targeted grant is more cost effective in saving employment in smaller firms and it creates fewer zombie firms in comparison to the untargeted PPP policy.

By focusing on the impact of business grants on firm entry, exit, investment and factor reallocation, our paper does not analyze how government-subsidized loans, possibly in combinations with different conditions for partial loan forgiveness, would affect business dynamics and macroeconomic aggregates. We also abstract from public debt and the timing of taxation by assuming that the government grants are financed by lump-sum taxes. Both are highly relevant issues for future research.

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Appendix

A PPP Loan Terms

Eligibility The following requirements apply to first draw PPP loans:

- The business was operational before February 15, 2020, and is still open and operational.
- There are no more than 500 employees. If a business has multiple locations, there must be no more than 500 employees per location.

Loan Amount The maximum amount of a PPP loan is determined based on the average monthly payroll up to a threshold. The maximum amount of first draw PPP loans is 2.5 times the average monthly payroll up to $10 million. In the Accommodation and Food services sector, it is 3.5 times the average monthly payroll up to $10 million. For non-employers, the maximum amount is based on the business net profit.

Interest Rate PPP loans have an interest rate of 1%.

Maturity PPP loans issued after June 5, 2020, have a maturity of five years. PPP loans issued prior to June 5, 2020, have a maturity of two years. However, by mutual agreement, the maturity can be extended to 5 years.

Covered Period The covered period is the 8- or 24-week after loan disbursement. The 24-week period applies to all borrowers that received forgiveness prior to December 27, 2020, but borrowers that received an SBA loan number before June 5, 2020, have the option to use an eight-week period.

Conditions for Loan Forgiveness Full loan forgiveness is possible if, during the 8- or 24-week covered period following loan disbursement, all of the following are achieved:

- Employee and compensation levels are maintained (temporary layoff is allowed if the workers are rehired),
- all loan proceeds are spent on payroll costs and other eligible expenses, and
- at least 60% of the proceeds are spent on payroll costs.

Partial Forgiveness "If a borrower uses less than 60% of the loan amount for payroll costs during the forgiveness covered period, the borrower will continue to be eligible for partial loan forgiveness, subject to at least 60 percent of the loan forgiveness amount having been used for payroll costs."[23]

B Targeted Grant

In addition to the baseline grant and the targeted grant described in Section 5.2, we consider two additional targeted grant policies with different grant amounts. In "targeted grant small", we set $X_p^{small} = 0.5X_p$ and in "targeted grant large", we set $X_p^{large} = 6.32X_p$, where X_p is the amount of the grant as a fraction of the quarterly payroll in the baseline grant and the targeted grant scenarios in Section 5.2. The grant amount in "targeted grant large" is chosen such that the fiscal cost of the grant is the same as that in the baseline grant.

Table 9 compares the cost-effectiveness of the baseline grant to the three targeted grants. As the grant amount increases, the targeted grant becomes less cost-effective. In particular, a targeted grant as costly as the untargeted baseline grant turns out to be less cost effective, even in comparison to the baseline grant. Intuitively, while a larger targeted grant saves more impacted firms and positively affects investment and labor in those firms, it does so with a diminishing impact at the margin.

	Baseline grant	Targeted grant small	Targeted grant	Targeted grant large
Cost (Frac. GDP)	4.99%	0.39%	0.78%	4.99%
Emp. save (Frac. Emp)	0.76%	0.32%	0.39%	0.64%
Cost per perc. jobs saved	6.54%	1.23%	2.02%	7.75%

Notes: The fiscal cost is computed as a fraction of GDP in the steady state. Employment saved is computed as the per-quarter difference in small-firm employment relative to the laissez-faire economy over a 10 year period from the onset of the pandemic.

Table 9: Cost per Job Saved in Small Firms

Table 10 shows that the large targeted grant is the most effective in rescuing firms that would had exited without any grant, but it also generates the most zombie firms.

Comparing the effect of the two policies on small firm investment, we find that the large targeted grant is less effective in preventing capital liquidation at the exit margin than the

[23]Source: https://home.treasury.gov/news/press-releases/sm1026

equally-costly baseline grant in the first two quarters of the pandemic. It is also not as effective in preventing active firms from under-investing as the baseline grant.

	Baseline grant	Targeted grant	Targeted grant large
Fraction of saved firms	47.1%	68.5%	89.8%
Fraction of zombie firms	1.35%	0.36%	1.80%

Notes: Fraction of saved firms is the measure of saved firms in period $t = 0$ divided by the measure of firms that exit under the laissez-faire environment, conditional on receiving the grant. Fraction of zombie firms is the measure of zombie firms divided by the measure of saved firms. See Section 5.3 for definitions of saved firms and zombie firms.

Table 10: Zombie Firms

C Conditional Grant

As a robustness check, we assume that the grant comes with a minimum employment requirement in period $t = 0$ that resembles the condition for forgiveness in the PPP program (see Appendix A).[24] That is, firms that choose to take up the grant must maintain employment at 60% of the pre-pandemic level. The grant is only given in period $t = 0$, as in our benchmark model, and the minimum employment condition only applies in period $t = 0$. This is consistent with the length of the covered period in the PPP program that ranges from 8 to 24 weeks. Other than the minimum employment requirement, all policy assumptions are the same as in Section 3.3.

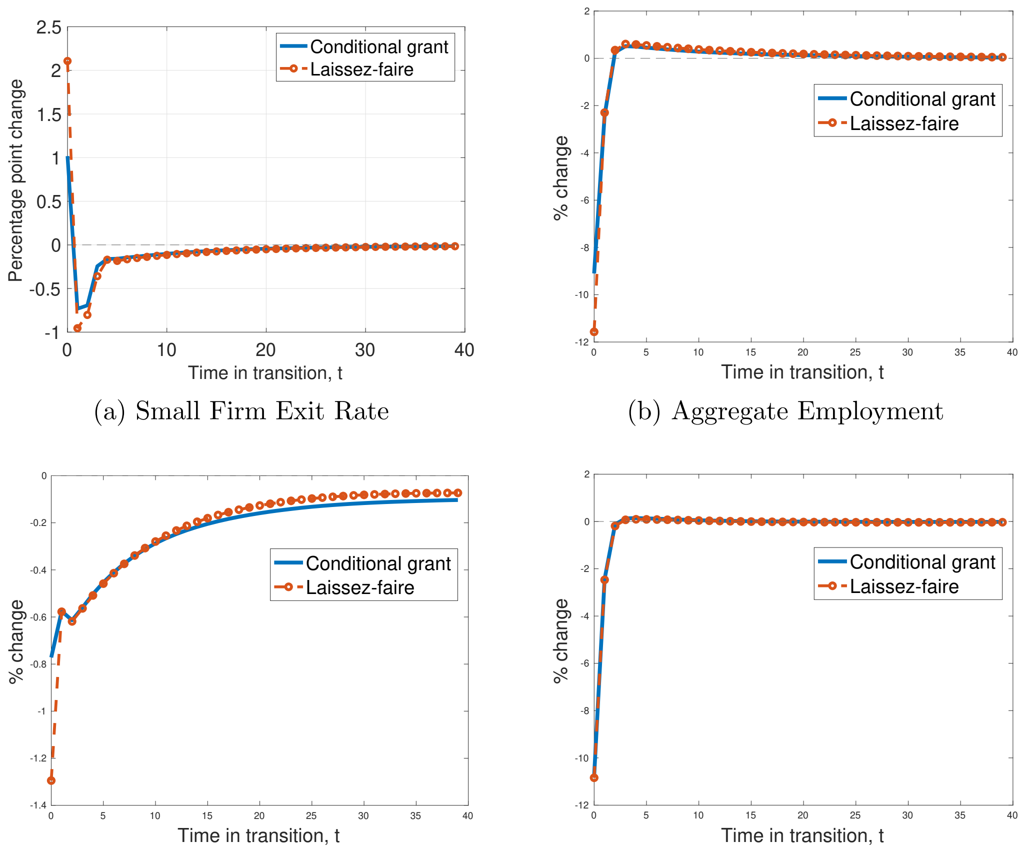
Specifically, an η fraction of small firms have the opportunity to take up the grant. They choose to do so if $v_0(x, b, k, \iota, s = 1) \geq v_0(x, b, k, \iota, s = 0)$, where $s \in \{0, 1\}$ indicates grant take-up and $\iota \in \{0, 1\}$ is the impact status. The profit of a firm that takes up the grant in $t = 0$ changes from equation (27) to

$$\begin{aligned} \pi_0(x, b, k, \iota, s = 1) = & \max_{\ell} (1 + \nu_0^n \iota) x f(k, \ell) + b_p(x, k) - w_0 \ell - c^f(k) \\ & \text{such that} \quad \ell \geq 0.6 \times \ell(x, k) \end{aligned}$$

where $b_p(x, k)$ is the grant amount defined in equation (26), $\ell(x, k)$ is the labor demand in the steady state, and $0.6 \times \ell(x, k)$ is the minimum employment requirement.

We simulate the economy with the conditional grant using the same calibrated parameter values reported in Section 4.2. The results indicate that, although the minimum employment

[24]For simplicity, we do not model the option of not fulfilling the condition, thus abandoning forgiveness.



(a) Small Firm Exit Rate

(b) Aggregate Employment

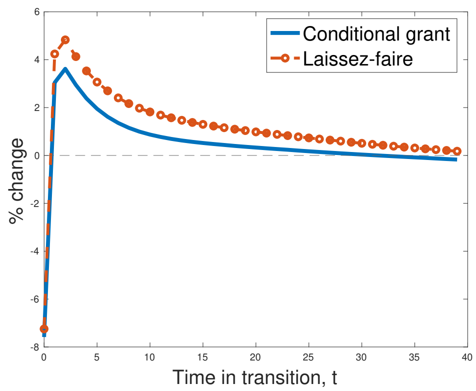
(c) Aggregate Capital

(d) Aggregate Output

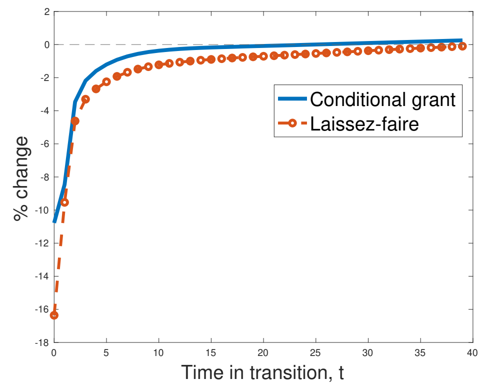
Figure 13: Impulse Response to the Pandemic Shock (Conditional Grant)

condition stimulates short-run employment in small firms, the conditional grant delivers similar medium- and long-run impacts on aggregate output, employment, and capital as the baseline, unconditional grant. Thus, the main conclusions of the paper carry over. The conditional grant modeled here can be viewed as an extreme case because, in practice, the conditionality of the PPP loan forgiveness is not strictly enforced and partial forgiveness is possible.

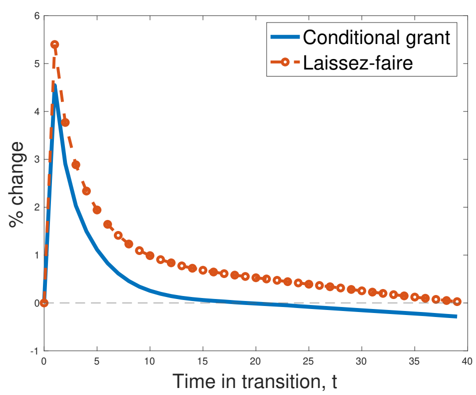
Figures 13 and 14 show the impulse responses to a number of variables of interest, comparing the laissez-faire to the conditional grant. Figures 15 and 16 show the short- and long-run impact of the conditional grant policy. In comparison to the impulse responses under the unconditional grant, the conditional grant dampens the initial employment decline while mitigating the initial decline in firm exits. This is because employment in staying impacted firms remains higher while other impacted firms prefer to exit rather than accepting the conditional grant. From period $t = 1$ onward, however, the observed reallocation effects are similar to those under the unconditional grant scenario.



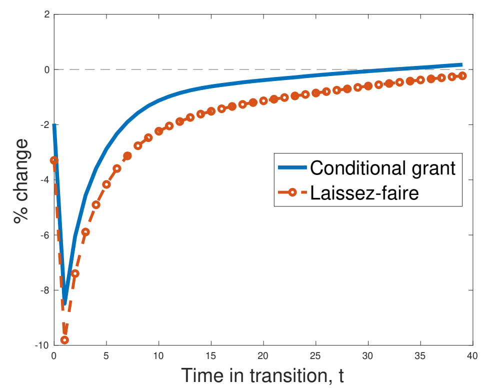
(a) Employment in the Corporate Sector



(b) Employment in Small Firms

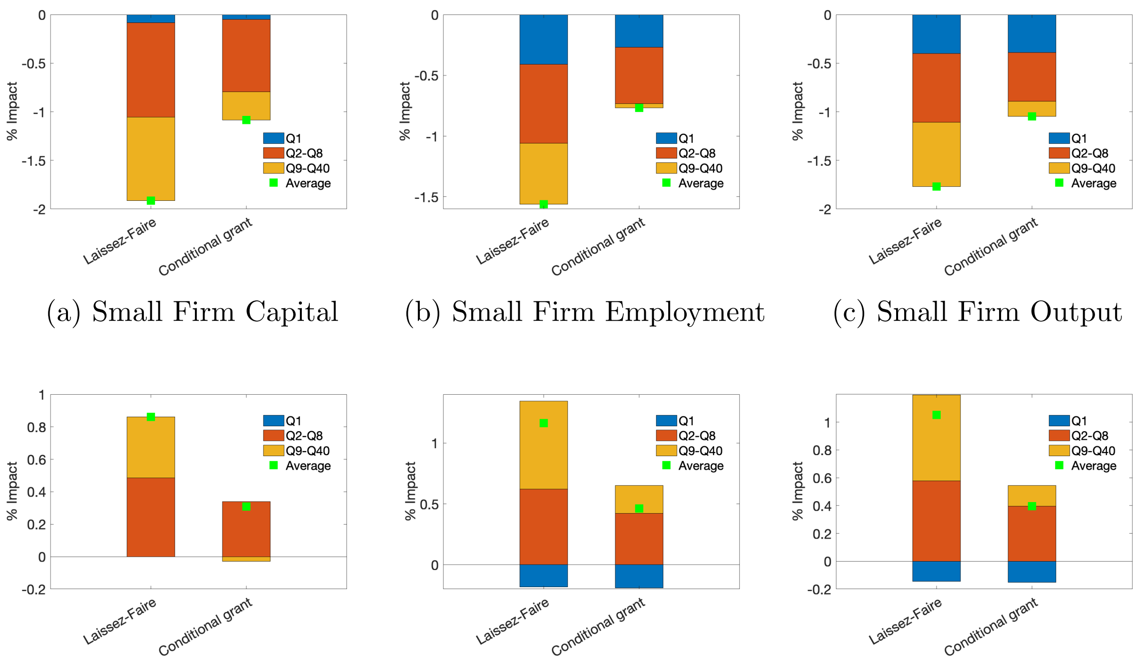


(c) Capital in the Corporate Sector



(d) Capital in Small Firms

Figure 14: Impulse Response to the Pandemic Shock by Sector (Conditional Grant)

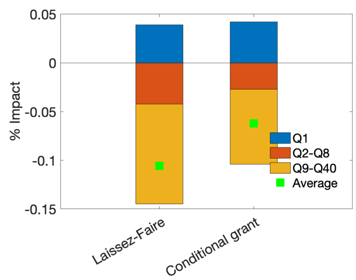


(a) Small Firm Capital (b) Small Firm Employment (c) Small Firm Output

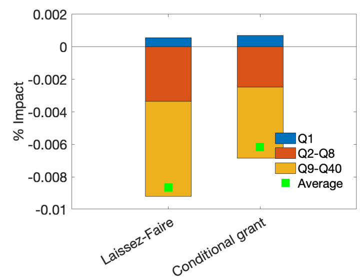
(d) Corporate Capital (e) Corporate Employment (f) Corporate Output

Notes: The net height of the bars represents the average quarterly effect of the pandemic over 10 years. Short-run: first two quarters; medium-run = quarter 3 to 8; long-run = quarter 9 to 40.

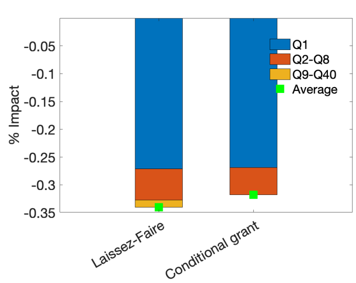
Figure 15: Short- and Long-Run Effects (Conditional Grant)



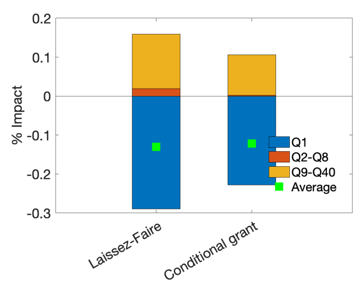
(a) Wage, w_t



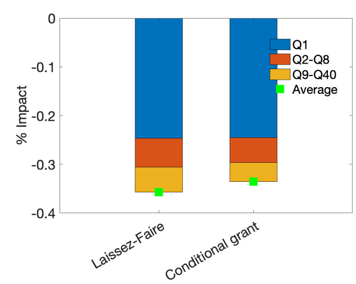
(b) Price of Financial Asset, q_t



(c) Aggregate Output



(d) Aggregate Employment

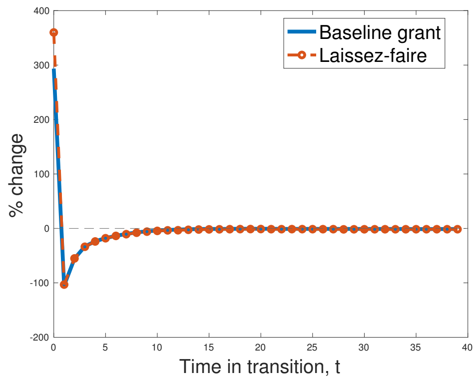


(e) Aggregate Consumption

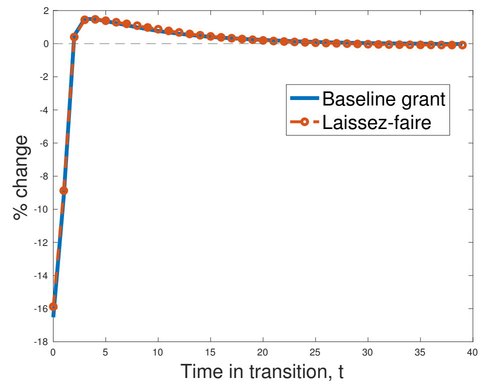
Figure 16: Short- and Long-Run Effects (Conditional Grant, Other Variables)

D Further Details

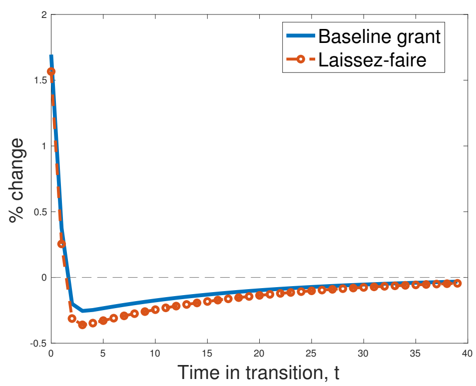
D.1 Additional Figures



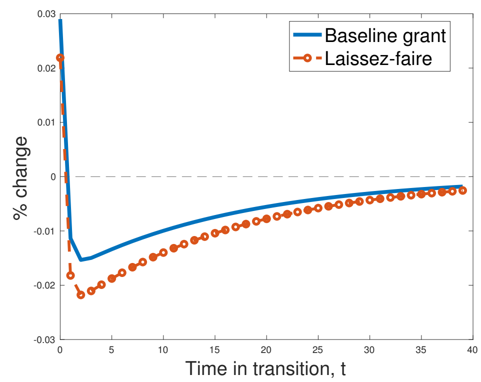
(a) Corporate Sector Investment, I_t^c



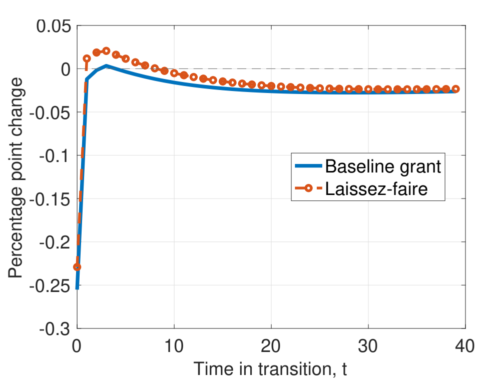
(b) Aggregate Investment, I_t



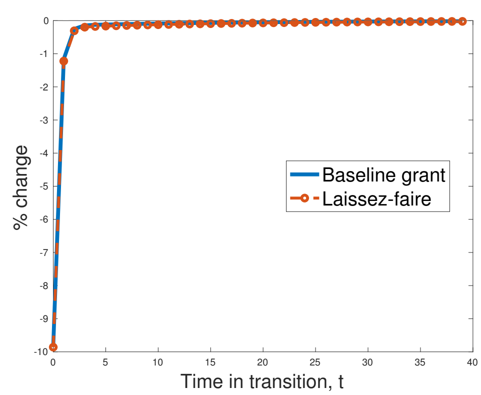
(c) Wage, w_t



(d) Price of Financial Asset, q_t



(e) Entry rate



(f) Aggregate Consumption, C_t

Figure 17: IRFs: Other variables of interest

D.2 Short- and Long-Run Decomposition

We decompose the total cumulative effect of the pandemic shock in a 10-year period into short-run, medium-run, and long-run effects. Specifically, let $irf_t = \frac{y_t - y_{ss}}{y_{ss}}$ be the percentage effect of the pandemic on variable y in period t, and let $\overline{irf}$ be the average effect of the pandemic on a variable over a period of T_{LR} periods, i.e.

$$\overline{irf} = \frac{1}{T_{LR}} \sum_{t=0}^{T_{LR}-1} irf_t \ .$$

We compute $\overline{irf}_{SR}, \overline{irf}_{MR}$ and $\overline{irf}_{LR}$ such that

$$\overline{irf} = \overline{irf}_{SR} + \overline{irf}_{MR} + \overline{irf}_{LR},$$

where

$$\overline{irf}_{SR} = \frac{1}{T_{LR}} \sum_{t=0}^{T_{SR}-1} irf_t \ ,$$

$$\overline{irf}_{MR} = \frac{1}{T_{LR}} \sum_{t=T_{SR}}^{T_{MR}-1} irf_t \ ,$$

$$\overline{irf}_{LR} = \frac{1}{T_{LR}} \sum_{t=T_{MR}}^{T_{LR}-1} irf_t \ .$$

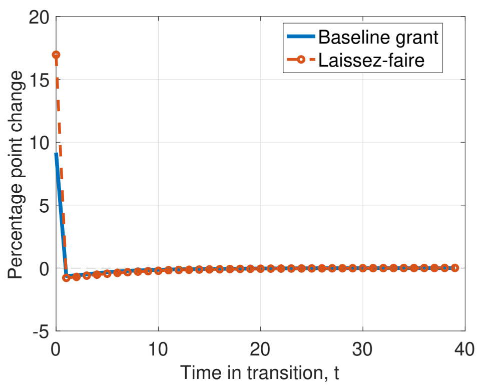
We consider the short-run to be the first quarter post-pandemic, the medium-run to be quarter 2 to year 3, and the long-run to be year 4 to year 10.

E Robustness: Alternative Calibration of the Productivity Process

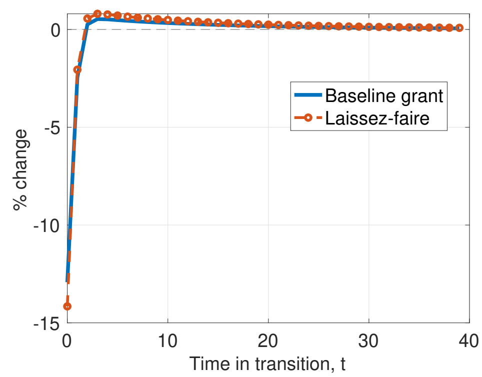
We vary ε_x and $\bar{x}$ to match the job creation and job destruction rates, while holding other parameter values the same. The new ε_x is 0.025, smaller that in the original calibration. We change the value of $\bar{x}$ to 1.40 so that the small-firm share of employment is consistent with the data. Table 11 shows the fit of this alternative calibration. With a smaller ε_x, we are not able to generate enough dispersion in the firm size distribution such that the average firm size is significantly smaller than in the benchmark model. Nevertheless, the main conclusions of the model remain. Figure 18 shows that the baseline grant reduces the sharp increase in firm exit at the onset of the pandemic, leading to a smaller drop in aggregate capital. But the grant has only modest effects on aggregate employment and output at the onset of the pandemic, and small effects after the initial impact of the pandemic. As in the benchmark calibration, the grant also significantly dampens the medium- and long-run reallocation of production factors from the small-firm sector to the corporate sector (Figure 19).

Moment	Data	Model
Job creation rate	0.0250	0.0268
Job destruction rate	0.0215	0.0298
Average employment in small firms	9.2516	4.1666
Small firm share of employment	0.4895	0.4504
Small firm exit rate	0.0198	0.0126
Average employment, age 0	5.2935	3.4815
Fixed expense to revenue ratio	0.2448	0.3994
Autocorr. employment	0.9667	0.9882
Time spent in market work	0.3300	0.3302
Exit rate, emp. size 0 to 9	0.0246	0.0127
serial corr. investment rate	0.0580	0.1588

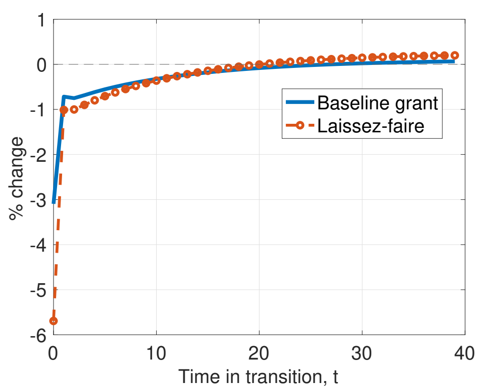
Table 11: Model Fit (Alternative Calibration)



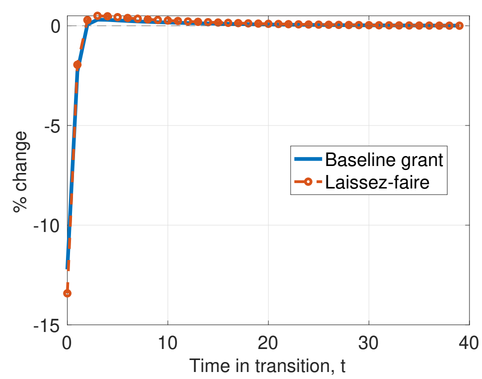
(a) Small Firm Exit Rate



(b) Aggregate Employment

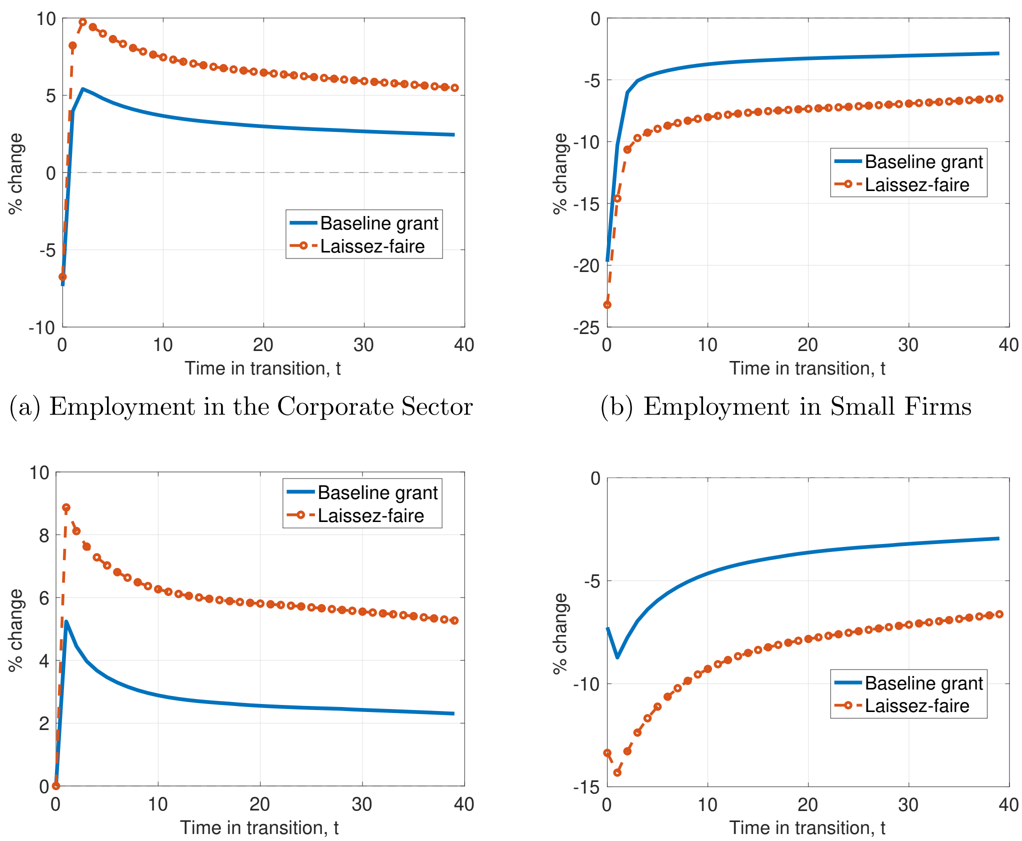


(c) Aggregate Capital



(d) Aggregate Output

Figure 18: Impulse Response to the Pandemic Shock (Alternative Calibration)



(a) Employment in the Corporate Sector

(b) Employment in Small Firms

(c) Capital in the Corporate Sector

(d) Capital in Small Firms

Figure 19: Impulse Response to the Pandemic Shock by Sector (Alternative Calibration)

F Computational Appendix

F.1 Computation of the Stationary Equilibrium

The financial discount factor is $q = \beta$. The capital-labor ratio in the corporate sector is pinned down from $1 = \beta[1 - \delta + F_K]$. Using the functional form $F(K^c, L^c) = A(K^c)^\alpha (L^c)^{1-\alpha}$, we obtain

$$1 = q\left[1 - \delta + A\alpha\left(\frac{K^c}{L^c}\right)^{\alpha-1}\right] ,$$

from which we obtain the capital-labor ratio in corporate firms

$$\kappa^* = \frac{K^c}{L^c} = \left(\frac{A\alpha}{\frac{1}{q} - 1 + \delta}\right)^{\frac{1}{1-\alpha}} .$$

The real wage follows from $w = F_L$, i.e.

$$w = A\left(1-\alpha\right)\left(\kappa^*\right)^\alpha .$$

Consumption follows from the first order condition $u_C\left(C, 1-L\right) w = u_{1-L}\left(C, 1-L\right)$, i.e.

$$C^\sigma = \frac{w}{\zeta} .$$

This permits the calculation of $\pi\left(x, k\right) = \max_{\ell \geq 0} x f(k, \ell) - w\ell - c^f(x)$. The optimal labor demand can be derived as

$$\ell\left(x, k\right) = \left(\frac{w}{xA\left(1-\gamma_1\right)\gamma_2}\right)^{\frac{1}{(1-\gamma_1)\gamma_2 - 1}} k^{-\frac{\gamma_1\gamma_2}{(1-\gamma_1)\gamma_2-1}} .$$

Given $\pi\left(x, k\right)$, the value of an unconstrained firm net of debt $V(x, k)$ can be obtained from (18), which allows us to compute $\tilde{x}(k)$ from (19), the investment policy $k'(x, k)$ from (20), and $\hat{B}(x, k)$ as the fixed point of (21) and (22). We discretize the state space for capital $k_i \in [0, \bar{k}]$ and define a flexible grid for debt which depends on the firm's capital stock such that $b \in [\tilde{B}(k_i), \lambda k_i]$ where $\tilde{B}(k) = \min_{x \geq \tilde{x}(k)} \hat{B}(x, k)$ is the maximum debt level at which the firm is unconstrained. As described in the main text, unconstrained firms keep the debt level at $\tilde{B}(k)$. That is, in a given period they invest $k'(x, k)$ and set $b' = \tilde{B}(k'(x, k))$. Value and policy functions are then defined on a finite grid G which is a subset of $\mathbb{X} \times \mathbb{R} \times \mathbb{R}_+$. Value functions for unconstrained firms are $v_u(x, b, k) = V(x, k) - b$. For constrained firms they follow from (23)–(25). Liquidation and entry policy functions follow from (12) and (13), and the investment and borrowing policy functions $k'(x,b,k)$ and $b'(x,b,k)$ follow from (25).

This permits calculation of the stationary measure of small firms. Combining (5) and (8) yields one equation for the stationary measure μ^0:

$$\mu^0(A) = \int \mathbb{I}_{(x',b'(x,b,k),k'(x,b,k))\in A}\, g\left(x'|x\right)(1-\psi)\left(1-d^l\left(x,b,k\right)\right)\, d\mu^0(x,b,k)$$
$$+ M\int \mathbb{I}_{(x',b'(x,b,k),k'(x,b,k))\in A}\, g\left(x'|x\right) d^e\left(x,b,k\right) d\Phi(x,b,k)\ ,$$

for all Borel subsets $A \subset \mathbb{X}\times\mathbb{R}\times\mathbb{R}_+$. After discretization, μ^0 is a vector with dimension equal to the cardinality of the grid G, and the above equation can be solved by matrix inversion. This also defines the measure of active firms $\mu(x,b,k) = (1-\psi)(1-d^l(x,b,k))\mu^0(x,b,k) + Md^e(x,b,k)\Phi(x,b,k)$ for $(x,b,k) \in G$. Note that both μ^0 and μ are linear in M; hence parameter M linearly scales the size of the non-corporate sector.

The capital stock in the corporate sector K^c can be backed out from the goods-market equilibrium condition

$$C - \int \left\{xf(k,\ell(x,k)) - c^f(k) - \xi[k'(x,b,k)-(1-\delta)k]\right\}\, d\mu(x,b,k)$$
$$+ M\int (k+c^e)d^e(x,b,k)d\Phi(x,b,k) - \theta(1-\delta)\int k\left(\psi+(1-\psi)d^l(x,b,k)\right)\, d\mu^0(x,b,k)$$
$$= K^c[F(1,1/\kappa^*)-\delta]\ ,$$

with capital-labor ratio in the corporate sector κ^* obtained before. Finally, employment in the corporate sector is $L^c = K^c/\kappa^*$ and labor supply is $L = L^c + \int \ell(x,k)\ d\mu(x,b,k)$.

F.2 Computation of the Transition Path

We describe how the transition path after the unexpected pandemic shock at $t = 0$ back to the original steady state can be calculated numerically. Choose large enough T and suppose that the economy has approximately reached the steady state from period $T+1$ onward. That is, all endogenous variables attain their steady state values for $t \geq T+1$.

1. Start with a guess for initial consumption C_0. Then use the following first-order condi-

tions of the household and corporate sector decision problems:

$$
\begin{aligned}
q_t &= \beta \frac{D_{t+1}}{D_t} \left(\frac{C_t}{C_{t+1}} \right)^{\sigma}, && (28) \\
w_t &= \zeta_t C_t^{\sigma} , && (29) \\
w_t &= (1-\alpha) A_t \kappa_t^{\alpha} , && (30) \\
1 &= q_t [1 - \delta + \alpha A_{t+1} \kappa_{t+1}^{\alpha-1}] , && (31)
\end{aligned}
$$

which hold for all $t = 0, \ldots, T$, where $A_t = A\,(1 + \nu_t^c)$, $D_t = \left(1 + \nu_t^d\right)$, and $\zeta_t = \left(1 + \nu_t^{\ell}\right)$ are time-varying multipliers of corporate productivity, utility of consumption, and utility of leisure, respectively. κ_t is the capital-labor ratio in the corporate sector. The guess for C_0 directly yields w_0 and κ_0. Then use all four equations to obtain a dynamic equation for κ_t:

$$
\beta \kappa_t^{\alpha} = \frac{A_{t+1} D_t \zeta_t}{A_t D_{t+1} \zeta_{t+1}} \cdot \frac{\kappa_{t+1}^{\alpha}}{1 - \delta + \alpha A_{t+1} \kappa_{t+1}^{\alpha-1}} \ .
$$

This equation can be inverted numerically to obtain iteratively $\kappa_1, \ldots, \kappa_T$.

Given this solution, the above equations can be used to back out C_t, w_t and q_t for all $t = 0, 1, \ldots, T$. This obtains operating profits of small firms $\pi_t(x, k, \iota)$ for all periods $t = 1, \ldots, T$ where dummy variable ι indicates whether the firm is impacted by the shock. In period 0, write $\pi_0(x, k, \iota, s)$, where index $s = 1$ indicates that these firms receive the grant in $t = 0$. We further write $\ell_t(x, k, \iota)$ for the employment decision of firm (x, k) in impact status ι which depends on t through the real wage.

2. Iterate (10) and (11) (adjusted for the pandemic shock) backwards from steady state value functions yields value functions $v_t(x, b, k, \iota)$ and $v_t^0(x, b, k, \iota)$ for $t = 0, \ldots, T$ and $\iota \in \{0, 1\}$. See below for a description how this can be done without optimization over assets. Regarding period $t = 0$, calculate $v_0(x, b, k, \iota, s)$ and $v_0^0(x, b, k, \iota, s)$ separately for firms with and without grant ($s = 0, 1$) as these have different profits in $t = 0$.

   Calculate entry, exit and investment policies $d_0^e(x, b, k, \iota)$, $d_0^l(x, b, k, \iota, s)$ and $k_0'(x, b, k, \iota, s)$ in period $t = 0$, and $d_t^e(x, b, k, \iota)$, $d_t^l(x, b, k, \iota)$ and $k_t'(x, b, k, \iota)$ for $t = 1, \ldots, T$. Note that the grant is only paid in period $t = 0$ to incumbent firms which is why s enters only the exit policy function d_0^l and the investment policy k_0', but not the entry policy d_0^e.

   Calculate the distribution of active firms in $t = 0$ after entry and exit, $\mu_0(x, b, k, \iota, s)$ from (5) and the given steady-state distribution of incumbent firms at the beginning of the period $\mu^0(x,b,k)$. Here we impose that fraction η_i of incumbent firms and potential entrants are hit by the pandemic and that fraction η of continuing firms in period $t=0$ receive the grant, while entrants in period $t=0$ are not eligible for the grant. Hence,

$$\mu_{t=0}(x,b,k,\iota,s) = \begin{cases} \eta_i \eta \mu^0(x,b,k)(1-\psi)(1-d_0^l(x,b,k,1,1)) & , \ \iota=1, \ s=1, \\ \eta_i \Big[(1-\eta)\mu^0(x,b,k)(1-\psi)(1-d_0^l(x,b,k,1,0)) & \\ +Md_0^e(x,b,k,1)\Phi(x,b,k)\Big] & , \ \iota=1, \ s=0, \\ (1-\eta_i)\eta\mu^0(x,b,k)(1-\psi)(1-d_0^l(x,b,k,0,1)) & , \ \iota=0, \ s=1, \\ (1-\eta_i)\Big[(1-\eta)\mu^0(x,b,k)(1-\psi)(1-d_0^l(x,b,k,0,0)) & \\ +Md_0^e(x,b,k,0)\Phi(x,b,k)\Big] & , \ \iota=0, \ s=0. \end{cases}$$

Continue to iterate over (5) and (8), adjusted for the additional state variable ι, to obtain firm distributions before and after entry/exit in all subsequent periods $t=1,\ldots,T$, $\mu_t^0(x,b,k,\iota)$ and $\mu_t(x,b,k,\iota)$ (note that state variable s only matters in $t=0$). Calculate the output of small firms net of investment expenditures:

$$\begin{aligned} Y_0^n = & \int \left\{(1-\nu_0^n\iota)xf(k,\ell_0(x,k,\iota)) - c^f(k) - \xi[k_0'(x,b,k,\iota,s)-(1-\delta)k]\right\} \ d\mu_0(x,b,k,\iota,s) \\ & + \theta(1-\delta)\int k\Big[\psi + (1-\psi)[\eta_i\big(\eta d_0^l(x,b,k,1,1) + (1-\eta)d_0^l(x,b,k,1,0)\big) \\ & \qquad + (1-\eta_i)\big(\eta d_0^l(x,b,k,0,1) + (1-\eta)d_0^l(x,b,k,0,0)\big)]\Big] \ d\mu^0(x,b,k) \\ & - M\int \big[\eta_i d_0^e(x,b,k,1) + (1-\eta_i)d_0^e(x,b,k,0)\big](k+c^e)d\Phi(x,b,k) \ , \end{aligned}$$

$$\begin{aligned} Y_t^n = & \int \left\{(1-\nu_t^n\iota)xf(k,\ell_t(x,k,\iota)) - c^f(k) - \xi[k_t'(x,b,k,\iota)-(1-\delta)k]\right\} \ d\mu_t(x,b,k,\iota) \\ & + \theta(1-\delta)\int k\left[\psi + (1-\psi)d_t^l(x,b,k,\iota)\right] \ d\mu_t^0(x,b,k,\iota) \\ & - M\int [\eta_i d_t^e(x,b,k,1) + (1-\eta_i)d_t^e(x,b,k,0)](k+c^e)d\Phi(x,b,k) \ , \ t=1,\ldots,T. \end{aligned}$$

Starting at $t=0$, given the initial steady-state capital stock in the corporate sector $K_0^c = K^{c*}$ and the previously obtained path for κ_t, calculate L_t^c, output Y_t^c, investment I_t^c, and K_{t+1}^c iteratively for all $t=0,\ldots,T$:

$$L_t^c = K_t^c/\kappa_t \ \Rightarrow \ Y_t^c = A_tF(K_t^c, L_t^c)$$

$$\Rightarrow I_t^c = Y_t^n + Y_t^c - C_t \ \Rightarrow K_{t+1}^c = (1-\delta)K_t^c + I_t^c \ .$$

If $K_{T+1}^c > K^{c*}$, increase C_0; if $K_{T+1}^c < K^{c*}$, decrease C_0. Repeat until K_{T+1}^c is reasonably close to K^{c*} (saddle path).

Regarding value function iteration (step 2), investment and borrowing/savings policies can be obtained using a similar logic as explained in the main text for the stationary equilibrium. Firms become unconstrained when their financial savings exceed a certain threshold level after which they are able to pay positive dividends. The value function of these unconstrained firms takes the form $v_{ut}(x,b,k,\iota) = V_t(x,k,\iota) - b$ where

$$\begin{aligned} V_t(x,k,\iota) = & \max_{k' \geq 0} \pi_t(x,k,\iota) - \xi[k' - (1-\delta)k] \\ & + q_t \left[\psi\theta(1-\theta)k' + (1-\psi)\mathbb{E}_{x'|x} \max[\theta(1-\delta)k', V_{t+1}(x',k',\iota)] \right] \ . \end{aligned} \tag{32}$$

This equation can be solved backwards starting from $V_{T+1} = V$ (steady state). Define cutoff productivity levels $\tilde{x}_t(k,\iota) = \min\{x \in \mathbb{X} | V_t(x,k,\iota) \geq \theta(1-\delta)k\}$ such that unconstrained firms with productivity below $\tilde{x}_t(k,\iota)$ liquidate the firm in period t, while all others stay. Investment policies of unconstrained firms are

$$k_t'(x,k,\iota) = \begin{cases} k_t^{**}(x,\iota) \ , & \text{if } (1-\delta)k > k_t^{**}(x,\iota) \ , \\ (1-\delta)k \ , & \text{if } (1-\delta)k \in [k_t^*(x,\iota), k_t^{**}(x,\iota)] \ , \\ k_t^*(x,\iota) \ , & \text{if } (1-\delta)k < k_t^*(x,\iota) \ , \end{cases}$$

where the downward and upward investment levels are

$$\begin{aligned} k_t^{**}(x,\iota) &= \arg\max_{k' \geq 0} -(1 - q_t\psi(1-\delta))\theta k' + q_t(1-\psi)\mathbb{E}_{x'|x} \max[\theta(1-\delta)k', V_{t+1}(x',k',\iota)] \ , \\ k_t^*(x,\iota) &= \arg\max_{k' \geq 0} -(1 - q_t\psi\theta(1-\delta))k' + q_t(1-\psi)\mathbb{E}_{x'|x} \max[\theta(1-\delta)k', V_{t+1}(x',k',\iota)] \ . \end{aligned}$$

Borrowing policies of unconstrained firms and maximum debt levels consistent with non-negative dividends follow similarly as in the stationary equilibrium. The borrowing policy in period t is

$$b_t'(x,k,\iota) = \min \left[\lambda k_t'(x,k,\iota), \min_{x' \geq \tilde{x}_{t+1}(k_t'(x,k,\iota),\iota)} \hat{B}_{t+1}(x', k_t'(x,k,\iota), \iota) \right] \ ,$$

with maximum debt levels defined by

$$\hat{B}_t(x,k,\iota) = \pi_t(x,k,\iota) + q_t b'_t(x,k,\iota) - \xi[k'_t(x,k,\iota) - (1-\delta)k] \ .$$

These equations can be solved recursively, starting from the steady-state in period $T+1$, i.e. $\hat{B}_{T+1}(x,k,\iota) = \hat{B}(x,k)$. Note that in period $t=0$, $\hat{B}_0$ also depends on the receipt of the grant which impacts the profit π_0.

In any period t, a firm is unconstrained if $b \leq \hat{B}_t(x,k,\iota)$ in which case it chooses $k'_t(x,k,\iota)$ and $b'_t(x,k,\iota)$ as specified above, conditional on survival which necessitates $x \geq \tilde{x}_t(k,\iota)$. Otherwise, if $x < \tilde{x}_t(k,\iota)$, the firm is liquidated, $d^l_t(x,b,k,\iota) = 1$. Because $\hat{B}_0(x,k,\iota,s)$ also depends on the receipt status of the grant s in the pandemic period $t=0$, the debt cutoff that differentiates constrained from unconstrained firms depends on the receipt of the grant.

Any firm with $b > \hat{B}_t(x,k,\iota)$ is constrained. The value function of a firm at the beginning of period t is then

$$v^0_t(x,b,k,\iota) = \begin{cases} v^0_{ut}(x,b,k,\iota) \ , & \text{if } b \leq \hat{B}_t(x,k,\iota) \ , \\ v^0_{ct}(x,b,k,\iota) \ , & \text{otherwise} \ , \end{cases}$$

where the value of a constrained firm before exit is

$$v^0_{ct}(x,b,k,\iota) = \begin{cases} \theta(1-\delta)k - b & \text{if } \pi_t(x,k,\iota) - b + \theta(1-\delta)k < 0 \text{ or } v_{ct}(x,b,k,\iota) < \theta(1-\delta)k - b \ , \\ v_{ct}(x,b,k,\iota) & \text{else} \ , \end{cases}$$

with $v_{ct}(x,b,k,\iota)$ denoting the value of a continuing constrained firm in period t. This firm's Bellman equation is

$$\begin{aligned} v_{ct}(x,b,k,\iota) = &\max_{k' \geq 0} q_t \left[\psi(\theta(1-\delta)k' - b') + (1-\psi)\mathbb{E}_{x'|x} v^0_{t+1}(x',b',k',\iota) \right] \ , \\ &\text{s.t. } b' = \frac{1}{q_t}\left\{ b - \pi_t(x,k,\iota) + \xi[k' - (1-\delta)k] \right\} \leq \lambda k' \ . \end{aligned}$$

The borrowing constraint implies an upper bound for investment in period t:

$$k' \leq \bar{k}'_t(x,b,k,\iota) = \begin{cases} \frac{\pi_t(x,k,\iota) + (1-\delta)k - b}{1 - q_t\lambda} & \text{if } \pi_t(x,k,\iota) + q_t\lambda(1-\delta)k - b \geq 0 \ , \\ \frac{\pi_t(x,k,\iota) + \theta(1-\delta)k - b}{\theta - q_t\lambda} & \text{else} \ . \end{cases}$$

Again, these value functions can be solved by backward induction starting at the steady-state value function of constrained firms v_c in period $T+1$. Note again that in period $t=0$, the value and policy functions of constrained firms differs between firms with grant and firms without grant via the profit function $\pi_0(x,k,\iota,s)$ in equation (27).

G Moment Computation

Firm exit rate. The firm exit rate is the quarterly rate at which small firms permanently exit. In the model, we calculate the firm exit rate as the measure of firms that exit in a period divided by the measure of firms at the beginning of the period. In the steady state, the exit rate is

$$r_{exit} = \frac{\int_{x,b,k} d^l(x,b,k)\ d\mu^0(x,b,k)}{\int_{x,b,k} d\mu^0(x,b,k)}.$$

The data counterpart is computed based on data from the BDS. Since the BDS data is annual, we first compute the annual exit rate by dividing the number of establishment death in year t, and then convert them into the quarterly rate. The relationship between annual exit rate (r^y_{exit}) and quarterly exit rate (r^q_{exit}) is as follows.

$$r^y_{exit} = 1-\prod_{q=1}^{4}(1-r^q_{exit}).$$

We use the establishment exit rate to proxy for the firm exit rate since we do not have high-quality micro data on firm exits.

Fixed expense to revenue ratio. Fixed expenses are firms' non-payroll overhead expense. We compute the data target based on a table prepared by Sageworks published in the Washington Post, which is constructed based on the 2007 U.S. Economic Census.[25] We weight the expense-to-revenue ratios of the nine types of small businesses by their yearly revenues.

The model counterpart of fixed expenses of small firms include the maintenance cost of capital and the additional fixed cost, and the model counterpart of small firm revenue is their output. We compute fixed expense to revenue ratio in the steady state as

$$\frac{\int_{x,b,k}\left[\delta k+c^f(x,k)\right]\ d\mu(x,b,k)}{\int_{x,b,k} xf(k,\ell(x,k))\ d\mu(x,b,k)}.$$

[25]Link: https://www.washingtonpost.com/wp-srv/special/business/costofrunningabusiness.html, accessed on May 28, 2021.

Debt to Asset Ratio. The data target for the debt-to-asset ratio is computed based on the KFS. It is computed as the ratio between the sum of positive net debt of small firms divided by the sum of their real assets. The net debt of a firm is its total debt minus financial assets; we treat firms with a negative net debt (positive financial asset) as having zero debt. The model counterpart of the debt to asset ratio is

$$\frac{\int_{x,b,k} \max\{b,0\}\ d\mu(x,b,k)}{\int_{x,b,k} k\ d\mu(x,b,k)}.$$

H Consumption-Equivalent Variation (CEV)

We measure the welfare effect of government policies by a one-year consumption-equivalent variation. Specifically, CEV^G is defined as the percentage increase in consumption for the first four quarters in the pandemic in the laissez-faire (LF) environment that would make the representative household indifferent between the laissez-faire economy and the economy with grant G. That is,

$$\sum_{t=0}^{3} \beta^t U_t \left((1+CEV^G)C_t^{LF}, L_t^{LF}\right) + \sum_{t=4}^{\infty} \beta^t U_t \left(C_t^{LF}, L_t^{LF}\right) = V_0^G \ , \tag{33}$$

where $\{C_t^{LF}, L_t^{LF}\}$ are consumption and labor supply in period t in the laissez-faire economy, and V_0^G is the value of the household in period 0 under the economy with grant G, which can be either the baseline grant or the targeted grant.

Since the utility function is additively separable (see Section 4.1.1), we can rewrite the left-hand side of (33) as

$$\sum_{t=0}^{3} \beta^t (1+CEV)^{1-\sigma} \left(1+\nu_t^d\right) \frac{(C_t^{LF})^{1-\sigma}}{1-\sigma} + \sum_{t=4}^{\infty} \beta^t \left(1+\nu_t^d\right) \frac{(C_t^{LF})^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} (1+\nu_t^\ell)\zeta(1-L_t^{LF}) \ , \tag{34}$$

which can be further simplified as

$$[(1+CEV^G)^{1-\sigma} - 1]\tilde{V}^C + V_0^{LF} \ , \tag{35}$$

where V_0^{LF} is the value in period 0 in the laissez-faire economy and

$$\tilde{V}^C \equiv \sum_{t=0}^{3} \beta^t \left(1+\nu_t^d\right) \frac{(C_t^{LF})^{1-\sigma}}{1-\sigma} \ .$$

Substituting the left-hand side of (33) with (35) and rearranging, we obtain

$$CEV^{G} = \left(\frac{V_0^{G} - V_0^{LF}}{\tilde{V}^{C}} + 1 \right)^{\frac{1}{1-\sigma}} - 1.$$