Problem Set 1

Due on Tuesday, April 23, h 13:00. Mail box F230.

Hansen's RBC model (30 points)

This exercise asks you to derive some results of Hansen's real business cycle model that we have seen in Lecture 1.

Preferences. A representative consumer has preferences defined over consumption C_t and hours worked H_t as described by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t - \gamma H_t\right], \tag{1}$$

where $\gamma > 0$ and $\beta \in (0,1)$. The linearity of utility in hours worked can be motivated with *indivisible labor*,[1] along the lines of Hansen (1985) and Rogerson (1988).

Technology. Output Y_t is produced using capital K_t and labor H_t according to the CRS technology described by

$$Y_t = A_t K_t^{\theta} \left(\eta^t H_t\right)^{1-\theta}, \tag{2}$$

where $\eta > 1$ is the gross rate of labor-augmenting technological progress and where $\theta \in (0,1)$. The technology shock A_t follows the AR(1) process:

$$\log A_t = (1-\rho)\log A + \rho \log A_{t-1} + \varepsilon_t, \tag{3}$$

where $A > 0$, $\rho \in (-1,1)$, and $\varepsilon_t \sim N\left(0, \sigma^2\right)$. The representative consumer divides output between consumption and investment:

$$Y_t = C_t + I_t. \tag{4}$$

Investment increases the capital stock according to

$$K_{t+1} = (1-\delta) K_t + I_t, \tag{5}$$

where $\delta \in (0,1)$.

Notation. Using the notation developed in Lecture 1, we can summarize all endogenous variables in the 6×1 vector $\mathbf{z}_t$:

$$\mathbf{z}_t = \left[Y_t, C_t, I_t, H_t, K_t, A_t\right]'.$$

[1] We will cover this topic later in the course.

There is only one exogenous shock, so that

$$\mathbf{v}_t = \varepsilon_t.$$

Finally, the model has 8 parameters, hence:

$$\boldsymbol{\mu} = \left[\beta, \gamma, \theta, \eta, \delta, A, \rho, \sigma^2\right]'.$$

Since the economy is undistorted and there is no heterogeneity, the competitive equilibrium allocation (the object we are ultimately interested in) coincides with the allocation chosen by a benevolent planner who maximizes the discounted utility (1) subject to resource and technological constraints (2)-(5).

1. Derive the first-order conditions for the planner's problem.

2. The equilibrium behavior of the 6 endogenous variables is determined by a system of 6 non-linear stochastic difference equations. List these equations.

3. Due to the presence of exogenous technological progress (i.e. $\eta > 1$), the system derived above is non-stationary. Define the detrended variables as
$$y_t = \frac{Y_t}{\eta^t}, c_t = \frac{C_t}{\eta^t}, i_t = \frac{I_t}{\eta^t}, h_t = H_t, k_t = \frac{K_t}{\eta^t}, a_t = A_t.$$
Note that we don't need to detrend hours worked and the technology shock, since they are already stationary. Write down the transformed system in $\{y_t, c_t, i_t, h_t, k_t, a_t\}_{t=0}^{\infty}$.

4. In the absence of shocks (i.e. if $\varepsilon_t = 0$, for all $t = 0, 1, 2, \ldots$), the economy converges to a steady state in which each of the 6 stationary variables is constant. Denote the steady state values as $\overline{y}, \overline{c}, \overline{i}, \overline{h}, \overline{k}$, and $\overline{a}$. Equation (3) immediately delivers $\overline{a} = A$. Find the remaining steady state values as functions of the model parameters. Do the parameters ρ and σ have any impact on the model's steady state?

5. Define $\widetilde{y}_t = \log \frac{y_t}{\overline{y}}$, $\widetilde{c}_t = \log \frac{c_t}{\overline{c}}$ and similarily for the remaining variables. Log-linearize the equilibrium equations derived in question 3 above. Again, you should obtain a system of 6 *linear* stochastic difference equations.