Dynamic Macroeconomic Models

Lecture 1

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General Information

- Lecturer: Alessandro Di Nola
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- Office hours: Friday, 14:00-15:00
- Class times: Lecture (weekly) on Tuesday, 13:30-15:00 (M1001 M).
- Tutorial (biweekly) on Tuesday, 15:15-16:45 (M1001 M).

Grading

- Final exam 60%.
- Problem sets (mostly numerical/computational questions) 40%.
- Submit all 5 assignments and be ready to present their solutions in the tutorials.
- Printouts (also computer code and figures if required) of the assignments are to be submitted before the respective tutorial.
- Students are allowed to cooperate with each other but must submit an individual copy of each problem set.
- Only 4 best assignments will be considered for the final grade.

Outline of the Course

- Brief overview of DSGE models
  - Real business cycle (RBC)
  - Monopolistic competition and sticky prices and/or sticky wages (NK)
  - Models with financial frictions (with occasionally binding constraints)
- Solving DSGE models with perturbation methods
  - (Log)-Linearization
  - Higher-order perturbation
- Solving DSGE models with global methods (if time permits)
  - Value/policy function iteration, endogenous gridpoint method
  - Finite element methods, orthogonal polynomials

Outline of the Course

- Estimating DSGE models
  - Calibration
  - Generalized method of moments
  - Kalman filter and likelihood methods
- Applications
  - RBC and NK models with financial frictions
  - Fiscal policy
  - International macro

Casting models in canonical form

- Most of DSGE models are described by a set of equations that can be written as:

$$\Gamma\left(E_t \mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{v}_{t+1}; \mu\right) = 0 \tag{1}$$

  where
  - $\mathbf{z}_t$ denotes the collection of all endogenous model variables,
  - $\mathbf{v}_t$ collects all exogenous shocks and
  - μ collects all model parameters.
- How general is this 1st-order specification?
- Higher-order specifications can also be written as 1st-order systems.
- Suppose you want to fit 2nd-order difference equation

$$y_{t+1} = \alpha_1 y_t + \alpha_2 y_{t-1} + \varepsilon_t$$

  in the canonical form (1).

Casting models in canonical form

- Let
$$\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \varepsilon_t. \tag{2}$$
Define $\mathbf{z}_t = [y_t, y_{t-1}]'$ and $\mathbf{v}_t = \varepsilon_{t-1}$. It is clear that (2) fits into (1).
- Denoting η_t the collection of expectational errors (where $\mathbf{z}_{t+1} = E_t \mathbf{z}_{t+1} + \eta_{t+1}$, and η_{t+1} is orthogonal to $E_t \mathbf{z}_{t+1}$) we can also write:
$$\Gamma\left(\mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{v}_{t+1}, \eta_{t+1}; \mu\right) = 0 \tag{3}$$
- Typically (1) or (3) are a system of highly nonlinear stochastic difference equations.

Casting models in canonical form

A familiar example from Macro I

- Stochastic growth model with inelastic labor supply.
- Equilibrium equations for $\{c_t, k_t, A_t \,|\, k_0, A_0\}_{t=0}^{\infty}$ are:

$$
\begin{aligned}
c_t^{-\sigma} &= \beta E_t \left\{ c_{t+1}^{-\sigma} \left[\alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right] \right\}, \\
c_t + k_{t+1} &= (1-\delta) k_t + A_t k_t^{\alpha}, \\
\log(A_{t+1}) &= \rho \log(A_t) + \varepsilon_{t+1}, \;\; \varepsilon_{t+1} \sim N\left(0, \sigma_\varepsilon^2\right).
\end{aligned}
$$

  where c_t is consumption, k_t is capital and A_t is technology/TFP shock.
- Vector with structural parameters $\mu = \left[\sigma, \beta, \alpha, \delta, \rho, \sigma_\varepsilon^2\right]'$.
- Endogenous variables: $\mathbf{z}_t = [k_t, A_t, c_t]'$, exogenous shocks: $\mathbf{v}_t = \varepsilon_t$.

Casting models in canonical form

- It is often convenient to take a (log)-linear approximation of $\Gamma\left(\mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{v}_{t+1}, \eta_{t+1}; \mu\right) = 0$.
- Vector $\mathbf{x}_t$ collects all model variables written in terms of logged deviations from steady-state values, i.e. for each element $i = 1, \ldots, n$

$$x_{it} = \log\left(\frac{z_{it}}{\bar{z}_i}\right)$$

  where the steady state value of each variable $i = 1, \ldots, n$ is denoted as $\bar{z}_i$.
- Therefore

$$\mathbf{A}\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t + \mathbf{C}\mathbf{v}_{t+1} + \mathbf{D}\eta_{t+1}, \tag{4}$$

  where the elements of the matrices $\mathbf{A}$, $\mathbf{B}, \mathbf{C}, \mathbf{D}$ are functions of the structural parameters μ.
- Solutions of (4) are written as:

$$\boxed{\mathbf{x}_{t+1} = F\left(\mu\right)\mathbf{x}_t + G\left(\mu\right)\mathbf{v}_{t+1}}. \tag{5}$$

- Ex: cast the stochastic growth model in the canonical form given by (4)

Casting models in canonical form

- In the stochastic growth model, let $\mathbf{x}_t = \left[\widetilde{k}_t, \widetilde{A}_t, \widetilde{c}_t\right]'$ and $\widetilde{k}_t = \log\left(\frac{k_t}{\bar{k}}\right)$, $\widetilde{A}_t = \log\left(\frac{A_t}{\bar{A}}\right)$ and $\widetilde{c}_t = \log\left(\frac{c_t}{\bar{c}}\right)$.
- The solution is given by the policy functions

$$\begin{aligned}\widetilde{k}_{t+1} &= \pi_{kk}\widetilde{k}_t + \pi_{kA}\widetilde{A}_t \\ \widetilde{c}_t &= \pi_{ck}\widetilde{k}_t + \pi_{cA}\widetilde{A}_t\end{aligned}$$

  and by the stochastic process

$$\widetilde{A}_{t+1} = \rho\widetilde{A}_t + \varepsilon_{t+1}.$$

- The above solution can be recast to fit into (5) as follows:

$$\begin{bmatrix}\widetilde{k}_{t+1}\\ \widetilde{A}_{t+1}\\ \widetilde{c}_{t+1}\end{bmatrix} = \begin{bmatrix}\pi_{kk} & \pi_{kA} & 0\\ 0 & \rho & 0\\ \pi_{ck}\pi_{kk} & \pi_{ck}\pi_{kA} + \pi_{cA}\rho & 0\end{bmatrix}\begin{bmatrix}\widetilde{k}_t\\ \widetilde{A}_t\\ \widetilde{c}_t\end{bmatrix} + \begin{bmatrix}0\\ 1\\ \pi_{cA}\end{bmatrix}\varepsilon_{t+1}.$$

Example I: Real Business Cycle

- Standard RBC model.
- Main reference: Hansen, Gary D., 1985. *"Indivisible Labor and The Business Cycle,"* Journal of Monetary Economics.
- Textbook treatments:
  - McCandless, G., 2008. *"The ABCs of RBCs,"* Harvard University Press.
  - DeJong, D. and Dave, C, 2012. *"Structural Macroeconometrics."* Princeton University Press.

Economic Environment

Preferences

- A representative consumer has preferences defined over consumption C_t and hours worked H_t as described by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t - \gamma H_t\right],$$

where $\gamma > 0$ and $\beta \in (0, 1)$.

Technologies

- Output is produced using capital K_t and labor H_t according to the CRS technology described by

$$Y_t = A_t K_t^{\theta} \left(\eta^t H_t\right)^{1-\theta},$$

where $\eta > 1$ is the gross rate of labor-augmenting technological progress and where $\theta \in (0, 1)$.

Economic Environment

- Technology shock A_t follows the AR(1) process:

$$\log A_t = (1-\rho)\log A + \rho \log A_{t-1} + \varepsilon_t,$$

where $A > 0$, $\rho \in (-1, 1)$, and ε_t is an *i.i.d.* innovation such that

$$\varepsilon_t \sim N\left(0, \sigma^2\right).$$

- Representative consumer divides output between consumption and investment:

$$Y_t = C_t + I_t$$

- Investment increases the capital stock according to

$$K_{t+1} = (1-\delta) K_t + I_t,$$

where $\delta \in (0, 1)$.

Variables and parameters

- The model has implications for 6 *endogenous* variables:

$$\mathbf{z}_t = \left[Y_t, C_t, I_t, H_t, K_t, A_t\right]'.$$

- Only 1 *exogenous* shock:

$$\mathbf{v}_t = \varepsilon_t.$$

- The model has 8 *structural* parameters:

$$\mu = \left[\beta, \gamma, \theta, \eta, \delta, A, \rho, \sigma^2\right]'.$$

Equilibrium Allocations

- **Welfare theorems**: the equilibrium allocations solve the social planner's problem.
- The equilibrium behaviour of the 6 endogenous variables $[Y_t, C_t, I_t, H_t, K_t, A_t]'$ is fully determined by 6 equilibrium equations (plus an appropriate TVC).
- What about prices?
- **Exercise**: Derive the equilibrium equations by solving the social planner's problem. In addition, write down 2 extra equations for real interest rate R_t and for wage rate w_t.
- Is the model *stationary*?

Example II: New-Keynesian model

- RBC plus “frictions”:
  - Monopolistic competition.
  - Sticky prices.
- Main reference: Ireland, P., 2004. *"Technology Shocks in the New Keynesian Model,"* Review of Economics and Statistics.
- Textbook treatments:
  - DeJong, D. and Dave, C, 2012. *“Structural Macroeconometrics.”* Princeton University Press.

Example II: New-Keynesian model

- Continuum of identical households.
- Two distinct production sectors:
  - Intermediate-goods sector.
  - Final-goods sector.
- Intermediate-goods sector is imperfectly competitive
  - It consists of many firms that produce differentiated products sold to the final-goods sector.
  - Intermediate-goods producers can set prices but face quadratic adjustment costs.
- There is a central bank $\rightarrow$ Taylor rule.

Households

- The representative household solves:

$$\max_{c_t, b_t, m_t, n_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ a_t \log c_t + \log \frac{m_t}{p_t} - \frac{n_t^{\xi}}{\xi} \right\}$$

s.t.

$$p_t c_t + \frac{b_t}{r_t} + m_t = m_{t-1} + b_{t-1} + \tau_t + w_t n_t + d_t$$

where $\beta \in (0, 1)$ and $\xi \geq 1$.

- r_t is the gross *nominal* interest rate, m_t and b_t are holdings of money and bonds.
- Household receives a transfer τ_t from the monetary authority.
- Household owns intermediate-goods firms receiving dividend payments $d_t = \int_0^1 d_{it} di$.
- Exogenous demand shock a_t (follows AR(1) process, to be defined later).
- **Exercise**: Derive the optimality conditions for the household (3 equations).

Firms

- **Final-good firm**:
  - operates in a competitive environment; takes prices $(p_{it})_{i\in[0,1]}$ as given.
  - buys y_{it} units of each intermediate good $i \in [0,1]$ at price p_{it}, to produce y_t units of the final good.
- Profit maximization:

$$\max_{y_{it}} p_t y_t - \int_0^1 p_{it} y_{it} di$$

s.t.

$$y_t = \left[\int_0^1 y_{it}^{\frac{\theta_t - 1}{\theta_t}} di\right]^{\frac{\theta_t}{\theta_t - 1}},$$

where θ_t follows an AR(1) process with unconditional mean $\overline{\theta} > 1$.
- First-order condition wrt $y_{it} \implies$ Demand function for variety $i \in [0,1]$:

$$y_{it} = y_t \left(\frac{p_{it}}{p_t}\right)^{-\theta_t}.$$

Algebra

Firms

- Note that θ_t is the markup of price above marginal cost; it is a "cost-push" shock.
- Zero profit condition implies an expression for price aggregator:

$$p_t = \left[\int_0^1 p_{it}^{1-\theta_t} di \right]^{\frac{1}{1-\theta_t}}.$$

Firms

- **Intermediate-goods producers**, indexed by $i \in [0, 1]$, solve:

$$\max_{p_{it}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{a_t}{c_t} \right) \left(\frac{d_{it}}{p_t} \right)$$

subject to:

$$y_{it} = z_t n_{it}$$

$$y_{it} = y_t \left(\frac{p_{it}}{p_t} \right)^{-\theta_t}$$

$$\chi(p_{it}, p_{it-1}) = \frac{\phi}{2} \left[\frac{p_{it}}{\overline{\pi} p_{it-1}} - 1 \right]^2 y_t, \; \phi > 0,$$

where real dividends are defined as:

$$\frac{d_{it}}{p_t} = \left\{ \frac{p_{it} y_{it} - w_t n_{it}}{p_t} - \chi(p_{it}, p_{it-1}) \right\}.$$

Firms

- Maximization problem of intermediate-goods producers can be simplified as:

$$\max_{p_{it}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{a_t}{c_t}\right)\left(\frac{d_{it}}{p_t}\right)$$

subject to

$$\frac{d_{it}}{p_t} = \left(\frac{p_{it}}{p_t}\right)^{1-\theta_t} y_t - \left(\frac{p_{it}}{p_t}\right)^{-\theta_t}\left(\frac{w_t}{p_t}\right)\left(\frac{y_t}{z_t}\right) - \frac{\phi}{2}\left[\frac{p_{it}}{\overline{\pi} p_{it-1}} - 1\right]^2 y_t.$$

- First-order condition wrt p_{it}:

$$\begin{aligned} 0 \quad = \quad & (1-\theta_t)\left(\frac{a_t}{c_t}\right)\left(\frac{p_{it}}{p_t}\right)^{-\theta_t}\left(\frac{y_t}{p_t}\right) \\ & +\theta_t\left(\frac{a_t}{c_t}\right)\left(\frac{p_{it}}{p_t}\right)^{-\theta_t-1}\left(\frac{w_t}{p_t}\right)\left(\frac{y_t}{z_t}\right)\left(\frac{1}{p_t}\right) \\ & -\phi\left(\frac{a_t}{c_t}\right)\left[\frac{p_{it}}{\overline{\pi} p_{it-1}} - 1\right]\left[\frac{y_t}{\overline{\pi} p_{it-1}}\right] \\ & +\beta\phi E_t\left\{\left(\frac{a_{t+1}}{c_{t+1}}\right)\left[\frac{p_{it+1}}{\overline{\pi} p_{it}} - 1\right]\left[\frac{y_{t+1}}{p_{it}}\right]\left[\frac{p_{it+1}}{\overline{\pi} p_{it}}\right]\right\}. \end{aligned} \tag{6}$$

Firms

- If prices are fully flexible, i.e. $\phi = 0$, price-setting rule (6) boils down to:

$$
\begin{aligned}
0 \quad = \quad & (1-\theta_t)\left(\frac{a_t}{c_t}\right)\left(\frac{p_{it}}{p_t}\right)^{-\theta_t}\left(\frac{y_t}{p_t}\right) \\
& +\theta_t\left(\frac{a_t}{c_t}\right)\left(\frac{p_{it}}{p_t}\right)^{-\theta_t-1}\left(\frac{w_t}{p_t}\right)\left(\frac{y_t}{z_t}\right)\left(\frac{1}{p_t}\right)
\end{aligned} \tag{7}
$$

$$
\implies p_{it} = \underbrace{\frac{\theta}{\theta-1}}_{\text{markup}} \times \underbrace{\frac{w_t}{z_t}}_{\text{MC}} .
$$

- "Price = markup over marginal cost".
- Markup is $\frac{\theta}{\theta-1} > 1$ for all $\theta > 1$.
- Markup depends negatively on the elasticity of substitution θ.
- Markup $\frac{\theta}{\theta-1} \to 1$ as $\theta \to \infty$ (perfect competition).

The Central Bank

- The CB chooses the nominal interest rate according to a Taylor rule:

$$r_t = \rho_r r_{t-1} + \rho_\pi \pi_t + \varepsilon_{rt}$$

where

$$\varepsilon_{rt} \sim i.i.d.\ N\left(0, \sigma_r^2\right).$$

- Can include also reaction to *output gap*.
- Output gap is deviation of actual output wrt **natural** level of output (to be defined later).

Stochastic Specification

- In addition to the monetary policy shock ε_{rt}, the model features a demand shock a_t, a technology shock z_t, and a cost-push shock θ_t.
- They evolve according to:

$$
\begin{aligned}
\log a_t &= (1-\rho_a)\log \overline{a} + \rho_a \log a_{t-1} + \varepsilon_{at}, \ \ \overline{a} > 1 \\
\log z_t &= \log \overline{z} + \log z_{t-1} + \varepsilon_{zt}, \ \ \overline{z} > 1 \\
\log \theta_t &= (1-\rho_\theta)\log \overline{\theta} + \rho_\theta \log \theta_{t-1} + \varepsilon_{\theta t}, \ \ \overline{\theta} > 1
\end{aligned}
$$

- Note: the technology shock z_t is *non-stationary*. It evolves as a random-walk with drift.

Market Clearing

- Focus on *symmetric* equilibrium in which all intermediate-goods firms make identical decisions:

$$y_{it} = y_t,\ n_{it} = n_t,\ p_{it} = p_t,\ d_{it} = d_t.$$

- Labor market clears:

$$n_t = \int_0^1 n_{it} di \tag{8}$$

- Bond market clears:

$$b_t = 0. \tag{9}$$

- Money market clears:

$$m_t = m_{t-1} + \tau_t. \tag{10}$$

- Aggregate resource constraint (implied by Walras's law):

$$c_t = y_t - \frac{\phi}{2}\left[\frac{\pi_t}{\pi} - 1\right]^2 y_t,\ \text{where } \pi_t \equiv p_t / p_{t-1}. \tag{11}$$

Aside on Walras's Law

- Household b.c. $+$ market clearing conditions (8), (9) and (10) $\Longrightarrow$ aggregate resource constraint (11).
- After imposing symmetry:

$$p_t c_t + \frac{b_t}{r_t} + m_t = m_{t-1} + b_{t-1} + \tau_t + w_t n_t + d_t$$

- Then

$$\begin{aligned} c_t &= \frac{w_t}{p_t} n_t + \frac{d_t}{p_t} \\ &= \frac{w_t}{p_t} n_t + \frac{p_{it} y_{it} - w_t n_{it}}{p_t} - \chi\left(p_{it}, p_{it-1}\right) \\ &= \frac{w_t}{p_t} n_t + y_t - \frac{w_t}{p_t} n_t - \frac{\phi}{2}\left[\frac{p_{it}}{\overline{\pi} p_{it-1}} - 1\right]^2 y_t \\ &= y_t - \frac{\phi}{2}\left[\frac{p_t}{\overline{\pi} p_{t-1}} - 1\right]^2 y_t . \end{aligned}$$

Equilibrium Equations

- The 11 endogenous variables are $\{y_t, c_t, n_t, d_t, r_t, w_t, p_t, m_t, a_t, z_t, \theta_t\}$ and satisfy the following 11 conditions:
- Household's FOCs:

$$\left(\frac{w_t}{p_t}\right)\left(\frac{a_t}{c_t}\right) = n_t^{\xi-1}$$

$$\frac{a_t}{c_t} = \beta r_t E_t \left\{ \frac{a_{t+1}}{c_{t+1}} \frac{p_t}{p_{t+1}} \right\}$$

$$\left(\frac{m_t}{p_t}\right)^{-1} + \beta E_t \left\{ \frac{a_{t+1}}{c_{t+1}} \frac{1}{p_{t+1}} \right\} = \left(\frac{a_t}{c_t}\right)\left(\frac{1}{p_t}\right)$$

- Aggregate resource constraint:

$$c_t = y_t - \frac{\phi}{2}\left[\frac{p_t}{\overline{\pi} p_{t-1}} - 1\right]^2 y_t$$

- Production function:

$$y_t = z_t n_t$$

- Dividends:

$$\frac{d_t}{p_t} = y_t - \frac{w_t}{p_t} n_t - \frac{\phi}{2}\left[\frac{p_t}{\overline{\pi} p_{t-1}} - 1\right]^2 y_t$$

Equilibrium Equations

- Price setting rule:

$$0 = (1-\theta_t) + \theta_t \left(\frac{w_t}{p_t}\right)\left(\frac{1}{z_t}\right) - \phi\left[\frac{p_t}{\overline{\pi} p_{t-1}} - 1\right]\left[\frac{p_t}{\overline{\pi} p_{t-1}}\right] \tag{12}$$

$$+\beta\phi E_t\left\{\left(\frac{a_{t+1}}{a_t}\right)\left(\frac{c_t}{c_{t+1}}\right)\left[\frac{p_{t+1}}{\overline{\pi} p_t} - 1\right]\left[\frac{p_{t+1}}{\overline{\pi} p_t}\right]\left[\frac{y_{t+1}}{y_t}\right]\right\}.$$

- Stochastic processes for shocks:

$$\log a_t = (1-\rho_a)\log \overline{a} + \rho_a \log a_{t-1} + \varepsilon_{at},\ \ \overline{a} > 1$$

$$\log z_t = \log \overline{z} + \log z_{t-1} + \varepsilon_{zt},\ \ \overline{z} > 1$$

$$\log \theta_t = (1-\rho_\theta)\log \overline{\theta} + \rho_\theta \log \theta_{t-1} + \varepsilon_{\theta t},\ \ \overline{\theta} > 1$$

- Taylor rule:

$$r_t = \rho_r r_{t-1} + \rho_\pi \pi_t + \varepsilon_{rt},\ \varepsilon_{rt} \sim i.i.d.\ N\left(0, \sigma_r^2\right).$$

Towards the canonical form

- Before we can write the system of equilibrium equations in canonical form, we still need to:
  - Normalize the system to eliminate unit root implied by the random walk for z_t;
  - Find the (deterministic) steady-state;
  - Log-linearize the equilibrium equations around the steady-state.
- For details on the above steps, see Dejong and Dave (2012) or Ireland (2004).
- As usual, define variables in logged deviations as

$$\widetilde{x}_t = \log\left(\frac{x_t}{\overline{x}}\right)$$

where $x_t = y_t,\ c_t$, etc.

The Linearized System

- The (approximated) equilibrium for $\{x_t, \pi_t, c_t, y_t, r_t, a_t, \theta_t, z_t\}_{t=0}^{\infty}$ is given by the following 8 linear equations:

$$\widetilde{x}_t = E_t \widetilde{x}_{t+1} - (\widetilde{r}_t - E_t \widetilde{\pi}_{t+1}) + \left(1 - \frac{1}{\xi}\right)(1 - \rho_a)\,\widetilde{a}_t$$

$$\phi \widetilde{\pi}_t = \beta \phi E_t \widetilde{\pi}_{t+1} + \eta\,(\theta - 1)\,\widetilde{x}_t - \widetilde{\theta}_t$$

$$\widetilde{c}_t = \widetilde{y}_t$$

$$\widetilde{x}_t = \widetilde{y}_t - \frac{1}{\xi}\widetilde{a}_t$$

$$\widetilde{z}_t = \varepsilon_{z,t}$$

$$\widetilde{a}_t = \rho_a \widetilde{a}_{t-1} + \varepsilon_{a,t}$$

$$\widetilde{\theta}_t = \rho_\theta \widetilde{\theta}_{t-1} + \varepsilon_{\theta,t}$$

$$\widetilde{r}_t = \rho_r \widetilde{r}_{t-1} + \rho_\pi \widetilde{\pi}_t + \varepsilon_{rt}$$

The Linearized System in Canonical Form

- Exercise: recast the 8 equations above in the canonical form

$$Ax_{t+1} = Bx_t + Cv_{t+1} + D\eta_{t+1}$$

where x_t collects all 8 endogenous variables, v_t is a vector of structural shocks (with mean zero) and η_t is a vector of expectation errors.

Model III: Financial Frictions

References

- Main reference for this section:
  - Pintus, P., and Wen, Y., 2013. *"Leveraged borrowing and boom–bust cycles,"* Review of Economic Dynamics.
- Additional references:
  - Bernanke, Gertler and Gilchrist (1994), Kiyotaki and Moore (1997).
  - Gertler and Karadi (2011), Iacoviello (2017).

Models with Financial Frictions

- Introduce collateralized borrowing in a simple RBC model.
- Aim: reduce the role of TFP shocks in explaining business cycles.
- Understanding U.S. Great Recession:
    - small "shock" (to subprime markets) transformed into big recession.
- What are the mechanisms than may amplify initial shock?
- Borrowers face a collateral constraint
    - Housing/land as collateral
    - Fluctuations in the price of housing/land $\implies$ ability to borrow
    - Applications to study financial crisis (e.g. Great Recession).

Economic Environment

- Economy consists of two types of agents:
  - Saver/lender (variables with tilde)
  - Borrower/entrepreneur (variables w/out tilde)
- Lenders:

$$\max_{\left\{\widetilde{C}_t, \widetilde{L}_{t+1}, B_{t+1}\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \widetilde{\beta}^t \left[\widetilde{C}_t + b\widetilde{L}_t\right] \qquad \text{(L)}$$

subject to

$$\widetilde{C}_t + \underbrace{Q_t\left(\widetilde{L}_{t+1} - \widetilde{L}_t\right)}_{\text{land invest.}} + \underbrace{B_{t+1}}_{\text{new loans}} \leq (1 + R_t)\underbrace{B_t}_{\text{loans}}, \qquad (1)$$

where $\widetilde{C}_t$ consumption, $\widetilde{L}_t$ land stock, Q_t land price, B_t amount of loans (credit lending), R_t real interest rate.

Lender

- Let $\widetilde{\lambda}_t$ denote the marginal utility of consumption.
- Lender's FOCs wrt $\left\{\widetilde{C}_t, \widetilde{L}_{t+1}, B_{t+1}\right\}$ are:

$$\widetilde{\lambda}_t = 1$$

- Consumption Euler:

$$\widetilde{\lambda}_t = \widetilde{\beta} E_t \left\{\widetilde{\lambda}_{t+1} \left(1 + R_t\right)\right\}$$

- Land investment:

$$\widetilde{\lambda}_t Q_t = \widetilde{\beta} \left[b + E_t \left\{\widetilde{\lambda}_{t+1} Q_{t+1}\right\}\right]$$

Lender

- We can simplify and obtain:
  - Constant interest rate
$$R_t = \frac{1}{\widetilde{\beta}} - 1$$
  - No-arbitrage equation for the price of land Q_t
$$Q_t = \widetilde{\beta}\left(\underbrace{b + E_t\{Q_{t+1}\}}_{\text{dividend+resell value}} \right) \tag{13}$$
- Solving (13) forward:
$$Q_t = \frac{\widetilde{\beta}}{1-\widetilde{\beta}} b = \frac{b}{R}$$
- *Remark*: land price Q_t is present value of constant "dividend" stream.

Borrowers: Maximization

- Entrepreneurs/borrowers:

$$\max_{\{C_t, L_{t+1}, B_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log C_t$$

subject to:

$$C_t + Q_t \left(L_{t+1} - L_t\right)_. + \left(1 + R_t\right) B_t \leq B_{t+1} + Y_t$$

where Y_t is output from production function $A_t L_t^{\gamma}$ with $\gamma \in (0, 1)$, and debt/collateral constraint:

$$\left(1 + R_{t+1}\right) B_{t+1} \leq Q_{t+1} L_{t+1}.$$

- Define marginal product of land as $MPL_t \equiv \gamma \frac{Y_t}{L_t}$.
- Assumption: borrowers are more impatient than lenders, i.e. $\beta < \widetilde{\beta}$.

Intuition for collateral constraint

- Collateral constraint: amount the borrower agrees to repay, $(1 + R_{t+1})\, B_{t+1}$, is tied to the future value of the durable asset (land), $Q_{t+1} L_{t+1}$.
- Why is this the case?

Intuition for collateral constraint

- Collateral constraint: amount the borrower agrees to repay, $(1 + R_{t+1})\, B_{t+1}$, is tied to the future value of the durable asset (land), $Q_{t+1} L_{t+1}$.
- Why is this the case?
- Credit market friction: lender cannot force borrower to repay the loan.
- Since the borrower can walk away with the loan, lender requires collateral to secure borrowing.
- Only durable assets (i.e. land or housing) can be used as collateral.

Borrowers: FOCs

- Let Φ_t denote the Lagrangian multiplier on collateral constraint.
- Euler equation for one-period debt:

$$\underbrace{\frac{1}{C_t}}_{\text{MU cons.}} = \underbrace{\beta E_t \left[\frac{1}{C_{t+1}} (1 + R_{t+1}) \right]}_{\text{MU saving}} + \Phi_t E_t (1 + R_{t+1})$$

- Note: $\Phi_t > 0 \rightarrow$ collateral constraint binding $\rightarrow$ MU consumption $>$ MU saving.

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- Note: $\Phi_t > 0 \rightarrow$ collateral constraint binding $\rightarrow$ MU consumption $>$ MU saving.
- Euler equation for land investment:

$$\underbrace{\frac{1}{C_t} Q_t}_{\text{MC of land}} = \underbrace{\beta E_t \left[\frac{1}{C_{t+1}} \left(Q_{t+1} + MPL_{t+1}\right) \right]}_{\text{resell value+marg.product}} + \underbrace{\Phi_t E_t \left[Q_{t+1}\right]}_{\text{shadow value of borrowing}}$$

Borrowers

- Recall that $R_t = R$ and $Q_t = Q$ for all t.
- Simplifying we get

$$\frac{1}{C_t} = (1+R)\, E_t \left(\frac{\beta}{C_{t+1}} \right) + (1+R)\, \Phi_t$$

$$\frac{1}{C_t} = E_t \left\{ \frac{\beta}{C_{t+1}} \left(\frac{Q + MPL_{t+1}}{Q} \right) \right\} + \Phi_t$$

- Equating the last two expressions yields:

$$\Phi_t = \beta E_t \left\{ \frac{1}{C_{t+1}} \left[\frac{MPL_{t+1}}{b} - 1 \right] \right\}$$

- Two cases:
    1. Collateral constraint is slack: $\Phi_t = 0$
    2. Collateral constraint is binding: $\Phi_t > 0$

Case 1: debt constraint is slack

- There are no credit market frictions $\implies$ CE is Pareto-efficient. Planner

Case 1: debt constraint is slack

- There are no credit market frictions $\implies$ CE is Pareto-efficient. Planner
- This means $MPL_t = b$ for all t: land allocation is efficient!
- In equilibrium

$$L_t = \left(\frac{\gamma A_t}{b}\right)^{\frac{1}{1-\gamma}} \equiv L^{FB}.$$

- Equations for land holdings and output in logs:

$$\log L_t = const. + \frac{1}{1-\gamma}\log A_t$$

$$\log Y_t = \gamma \log L_t + \log A_t$$

- Combining:

$$\log Y_t = const. + \frac{1}{1-\gamma}\log A_t.$$

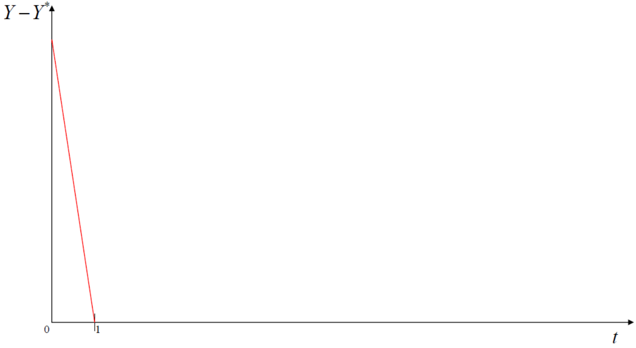
- Bottom line: if shock to A (TFP shock) i.i.d. then output Y is also i.i.d.
- More generally: without binding collateral constraint, output as persistent as shock (NO amplification).

Lack of Persistence

- Example: suppose shock to A is i.i.d.
- Compare impulse-response function of ouput in
  - model
  - data

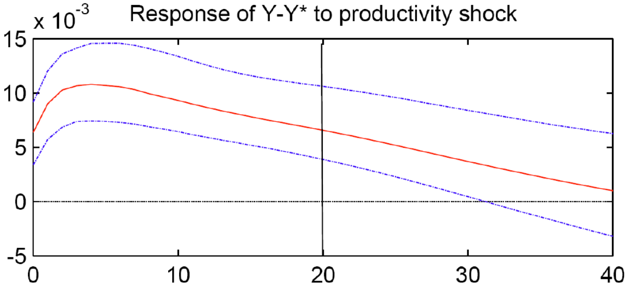
Lack of Persistence

- Model: IRF of output to a TFP shock (with slack debt constraint)



Lack of Persistence

- Data (VAR): IRF of output to a TFP shock



Case 2: debt constraint is binding

- Then $\Phi_t > 0 \rightarrow MPL_t > b \rightarrow L_t < L^{FB}$.
- Misallocation: borrower/entrepreneurs has too little land!
- Why? Because debt constraint binds: borrowing is limited.
- What are the consequences of binding debt limits on output persistence?
- We need to solve for the competitive equilibrium.

Equilibrium

- A CE is a sequence of allocations $\left\{C_t, \widetilde{C}_t, B_{t+1}, L_{t+1}, \widetilde{L}_{t+1}\right\}_{t=0}^{\infty}$ and prices $\{Q_t, R_t\}_{t=0}^{\infty}$ such that
  1. $\left\{\widetilde{C}_t, \widetilde{L}_{t+1}, B_{t+1}\right\}_{t=0}^{\infty}$ solve the lender's problem, given $\{Q_t, R_t\}_{t=0}^{\infty}$ and the initial endowments.
  2. $\{C_t, L_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ solve the borrower's problem, given $\{Q_t, R_t\}_{t=0}^{\infty}$ and the initial endowments.
  3. The goods and the asset (land) markets clear:

$$C_t + \widetilde{C}_t = Y_t,$$

$$L_t + \widetilde{L}_t = 1.$$

Equilibrium equations

- 9 Endogenous variables: $\left\{\widetilde{C}_t, \widetilde{L}_{t+1}, B_{t+1}, C_t, L_{t+1}, \Phi_t, Y_t, R_t, Q_t\right\}$
- In equilibrium prices are constant:

$$R_t = R = \frac{1}{\widetilde{\beta}} - 1,\ \ Q_t = Q = \frac{b}{R}.$$

- Lender's budget constraint:

$$\widetilde{C}_t + Q\left(\widetilde{L}_{t+1} - \widetilde{L}_t\right) + B_{t+1} = (1+R)\,B_t$$

Equilibrium equations

- Borrower's equations:
  - Budget constraint
$$C_t + Q_t\left(L_{t+1} - L_t\right)_. + (1+R)\,B_t = B_{t+1} + Y_t$$
  - Collateral constraint/debt limit:
$$(1+R)\,B_{t+1} \leq QL_{t+1}$$
  - Euler equation debt
$$\frac{1}{C_t} = (1+R)\left(\frac{\beta}{C_{t+1}} + \Phi_t\right) \tag{14}$$
  - Euler equation land
$$\frac{1}{C_t} = \frac{\beta}{C_{t+1}}\left(\frac{Q + MPL_{t+1}}{Q}\right) + \Phi_t \tag{15}$$
- Production function:
$$Y_t = A_t L_t^{\gamma}$$
- Market clearing for land:
$$L_t + \widetilde{L}_t = 1$$
- Note: the market clearing condition for goods is redundant (Walras' law).

Steady State

- From borrowers' Euler equation (14):

$$\frac{1}{C} = (1+R)\left(\frac{\beta}{C} + \Phi\right)$$

$$\rightarrow \Phi = \left(1 - \frac{\beta}{\widetilde{\beta}}\right)\frac{1}{C}$$

- Hence

$$\Phi > 0 \iff \beta < \widetilde{\beta}$$

- Collateral constraint binding in S.S. requires $\beta < \widetilde{\beta}$ i.e. borrowers/entrepreneurs more impatient than lenders.

Equilibrium with binding debt limit

- Assume $\Phi_t > 0$ for all t.
- Eliminate Φ_t by combining (14) and (15) gives:

$$\frac{C_{t+1}}{C_t} = \frac{\beta}{\widetilde{\beta}} \frac{\gamma A_{t+1} L_{t+1}^{\gamma-1}}{b} \tag{16}$$

- Substitute binding debt limit into budget constraint:

$$C_t + b\widetilde{\beta} L_{t+1} = A_t L_t^{\gamma} \tag{17}$$

- (16)-(17) form a system of two difference equations in two unknowns $\{C_t, L_{t+1}\}$ with $\{A_t, L_t\}$ describing the state.
- Guess a solution of the form

$$C_t = (1-m)\, Y_t = (1-m)\, A_t L_t^{\gamma},$$

where $m \in (0,1)$ is a constant to be determined.

Equilibrium with binding debt limit

- Replacing the guess into (16) (17) delivers:

$$mA_tL_t^\gamma = \beta\gamma A_tL_t^\gamma.$$

  The above restriction is valid for any $\{A_t, L_t\}$ iff $m = \beta\gamma$.

- Hence policy functions for consumption and land holdings are:

$$C_t = (1 - \beta\gamma)\, A_tL_t^\gamma,$$

$$L_{t+1} = \frac{\beta\gamma}{\widetilde{\beta} b} A_tL_t^\gamma$$

  and

$$Y_t = A_tL_t^\gamma.$$

Persistence

- Production function in logs:

$$\log Y_t = \gamma \log L_t + \log A_t$$

- Policy function of land in logs:

$$\log L_{t+1} = const. + \log Y_t$$

$$\rightarrow \log Y_t = const. + \gamma \log Y_{t-1} + \log A_t$$

- Key result: With collateral constraint obtain serial correlation in output (**endogenous** persistence).
  - TFP shock i.i.d. $\rightarrow$ output follows AR(1)
  - TFP shock AR(1) $\rightarrow$ output follows AR(2)

Persistence: Intuition

- Intuition for endogenous persistence:
  - Positive TFP shock increases income, hence:
  - price of collateral (land) goes up
  - debt constraint relaxed
  - consumption and investment go up
  - etc.
- Bottom line:
  - Debt limits create endogenous persistence: i.i.d. shocks have persistent impact on output
  - No hump-shaped response. Boom-bust cycles require additional mechanism.

Appendix

Final-goods Firm

- The profit maximization problem is:

$$\max_{p_{it}} p_t \left[\int_0^1 y_{it}^{\frac{\theta_t - 1}{\theta_t}} di \right]^{\frac{\theta_t}{\theta_t - 1}} - \int_0^1 p_{it} y_{it} di$$

- FOC w.r.t. p_{it}:

$$p_t \left(\frac{\theta_t}{\theta_t - 1} \right) \left[\int_0^1 y_{it}^{\frac{\theta_t - 1}{\theta_t}} di \right]^{\frac{1}{1-\theta_t}} \left(\frac{\theta_t - 1}{\theta_t} \right) y_{it}^{-\frac{1}{\theta}} = p_{it}$$

for all $i \in [0, 1]$ and for all $t \geq 0$.

- Note:

$$\left[\int_0^1 y_{it}^{\frac{\theta_t - 1}{\theta_t}} di \right]^{\frac{1}{1-\theta_t}} = y_t^{\frac{1}{\theta}}.$$

- Using the above result and rearranging delivers

$$y_{it} = y_t \left(\frac{p_{it}}{p_t} \right)^{-\theta_t}.$$

Back

Planner's problem: Allocation without debt limits

- Absent credit constraints, the competitive equilibrium is Pareto-efficient
- CE allocation solves the following social planner's problem:

$$\max_{C_t, \widetilde{C}_t, L_t, \widetilde{L}_t} \sum_{t=0}^{\infty} \left\{ \beta^t \log C_t + \widetilde{\beta}^t \left[\widetilde{C}_t + b\widetilde{L}_t \right] \right\}$$

subject to

$$\begin{aligned} C_t + \widetilde{C}_t &\leq Y_t \\ L_t + \widetilde{L}_t &\leq 1 \\ Y_t &\leq A_t L_t^{\gamma}. \end{aligned}$$

- FOCs are

$$\text{(i) } C_t = \left(\frac{\beta}{\widetilde{\beta}} \right)^t \text{ and (ii) } b = \gamma \frac{Y_t}{L_t}.$$

Back