Takehome Exam

Due on August 22, h 13:00. Please send your pdf and matlab/dynare codes in a single zip file as `firstname_surname.zip`

Pintus and Wen (2013)

The purpose of this exercise is to replicate the numerical findings of the paper "**Leveraged Borrowing and Boom-Bust Cycles**" by Patrick A. Pintus and Yi Wen, published in *Review of Economic Dynamics (2013).*

The authors enrich the standard Kiyotachi-Moore model (see Lecture 1) to build a quantitative model capable to explain the business cycle fluctuations associated with a credit boom.

(1) Read the paper.

(2) Focusing on sections 2.3 and 2.4 of the paper, the endogenous variables[1] are the following:

- consumption for borrowers and lenders: $C_t, \widetilde{C}_t$,
- capital stock: K
- Assets/Debt: B_t
- land holdings for borrowers and lenders: $L_t, \widetilde{L}_t$
- marginal utility of wealth for borrowers and lenders: $\Lambda_t, \widetilde{\Lambda}_t$
- multiplier on the credit constraint: Φ_t
- Relative prices of land and assets/debt: Q_t, R_t
- Production: Y_t

List the equilibrium conditions that characterize the optimal path of these 12 variables. Assume that productivity shock A_t follows an exogenous stochastic process.

(3) Find the steady state values of the model variables (*Hint: the steps are described in the paper; notice that to find the steady state value of* L_t *you need to solve numerically eq(20)*).

(4) Show that the credit constraint (equation 7 of the paper) is binding in the non-stochastic steady state. (*Hint: invoking the complementarity condition stated in the paper, section 2.3, it is enough to show that the multiplier* Φ_t *is strictly positive in the non-stochastic steady state of the model*).

[1] I use the same notation as in the paper.

(5) In the paper's appendix you can find the log-linearized version of the equilibrium equations (there are only 10 equations since the authors eliminated R_t and Φ_t). What you have to do is to write down these equations in Dynare and solve the model using a first-order approximation.

(6) Using the parameter values reported in Table 1 section 2.4, reproduce the findings in Figure 1, focusing just on the case of risk averse lender. You have to compute the impulse responses of output, aggregate consumption, land price and interest rate under an i.i.d. TFP shock (Calibration 1 in Table 1) and under a persistent AR(1) TFP shock (calibration 2 in Table 1).

(7) Notice that land distribution between lenders and borrowers is constant (in other words, the impulse response function of L_t is just a flat line at zero, and this is the reason why is not showed in the right-column windows of fig.1). Can you explain why it is so?

Solution

Here I provide a full replication of the paper's results. It is much more detailed than what was expected by the students.

Lenders' Problem

Lenders solve the following problem:

$$\max_{\left\{\widetilde{C}_t,\widetilde{L}_{t+1},B_{t+1}\right\}} \sum_{t=0}^{\infty} \widetilde{\beta}^t \left[\frac{\widetilde{C}_t^{1-\sigma_L}-1}{1-\sigma_L} + b\frac{\widetilde{L}_t^{1-\sigma_W}-1}{1-\sigma_W}\right] \tag{L}$$

subject to

$$\widetilde{C}_t + Q_t\left(\widetilde{L}_{t+1}-\widetilde{L}_t\right) + B_{t+1} \leq (1+R_t)\,B_t \tag{1}$$

where Q_t is the relative price of land (in terms of consumption; the price of consumption is normalized to 1).

Remark: Introducing land in the utility function of savers is a short-cut for generating a demand for land holdings by the savers.

Denoting with $\widetilde{\Lambda}_t$ the multiplier to constraint (1) we get the following focs:

$$\widetilde{\Lambda}_t = C_t^{-\sigma_l} \tag{13}$$

$$Q_t\widetilde{\Lambda}_t = \widetilde{\beta}Q_{t+1}\widetilde{\Lambda}_{t+1} + \widetilde{\beta}b\widetilde{L}_{t+1}^{-\sigma_W} \tag{14}$$

$$\widetilde{\Lambda}_t = \widetilde{\beta}\,(1+R_{t+1})\,\widetilde{\Lambda}_{t+1} \tag{15}$$

Borrowers Problem (w/out tilde)

Borrowers (or impatient agents) solve the following problem

$$\max_{\{C_t, L_{t+1}, K_{t+1}, B_{t+1}\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{\left(C_t - \rho \overline{C}_{t-1}\right)^{1-\sigma_B} - 1}{1-\sigma_B} \right\} \tag{B}$$

subject to

$$C_t + K_{t+1} - (1-\delta) K_t + Q_t (L_{t+1} - L_t) + (1 + R_t) B_t \leq B_{t+1} + Y_t, \tag{5}$$

and

$$(1 + R_{t+1}) B_{t+1} \leq \theta_t Q_{t+1} L_{t+1}, \tag{7}$$

where

$$Y_t = A_t K_t^{\alpha} L_t^{\gamma} \tag{3}$$

Attaching $\beta^t \Lambda_t$ to (5) and $\beta^t \Phi_t$ to (7), I can set up the following Lagrangian:

$$\begin{aligned} L \;=\; & \sum_{t=0}^{\infty} \beta^t \frac{\left(C_t - \rho C_{t-1}\right)^{1-\sigma_B} - 1}{1-\sigma_B} \\ & + \sum_{t=0}^{\infty} \beta^t \Lambda_t \left[B_{t+1} + A_t K_t^{\alpha} L_t^{\gamma} - C_t - K_{t+1} + (1-\delta) K_t - Q_t (L_{t+1} - L_t) - (1 + R_t) B_t\right] \\ & + \sum_{t=0}^{\infty} \beta^t \Phi_t \left[\theta_t Q_{t+1} L_{t+1} - (1 + R_{t+1}) B_{t+1}\right] \end{aligned}$$

The FOCS are:

$$C_t : \Lambda_t = \left(C_t - \rho C_{t-1}\right)^{-\sigma_B} \tag{16}$$

$$L_{t+1} : Q_t \Lambda_t = \beta Q_{t+1} \Lambda_{t+1} + \beta \gamma \frac{Y_{t+1}}{L_{t+1}} \Lambda_{t+1} + \theta_t \Phi_t Q_{t+1} \tag{17}$$

$$K_{t+1} : \Lambda_t = \beta \Lambda_{t+1} \left[\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right] \tag{18}$$

$$B_{t+1} : \Lambda_t = \beta (1 + R_{t+1}) \Lambda_{t+1} + \Phi_t (1 + R_{t+1}) \tag{19}$$

Remark: From (19) you see that if $\Phi_t > 0$ (so that the credit constraint binds) then

$$\Lambda_t > \beta (1 + R_{t+1}) \Lambda_{t+1}$$

i.e. the marginal utility of consumption today is higher than the marginal utility of saving: the borrowers would like to increase consumption today by borrowing more (i.e. by issuing more debt) but they cannot do it since they hit their maximum borrowing ability.

Bellman Equation

Alternatively, the optimality conditions can also be derived using a recursive approach. The endogenous state variables are clearly capital, land investment and borrowing/lending. For the sake of brevity, I focus only on the borrower's problem. The Bellman equation reads as

$$V\left(K_t, L_t, B_t\right) = \max_{C_t, K_{t+1}, L_{t+1}, B_{t+1}} \left\{ \frac{\left(C_t - \rho C_{t-1}\right)^{1-\sigma_B} - 1}{1-\sigma_B} + \beta E_t V\left(K_{t+1}, L_{t+1}, B_{t+1}\right) \right\}$$

s.t.

$$C_t + K_{t+1} - (1-\delta) K_t + Q_t \left(L_{t+1} - L_t\right) + \left(1 + R_t\right) B_t = B_{t+1} + Y_t$$
$$\left(1 + R_{t+1}\right) B_{t+1} = \theta_t Q_{t+1} L_{t+1}$$

This is a constrained maximization problem that can be solved using Lagrangian. The only difficulty is to find the derivative of the value function (envelope).

$$\begin{aligned} L \quad = \quad & \frac{\left(C_t - \rho C_{t-1}\right)^{1-\sigma_B} - 1}{1-\sigma_B} + \beta E_t V\left(K_{t+1}, L_{t+1}, B_{t+1}\right) \\ & + \lambda_t \left[B_{t+1} + A_t K_t^{\alpha} L_t^{\gamma} - C_t - K_{t+1} + (1-\delta) K_t - Q_t \left(L_{t+1} - L_t\right) - \left(1 + R_t\right) B_t\right] \\ & + \Phi_t \left[\theta_t Q_{t+1} L_{t+1} - \left(1 + R_{t+1}\right) B_{t+1}\right] \end{aligned}$$

The FOCs are:

$$C_t :$$
$$K_{t+1} : -\lambda_t + \beta \frac{\partial V\left(K_{t+1}, L_{t+1}, B_{t+1}\right)}{\partial K_{t+1}} = 0$$
$$B_{t+1} : \lambda_t - \Phi_t \left(1 + R_{t+1}\right) + \beta \frac{\partial V\left(K_{t+1}, L_{t+1}, B_{t+1}\right)}{\partial B_{t+1}} = 0$$
$$L_{t+1} :$$

Envelope

$$\frac{\partial V\left(K_t, L_t, B_t\right)}{\partial K_t} = \frac{\partial L}{\partial K_t} = \lambda_t \left[\alpha \frac{Y_t}{K_t} + (1-\delta)\right]$$
$$\frac{\partial V\left(K_t, L_t, B_t\right)}{\partial B_t} = \frac{\partial L}{\partial B_t} = -\lambda_t \left(1 + R_t\right)$$

Hence updating one period and substituting into the focs we get:

$$\lambda_t = \beta E_t \lambda_{t+1} \left[\alpha \frac{Y_{t+1}}{K_{t+1}} + (1-\delta)\right]$$

$$\lambda_t + \beta E_t \left\{ \frac{\partial V\left(K_{t+1}, L_{t+1}, B_{t+1}\right)}{\partial B_{t+1}} \right\} = \Phi_t \left(1 + R_{t+1}\right)$$
$$\Longrightarrow \lambda_t + \beta E_t \left\{-\lambda_{t+1} \left(1 + R_{t+1}\right)\right\} = \Phi_t \left(1 + R_{t+1}\right)$$
$$\Longrightarrow \lambda_t = \Phi_t \left(1 + R_{t+1}\right) + \beta E_t \left\{\lambda_{t+1} \left(1 + R_{t+1}\right)\right\}$$

Equilibrium

I start by defining the decentralized equilibrium for this economy:

Definition 1 *A competitive equilibrium is an allocation* $\left\{\widetilde{C}_t, \widetilde{L}_{t+1}, C_t, L_{t+1}, K_{t+1}\right\}$ *and a sequence of prices* $\{Q_t, R_t\}$ *such that: (i)* $\left\{\widetilde{C}_t, \widetilde{L}_{t+1}\right\}$ *solve (L) given* $\{Q_t, R_t\}$ *; (ii)* $\{C_t, L_{t+1}, K_{t+1}\}$ *solve (B) given* $\{Q_t, R_t\}$ *and the complementary slackness condition*

$$\Phi_t \left[\theta_t Q_{t+1} L_{t+1} - (1 + R_{t+1}) B_{t+1}\right] = 0$$

holds for all $t \geq 0$*;*

(iii) The good market and the market for land clear for all t*, that is* $C_t + \widetilde{C}_t + K_{t+1} - (1-\delta) K_t = Y_t$ *and* $L_t + \widetilde{L}_t = \overline{L}$*.*

Claim 2 *The market clearing equation for consumption is redundant (Walras' Law)*

Proof. Summing up the two budget constraints (1) and (5) delivers

$$\begin{aligned} \widetilde{C}_t + Q_t\left(\widetilde{L}_{t+1} - \widetilde{L}_t\right) + B_{t+1} &= (1 + R_t) B_t \\ C_t + K_{t+1} - (1-\delta) K_t + Q_t (L_{t+1} - L_t) + (1+R_t) B_t &\leq B_{t+1} + Y_t \end{aligned}$$

$$\begin{aligned} \Longrightarrow &\quad \widetilde{C}_t + Q_t\left(\widetilde{L}_{t+1} - \widetilde{L}_t\right) + B_{t+1} + C_t + K_{t+1} - (1-\delta) K_t + Q_t (L_{t+1} - L_t) + (1+R_t) B_t = \\ = &\quad (1+R_t) B_t + B_{t+1} + Y_t \end{aligned}$$

Rearranging terms we get

$$\widetilde{C}_t + Q_t\left(\widetilde{L}_{t+1} - \widetilde{L}_t\right) + C_t + K_{t+1} - (1-\delta) K_t + Q_t (L_{t+1} - L_t) = Y_t$$

$$\Longrightarrow \widetilde{C}_t + C_t + Q_t \Big(\underbrace{\widetilde{L}_{t+1} + L_{t+1}}_{\overline{L}}\Big) - Q_t \Big(\underbrace{\widetilde{L}_t + L_t}_{\overline{L}}\Big) + K_{t+1} - (1-\delta) K_t = Y_t$$

by market clearing for land. Hence we get:

$$\widetilde{C}_t + C_t + K_{t+1} - (1-\delta) K_t = Y_t.$$

■

Listing Equilibrium Equations

Lenders first-order conditions:

$$\widetilde{\Lambda}_t = C_t^{-\sigma_l} \tag{1}$$

$$Q_t\widetilde{\Lambda}_t = \widetilde{\beta} Q_{t+1}\widetilde{\Lambda}_{t+1} + \widetilde{\beta} b \widetilde{L}_{t+1}^{-\sigma_W} \tag{2}$$

$$\widetilde{\Lambda}_t = \widetilde{\beta}\left(1 + R_{t+1}\right)\widetilde{\Lambda}_{t+1} \tag{3}$$

Borrowers first-order conditions (with binding credit constraint):

$$\Lambda_t = \left(C_t - \rho C_{t-1}\right)^{-\sigma_B} \tag{4}$$

$$Q_t\Lambda_t = \beta Q_{t+1}\Lambda_{t+1} + \beta\gamma\frac{Y_{t+1}}{L_{t+1}}\Lambda_{t+1} + \theta_t\Phi_t Q_{t+1} \tag{5}$$

$$\Lambda_t = \beta\Lambda_{t+1}\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right] \tag{6}$$

$$\Lambda_t = \beta\left(1 + R_{t+1}\right)\Lambda_{t+1} + \Phi_t\left(1 + R_{t+1}\right) \tag{7}$$

$$\left(1 + R_{t+1}\right)B_{t+1} = \theta_t Q_{t+1} L_{t+1} \tag{8}$$

Production function:

$$Y_t = A_t K_t^{\alpha} L_t^{\gamma} \tag{9}$$

Market clearing conditions (with lenders budget constraint):

$$\widetilde{C}_t + Q_t\left(\widetilde{L}_{t+1} - \widetilde{L}_t\right) + B_{t+1} = \left(1 + R_t\right)B_t \tag{10}$$

$$C_t + \widetilde{C}_t + K_{t+1} - \left(1 - \delta\right)K_t = Y_t \tag{11}$$

$$L_t + \widetilde{L}_t = \overline{L} \tag{12}$$

Stochastic process for the TFP shock:

$$\log A_t = \rho_a \log A_{t-1} + \varepsilon_{a,t} \tag{13}$$

The endogenous variables are:

- consumption for borrowers and lenders: $C_t, \widetilde{C}_t$,
  - capital stock: K_t,
  - Assets/Debt: B_t
  - land holdings for borrowers and lenders: $L_t, \widetilde{L}_t$
  - marginal utility of wealth for borrowers and lenders: $\Lambda_t, \widetilde{\Lambda}_t$

- multiplier on the credit constraint: Φ_t
- Relative prices of land and assets/debt: Q_t, R_t
- Production: Y_t, A_t

We have 13 equations and 13 variables, OK!

Steady State

The 12 steady state values $C, \widetilde{C}, K,\ B,\ L,\ \widetilde{L},\ \Lambda,\ \widetilde{\Lambda},\ \Phi,\ Q,\ R,\ Y$ are the solutions to the following non-linear system:

$$\widetilde{\Lambda} = C^{-\sigma_l} \tag{1}$$

$$Q\widetilde{\Lambda} = \widetilde{\beta}Q\widetilde{\Lambda} + \widetilde{\beta}b\widetilde{L}^{-\sigma_W} \tag{2}$$

$$\widetilde{\Lambda} = \widetilde{\beta}\left(1+R\right)\widetilde{\Lambda} \tag{1}$$

$$\Lambda = \left[C\left(1-\rho\right)\right]^{-\sigma_B} \tag{4}$$

$$Q\Lambda = \beta Q\Lambda + \beta\gamma\frac{Y}{L}\Lambda + \Phi Q \tag{5}$$

$$\Lambda = \beta\Lambda\left[\alpha\frac{Y}{K} + 1 - \delta\right] \tag{6}$$

$$\Lambda = \beta\left(1+R\right)\Lambda + \Phi\left(1+R\right) \tag{7}$$

$$\left(1+R\right)B = QL \tag{8}$$

$$Y = K^{\alpha}L^{\gamma} \tag{9}$$

$$\widetilde{C} + Q\left(\widetilde{L} - \widetilde{L}\right) + B = \left(1+R\right)B \tag{10}$$

$$C + \widetilde{C} + K - \left(1-\delta\right)K = Y \tag{11}$$

$$L + \widetilde{L} = \overline{L} \tag{12}$$

The steady-state values for the shocks are $\theta = 1$ and $A = 1$.

Claim 3 *The credit constraint (7) is always binding in a neighborhood of the steady state*
Proof. *Evaluating (7) in the steady state delivers:*

$$\Phi = \Lambda\left(\widetilde{\beta} - \beta\right) \tag{ss}$$

Hence $\Phi > 0$ if $\widetilde{\beta} - \beta > 0$, which is true by assumption. ■

From equation (6) I get an expression for the capital-output ratio:

$$\frac{K}{Y}=\frac{\alpha\beta}{1-\beta\left(1-\delta\right)}$$

Manipulating equation (5) I get:

$$Q=\beta Q+\beta\gamma\frac{Y}{L}+\frac{\Phi}{\Lambda}Q$$

But from (ss) I know that $\frac{\Phi}{\Lambda}=\widetilde{\beta}-\beta$, hence the previous equation becomes:

$$\begin{aligned}
Q &= \beta Q+\beta\gamma\frac{Y}{L}+\left(\widetilde{\beta}-\beta\right)Q\\
Q\left(1-\beta-\widetilde{\beta}+\beta\right) &= \beta\gamma\frac{Y}{L}\\
Q &= \left(1-\widetilde{\beta}\right)^{-1}\beta\gamma\frac{Y}{L}
\end{aligned}$$

But notice that $Q=\left(1-\widetilde{\beta}\right)^{-1}\beta\gamma\frac{Y}{L}=\sum_{i=0}^{\infty}\widetilde{\beta}^{i}\beta\gamma\frac{Y}{L}$, hence the price of land in the steady state is equal to the present discounted value of land's marginal product.

Solving equation (2) for Q I get:

$$Q=\left(1-\widetilde{\beta}\right)^{-1}\widetilde{\beta}b\widetilde{L}^{-\sigma_w}\widetilde{C}^{\sigma_L}$$

From (8) I get:

$$\begin{aligned}
\left(1+R\right)B &= QL\\
B &= \frac{1}{1+R}QL
\end{aligned}$$

But since $\frac{1}{1+R}=\widetilde{\beta}$ I have

$$B=\widetilde{\beta}QL$$

Then from (10) i obtain

$$\begin{aligned}
\widetilde{C} &= RB\\
&= R\widetilde{\beta}QL
\end{aligned}$$

But since $R\widetilde{\beta}=1-\widetilde{\beta}$, I get

$$\widetilde{C}=\left(1-\widetilde{\beta}\right)QL$$

Now plugging x into y I get:

$$\begin{aligned}
\widetilde{C} &= \left(1-\widetilde{\beta}\right) QL \\
&= \left(1-\widetilde{\beta}\right)\left(1-\widetilde{\beta}\right)^{-1} \beta\gamma \frac{Y}{L} L \\
&= \beta\gamma Y
\end{aligned}$$

Hence

$$\widetilde{C} = \beta\gamma Y$$

Armed with these computations, I can greatly simplify the above nonlinear system. I have now the following system:

$$\begin{aligned}
Q &= \left(1-\widetilde{\beta}\right)^{-1} \beta\gamma \frac{Y}{L} \\
Q &= \left(1-\widetilde{\beta}\right)^{-1} \widetilde{\beta} b \widetilde{L}^{-\sigma_w} \widetilde{C}^{\sigma_L} \\
\frac{K}{Y} &= \frac{\alpha\beta}{1-\beta(1-\delta)} \\
\widetilde{C} &= \left(1-\widetilde{\beta}\right) QL \\
C + \widetilde{C} &= Y - \delta K \\
L + \widetilde{L} &= \overline{L} \\
Y &= K^{\alpha} L^{\gamma}
\end{aligned}$$

The variables are:$Q, Y, L, \widetilde{L}, K, C, \widetilde{C}$. Seven equations and seven variables, OK!

I can further simplify by eliminating Q and thus getting:

$$\beta\gamma \frac{Y}{L} = \widetilde{\beta} b \widetilde{L}^{-\sigma_w} \widetilde{C}^{\sigma_L}$$
$$\Longrightarrow \beta\gamma \frac{Y}{L} = \widetilde{\beta} b \left(\overline{L} - L\right)^{-\sigma_w} \widetilde{C}^{\sigma_L}$$

which is equation (20) in the paper. Hence the steady state values Y, L, and K satisfy:

$$\beta\gamma \frac{Y}{L} = \widetilde{\beta} b \left(\overline{L} - L\right)^{-\sigma_w} (\beta\gamma Y)^{\sigma_L} \tag{2}$$
$$Y = K^{\alpha} L^{\gamma} \tag{3}$$
$$\frac{K}{Y} = \frac{\alpha\beta}{1-\beta(1-\delta)} \tag{4}$$

The authors calibrate the utility weight parameter b to match a steady state ratio $\frac{\widetilde{L}}{L} = 1$. Remark: **In the takehome exam is asked to solve the implicit equation numerically**

and find the value of b that delivers $\frac{\widetilde{L}}{L} = 1$. Note however that the exact value of b does not really matter for solving the model. Also the exact value of $\bar{L}$ does not matter. For simplicity, I set it equal to one.

Pintus and Wen Appendix (2012) - Log-linear equations

The authors use the following equations (the numbering refers to the paper):

$$\widetilde{C}_t + Q_t\left(\widetilde{L}_{t+1} - \widetilde{L}_t\right) + B_{t+1} = (1 + R_t) B_t \tag{1}$$

$$Y_t = A_t K_t^{\alpha} L_t^{\gamma} \tag{3}$$

$$L_t + \widetilde{L}_t = \overline{L} \tag{4}$$

$$C_t + K_{t+1} - (1-\delta) K_t + Q_t (L_{t+1} - L_t) + (1 + R_t) B_t = B_{t+1} + Y_t, \tag{5}$$

$$(1 + R_{t+1}) B_{t+1} = \theta_t Q_{t+1} L_{t+1}, \tag{7}$$

$$\widetilde{\Lambda}_t = C_t^{-\sigma_l} \tag{13}$$

$$Q_t \widetilde{\Lambda}_t = \widetilde{\beta} Q_{t+1} \widetilde{\Lambda}_{t+1} + \widetilde{\beta} b \widetilde{L}_{t+1}^{-\sigma_W} \tag{14}$$

$$\Lambda_t = (C_t - \rho C_{t-1})^{-\sigma_B} \tag{16}$$

$$Q_t \Lambda_t = \beta Q_{t+1} \Lambda_{t+1} + \beta\gamma \frac{Y_{t+1}}{L_{t+1}} \Lambda_{t+1} + \theta_t \Phi_t Q_{t+1} \tag{17}$$

$$\Lambda_t = \beta \Lambda_{t+1}\left[\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right] \tag{18}$$

Then eliminate R_{t+1} and Φ_{t+1} by using (15) and (19) respectively

$$\widetilde{\Lambda}_t = \widetilde{\beta} (1 + R_{t+1}) \widetilde{\Lambda}_{t+1} \tag{15}$$

$$\Lambda_t = \beta (1 + R_{t+1}) \Lambda_{t+1} + \Phi_t (1 + R_{t+1}) \tag{19}$$

Notice that the authors use the two budget constraints (1)-(5) and omit the aggregagte resource constraint (11) which is redundant by Walras' Law. Then they loglinearize these equations. I will show you how the detailed log-linearization of some of them and I will highlight a mistake (which is still present in the published version of the paper!). Start from equation (1), using Uhlig's method[2]

$$\widetilde{C} e^{c_t} + Q e^{q_t}\left(\widetilde{L} e^{l_{t+1}} - \widetilde{L} e^{l_t}\right) + B e^{b_{t+1}} = (1 + R^{r_t}) B e^{b_t}$$

[2]Recall $x_t = x e^{\widehat{x}_t} \approx x(1 + \widehat{x}_t)$, where $\widehat{x}_t \equiv \log\left(\frac{x_t}{x}\right)$. For further details, see "A toolkit for analysing nonlinear dynamic stochastic models easily" by H.Uhlig or my handout on log-linearization.

$$\widetilde{C}\left(1+c_t\right)+QL\left(1+q_t+l_{t+1}\right)-QL\left(1+q_t+l_t\right)+B\left(1+b_{t+1}\right)=B\left(1+b_t\right)+RB\left(1+r_t+b_t\right)$$

$$\frac{\widetilde{C}}{B}c_t+\frac{Q\widetilde{L}}{B}\left(\widetilde{l}_{t+1}-\widetilde{l}_t\right)+b_{t+1}=\left(1+R\right)b_t+Rr_t$$

Then we have to replace r_t using log-linear version of (15):

$$r_{t+1}=\left(\frac{1+R}{R}\right)\left(\widetilde{\lambda}_t-E_t\widetilde{\lambda}_{t+1}\right) \tag{5}$$

Notice furthermore that all lowercase variables are defined as percentage deviation from steady state. Replacing (5) into () we get:

$$\frac{\widetilde{C}}{B}c_t+\frac{Q\widetilde{L}}{B}\left(\widetilde{l}_{t+1}-\widetilde{l}_t\right)+b_{t+1}=\left(1+R\right)\left(b_t+\widetilde{\lambda}_{t-1}-\widetilde{\lambda}_t\right)$$

which is equation (35) in Pintus' Appendix. Equations (36) and (37) are trivial to derive. Equation (5) when log-linearized becomes:

$$\frac{C}{Y}c_t+\frac{K}{Y}k_{t+1}-\left(1-\delta\right)\frac{K}{Y}k_t+\frac{QL}{Y}\left(l_{t+1}-l_t\right)+\frac{\left(1+R\right)B}{Y}b_t+\frac{\left(1+R\right)B}{Y}\left(\widetilde{\lambda}_{t-1}-\widetilde{\lambda}_t\right)=\frac{B}{Y}b_{t+1}+y_t \tag{6}$$

As you see this corresponds to equation (38) of Pintus' Appendix but for the term $\frac{(1+R)B}{Y}\left(\widetilde{\lambda}_{t-1}-\widetilde{\lambda}_t\right)$ which is missing! Of course in the case of risk-neutral lender the marginal utility of consumption $\widetilde{\Lambda}_t$ is constant over time (i.e. the deviation $\widetilde{\lambda}_t$ is 0) and therefore the omitted term does not affect the result.

The other equations are correct. Hence to solve the model you should have used equations (35)-(44) in the paper's appendix but with (38) replaced with (6).

Dynare notation

Bear in mind that Dynare uses a different time convention for predetermined variables. Here the predetermined variables are borrowing, land investment and capital. For example, equation (1) in Dynare reads as

$$\widetilde{C}_t+Q_t\left(\widetilde{L}_t-\widetilde{L}_{t-1}\right)+B_t=\left(1+R_t\right)B_{t-1}$$

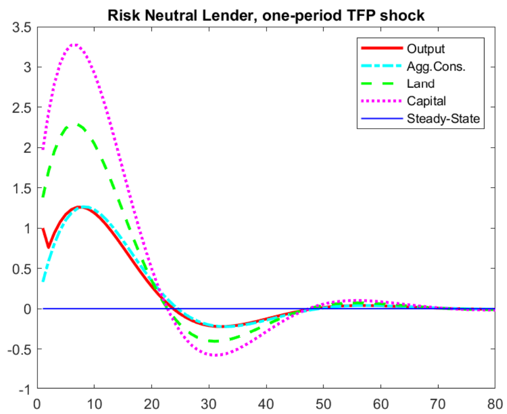
Likewise, the production function becomes

$$Y_t=A_tK_{t-1}^{\alpha}L_{t-1}^{\gamma}$$

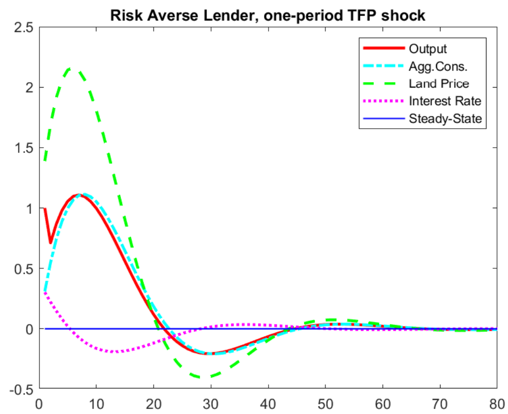
Replication Files

In the takehome, the students were asked to replicate only the right panel of Figure 1 (Risk Averse Lender). For the sake of completeness, here I replicated also the left panel of the same figure (Risk Neutral Lender). To execute the replication, run the file `main_takehome.m`. Set `lender = 'RN'` if you want to simulate the model with risk neutral lender, while `'RA'` if the risk averse (with log-utility) case is desired. Set 1 in `calib` for calibration 1, while 2 for calibration 2.

All in all, the replication exercise was successful: for calibration 1, the IRF of my replication are identical to those in the paper (see Figure 1), for calibration 2 there is a slight difference in terms of timing. In particular I was not able to reproduce *exactly* the case of Risk averse lender with Calibration 2: compare Figure (2b) with Pintus and Wen Figure 1, right-bottom panel. I thank some students for noting that if you set $\sigma_B = 4$, you can replicate exactly the figure in the paper. This suggests that the authors probably forgot to change the value of this parameter when going from Calibration 1 to Calibration 2.

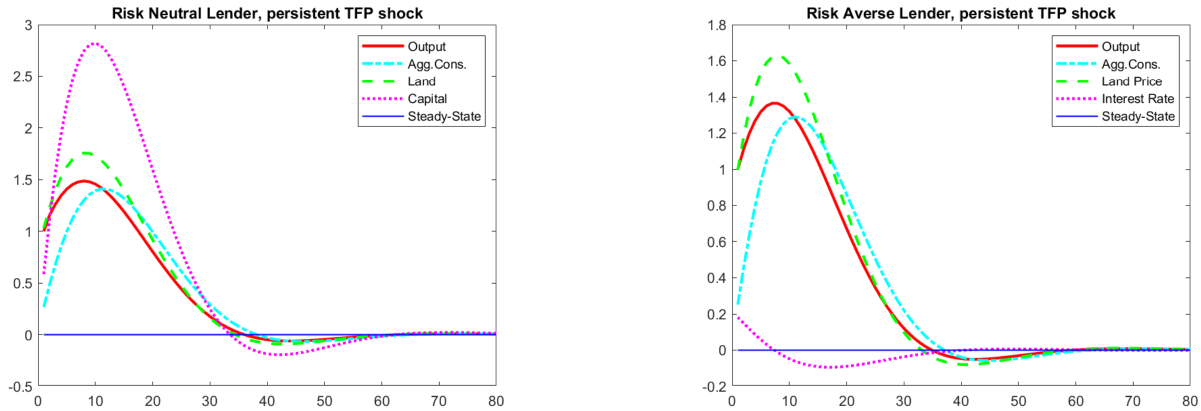


(a) Risk neutral lender



(b) Risk averse lender

Figure 1: IRFs to a positive i.i.d. TFP shock, Calibration 1 (see first row of Table 1 in Pintus and Wen 2013).



(a) Risk neutral lender

(b) Risk averse lender

Figure 2: IRFs to a positive persistent TFP shock, Calibration 2 (see second row of Table 1 in Pintus and Wen 2013).

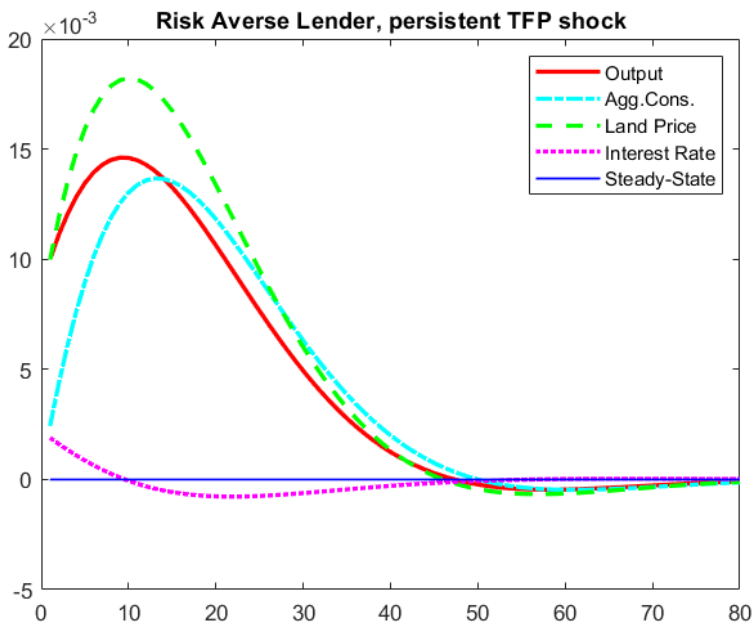


Figure 3: Risk averse lender, calibration 2 **with** $\sigma_B = 4$.