LH Advanced Financial Markets - Part B

Topic 2: Making Choices in Risky Situations

Dr. Alessandro Di Nola

University of Birmingham

Spring 2025

Outline

Criteria for Choice Over Risky Prospects

- Reference for this set of slides: Ch.3 Danthine and Donaldson
- In the broadest sense, "risk" refers to uncertainty about the future cash flows provided by a financial asset.
- A more specific way of modeling risk is to think of those cash flows as varying across different states of the world in future periods...
- ...that is, to describe future cash flows as random variables.

Criteria for Choice Over Risky Prospects

Consider three assets:

		Payoffs Next Year in	
	Price Today	*Good State*	*Bad State*
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

where the good and bad states occur with equal probability ($\pi = 1 - \pi = 1/2$).

Criteria for Choice Over Risky Prospects

	Price Today	Payoffs Next Year in *Good State*	*Bad State*
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

Asset 3 exhibits state-by-state dominance over assets 1 and 2. Any investor who prefers more to less would always choose asset 3 above the others.

Criteria for Choice Over Risky Prospects

In general, one asset displays state-by-state dominance over another if:

1. It pays off at least as much in all states

AND

2. It pays off more in at least one state,

so investors who prefer more to less will never regret buying it.

Criteria for Choice Over Risky Prospects

		Payoffs Next Year in	
	Price Today	*Good State*	*Bad State*
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

But the choice between assets 1 and 2 is not as clear cut.
Asset 2 provides a larger gain in the good state, but exposes the investor to a loss in the bad state.

Criteria for Choice Over Risky Prospects

It can often be helpful to convert prices and payoffs to percentage returns:

		Payoffs Next Year in	
	Price Today	*Good State*	*Bad State*
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

	Percentage Return in	
	Good State	*Bad State*
Asset 1	20	5
Asset 2	60	-50
Asset 3	60	5

Criteria for Choice Over Risky Prospects

- In probability theory, if a random variable $\tilde{X}$ can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$,
- then the expected value of $\tilde{X}$ is

$$E(\tilde{X}) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n$$

- the variance of $\tilde{X}$ is

$$\begin{aligned}\sigma^2(\tilde{X}) =& \pi_1 \left[X_1 - E(\tilde{X})\right]^2 \\ & + \pi_2 \left[X_2 - E(\tilde{X})\right]^2 + \ldots + \pi_n \left[X_n - E(\tilde{X})\right]^2\end{aligned}$$

- and the standard deviation of $\tilde{X}$ is $\sigma(\tilde{X}) = \left[\sigma^2(\tilde{X})\right]^{1/2}$.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	*Bad State*	$E(R)$	$\sigma(R)$
Asset 1	20	5		
Asset 2	60	-50		
Asset 3	60	5		

Asset 1

- Expected value :

$$E(R_1) = (1/2)20 + (1/2)5 = 12.5$$

- Standard deviation:

$$\sigma(R_1) = \left[(1/2)(20 - 12.5)^2 + (1/2)(5 - 12.5)^2\right]^{1/2} = 7.5$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	*Bad State*	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50		
Asset 3	60	5		

Asset 2

- Expected value :

$$E(R_2) = (1/2)60 + (1/2)(-50) = 5$$

- Standard deviation:

$$\sigma(R_2) = \left[(1/2)(60-5)^2 + (1/2)(-50-5)^2\right]^{1/2} = 55$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	*Bad State*	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50	5	55
Asset 3	60	5		

Asset 3

- Expected value :

$$E\left(R_3\right) = (1/2)60 + (1/2)5 = 32.5$$

- Standard deviation:

$$\sigma\left(R_3\right) = \left[(1/2)(60 - 32.5)^2 + (1/2)(5 - 32.5)^2\right]^{1/2} = 27.5$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	*Bad State*	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50	5	55
Asset 3	60	5	32.5	27.5

Asset 1 exhibits mean-variance dominance over asset 2, since it offers a higher expected return with lower variance.

Criteria for Choice Over Risky Prospects

In general, one asset displays mean-variance dominance over another if:

1. $E(R_1) > E(R_2)$ and $\sigma(R_1) \leq \sigma(R_2)$
   so that it offers a higher expected return with no greater standard deviation,

OR

2. $E(R_1) \geq E(R_2)$ and $\sigma(R_1) < \sigma(R_2)$
   so that it offers a smaller standard deviation and no less expected return.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	*Bad State*	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50	5	55
Asset 3	60	5	32.5	27.5

But notice that by the mean-variance criterion, asset 3 dominates asset 2 but not asset 1, even though on a state-by-state basis, asset 3 is clearly to be preferred.

Criteria for Choice Over Risky Prospects

Consider two more assets:

	Percentage Return in			
	Good State	*Bad State*	$E(R)$	$\sigma(R)$
Asset 4	5	3		
Asset 5	8	2		

Again, neither exhibits state-by-state dominance, so let's try to use the mean-variance criterion again.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	*Bad State*	$E(R)$	$\sigma(R)$
Asset 4	5	3		
Asset 5	8	2		

- Asset 4:

$$E(R_4) = (1/2)5 + (1/2)3 = 4$$

$$\sigma(R_4) = \left[(1/2)(5-4)^2 + (1/2)(3-4)^2\right]^{1/2} = 1$$

- Asset 5:

$$E(R_5) = (1/2)8 + (1/2)2 = 5$$

$$\sigma(R_5) = \left[(1/2)(8-5)^2 + (1/2)(2-5)^2\right]^{1/2} = 3$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	*Bad State*	$E(R)$	$\sigma(R)$
Asset 4	5	3	4	1
Asset 5	8	2	5	3

Neither asset exhibits mean-variance dominance either.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	*Bad State*	$E(R)$	$\sigma(R)$
Asset 4	5	3	4	1
Asset 5	8	2	5	3

William Sharpe (US, b.1934, Nobel Prize 1990) suggested that in these circumstances, it can help to compare the two assets' Sharpe ratios, defined as $E(R)/\sigma(R)$.

Criteria for Choice Over Risky Prospects

- Note: in practice, the Sharpe ratio is usually defined as the expected "excess return" above the risk-free rate r_f divided by the standard deviation:

$$\frac{E(R) - r_f}{\sigma(R)}$$

- For these preliminary examples, we are either assuming that $r_f = 0$ or using $E(R)/\sigma(R)$ as a simplified definition of the Sharpe ratio.

Criteria for Choice Over Risky Prospects

	Percentage Return in				
	Good State	*Bad State*	$E(R)$	σ	$E/\sigma(R)$
Asset 4	5	3	4	1	4
Asset 5	8	2	5	3	1.67

Comparing Sharpe ratios, asset 4 is preferred to asset 5 .

Criteria for Choice Over Risky Prospects

	Percentage Return in				
	Good State	*Bad State*	$E(R)$	$\sigma(R)$	E/σ
Asset 4	5	3	4	1	4
Asset 5	8	2	5	3	1.67

- But using the Sharpe ratio to choose between assets assumes that investors “weight” the mean and standard deviation equally, in the sense that a doubling of $\sigma(R)$ is adequately compensated by a doubling of $E(R)$.
- Investors who are more or less averse to risk will disagree.

Criteria for Choice Over Risky Prospects

1. State-by-state dominance is the most robust criterion, but often cannot be applied.
2. Mean-variance dominance is more widely-applicable, but can sometimes be misleading and cannot always be applied.
3. The Sharpe ratio can always be applied, but requires a very specific assumption about consumer attitudes towards risk.

We need a more careful and comprehensive approach to comparing random cash flows.

Outline

Preferences and Utility Functions

- Of course, economists face a more general problem of this kind.

- Even if we accept that more (of everything) is preferred to less, how do consumers compare different "bundles" of goods that may contain more of one good but less of another?

- Microeconomists have identified a set of conditions that allow a consumer's preferences to be described by a utility function.

Preferences and Utility Functions

- Let a, b, and c represent three bundles of goods.
- These may be arbitrarily long lists, or vectors $\left(a \in \mathbb{R}^N\right)$, indicating how much of each of an arbitrarily large number of goods is included in the bundle.
- A preference relation $\succeq$ can be used to represent the consumer's preferences over different consumption bundles.

Preferences and Utility Functions

The expression

$$a \succ b$$

indicates that the consumer strictly prefers a to b,

$$a \sim b$$

indicates that the consumer is indifferent between a and b, and

$$a \succeq b$$

indicates that the consumer either strictly prefers or is indifferent between a and b.

Preferences and Utility Functions

A1 The preference relation is assumed to be complete: For any two bundles a and b, either $a \succeq b$, $b \succeq a$, or both, and in the latter case $a \sim b$.

The consumer has to decide whether he or she prefers one bundle to another or is indifferent between the two.
Ambiguous tastes are not allowed.

Preferences and Utility Functions

A2 The preference relation is assumed to be transitive: For any three bundles a, b, and c, if $a \succeq b$ and $b \succeq c$, then $a \succeq c$.

The consumer's tastes must be consistent in this sense. Together, (A1) and (A2) require the consumer to be fully informed and rational.

A3 The preference relation is assumed to be continuous: If $\{a_n\}$ and $\{b_n\}$ are two sequences of bundles such that $a_n \to a$, $b_n \to b$, and $a_n \succeq b_n$ for all n, then $a \succeq b$.

Very small changes in consumption bundles cannot lead to large changes in preferences over those bundles.

Preferences and Utility Functions

- A two-good example that violates (A3) is the case of lexicographic preferences:

$$a = (a_1, a_2) \succ b = (b_1, b_2) \quad \begin{array}{ll} \text{if} & a_1 > b_1 \\ \text{or} & a_1 = b_1 \text{ and } a_2 > b_2. \end{array}$$

- To see that these preferences violate (A3), consider sequences $a_n = (1, 1)$ and $b_n = (1 + \frac{1}{n}, 0)$. Clearly $a_n \to a = (1, 1)$ and $b_n \to b = (1, 0)$
- Note that $b_n = (1 + \frac{1}{n}, 0) \succ (1, 1) = a_n$ for all n, but $a = (1, 1) \succ b = (1, 0)$! Draw graph.
- It is not possible to represent these preferences with a utility function, since the preferences are fundamentally two-dimensional and the value of the utility function has to be one-dimensional.

Preferences and Utility Functions

- The following theorem was proven by **Gerard Debreu** in 1954.

Theorem If preferences are complete, transitive, and continuous, then they can be represented by a continuous, real-valued utility function. That is, if (A1)-(A3) hold, there is a continuous function $u : \mathbb{R}^n \mapsto \mathbb{R}$ such that for any two consumption bundles a and b,

$$a \succeq b \text{ if and only if } u(a) \geq u(b).$$

Preferences and Utility Functions

- Note that if preferences are represented by the utility function u,

$$a \succeq b \text{ if and only if } u(a) \geq u(b)$$

then they are also represented by the utility function v, where

$$v(a) = F(u(a))$$

and $F : \mathbb{R} \mapsto \mathbb{R}$ is any increasing function.

- The concept of utility as it is used in standard microeconomic theory is ordinal, as opposed to cardinal.

Outline

Expected Utility Functions

- Under certainty, the "goods" are described by consumption baskets with known characteristics.

- Under uncertainty, the "goods" are random (state-contingent) payoffs.

- The problem of describing preferences over these state-contingent payoffs, and then summarizing these preferences with a utility function, is similar in overall spirit but somewhat different in its details to the problem of describing preferences and utility functions under certainty.

Expected Utility Functions

Consider shares of stock in two companies:

		Price Next Year in	
	Price Today	*Good State*	*Bad State*
AT&T	-100	150	100
Verizon	-100	150	100

where the good state occurs with probability π and the bad state occurs with probability $1 - \pi$.

Expected Utility Functions

		Price Next Year in	
	Price Today	*Good State*	*Bad State*
AT&T	-100	150	100
Verizon	-100	150	100
	probability	π	$1 - \pi$

We will assume that if the two assets provide exactly the same state-contingent payoffs, then investors will be indifferent between them.

Expected Utility Functions

		Price Next Year in	
	Price Today	*Good State*	*Bad State*
AT&T	-100	150	100
Verizon	-100	150	100
	probability	π	$1 - \pi$

1. Investors care only about payoffs and probabilities.

Expected Utility Functions

Consider another comparison:

	Price Today	Price Next Year in *Good State*	Price Next Year in *Bad State*
AT&T	-100	150	100
Apple	-100	160	110
	probability	π	$1-\pi$

We will also assume that investors will prefer any asset that exhibits state-by-state dominance over another.

Expected Utility Functions

		Price Next Year in	
	Price Today	*Good State*	*Bad State*
AT&T	-100	150	100
Apple	-100	160	110
	probability	π	$1-\pi$

2. If $u(p)$ measures utility from the payoff p in any particular state, then u is increasing.

Expected Utility Functions

Consider a third comparison:

		Price Next Year in	
	Price Today	*Good State*	*Bad State*
AT&T	-100	150	100
IBM	-100	160	90
	probability	π	$1 - \pi$

Here, there is no state-by-state dominance, but it seems reasonable to assume that a higher probability π will make investors tend to prefer IBM, while a higher probability $1 - \pi$ will make investors tend to prefer AT&T.

Expected Utility Functions

		Price Next Year in	
	Price Today	*Good State*	*Bad State*
AT&T	-100	150	100
IBM	-100	160	90
	probability	π	$1-\pi$

3. Investors should care more about states of the world that occur with greater probability.

Expected Utility Functions

A criterion that has all three of these properties was suggested by **Blaise Pascal** (France, 1623-1662): base decisions on the expected payoff,

$$E(p) = \pi p_G + (1 - \pi) p_B$$

where p_G and p_B, with $p_G > p_B$, are the payoffs in the good and bad states.

Expected payoff

$$E(p) = \pi p_G + (1 - \pi) p_B$$

1. Depends only on payoffs and probabilities.
2. Increases whenever p_G or p_B rises.
3. Attaches higher weight to states with higher probabilities.

Expected Utility Functions

Nicolaus Bernoulli (Switzerland, 1687-1759) pointed to a problem with basing investment decisions exclusively on expected payoffs: it ignores risk. To see this, specialize the previous example by setting $\pi = 1 - \pi = 1/2$ but add, as well, a third asset:

		Price Next Year in	
	Price Today	*Good State*	*Bad State*
AT&T	-100	150	100
IBM	-100	160	90
Gov't Bond	-100	125	125
	probability	$\pi = 1/2$	$1 - \pi = 1/2$

Expected Utility Functions

		Price Next Year in	
	Price Today	*Good State*	*Bad State*
AT&T	-100	150	100
IBM	-100	160	90
Gov't Bond	-100	125	125
	probability	$\pi = 1/2$	$1 - \pi = 1/2$

AT&T: $E(p) = (1/2)150 + (1/2)100 = 125$

IBM: $E(p) = (1/2)160 + (1/2)90 = 125$

Govt́ Bond: $E(p) = (1/2)125 + (1/2)125 = 125$

Expected Utility Functions

$$\begin{aligned}
\text{AT\&T: } E(p) &= (1/2)150 + (1/2)100 = 125 \\
\text{IBM: } E(p) &= (1/2)160 + (1/2)90 = 125 \\
\text{Gov\'{t} Bond: } E(p) &= (1/2)125 + (1/2)125 = 125
\end{aligned}$$

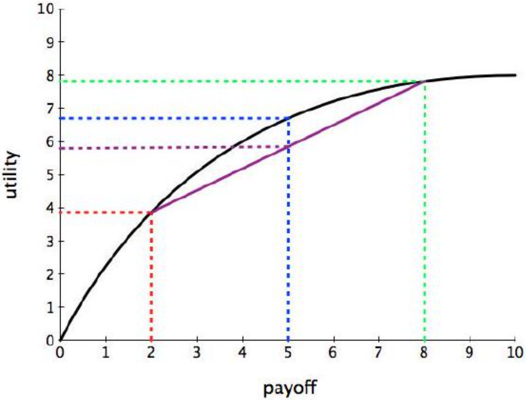
All three assets have the same expected payoff, but the bond is less risky than both stocks and AT&T stock is less risky than IBM stock.

Expected Utility Functions

- **Daniel Bernoulli** (Switzerland, 1700-1782) suggested that more reliable comparisons could be made by assuming that the utility function u over payoffs in any given state is concave as well as increasing.

- This implies that investors prefer more to less, but have diminishing marginal utility as payoffs increase.

Expected Utility Functions



When u is concave, a payoff of 5 for sure is preferred to a payoff of 8 with probability $1/2$ and 2 with probability $1/2$.

Expected Utility Functions

About two centuries later, **John von Neumann** (Hungary, 1903-1957) and **Oskar Morgenstern** (Germany, 1902-1977) worked out the conditions under which investors' preferences over risky payoffs could be described by an expected utility function such as

$$U(p) = E[u(p)] = \pi u(p_G) + (1 - \pi) u(p_B)$$

where the Bernoulli utility function over payoffs u is increasing and concave and the von Neumann-Morgenstern expected utility function U is linear in the probabilities.

Expected Utility Functions

- von Neumann and Morgenstern's axiomatic derivation of expected utility appeared in the second edition of their book, *Theory of Games and Economic Behavior*, published in 1947.

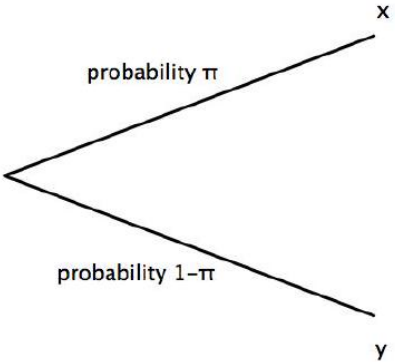
$$U(p) = E[u(p)] = \pi u\,(p_G) + (1 - \pi)u\,(p_B)$$

- Linearity in the probabilities is the "defining characteristic" of the expected utility function $U(p)$.

Outline

1. Criteria for Choice Over Risky Prospects
2. Preferences and Utility Functions
3. Expected Utility Functions
4. The Expected Utility Theorem
5. The Allais Paradox
6. Generalizations of Expected Utility

The Expected Utility Theorem



The simple lottery (x, y, π) offers payoff x with probability π and payoff y with probability $1 - \pi$.

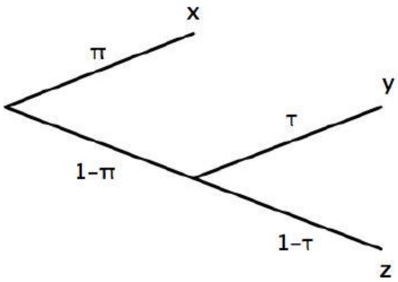
The Expected Utility Theorem

- The simple lottery (x, y, π) offers payoff x with probability π and payoff y with probability $1 - \pi$.

- In this definition, x and y can be monetary payoffs, as in the stock and bond examples from before.

- Alternatively, they can be additional lotteries!

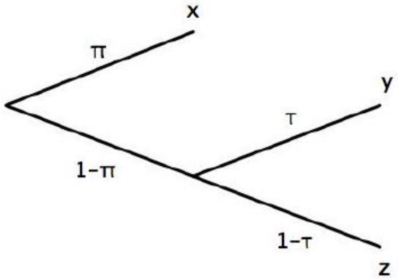
The Expected Utility Theorem



The compound lottery $(x, (y, z, \tau), \pi)$ offers payoff x with probability π and lottery (y, z, τ) with probability $1 - \pi$.

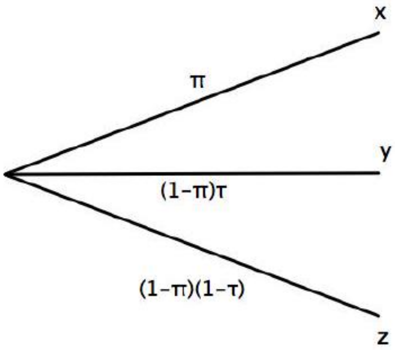
The Expected Utility Theorem

Notice that a simple lottery with more than two outcomes can always be reinterpreted as a compound lottery where each individual lottery has only two outcomes.



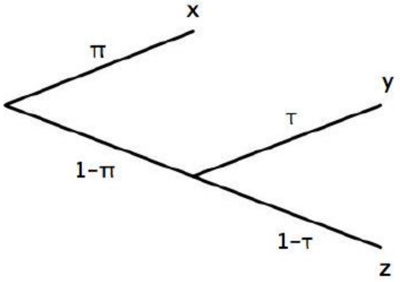
The Expected Utility Theorem

Notice that a simple lottery with more than two outcomes can always be reinterpreted as a compound lottery where each individual lottery has only two outcomes.



The Expected Utility Theorem

Notice that a simple lottery with more than two outcomes can always be reinterpreted as a compound lottery where each individual lottery has only two outcomes.



The Expected Utility Theorem

- Notice that a simple lottery with more than two outcomes can always be reinterpreted as a compound lottery where each individual lottery has only two outcomes.

- So restricting ourselves to lotteries with only two outcomes does not entail any loss of generality in terms of the number of future states that are possible.

- But to begin describing preferences over lotteries, we need to make additional assumptions.

The Expected Utility Theorem

- B1a A lottery that pays off x with probability one is the same as getting x for sure: $(x, y, 1) = x$.

- B1b Investors care about payoffs and probabilities, but not the specific ordering of the states: $(x, y, \pi) = (y, x, 1 - \pi)$

- B1c In evaluating compound lotteries, investors care only about the probabilities of each final payoff:

$$(x, z, \pi) = (x, y, \pi + (1 - \pi)\tau) \quad \text{if } z = (x, y, \tau)$$

The Expected Utility Theorem

- B2 There exists a preference relation $\succeq$ defined on lotteries that is complete and transitive.

Again, this amounts to requiring that investors are fully informed and rational.

The Expected Utility Theorem

- B3 The preference relation $\succeq$ defined on lotteries is continuous.

Hence, very small changes in lotteries cannot lead to very large changes in preferences over those lotteries.

The Expected Utility Theorem

- By the previous theorem, we already know that (B2, complete and transitive) and (B3, continuity) are sufficient to guarantee the existence of a utility function over lotteries and, by (B1a), payoffs received with certainty as well.

- What remains is to identify the extra assumptions that guarantee that this utility function is linear in the probabilies, that is, of the von Neumann-Morgenstern (vN-M) form.

The Expected Utility Theorem

- B4 Independence axiom: For any two lotteries (x, y, π) and (x, z, π), $y \succeq z$ if and only if $(x, y, \pi) \succeq (x, z, \pi)$.

This assumption is controversial and unlike any made in traditional microeconomic theory: you would not necessary want to assume that a consumer's preferences over sub-bundles of any two goods are independent of how much of a third good gets included in the overall bundle. But it is needed for the utility function to take the vN-M form.

The Expected Utility Theorem

There is a technical assumption that makes the expected utility theorem easier to prove.

- B5 There is a best lottery b and a worst lottery w.

This assumption will automatically hold if there are only a finite number of possible payoffs and if the independence axiom holds.

The Expected Utility Theorem

Finally, there are two additional assumptions that, strictly speaking, follow from those made already:

- B6 (implied by (B3, continuity)) Let x, y, and z satisfy $x \succ y \succ z$. Then there exists a probability π such that $(x, z, \pi) \sim y$.

- B7 (implied by (B4, independence)) Let $x \succ y$. Then $(x, y, \pi_1) \succ (x, y, \pi_2)$ if and only if $\pi_1 > \pi_2$.

The Expected Utility Theorem

Theorem (Expected Utility Theorem) If (B1)-(B7) hold, then there exists a utility function U defined over lotteries such that

$$U((x, y, \pi)) = \pi u(x) + (1 - \pi) u(y).$$

Note that we can prove the theorem simply by "constructing" the utility functions U and u with the desired properties.

The Expected Utility Theorem

Begin by setting

$$U(b) = 1$$
$$U(w) = 0$$

For any lottery z besides the best and worst, (B6) implies that there exists a probability π_z such that $(b, w, \pi_z) \sim z$ and (B7) implies that this probability is unique. For this lottery, set

$$U(z) = \pi_z$$

The Expected Utility Theorem

Condition (B7) also implies that with U so constructed, $z \succ z'$ implies

$$U(z) = \pi_z > \pi_{z'} = U\left(z'\right)$$

and $z \sim z'$ implies

$$U(z) = \pi_z = \pi_{z'} = U\left(z'\right)$$

so that U is a utility function that represents the underlying preference relation $\succeq$.

The Expected Utility Theorem

- Now let x and y denote two payoffs.
- By (B1a), each of these payoffs is equivalent to a lottery in which x or y is received with probability one.
- With this in mind, let

$$u(x) = U(x) = \pi_x$$
$$u(y) = U(y) = \pi_y$$

The Expected Utility Theorem

- Finally, let π denote a probability and consider the lottery $z = (x, y, \pi)$.

- Condition (B1c) implies

$$(x, y, \pi) \sim ((b, w, \pi_x), (b, w, \pi_y), \pi) \sim (b, w, \pi\pi_x + (1 - \pi)\pi_y)$$

- But this last expression is equivalent to

$$U(z) = U(x, y, \pi) = \pi\pi_x + (1 - \pi)\pi_y = \pi u(x) + (1 - \pi)u(y),$$

confirming that U has the vN-M form.

The Expected Utility Theorem

- Note that the key property of the $\mathrm{vN-M}$ utility function

$$U(z) = U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y)$$

- its linearity in the probabilities π and $1 - \pi$, is not preserved by all transformations of the form

$$V(z) = F(U(z))$$

where F is an increasing function.

- In this sense, vN-M utility functions are cardinal, not ordinal.

The Expected Utility Theorem

- On the other hand, given a vN-M utility function

$$U(z) = U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y)$$

consider an affine transformation

$$V(z) = \alpha U(z) + \beta$$

and define

$$v(x) = \alpha u(x) + \beta \text{ and } v(y) = \alpha u(y) + \beta$$

- We will show that also $V(z)$ is a vN-M utility function (in next slide)

The Expected Utility Theorem

- By assumption,

$$U(z) = U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y),$$
$$V(z) = \alpha U(z) + \beta.$$
$$v(x) = \alpha u(x) + \beta \text{ and } v(y) = \alpha u(y) + \beta$$

- Then

$$\begin{aligned} V(x, y, \pi) &= \alpha U(x, y, \pi) + \beta \\ &= \alpha[\pi u(x) + (1 - \pi)u(y)] + \beta \\ &= \pi[\alpha u(x) + \beta] + (1 - \pi)[\alpha u(y) + \beta] \\ &= \pi v(x) + (1 - \pi)v(y). \end{aligned}$$

- In this sense, the vN-M utility function that represents any given preference relation is not unique.

Exercise

- Under certainty, any increasing transformation of a utility function is also a utility function representing the same preferences
- Under uncertainty, we must restrict this statement to **linear transformations** if we are to keep the same preference representation.
- Let's check this property with a simple example

Exercise, cont'd

- Assume an initial utility function $u(\cdot)$ gives the following values to three prospects:

$$\begin{array}{ll} B & u(B) = 100 \\ M & u(M) = 10 \\ P & u(P) = 50 \end{array}$$

(a) Check that with this initial utility function, the lottery $L = (B, M, 0.50) \succ P$

(b) The proposed transformations are $f(x) = a + bx$ with $b > 0$ and $g(x) = \ln(x)$. Check that under f, $L \succ P$, but that under g, $P \succ L$. Discuss your results.

Exercise: Answers

(a) We have to show that $L = (B, M, 0.5) \succ P$. We have $U(L) = 0.5u(B) + 0.5u(M) = 0.5 \times 100 + 0.5 \times 10 = 55 > U(P) = 50$. But $U(L) > U(P) \implies L \succ P$.

Exercise: Answers, cont'd

(b1) Under transformation f, we have that lottery L has expected utility $f(U(L)) = 0.5(a + bu(B)) + 0.5(a + bu(M)) = a + b(0.5u(B) + 0.5u(M)) = a + b \times 55$, while "sure lottery" P has expected utility $f(U(P)) = a + bu(P) = a + b \times 50$. For any a and for any $b > 0$, $f(U(L)) > f(U(P))$ hence the ordering $L \succ P$ is preserved under f.

(b2) Under transformation g, $g(U(L)) = 0.5 \log u(B) + 0.5 \log u(M) = 0.5 \log(100) + 0.5 \log(10) = 3.45$. But $g(U(P)) = \log u(P) = \log 50 = 3.91$, hence $P \succ L$! The problem is that transformation g is not a linear transformation, hence it does not preserve the same preferences.

Exercise: Answers, cont'd

(b1) Under transformation f, we have that lottery L has expected utility $f(U(L)) = 0.5(a + bu(B)) + 0.5(a + bu(M)) = a + b(0.5u(B) + 0.5u(M)) = a + b \times 55$, while "sure lottery" P has expected utility $f(U(P)) = a + bu(P) = a + b \times 50$. For any a and for any $b > 0$, $f(U(L)) > f(U(P))$ hence the ordering $L \succ P$ is preserved under f.

(b2) Under transformation g, $g(U(L)) = 0.5 \log u(B) + 0.5 \log u(M) = 0.5 \log(100) + 0.5 \log(10) = 3.45$. But $g(U(P)) = \log u(P) = \log 50 = 3.91$, hence $P \succ L$! The problem is that transformation g is not a linear transformation, hence it does not preserve the same preferences.

Outline

The Allais Paradox

- As mentioned previously, the independence axiom has been and continues to be a subject of controversy and debate.

- **Maurice Allais** (France, 1911-2010, Nobel Prize 1988) constructed a famous example. Econometrica Vol. 21 (October 1953): pp.503-546.

Exercise

- We have seen that under certainty, any increasing monotone transformation of a utility function is also a utility function representing the same preferences. Under uncertainty, we must restrict this statement to linear transformations if we are to keep the same preference representation
- Let's do a simple exercise to check your understanding

The Allais Paradox

Consider two lotteries:

$$L_1 = \begin{cases} \$10000 & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_2 = \begin{cases} \$15000 & \text{with probability } 0.09 \\ \$0 & \text{with probability } 0.91 \end{cases}$$

Which would you prefer?

The Allais Paradox

Consider two lotteries:

$$L_1 = \begin{cases} \$10000 & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_2 = \begin{cases} \$15000 & \text{with probability } 0.09 \\ \$0 & \text{with probability } 0.91 \end{cases}$$

People tend to say $L_2 \succ L_1$.

But now consider other two lotteries:

$$L_3 = \begin{cases} \$10000 & \text{with probability } 1.00 \\ \$0 & \text{with probability } 0.00 \end{cases}$$

$$L_4 = \begin{cases} \$15000 & \text{with probability } 0.90 \\ \$0 & \text{with probability } 0.10 \end{cases}$$

Which would you prefer?

But now consider other two lotteries:

$$L_3 = \begin{cases} \$10000 & \text{with probability } 1.00 \\ \$0 & \text{with probability } 0.00 \end{cases}$$

$$L_4 = \begin{cases} \$15000 & \text{with probability } 0.90 \\ \$0 & \text{with probability } 0.10 \end{cases}$$

Many of the same people who say $L_2 \succ L_1$ often say $L_3 \succ L_4$.

Finally, consider

$$L_5 = \begin{cases} L_3 \text{ (\$10000 for sure)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_6 = \begin{cases} L_4 \text{ (\$15000 w/ prob 0.90)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

The independence axiom requires anyone who says $L_3 \succ L_4$ to also say that $L_5 \succ L_6$.

But notice that

$$L_5 = \begin{cases} L_3 \text{ (\$10000 for sure)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_6 = \begin{cases} L_4 \text{ (\$15000 w/ prob 0.90)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

are equivalent to

$$L_5 = \left\{ \begin{array}{ll} \$10000 & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{array} \right\} = L_1$$

$$L_6 = \left\{ \begin{array}{ll} \$15000 & \text{with probability } 0.09 \\ \$0 & \text{with probability } 0.91 \end{array} \right\} = L_2$$

The Allais Paradox

$$L_5 = \begin{cases} L_3 \text{ (\$10000 for sure)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

$$L_6 = \begin{cases} L_4 \text{ (\$15000 w/ prob 0.90)} & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{cases}$$

are equivalent to

$$L_5 = \left\{ \begin{array}{ll} \$10000 & \text{with probability } 0.10 \\ \$0 & \text{with probability } 0.90 \end{array} \right\} = L_1$$

$$L_6 = \left\{ \begin{array}{ll} \$15000 & \text{with probability } 0.09 \\ \$0 & \text{with probability } 0.91 \end{array} \right\} = L_2$$

The independence axiom requires anyone who says $L_3 \succ L_4$ to also say that $L_5 \succ L_6$. But some of these people say $L_6 = L_2 \succ L_5 = L_1$.

Outline

Generalizations of Expected Utility

- Another potential limitation of expected utility is that it does not capture preferences for early or late resolution of uncertainty.

- A generalization of expected utility that makes this distinction is proposed by **David Kreps** and **Evan Porteus**, "*Temporal Resolution of Uncertainty and Dynamic Choice Theory*," Econometrica Vol. 46 (January 1978): pp.185-200.

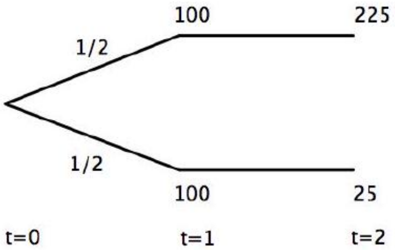
Generalizations of Expected Utility

- To model preferences for the temporal resolution of uncertainty, consider two assets.

- Both assets pay off \$100 next year for sure. And both assets pay off \$225 with probability $1/2$ and \$25 with probability $1/2$ two years from now.

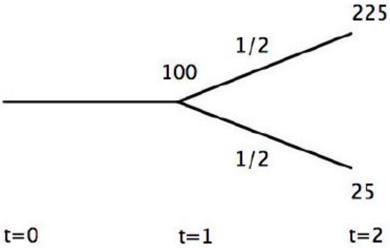
- But for asset 1, the payoff two years from now is revealed one year from now, whereas for asset 2, the payoff two years from now does not get revealed until the beginning of the second year.

Generalizations of Expected Utility



Asset 1 has early resolution of uncertainty.

Generalizations of Expected Utility



Asset 2 has late resolution of uncertainty.

Generalizations of Expected Utility

Kreps and Porteus allow the investor's utility function to take the form

$$E_0\left[u\left(p_1\right)\right] + E_0\left\{\left[E_1\left(u\left(p_2\right)\right)\right]^{\gamma}\right\},$$

where p_1 and p_2 are the payoffs one and two years from now, E_0 and E_1 are expected values based on information possessed today and one year from now, and the parameter γ is such that:

- if $\gamma = 1$ the investor has expected utility
- if $\gamma > 1$ the investor prefers early resolution (asset 1)
- if $\gamma < 1$ the investor prefers late resolution (asset 2)

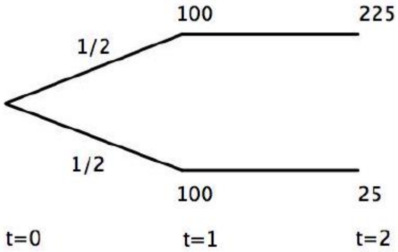
Generalizations of Expected Utility

To see how this works, let

$$u(p) = p^{1/2}$$

and call the state that leads to the 225 payoff two years from now the "good state" and the state that leads to the 25 payoff two years from now the "bad state."

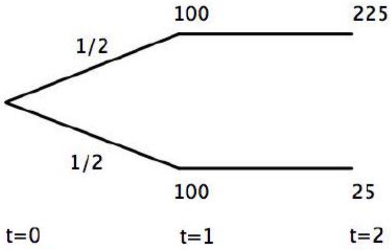
Generalizations of Expected Utility



For asset 1, $E_1(u(p_2))$ depends on the state:

$$E_1^G(u(p_2)) = (225)^{1/2} = 15 \text{ and } E_1^B(u(p_2)) = (25)^{1/2} = 5$$
$$E_0\{[E_1(u(p_2))]^{\gamma}\} = (1/2)15^{\gamma} + (1/2)5^{\gamma}$$

Generalizations of Expected Utility

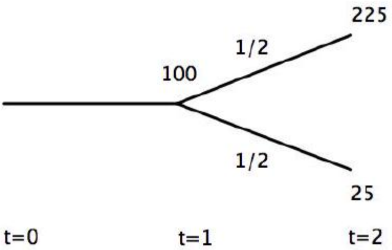


For asset 1:

$$E_0\left\{\left[E_1\left(u\left(p_2\right)\right)\right]^{\gamma}\right\} = (1/2)15^{\gamma} + (1/2)5^{\gamma}$$

$$E_0\left[u\left(p_1\right)\right] = (1/2)(100)^{1/2} + (1/2)(100)^{1/2} = 10$$

Generalizations of Expected Utility

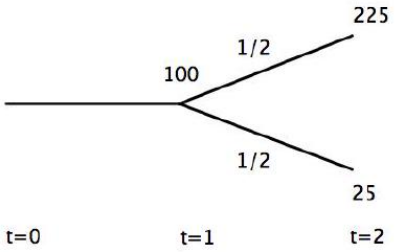


For asset 2:

$$E_1\left(u\left(p_2\right)\right) = (1/2)(225)^{1/2} + (1/2)(25)^{1/2} = (1/2)15 + (1/2)5 = 10$$

$$E_0\left\{\left[E_1\left(u\left(p_2\right)\right)\right]^{\gamma}\right\} = 10^{\gamma}$$

Generalizations of Expected Utility



For asset 2:

$$E_0 \left\{ \left[E_1 \left(u \left(p_2 \right) \right) \right]^{\gamma} \right\} = 10^{\gamma}$$

$$E_0 \left[u \left(p_1 \right) \right] = (100)^{1/2} = 10$$

Generalizations of Expected Utility

Hence, for asset 1, utility today is

$$U_1 = 10 + (1/2)15^{\gamma} + (1/2)5^{\gamma}$$

and for asset 2 , utility today is

$$U_2 = 10 + 10^{\gamma}$$

γ	U_1	U_2	
1	20.00	20.00	expected utility
1.5	44.64	41.62	prefers early resolution
0.5	13.05	13.16	prefers late resolution

Generalizations of Expected Utility

Hence, the Kreps-Porteus utility function "nests" vN-M expected utility as a special case, but allows for more general preferences over the timing of the resolution of uncertainty,

$$E_0 \left[u \left(p_1 \right) \right] + E_0 \left\{ \left[E_1 \left(u \left(p_2 \right) \right) \right]^{\gamma} \right\}$$

depending on whether utility today is linear (expected utility), convex (preference for early resolution), or concave (preference for late resolution) in $E_1 \left(u \left(p_2 \right) \right)$.

Generalizations of Expected Utility

- Another alternative to expected utility is proposed by **Daniel Kahneman** (Nobel Prize 2002) and **Amos Tversky**, "*Prospect Theory: An Analysis of Decision under Risk*," Econometrica Vol. 47 (March 1979): pp.263-292.

- Prospect theory suggests that investors may care not just about final payoffs but about whether those final payoffs represent gains or losses.

Generalizations of Expected Utility

Suppose that you already have \$1000 and can choose between two lotteries:

$$L_1 = \begin{cases} \$1000 & \text{with probability } 0.50 \\ \$0 & \text{with probability } 0.50 \end{cases}$$

$$L_2 = \begin{cases} \$500 & \text{with probability } 1 \\ \$0 & \text{with probability } 0 \end{cases}$$

Which would you prefer?

Generalizations of Expected Utility

Suppose that you are given \$1000 and must then choose between two lotteries:

$$L_1 = \begin{cases} \$1000 & \text{with probability } 0.50 \\ \$0 & \text{with probability } 0.50 \end{cases}$$

$$L_2 = \begin{cases} \$500 & \text{with probability } 1 \\ \$0 & \text{with probability } 0 \end{cases}$$

Most people say $L_2 \succ L_1$.

Generalizations of Expected Utility

Suppose instead that you are given \$2000 and must then choose between two lotteries:

$$L_3 = \begin{cases} -\$1000 & \text{with probability } 0.50 \\ \$0 & \text{with probability } 0.50 \end{cases}$$

$$L_4 = \begin{cases} -\$500 & \text{with probability } 1 \\ \$0 & \text{with probability } 0 \end{cases}$$

Which would you prefer?

Generalizations of Expected Utility

Suppose instead that you are given \$2000 and must then choose between two lotteries:

$$L_3 = \begin{cases} -\$1000 & \text{with probability } 0.50 \\ \$0 & \text{with probability } 0.50 \end{cases}$$

$$L_4 = \begin{cases} -\$500 & \text{with probability } 1 \\ \$0 & \text{with probability } 0 \end{cases}$$

Many people say $L_3 \succ L_4$.

Generalizations of Expected Utility

But in terms of final payoffs, L_1 is identical to L_3 and L_2 is identical to L_4 :

$$L_1 = \begin{cases} \$1000 + \$1000 & \text{with probability } 0.50 \\ \$1000 + \$0 & \text{with probability } 0.50 \end{cases}$$

$$L_2 = \begin{cases} \$1000 + \$500 & \text{with probability } 1 \\ \$1000 + \$0 & \text{with probability } 0 \end{cases}$$

$$L_3 = \begin{cases} \$2000 - \$1000 & \text{with probability } 0.50 \\ \$2000 + \$0 & \text{with probability } 0.50 \end{cases}$$

$$L_4 = \begin{cases} \$2000 - \$500 & \text{with probability } 1 \\ \$2000 + \$0 & \text{with probability } 0 \end{cases}$$

suggesting that respondents do care about gains versus losses.

Generalizations of Expected Utility

- Expected utility remains the dominant framework for analyzing economic decision-making under uncertainty.

- But a very active line of ongoing research continues to explore alternatives and generalizations.