# Productivity, Taxation and Evasion:

## A Quantitative Exploration of the Determinants of the Informal Economy

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Macroeconomics Seminar, June 6, 2016

# This topic

- Problems with Solow model
- Ramsey *optimal* growth model in discrete time

**Key elements:**

- Optimal savings, not an exogenous constant
- Intertemporal utility maximization

**Analysis:**

- Characterization of solution, two-dimensional phase diagram
- Saddle-path dynamics (sketch)

# Neoclassical Growth Model

- Optimal savings, not an exogenous constant $s$
  - In the Solow model, households don't optimize $\longrightarrow$ Bad for policy evaluation!
  - E.g. If productivity increases for a few periods, shouldn't $s$ respond? Maybe households would like to save some of the temporary increase in their income
- Solve the problem of a benevolent social planner
  - How should society save?
  - Without frictions, the allocation chosen by the social planner can be implemented using market arrangements (second welfare theorem)
  - Will see how to do this decentralization later

# Setup

- Discrete time $t = 0, 1, 2, \ldots$
- Aggregate production function with *constant returns to scale*

$$Y_t = A_t F(K_t, L_t)$$

(For now, keep things simple by setting $A_t = 1$ and $L_t = L$)

- Physical capital depreciates at rate $\delta$

$$K_{t+1} = (1-\delta) K_t + I_t, \quad 0 < \delta < 1, \quad K_0 > 0 \qquad (1)$$

- Goods may be either consumed or invested

$$C_t + I_t = Y_t \qquad (2)$$

- Combining (1) with (2), we get a sequence of resource constraints, one for each date $t$:

$$C_t + K_{t+1} = F(K_t, L) + (1-\delta) K_t, \quad K_0 > 0$$

# Intensive Form

- Resource constraints

$$C_t + K_{t+1} = F\left(K_t, L\right) + \left(1-\delta\right) K_t$$

Capital chosen at end of period $t$, $K_{t+1}$, will be used in production at time $t+1$.

- Dividing by employment, we get

$$\frac{C_t}{L} + \frac{K_{t+1}}{L} = \frac{F\left(K_t, L\right)}{L} + \left(1-\delta\right) \frac{K_t}{L},$$

hence

$$c_t + k_{t+1} = f\left(k_t\right) + \left(1-\delta\right) k_t$$

where $c_t = C_t/L$, $k_t = K_t/L$ and $f\left(k_t\right) = F\left(\frac{K_t}{L}, 1\right)$.

## Intertemporal Utility

- Social planner seeks to maximize intertemporal utility

$$U\left(\{c_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right), \quad 0 < \beta < 1$$

  with increasing and strictly concave period utility $u'\left(c\right) > 0$, $u''\left(c\right) < 0$.

- Future is discounted by constant factor $\beta$:

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) = u\left(c_0\right) + \beta u\left(c_1\right) + \beta^2 u\left(c_t\right) \ldots + \beta^t u\left(c_t\right) + \ldots$$

## Social Planner's problem

- The social planner solves the following constrained maximization problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the resource constraints

$$c_t + k_{t+1} = f(k_t) + (1-\delta) k_t, \quad k_0 > 0$$

- Note: the planner chooses *infinite* sequences of consumption, $\{c_0, c_1, c_2, \ldots\}$, and capital, $\{k_1, k_2, \ldots\}$ subject to *infinitely* many constraints

$$c_0 + k_1 = f(k_0) + (1-\delta) k_0$$

$$c_1 + k_2 = f(k_1) + (1-\delta) k_1$$

$$\ldots$$

$$c_t + k_{t+1} = f(k_t) + (1-\delta) k_t$$

$$\ldots$$

## Social Planner's problem, cont'd

- Consumption $c_t$ is called a *non-predetermined* variable, while capital $k_t$ is a *predetermined* or *state* variable.
  - At the start of period $t$, the planner has capital $k_t$ from the previous period (predetermined) and chooses $c_t$, and, consequentely, $k_{t+1}$.
- Note: The initial condition for capital, $k_0 > 0$, is *taken as given*, while $c_0$ is optimally *chosen*.

## Social Planner's problem, solution

- Discounted Lagrangian with multiplier $\lambda_t \geq 0$ for each resource constraint:

$$\mathbb{L}\left(\{c_t, k_{t+1}, \lambda_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t \{u(c_t) + \lambda_t [f(k_t) + (1-\delta)k_t - c_t$$

- First-order conditions are obtained by taking derivatives of the Lagrangian with respect to $c_t$, $k_{t+1}$ and $\lambda_t$:

$$c_t: \; \beta^t u'(c_t) - \beta^t \lambda_t = 0$$

$$k_{t+1}: \; -\beta^t \lambda_t + \beta^{t+1}\left[f'(k_{t+1}) + (1-\delta)\right] = 0$$

$$\lambda_t: f(k_t) + (1-\delta)k_t - c_t - k_{t+1} = 0$$

- These must hold for every $t = 0, 1, 2, ...$

# Key Optimality Conditions

- Eliminating the Lagrange multipliers, we get the following two **necessary** conditions for optimality:

$$u'\left(c_t\right) = \beta u'\left(c_{t+1}\right)\left[f'\left(k_{t+1}\right) + \left(1-\delta\right)\right] \qquad \text{(EE)}$$

and

$$c_t + k_{t+1} - \left(1-\delta\right)k_t = f\left(k_t\right) \qquad \text{(RC)}$$

- Equation (RC) simply restates the resource constraints for the economy: consumption plus investment is equal to output in each period.
- Equation (EE) is called "Euler equation"
  - It is the key condition for *intertemporal* maximization
  - It plays such an important role in modern macroeconomics that we will see (at least) *two* ways of explaining it

# Euler Equation: First Interpretation

$$\text{EE: } u'(c_t) = \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right]$$

- EE is a **necessary** condition for optimality. If it does not hold, then planner is **not** optimizing.
- Suppose that, for some $t$,

$$u'(c_t) < \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right] \tag{3}$$

- Reduce consumption by $\Delta$ units at date $t$
- Loss in utility today is approx. $u'(c_t)\Delta$
- Plan: take $\Delta$, invest it in capital, consume the proceeds in $t+1$
- In $t+1$ output will increase by $\left[f'(k_{t+1}) + (1-\delta)\right]\Delta$
- Gain in utility tomorrow is approx. $u'(c_{t+1})\left[f'\left(k^*_{t+1}\right) + (1-\delta)\right]\Delta$

# Euler Equation: First Interpretation, cont'd

- This "alternative plan" generates
  - Utility loss approx. equal to $u'(c_t)\,\Delta$
  - Utility gain approx. equal to $\beta u'(c_{t+1})\,[f'(k_{t+1}) + (1-\delta)]\,\Delta$
- By assumption (see eq. 3),

$$\beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right]\Delta - u'(c_t)\,\Delta > 0$$

- Since the change in total utility is positive, $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ is not optimal.

# Euler Equation: Second Interpretation

$$\text{EE: } u'(c_t) = \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right]$$

- Euler equation can be rewritten as:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + (1-\delta)$$

- MRS between $t$ and $t+1$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})}$$

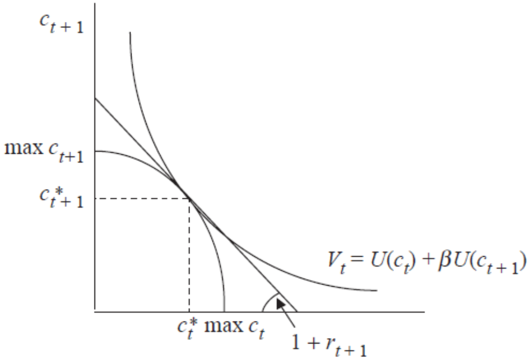
- MRT between $t$ and $t+1$

$$f'(k_{t+1}) + (1-\delta)$$

- Planner equates MRS and MRT.

# Euler Equation: Second Interpretation, Graph

$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{\text{MRS between } c_t \text{ and } c_{t+1}} = \underbrace{f'(k_{t+1}) + (1-\delta)}_{\text{MRT}}$$



# Full Solution

- Initial condition

$$k_0 > 0$$

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[ f'(k_{t+1}) + 1 - \delta \right]$$

- Resource constraint

$$c_t + k_{t+1} = f(k_t) + (1 - \delta) k_t$$

- Transversality condition (TVC)

$$\lim_{T \to \infty} u'(c_T) k_{T+1} = 0$$

(In the finite horizon version with $T < \infty$, TVC is simply $k_{T+1} = 0$)

# Characterizing the Solution

1. Steady state
   - Economy is in long-run equilibrium if $c_t = c_{t+1} = \overline{c}$ and $k_t = k_{t+1} = \overline{k}$
   - Solve for $\left(\overline{c}, \overline{k}\right)$ as a function of parameters of the model
   - If we change a parameter in the economy, how does the steady state change?
     - *Steady state comparative statics*
2. Dynamics
   - Is the steady-state *globally stable*?
   - What determines the *local dynamics* around the steady-state?
   - From steady state, if we change a parameter, how does the economy evolve?
     - Dynamic comparative statics

# 1. Steady State

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[ f'(k_{t+1}) + 1 - \delta \right]$$

- Resource constraint

$$c_t + k_{t+1} = f(k_t) + (1 - \delta) k_t$$

# 1. Steady State

- Euler equation

$$u'(\overline{c}) = \beta u'(\overline{c}) \left[ f'\left(\overline{k}\right) + 1 - \delta \right]$$

- Resource constraint

$$\overline{c} + \overline{k} = f\left(\overline{k}\right) + (1-\delta)\,\overline{k}$$

- Steady state Euler equation pins down $\overline{k}$,

$$1 = \beta \left[ f'\left(\overline{k}\right) + 1 - \delta \right]$$

- Resource constraint then determines $\overline{c}$,

$$\overline{c} = f\left(\overline{k}\right) - \delta \overline{k}$$

## 1. Steady State: Modified golden rule

- Steady state consumption, as a function of steady state capital

$$\bar{c} = f\left(\bar{k}\right) - \delta \bar{k}$$

- Note that $\bar{c}$ is maximized at the "golden rule" level, where

$$f'\left(k^{GR}\right) = \delta$$

- In this model, steady state capital is determined by

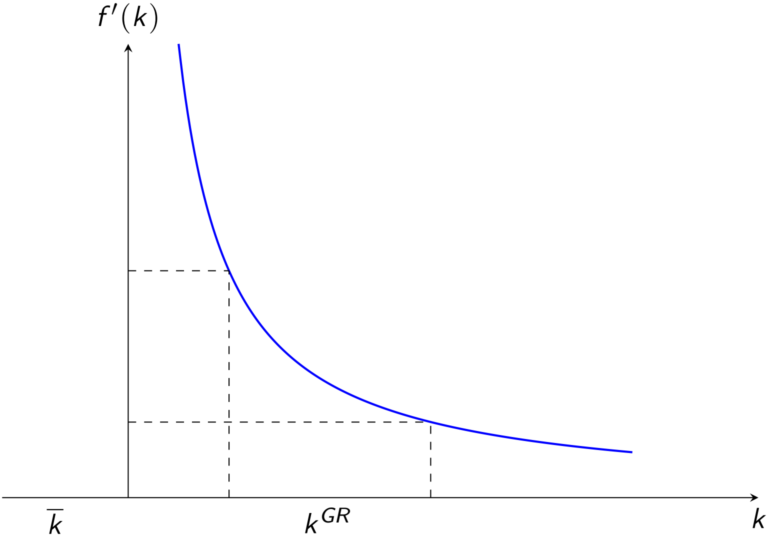
$$f'\left(\bar{k}\right) = \frac{1}{\beta} - 1 + \delta$$

- Steady state capital is less than the golden rule level

$$f'\left(\bar{k}\right) = \frac{1}{\beta} - 1 + \delta > \delta = f'\left(k^{GR}\right) \implies \bar{k} < k^{GR}$$

since marginal product of capital $f'\left(k\right)$ is decreasing.

# 1. Steady State: Modified golden rule



# 1. Steady State: comparative statics

$$f'\left(\overline{k}\right) = \frac{1}{\beta} - 1 + \delta$$
$$\overline{c} = f\left(\overline{k}\right) - \delta\overline{k}$$

- Decrease in discount factor: $\downarrow \beta$
  - Value future output less, so $\uparrow f'\left(\overline{k}\right)$ and $\downarrow \overline{k}$
  - Also lower $\downarrow \overline{c}$
- Increase in capital depreciation rate $\uparrow \delta$
  - Requires higher $\uparrow f'\left(\overline{k}\right)$, so $\downarrow \overline{k}$
  - Also lower $\downarrow \overline{c}$
- Why is $c$ increasing in $k$ around $\left(\overline{c}, \overline{k}\right)$?

# 2. Dynamics: Phase diagram

- Assuming $u\left(c_t\right) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$, Euler equation

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left[f'\left(k_{t+1}\right) + 1 - \delta\right]$$

- Resource constraint

$$c_t + k_{t+1} = f\left(k_t\right) + \left(1-\delta\right)k_t$$

- No change in consumption: $c_{t+1} = c_t$

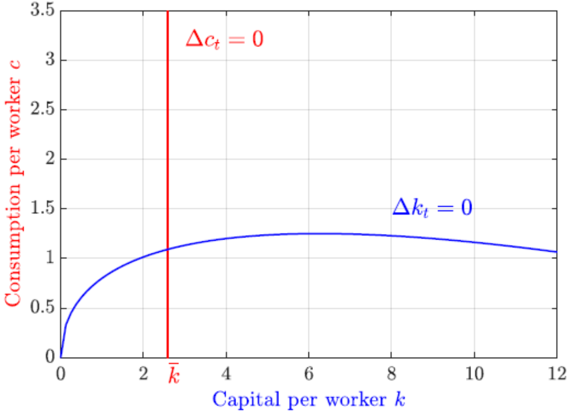
$$1 = \beta\left[f'\left(k_t\right) + 1 - \delta\right] \implies k_t = \overline{k}$$

- No change in capital: $k_{t+1} = k_t$

$$c_t = f\left(k_t\right) - \delta k_t$$

# 2. Dynamics: Phase diagram

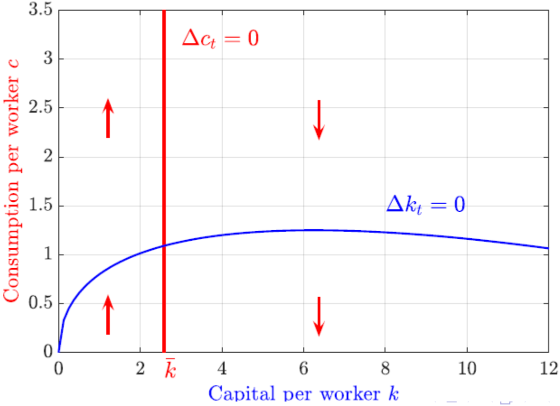
$$\Delta c_t = 0: \quad k_t = \overline{k}$$
$$\Delta k_t = 0: \quad c_t = f(k_t) - \delta k_t$$



# 2. Dynamics: Phase diagram

$$(c_{t+1}/c_t)^{\sigma} = \beta \left[ f'(k_{t+1}) + 1 - \delta \right]$$

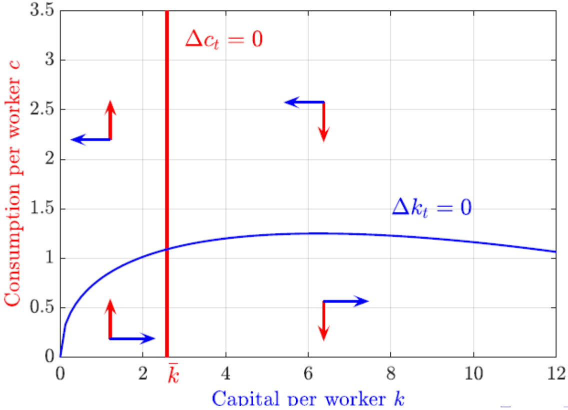
If $k_{t+1} > \overline{k}$, then low MPK $f'(k_{t+1})$, so $\uparrow c_t, \downarrow c_{t+1}$ so consumption is falling



## 2. Dynamics: Phase diagram

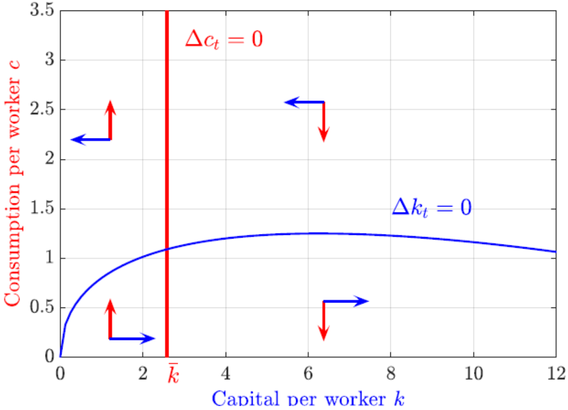
$$k_{t+1} - k_t = f\left(k_t\right) - \delta k_t - c_t, \quad \bar{c} = f\left(\bar{k}\right) - \delta \bar{k}$$

If $c_t > \bar{c}\left(\bar{k}\right)$, then consuming more than $f\left(k_t\right) - \delta k_t$ so $k_{t+1} < k_t$: capital is falling



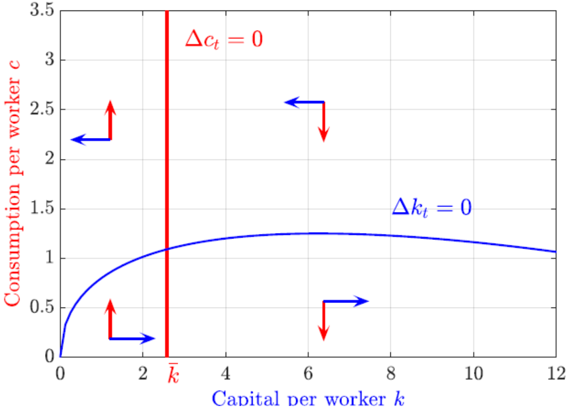
# 2. Dynamics: Phase diagram

- To the upper-left (↖) we violate the resource constraint
- Increasing marginal product of capital, increasing consumption, at some point $c_t > f(k_t) + (1-\delta) k_t$



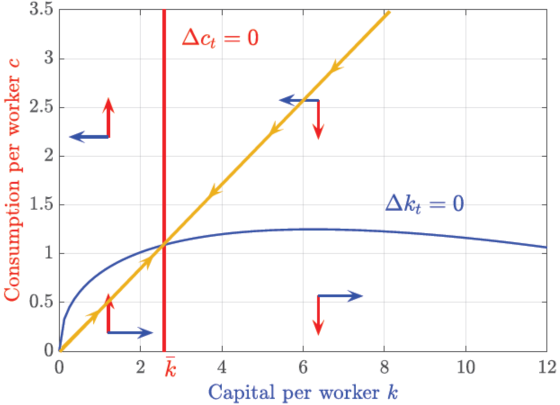
# 2. Dynamics: Phase diagram

- To the bottom-right ($\searrow$) we violate the transversality condition
- Increasing capital, falling consumption, $u'(c_t) \to \infty$



# 2. Dynamics: Saddle path

- On the *saddle-path* all equilibrium conditions hold
- Economy converges to the steady state



# 2. Dynamics: Comparative statics

- If initially in the steady state and we permanently change a parameter?
- What changes in $\Delta c_t = 0$ and $\Delta k_t = 0$ lines?
- Is there a new saddle path?
- Consumption jumps to new saddle path and economy converges towards (new) steady state.

## 2. Dynamics: Decrease in productivity $\downarrow A$

- $\Delta k_t = 0$ line:

$$\overline{c} = Af\left(\overline{k}\right) - \delta \overline{k}$$

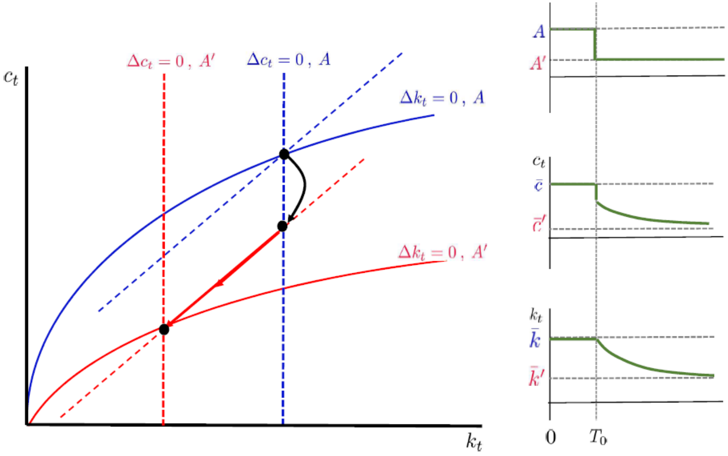
- $\Delta c_t = 0$ line

$$1 = \beta \left[ Af'\left(\overline{k}\right) + 1 - \delta \right]$$

- A decrease in $A$ will shift both $\Delta k_t = 0$ and $\Delta c_t = 0$ lines:
  - The line $\Delta c_t = 0$ will shift to the left
  - The line $\Delta k_t = 0$ will shift low and to the left
  - The "new" saddle-path lies below the "old" saddle-path

# 2. Dynamics: Decrease in productivity $\downarrow A$

$\downarrow c_0$ and decrease capital to smooth transition to lower $\overline{c}$, $\overline{k}$



# Decentralization: Bring In the Market

- See page 53 Mongey lecture 3