LH Advanced Financial Markets - Part B

Topic 3: Measuring Risk and Risk Aversion

Dr. Alessandro Di Nola

University of Birmingham

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Outline

Measuring Risk Aversion

- We've already seen that within the von Neumann-Morgenstern expected utility framework, risk aversion enters through the concavity of the Bernoulli utility function.
- More specifically, we say that a function u is concave if for any x and y and for any $\pi \in [0, 1]$,

$$u(\pi x + (1 - \pi)y) \geq \pi u(x) + (1 - \pi)u(y)$$

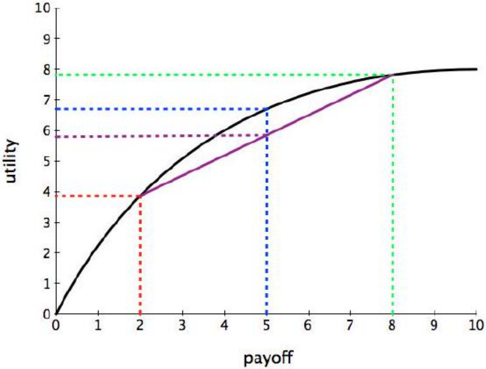
- Geometrically, the graph of the function u always lies above the line connecting the points $(x, u(x))$ and $(y, u(y))$

Measuring Risk Aversion

- When u is concave, a payoff of 5 for sure is preferred to a payoff of 8 with probability 1/2 and 2 with probability 1/2
- Indeed, if u is concave,

$$u(5) = u\left(\frac{1}{2}2 + \frac{1}{2}8\right) \geq \frac{1}{2}u(2) + \frac{1}{2}u(8)$$

Expected Utility Functions

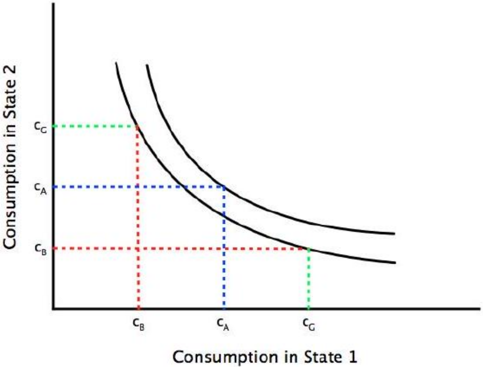


When u is concave, a payoff of 5 for sure is preferred to a payoff of 8 with probability $1/2$ and 2 with probability $1/2$.

Measuring Risk Aversion

- We've also seen previously that concavity of the utility function is related to convexity of indifference curves.
- In standard microeconomic theory, this feature of preferences represents a "taste for diversity."
- Under uncertainty, it represents a desire to smooth consumption across future states of the world.

Expected Utility Functions



A risk averse consumer prefers $c_A = (c_G + c_B)/2$ in both states to c_G in one state and c_B in the other.

Measuring Risk Aversion

- Mathematically, $u'(p) > 0$ means that an investor prefers higher payoffs to lower payoffs, and $u''(p) < 0$ means that the investor is risk averse.

- But is there a way of quantifying an investor's degree of risk aversion?

- And is there a criterion according to which we might judge one investor to be more risk averse than another?

Measuring Risk Aversion

- Since $u''(p) < 0$ makes an investor risk averse, one conjecture would be to say that an investor with Bernoulli utility function $v(p)$ is more risk averse than another investor with Bernoulli utility function $u(p)$ if $v''(p) < u''(p)$ for all payoffs p.

- But does a "more concave" Bernoulli utility function always correspond to greater risk aversion?

- Unfortunately, no.

Measuring Risk Aversion

Recall that the preference ordering of an investor with $\mathrm{vN-M}$ utility function

$$U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y)$$

is also represented by the $\mathrm{vN-M}$ utility function

$$V(x, y, \pi) = \alpha U(x, y, \pi)$$

for any value of $\alpha > 0$.

But with

$$\begin{aligned}
V(x,y,\pi) &= \alpha U(x,y,\pi) \\
&= \alpha\pi u(x) + \alpha(1-\pi)u(y) \\
&= \pi\alpha u(x) + (1-\pi)\alpha u(y) \\
&= \pi v(x) + (1-\pi)v(y)
\end{aligned}$$

where

$$v(p) = \alpha u(p)$$

for any payoff p.

Measuring Risk Aversion

Now

$$v(p) = \alpha u(p)$$

implies

$$v'(p) = \alpha u'(p)$$

and

$$v''(p) = \alpha u''(p)$$

By making α larger or smaller, the Bernoulli utility function can be made "more" or "less" concave without changing the underlying preference ordering.

Measuring Risk Aversion

In the mid-1960s, **Kenneth Arrow** and **John Pratt** proposed two alternative measures of risk aversion that are immune to this problem:

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

where Y measures the investor's income level.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

- Absolute risk aversion applies to bets over absolute dollar amounts: $\pm\$1000$.

- Relative risk aversion applies to bets expressed relative to (as a fraction of) income: ± 1 percent of Y.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

Since $v(p) = \alpha u(p)$ implies $v'(p) = \alpha u'(p)$ and $v''(p) = \alpha u''(p)$, these measures are **invariant to affine transformations** of the Bernoulli utility function.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

And since both measures have a minus sign out in front, both are **positive** and **increase when risk aversion rises**.

Measuring Risk Aversion

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

The notation $R_A(Y)$ and $R_R(Y)$ emphasizes that both measures of risk aversion can **depend on the investor's income** Y. A given bet can seem more or less risky, depending on the investor's income.

Outline

Interpreting the Measures of Risk Aversion

Recall from calculus the theorem stated by **Brook Taylor** (England, 1685-1731), regarding the approximation of a function f using its derivatives:

- "first-order" (linear) approximation

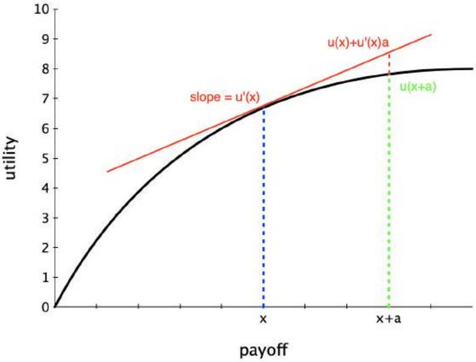
$$f(x+a) \approx f(x) + f'(x)a$$

- "second-order" (quadratic) approximation

$$f(x+a) \approx f(x) + f'(x)a + \frac{1}{2}f''(x)a^2$$

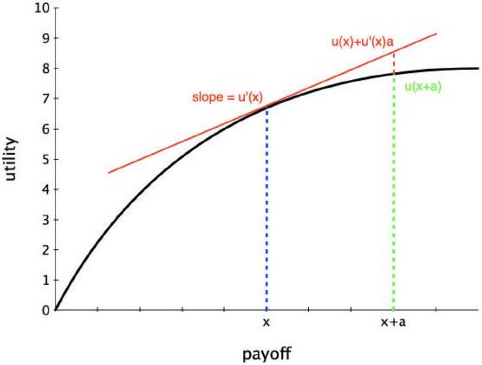
The second-order approximation is more accurate than the first, and both become more accurate as a becomes smaller.

Interpreting the Measures of Risk Aversion



The linear approximation $u(x+a) \approx u(x) + u'(x)a$ overstates $u(x+a)$ when u is concave.

Interpreting the Measures of Risk Aversion



Since $u''(x) < 0$, the quadratic approximation
$u(x+a) \approx u(x) + u'(x)a + (1/2)u''(x)a^2$ will be more accurate.

Interpreting the Measures of Risk Aversion

- Focusing first on the measure of absolute risk aversion, consider an investor with initial income Y who is offered a bet: win h with probability π and lose h with probability $1 - \pi$.

- A risk-averse investor with vN-M expected utility would never accept this bet if $\pi = 1/2$.

- The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

- Let π^* be the probability that is just high enough to get the investor to accept the bet.

- Then π^* must satisfy

$$u(Y) = \pi^* u(Y + h) + (1 - \pi^*)\, u(Y - h)$$

Interpreting the Measures of Risk Aversion

Take second-order Taylor approximations to $u(Y+h)$ and $u(Y-h)$:

$$u(Y+h) \approx u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$u(Y-h) \approx u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2$$

Interpreting the Measures of Risk Aversion

$$u(Y) = \pi^* u(Y+h) + (1-\pi^*)\, u(Y-h)$$

$$u(Y+h) \approx u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2$$

$$u(Y-h) \approx u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2$$

imply

$$\begin{aligned} u(Y) \approx \pi^* &\left[u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2\right] \\ &+ (1-\pi^*)\left[u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2\right] \end{aligned}$$

Interpreting the Measures of Risk Aversion

$$u(Y) \approx \pi^* \left[u(Y) + u'(Y)h + \frac{1}{2}u''(Y)h^2 \right]$$
$$+ (1 - \pi^*) \left[u(Y) - u'(Y)h + \frac{1}{2}u''(Y)h^2 \right]$$

implies

$$u(Y) \approx u(Y) + (2\pi^* - 1)\, u'(Y)h + \frac{1}{2}u''(Y)h^2$$

Interpreting the Measures of Risk Aversion

$$
\begin{aligned}
u(Y) &\approx u(Y) + (2\pi^* - 1)\, u'(Y)h + \frac{1}{2}u''(Y)h^2 \\
0 &\approx (2\pi^* - 1)\, u'(Y)h + \frac{1}{2}u''(Y)h^2 \\
0 &\approx (2\pi^* - 1)\, u'(Y) + \frac{1}{2}u''(Y)h \\
2\pi^* u'(Y) &\approx u'(Y) - \frac{1}{2}u''(Y)h \\
\pi^* &\approx \frac{1}{2} + \frac{1}{4}\left[-\frac{u''(Y)}{u'(Y)}\right] h
\end{aligned}
$$

Interpreting the Measures of Risk Aversion

- Since

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion,}$$

it follows from these calculations that

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{u''(Y)}{u'(Y)}\right] h = \frac{1}{2} + \frac{1}{4} h R_A(Y) > \frac{1}{2}.$$

- The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus h relates directly to the coefficient of absolute risk aversion.

Interpreting the Measures of Risk Aversion

- As an example, suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus $h = \$1000$?

- And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $h = \$1000$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} h R_A(Y)$$

implies

$$\begin{aligned} 0.75 &\approx 0.50 + \frac{1000}{4} R_A(Y) \\ 0.25 &\approx 250 R_A(Y) \\ R_A(Y) &\approx \frac{0.25}{250} = 0.001 \end{aligned}$$

Interpreting the Measures of Risk Aversion

- Realistically, a bet over $1000 is probably going to seem more risky to someone who starts out with less income.

- In general, $R_A(Y)$ can depend on Y.

- More specifically, it seems likely that $R_A(Y)$ decreases when Y goes up, so that
$$R_A'(Y) < 0$$

Interpreting the Measures of Risk Aversion

- Suppose, however, that the Bernoulli utility function takes the form
$$u(Y) = -\frac{1}{\nu}e^{-\nu Y}$$
where $\nu > 0$ and e^x is the exponential function ($e \approx 2.718$).
- Recall that exponential function has the special property that
$$f(x) = e^x \Rightarrow f'(x) = e^x$$
- By the chain rule
$$g(x) = e^{\alpha x} \Rightarrow g'(x) = \alpha e^{\alpha x}$$

Interpreting the Measures of Risk Aversion

With

$$u(Y) = -\frac{1}{\nu}e^{-\nu Y}$$

it follows that

$$u'(Y) = -\frac{1}{\nu}e^{-\nu Y}(-\nu) = e^{-\nu Y}$$
$$u''(Y) = -\nu e^{-\nu Y}$$
$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \frac{\nu e^{-\nu Y}}{e^{-\nu Y}} = \nu$$

so that this utility function displays constant absolute risk aversion, which does not depend on income.

Interpreting the Measures of Risk Aversion

So if we were willing to make the assumption of constant absolute risk aversion, we could use the results from our example, where an investor requires $\pi^* = 0.75$ to accept a bet with $h = \$1000$ to set $\nu = 0.001$ in

$$u(Y) = -\frac{1}{\nu} e^{-\nu Y}$$

and thereby tailor portfolio decisions specifically for this investor.

Interpreting the Measures of Risk Aversion

- Absolute risk aversion describes an investor's attitude towards absolute bets of plus or minus h.

- A similar analysis shows that relative risk aversion describes an investor's attitude towards relative bets of plus or minus kY, so that now, k is a fraction of total income.

Interpreting the Measures of Risk Aversion

- Consider an investor with initial income Y who is offered a bet: win kY with probability π and lose kY with probability $1 - \pi$.
- A risk-averse investor with $\mathrm{vN} - \mathrm{M}$ expected utility would never accept this bet if $\pi = 1/2$.
- The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

- Let π^* be the probability that is just high enough to get the investor to accept the bet.

- Now π^* must satisfy

$$u(Y) = \pi^* u(Y + Yk) + (1 - \pi^*)\, u(Y - Yk)$$

- As before, we can take second-order Taylor approximations to $u(Y + Yk)$ and $u(Y - Yk)$ and get an approximated formula that gives π^* as a function of the coefficient of relative risk aversion and k (left as an exercise)

Interpreting the Measures of Risk Aversion

- Recall

$$u(Y) = \pi^* u(Y + Yk) + (1 - \pi^*)\, u(Y - Yk)$$

- Since

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion,}$$

it follows that (proof omitted)

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{Yu''(Y)}{u'(Y)}\right] k = \frac{1}{2} + \frac{1}{4} k R_R(Y) > \frac{1}{2}.$$

- The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus kY relates directly to the coefficient of relative risk aversion.

Interpreting the Measures of Risk Aversion

- Suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus one percent ($k = 0.01$) of your income?

- And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $k = 0.01$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} k R_R(Y)$$

implies

$$0.75 \approx 0.50 + \frac{0.01}{4} R_R(Y)$$
$$0.25 \approx 0.0025 R_R(Y)$$
$$R_R(Y) = \frac{0.25}{0.0025} = 100$$

Interpreting the Measures of Risk Aversion

- Again, our notation $R_R(Y)$ allows relative risk aversion to depend on income Y.

- On the other hand, since the coefficient of relative risk aversion describes aversion to risk over bets that are expressed relative to income, it is more plausible to assume that investors have constant relative risk aversion.

Interpreting the Measures of Risk Aversion

- Suppose the Bernoulli utility function takes the form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

where $\gamma > 0$.

- For this function, **de l'Hôpital's rule** implies that when $\gamma = 1$

$$\frac{Y^{1-\gamma} - 1}{1 - \gamma} = \ln(Y),$$

where ln denotes the natural logarithm.

- This was the form that **Daniel Bernoulli** used to describe preferences over payoffs.

Interpreting the Measures of Risk Aversion

- By de l'Hôpital's rule

$$\lim_{\gamma\to 1}\frac{Y^{1-\gamma}-1}{1-\gamma} = \lim_{\gamma\to 1}\frac{\frac{d}{d\gamma}\left(Y^{1-\gamma}-1\right)}{\frac{d}{d\gamma}(1-\gamma)}$$
$$= \lim_{\gamma\to 1}\frac{-\ln(Y)Y^{1-\gamma}}{-1}$$
$$= \ln(Y)$$

- Note: Recall the following rule about derivatives:

$$\frac{d}{dx}a^x = \ln a \cdot \; a^x$$

Interpreting the Measures of Risk Aversion

With

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

it follows that

$$u'(Y) = Y^{-\gamma}$$
$$u''(Y) = -\gamma Y^{-\gamma-1}$$
$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \frac{Y\gamma Y^{-\gamma-1}}{Y^{-\gamma}} = \gamma,$$

so that this utility function displays constant relative risk aversion, which does not depend on income.

Interpreting the Measures of Risk Aversion

So if we were willing to make the assumption of constant relative risk aversion, we could use the results from our example, where an investor requires $\pi^* = 0.75$ to accept a bet with $k = 0.01$ to set $\gamma = 100$ in

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

and thereby tailor portfolio decisions specifically for this investor.

Interpreting the Measures of Risk Aversion

Finally, suppose that we do away with the concavity of the Bernoulli utility function and simply assume that

$$u(p) = \alpha p + \beta$$

where $\alpha > 0$, so that higher payoffs are preferred to lower payoffs. For this utility function,

$$u'(Y) = \alpha \text{ and } u''(Y) = 0$$

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = 0 \text{ and } R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = 0$$

This investor is risk-neutral and cares only about expected payoffs.

Outline

Risk Premium and Certainty Equivalent

- Our thought experiments so far have asked about how probabilities need to be boosted in order to induce a risk-averse investor to accept an absolute or relative bet.

- Let's take step away from gambling and towards investing by asking: suppose that an investor with income Y has the opportunity to buy an asset with a payoff $\tilde{Z}$ that is random and has expected value $E(\tilde{Z})$.

Risk Premium and Certainty Equivalent

- If this investor is risk-averse and has vN-M expected utility, he or she will always prefer an alternative asset that pays off $E(\tilde{Z})$ for sure. Mathematically,

$$u[Y + E(\tilde{Z})] \geq E[u(Y + \tilde{Z})]$$

"the utility of the expectation is greater than the expectation of utility."

Risk Premium and Certainty Equivalent

- This follows from an important mathematical result known as Jensen's inequality

- Theorem (Jensen's Inequality) Let g be a concave function and $\tilde{x}$ be a random variable. Then

$$g[E(\tilde{x})] \geq E[g(\tilde{x})].$$

  Furthermore, if g is strictly concave and the probability that $\tilde{x} \neq E(\tilde{x})$ is greater than zero, the inequality is strict.

- Examples of concave (increasing) functions: $\log(x)$, $\sqrt{x}$, $-1/x$, $-e^{-x}$ etc.

Risk Premium and Certainty Equivalent

- **Example**. Consider an asset $\tilde{Z}$ that pays off 8 with prob. 1/2 and 2 with prob. 1/2. Assume that the investor has Bernoulli utility function $u(x) = \sqrt{x}$ and initial wealth $Y = 0$.

- Then $E(\tilde{Z}) = \frac{1}{2} \cdot 8 + \frac{1}{2} \cdot 2 = 5$, so that

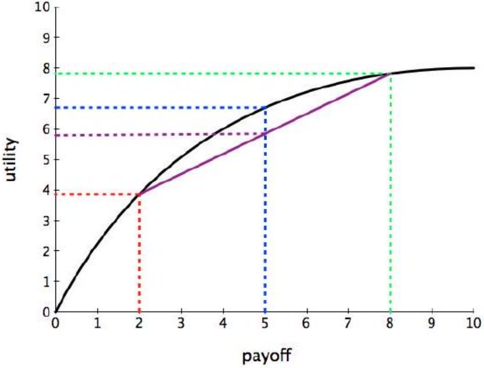
$$u[E(\tilde{Z})] = u[5] = \sqrt{5} = 2.24$$

and

$$E[u(\tilde{Z})] = \frac{1}{2} \cdot \sqrt{8} + \frac{1}{2}\sqrt{2} = 2.12$$

- Since u is concave, $E[u(\tilde{Z})] \leq u[E(\tilde{Z})]$, confirming Jensen's inequality

Risk Premium and Certainty Equivalent



This graph illustrates a special case of Jensen's inequality. The result holds much more generally.

Risk Premium and Certainty Equivalent

- Let $CE(\tilde{Z})$, or certainty equivalent of asset $\tilde{Z}$, be the maximum riskless payoff that a risk-averse investor is willing to exchange for that asset. Hence $CE(\tilde{Z})$ satisfies:

$$u[Y + CE(\tilde{Z})] = E[u(Y + \tilde{Z})]$$

- But since u is concave, from Jensen's inequality we know that

$$E[u(Y + \tilde{Z})] \leq u[Y + E(\tilde{Z})]$$

- Hence the certainty equivalent must also satisfy

$$u[Y + CE(\tilde{Z})] \leq u[Y + E(\tilde{Z})]$$

from which it follows that

$$CE(\tilde{Z}) \leq E(\tilde{Z})$$

Risk Premium and Certainty Equivalent

- We have just shown that for a risk averse investor, it always holds that $CE(\tilde{Z}) \leq E(\tilde{Z})$

- In plain English: A risk-averse investor dislikes uncertainty, so they are willing to accept a guaranteed amount that is less than the expected value of a risky investment

- The difference between the higher expected value $E(\tilde{Z})$ and the smaller certainty equivalent $CE(\tilde{Z})$ can then be used to define the positive risk premium $\Psi(\tilde{Z})$ for the asset:

$$\Psi(\tilde{Z}) = E(\tilde{Z}) - CE(\tilde{Z}) \geq 0$$

Risk Premium and Certainty Equivalent

- The certainty equivalent and risk premium are "two sides of the same coin"

$$\Psi(\tilde{Z}) = E(\tilde{Z}) - CE(\tilde{Z})$$

- $CE(\tilde{Z}) =$ lesser amount the investor is willing to accept to remain invested in the risk-free asset

- $\Psi(\tilde{Z}) =$ extra amount the investor needs to take on additional risk

Risk Premium and Certainty Equivalent

Combining the definitions of the certainty equivalent $CE(\tilde{Z})$,

$$E[u(Y+\tilde{Z})] = u[Y + CE(\tilde{Z})]$$

and the risk premium $\Psi(\tilde{Z})$,

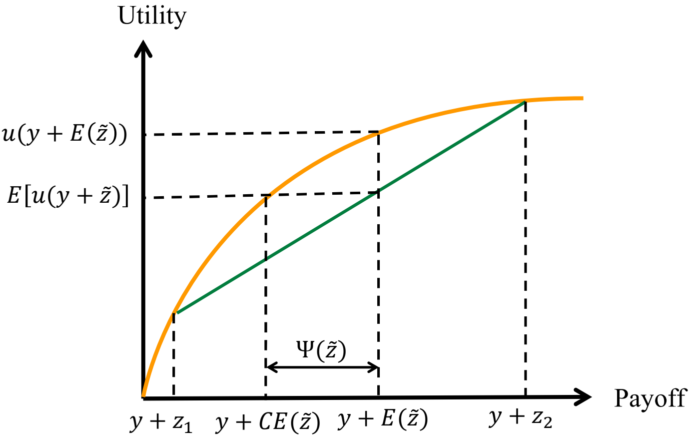
$$CE(\tilde{Z}) = E(\tilde{Z}) - \Psi(\tilde{Z}),$$

yields

$$E[u(Y+\tilde{Z})] = u[Y + E(\tilde{Z}) - \Psi(\tilde{Z})]$$

which we can use to link the risk premium $\Psi(\tilde{Z})$ to our measures of **risk aversion** (i.e. Arrow-Pratt coefficient(s))

Risk Premium and Certainty Equivalent



Risk Premium and Certainty Equivalent

- With this fact in mind, return to

$$E[u(Y + \tilde{Z})] = u[Y + E(\tilde{Z}) - \Psi(\tilde{Z})]$$

  but let's make our life easier and assume $\tilde{Z}$ is a zero-mean risky gamble, i.e. $E(\tilde{Z}) = 0$

- The definition of risk premium becomes

$$E\left[u(Y + \tilde{Z})\right] = u\left[Y - \Psi(\tilde{Z})\right]$$

Risk Premium and Certainty Equivalent

- Take a **second-order** Taylor approximation to $u(Y + \tilde{Z})$, viewing $\tilde{Z}$ as the "size of the bet"

$$u(Y + \tilde{Z}) \approx u(Y) + u'(Y)\tilde{Z} + \frac{1}{2}u''(Y)\tilde{Z}^2$$

- Note: we need a second-order or quadratic approximation to allow for the variability of $\tilde{Z}$

Risk Premium and Certainty Equivalent

Now take the expected value on both sides and simplify, using the fact that Y is not random:

$$
\begin{aligned}
E\left[u(Y+\tilde{Z})\right] &\approx E\left[u(Y)\right] + E\left[u'(Y)\tilde{Z}\right] + E\left[\frac{1}{2}u''(Y)\tilde{Z}^2\right] \\
&= u(Y) + u'(Y)E[\tilde{Z}] + \frac{1}{2}u''(Y)E[\tilde{Z}^2]
\end{aligned}
$$

Risk Premium and Certainty Equivalent

- Finally, use the fact that $E[\tilde{Z}] = 0$ and the definition of the variance of $\tilde{Z}$ to simplify further:

$$\begin{aligned} E\left[u(Y+\tilde{Z})\right] &\approx u(Y) + u'(Y)\,E[\tilde{Z}] + \frac{1}{2}u''(Y)\,E[\tilde{Z}^2] \\ &= u(Y) + \frac{1}{2}\sigma^2(\tilde{Z})u''(Y) \end{aligned}$$

- Note: Recall that for any random variable $\tilde{Z}$, the variance is defined as $\sigma^2(\tilde{Z}) = E\left[(\tilde{Z} - E[\tilde{Z}])^2\right]$ but in this case $E\left(\tilde{Z}\right) = 0$, therefore $\sigma^2(\tilde{Z}) = E[\tilde{Z}^2]$

Risk Premium and Certainty Equivalent

- On the other side of our original equation, consider a **first-order** Taylor approximation to $u\left[Y - \Psi(\tilde{Z})\right]$:

$$u\left[Y - \Psi(\tilde{Z})\right] \approx u\,(Y) - u'\,(Y)\,\Psi(\tilde{Z})$$

- Here a first-order (or linear) approximation is enough since $\Psi(\tilde{Z})$ is a fixed amount

Risk Premium and Certainty Equivalent

- Now substitute the approximations

$$E\left[u(Y+\tilde{Z})\right] \approx u\,(Y) + \frac{1}{2}\sigma^2(\tilde{Z})u''\,(Y)$$
$$u\left[Y-\Psi(\tilde{Z})\right] \approx u\,(Y) - u'\,(Y)\,\Psi(\tilde{Z})$$

in the equation defining the risk premium

$$E\left[u(Y+\tilde{Z})\right] = u\left[Y-\Psi(\tilde{Z})\right]$$

- We obtain

$$\frac{1}{2}\sigma^2(\tilde{Z})u''\,(Y) \approx -u'\,(Y)\,\Psi(\tilde{Z})$$

Risk Premium and Certainty Equivalent

- After some simplifications

$$\frac{1}{2}\sigma^2(\tilde{Z})u''(Y) \approx -u'(Y)\,\Psi(\tilde{Z})$$

$$\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z})\left[-\frac{u''(Y)}{u'(Y)}\right]$$

$$\boxed{\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z})R_A(Y)}$$

- This shows that the risk premium **depends directly on the coefficient of absolute risk aversion** $R_A(Y)$ and the absolute "size of the bet" $\sigma^2(\tilde{Z})$.

Risk Premium and Certainty Equivalent

- As an example, consider an investor with income $Y = 50000$ and utility function of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

  with $\gamma = 5$, who is considering buying an asset with random payoff $\tilde{Z}$ that equals 2000 with probability $1/2$ and 0 with probability $1/2$.

- The coefficient of absolute risk aversion is

$$R_A(Y) = -\frac{U''(Y)}{U'(Y)} = -\frac{(-\gamma)Y^{-\gamma-1}}{Y^{-\gamma}} = \frac{\gamma}{Y}$$

Risk Premium and Certainty Equivalent

- For this asset

$$E(\tilde{Z}) = (1/2)2000 + (1/2)0 = 1000$$
$$\sigma^2(\tilde{Z}) = (1/2)(2000 - 1000)^2 + (1/2)(0 - 1000)^2 = 1000^2$$

- Our approximation formula

$$\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z})R_A(Y + E(\tilde{Z}))$$

indicates that

$$\Psi(\tilde{Z}) \approx \frac{1}{2}(1000)^2 \left(\frac{5}{51000}\right) = 49.02$$

Risk Premium and Certainty Equivalent

The approximation $\Psi(\tilde{Z}) \approx 49.02$ implies that an investor with $Y = 50000$ and constant coefficient of relative risk aversion equal to 5 will give up a riskless payoff of up to about

$$CE(\tilde{Z}) = E(\tilde{Z}) - \Psi(\tilde{Z}) \approx 1000 - 49 = 951$$

for this risky asset with expected payoff equal to 1000 .

Outline

Assessing the Level of Risk Aversion

- We can use similar calculations to work through thought **experiments** that shed light on our own levels of risk aversion.

- Suppose your income is $Y = 50000$ and you have the chance to buy an asset that pays 50000 with probability $1/2$ and 0 with probability $1/2$.

- This asset has $E(\tilde{Z}) = (1/2)50000 + (1/2)0 = 25000$, but what is the maximum riskless payoff you would exchange for it?

Assessing the Level of Risk Aversion

- Suppose your utility function is of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

- Recall that the most you should pay for the asset is given by the certainty equivalent $CE(\tilde{Z})$ defined by

$$E[u(Y + \tilde{Z})] = u[Y + CE(\tilde{Z})].$$

- Rule of the game: you tell us the maximum you are willing to pay for the asset and we use this information to infer your coefficient of relative risk aversion γ

Assessing the Level of Risk Aversion

- The equation that defines the certainty equivalent

$$E[u(Y+\tilde{Z})] = u[Y + CE(\tilde{Z})]$$

becomes

$$(1/2)u(100000) + (1/2)u(50000) = u(50000 + CE(\tilde{Z}))$$

or

$$(1/2)\left(\frac{100000^{1-\gamma}}{1-\gamma}\right) + (1/2)\left(\frac{50000^{1-\gamma}}{1-\gamma}\right) = \frac{\left(50000 + CE(\tilde{Z})\right)^{1-\gamma}}{1-\gamma}$$

- Solving for $CE(\tilde{Z})$ gives:

$$CE(\tilde{Z}) = \left[(1/2)100000^{1-\gamma} + (1/2)50000^{1-\gamma}\right]^{1/(1-\gamma)} - 50000$$

More

Assessing the Level of Risk Aversion

Table below shows Certainty equivalent and Risk premium $\Psi(\tilde{Z}) = E(\tilde{Z}) - CE(\tilde{Z})$ for an asset that pays $(50000, 0, 1/2)$ when income is 50000 and the coefficient of relative risk aversion is γ.

γ	$CE(\tilde{Z})$	$\Psi(\tilde{Z})$	
0	25000	0	(risk neutrality)
1	20711	4289	(log utility)
2	16667	8333	
3	13246	11754	
4	10571	14429	
5	8566	16434	
10	3991	21009	
20	1858	23142	
50	712	24288	

Outline

The Concept of Stochastic Dominance

- It is important to recognize that the coefficients of absolute and relative risk aversion, $R_A(Y)$ and $R_R(Y)$, and the certainty equivalent $CE(\tilde{Z})$ and the risk premium $\Psi(\tilde{Z})$, all help describe or summarize investors' preferences over risky cash flows.

- They do not directly represent differences in market or equilibrium prices or rates of return across riskless and risky assets.

The Concept of Stochastic Dominance

- Since individuals will differ in their attitudes towards risk as in their preferences over everything else, it is useful to ask whether there are properties of payoff distributions that will allow "preference-free" comparisons to be made across risky cash flows.

- State-by-state dominance, as we've already seen, is one such property. But are there any others, which might be more widely applicable?

The Concept of Stochastic Dominance

Consider two assets, with random payoffs Z_1 and Z_2:

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

There may be no state-by-state dominance, if the payoffs $Z_1 = 10$ and $Z_2 = 100$ can occur together and the payoffs $Z_1 = 100$ and $Z_2 = 10$ can occur together.

The Concept of Stochastic Dominance

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

Because $E(Z_1) = 64, \sigma(Z_1) = 44, E(Z_2) = 244$, and $\sigma(Z_2) = 380$, there is no mean-variance dominance either.

The Concept of Stochastic Dominance

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

- But, intuitively, asset 2 "looks" better, because its distribution takes some of the probability of a payoff of 100 and "moves" that probability to the even higher payoff of 1000
- We can make this idea more concrete by looking at the distributions of these random payoffs in a different way.

The Concept of Stochastic Dominance

In probability theory, the cumulative distribution function (cdf) for a random variable X keeps track of the probability that the realized value of X will be less than or equal to x :

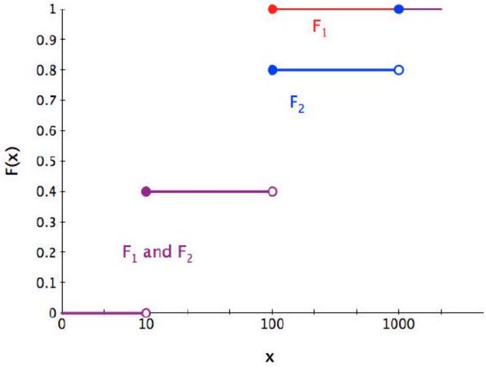
$$F(x) = \text{Prob}(X \leq x)$$

The Concept of Stochastic Dominance

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

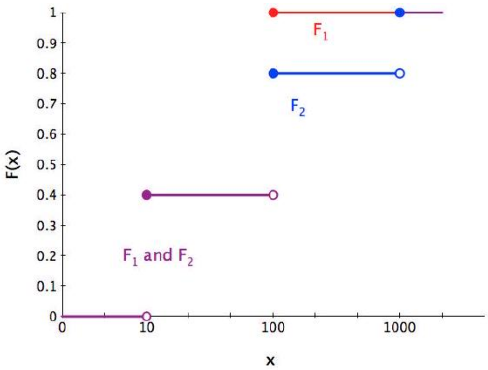
cdfs	$x < 10$	$10 \leq x < 100$	$100 \leq x < 1000$	$1000 \leq x$
$F_1(x)$	0.00	0.40	1.00	1.00
$F_2(x)$	0.00	0.40	0.80	1.00

The Concept of Stochastic Dominance



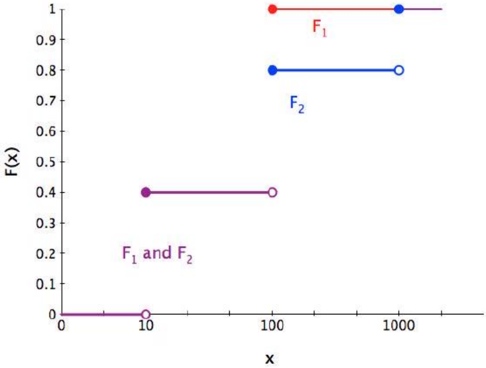
Cumulative distribution functions are always nondecreasing.

The Concept of Stochastic Dominance



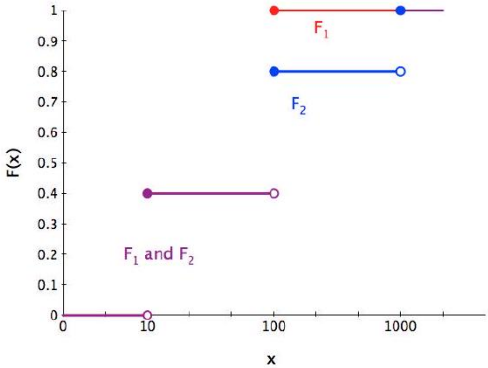
Cumulative distribution functions always satisfy $F(-\infty) = 0$, $F(\infty) = 1$ and $0 \leq F(x) \leq 1$.

The Concept of Stochastic Dominance



Cumulative distribution functions are always RCLL "right continuous with left limits."

The Concept of Stochastic Dominance



The fact that $F_2(x)$ always lies below $F_1(x)$ formalizes the **first-order stochastic dominance** of Z_2 over Z_1.

The Concept of Stochastic Dominance

cdfs	$x < 10$	$10 \leq x < 100$	$100 \leq x < 1000$	$1000 \leq x$
$F_1(x)$	0.00	0.40	1.00	1.00
$F_2(x)$	0.00	0.40	0.80	1.00

Asset 2 displays first-order stochastic dominance over asset 1 because $F_2(x) \leq F_1(x)$ for all possible values of x.

The Concept of Stochastic Dominance

Theorem Let $F_1(x)$ and $F_2(x)$ be the cumulative distribution functions for two assets with random payoffs Z_1 and Z_2. Then

$$F_2(x) \leq F_1(x) \text{ for all } x,$$

that is, asset 2 displays first-order stochastic dominance over asset 1 , if and only if

$$E\left[u\left(Z_2\right)\right] \geq E\left[u\left(Z_1\right)\right]$$

for any increasing Bernoulli utility function u.

The Concept of Stochastic Dominance

First-order stochastic dominance is a weaker condition than state-by-state dominance:

- state-by-state dominance implies first-order stochastic dominance
- but first-order stochastic dominance does not necessarily imply state-by-state dominance.

But first-order stochastic dominance remains quite a strong condition.

- An asset that displays first-order stochastic dominance over all others will be preferred by any investor with vN-M utility who prefers higher payoffs to lower payoffs
- the price of such an asset is likely to be bid up until the dominance goes away.

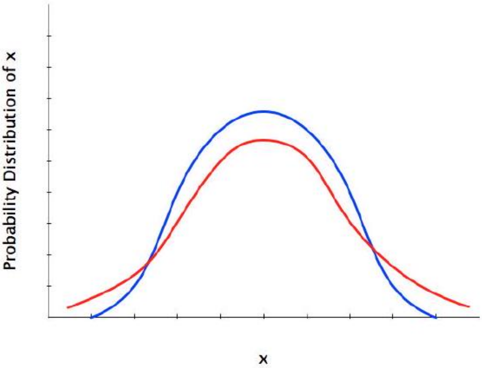
Outline

Mean Preserving Spreads

- Comparisons based on state-by-state dominance, or first-order stochastic dominance can reflect differences in the mean, or expected, payoff as well as in the standard deviation or variance of the payoff.

- It is also useful, therefore, to consider an alternative criterion that focuses entirely on the standard deviation of a random payoff, as a measure of the riskiness of the corresponding asset, holding the mean or expected value fixed.

Mean Preserving Spreads



Graphically, a mean preserving spread takes probability from the center of a distribution and shifts it to the tails.

Mean Preserving Spreads

- Mathematically, one way of producing a mean preserving spread is to take one random variable X_1 and define a second, X_2, by adding "noise" in the form of a third, zero-mean random variable Z :

$$X_2 = X_1 + Z$$

where $E(Z) = 0$.

Mean Preserving Spreads

As an example, suppose that

$$X_1 = \begin{cases} 5 & \text{with probability } 1/2 \\ 2 & \text{with probability } 1/2 \end{cases}$$

$$Z = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

then

$$X_2 = X_1 + Z = \begin{cases} 6 & \text{with probability } 1/4 \\ 4 & \text{with probability } 1/4 \\ 3 & \text{with probability } 1/4 \\ 1 & \text{with probability } 1/4 \end{cases}$$

$E(X_1) = E(X_2) = 3.5$, but if these are random payoffs, asset 2 seems riskier.

Mean Preserving Spreads

Theorem Consider two assets with random payoffs Z_1 and Z_2. If asset 1 is a **mean-preserving spread** over asset 2 then

$$E\left[u\left(Z_2\right)\right] \geq E\left[u\left(Z_1\right)\right]$$

for any **increasing and concave** Bernoulli utility function u.

- This theorem imply that any risk-averse investor with vN-M preferences will avoid "pure gambles," in the form of assets with payoffs that simply add more randomness to the payoff of another asset.

Appendix

Certainty equivalent

- The equation shown in the text for $CE(\tilde{Z})$ is valid only for $\gamma \neq 1$

- If $\gamma = 1$, then $u(Y) = \log(Y)$ and the equation for $CE(\tilde{Z})$ becomes

$$(1/2)\log(100000) + (1/2)\log(50000) = \log(50000 + CE(\tilde{Z}))$$

- Solving for $CE(\tilde{Z})$ gives:

$$CE(\tilde{Z}) = (100000)^{1/2}(50000)^{1/2} - 50000 \approx 20710$$

as shown in the table