LH Advanced Financial Markets - Part B

Topic 4: Risk Aversion and Investment Decisions

Dr. Alessandro Di Nola

University of Birmingham

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Outline

Risk Aversion and Portfolio Allocation

- Let's now put our framework of decision-making under uncertainty to use.
- Consider a risk-averse investor with vN-M expected utility and initial wealth Y_0
- The investor divides his or her initial wealth into
  - amount a allocated to a risky asset (say, the stock market) and
  - an amount $Y_0 - a$ allocated to a safe asset (say, a bank account or a government bond)

Risk Aversion and Portfolio Allocation

- Notation:

$$
\begin{aligned}
Y_0 &= \text{ initial wealth} \\
a &= \text{ amount allocated to stocks} \\
\tilde{r} &= \text{ random return on stocks} \\
r_f &= \text{ risk-free return} \\
\tilde{Y}_1 &= \text{ terminal wealth}
\end{aligned}
$$

- Investor's final wealth is random and equal to:

$$
\begin{aligned}
\tilde{Y}_1 &= (1 + r_f)\,(Y_0 - a) + a(1 + \tilde{r}) \\
&= Y_0\,(1 + r_f) + a\,(\tilde{r} - r_f)
\end{aligned}
$$

Risk Aversion and Portfolio Allocation

- The investor chooses a to maximize expected utility:

$$\max_a E\left[u\left(\tilde{Y}_1\right)\right] = \max_a E\left\{u\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\right\}$$

- If the investor is risk-averse, u is concave $\implies$ the first-order condition (FOC) is both a necessary and sufficient condition for maximization

- The FOC is found by differentiating the objective function by the choice variable a and equating to zero

Risk Aversion and Portfolio Allocation

- The investor's problem is

$$\max_{a} E\left\{u\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\right\}$$

- The first-order condition is

$$E\left\{u'\left[Y_0\left(1+r_f\right)+a^*\left(\tilde{r}-r_f\right)\right]\left(\tilde{r}-r_f\right)\right\}=0$$

- Note: we are allowing the investor to sell stocks short ($a^* < 0$) or to buy stocks on margin ($a^* > Y_0$) if he or she desires.

Risk Aversion and Portfolio Allocation

Theorem If the Bernoulli utility function u is increasing and concave, then

$$a^* > 0 \text{ if and only if } E(\tilde{r}) > r_f$$
$$a^* = 0 \text{ if and only if } E(\tilde{r}) = r_f$$
$$a^* < 0 \text{ if and only if } E(\tilde{r}) < r_f$$

Thus, a risk-averse investor will always allocate at least some funds to the stock market if the expected return on stocks exceeds the risk-free rate.

Risk Aversion and Portfolio Allocation

To prove the theorem, let

$$W(a) = E\left\{u'\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\left(\tilde{r}-r_f\right)\right\}$$

so that the investor's first-order condition can be written more compactly as

$$W\left(a^*\right) = 0$$

Risk Aversion and Portfolio Allocation

Next, note that with

$$W(a) = E\left\{u'\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\left(\tilde{r}-r_f\right)\right\}$$

it follows that

$$W'(a) = E\left\{u''\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\left(\tilde{r}-r_f\right)^2\right\} < 0$$

since u is concave. This means that W is a decreasing function of a.

Risk Aversion and Portfolio Allocation

Finally, note that with

$$W(a) = E\left\{u'\left[Y_0\left(1+r_f\right) + a\left(\tilde{r} - r_f\right)\right]\left(\tilde{r} - r_f\right)\right\}$$

$$\begin{aligned}
W(0) &= E\left\{u'\left[Y_0\left(1+r_f\right) + 0\left(\tilde{r} - r_f\right)\right]\left(\tilde{r} - r_f\right)\right\} \\
&= E\left\{u'\left[Y_0\left(1+r_f\right)\right]\left(\tilde{r} - r_f\right)\right\} \\
&= u'\left[Y_0\left(1+r_f\right)\right]E\left(\tilde{r} - r_f\right) \\
&= u'\left[Y_0\left(1+r_f\right)\right]\left[E(\tilde{r}) - r_f\right]
\end{aligned}$$

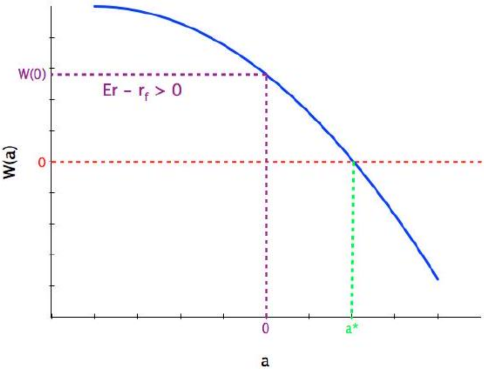
Since u is increasing, this means that $W(0)$ has the same sign as $E(\tilde{r}) - r_f$.

Risk Aversion and Portfolio Allocation

We now know that:

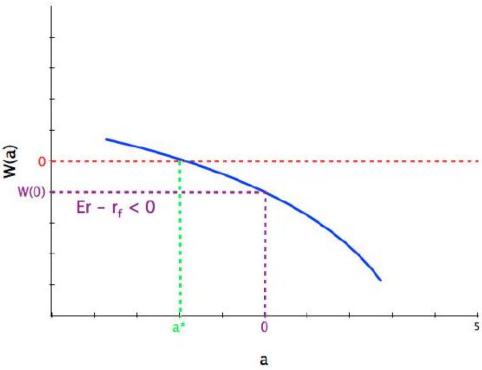
1. $W(a)$ is a decreasing function
2. $W(0)$ has the same sign as $E(\tilde{r}) - r_f$.
3. $W(a^*) = 0$

Risk Aversion and Portfolio Allocation



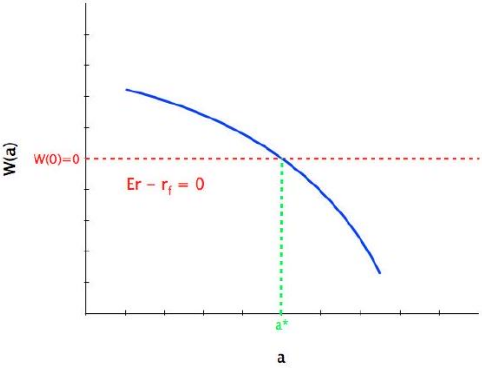
$E(\tilde{r}) - r_f > 0$ implies that $W(0) > 0$, and since W is decreasing, $W(a^*) = 0$ implies that $a^* > 0$.

Risk Aversion and Portfolio Allocation



$E(\tilde{r}) - r_f < 0$ implies that $W(0) < 0$, and since W is decreasing, $W(a^*) = 0$ implies that $a^* < 0$.

Risk Aversion and Portfolio Allocation



$E(\tilde{r}) - r_f = 0$ implies that $W(0) = 0$, and since W is decreasing, $W(a^*) = 0$ implies that $a^* = 0$.

Risk Aversion and Portfolio Allocation

Theorem If the Bernoulli utility function u is increasing and concave, then

$$a^* > 0 \text{ if and only if } E(\tilde{r}) > r_f$$
$$a^* = 0 \text{ if and only if } E(\tilde{r}) = r_f$$
$$a^* < 0 \text{ if and only if } E(\tilde{r}) < r_f$$

Thus, a risk-averse investor will always allocate at least some funds to the stock market if the expected return on stocks exceeds the risk-free rate.

Risk Aversion and Portfolio Allocation

- Danthine and Donaldson (3rd ed., p.41) report that in the United States, 1889-2010, average real (inflation-adjusted) returns on stocks and risk-free bonds are

$$E(\tilde{r}) = 0.075 (7.5 \text{ percent per year })$$
$$r_f = 0.011 (1.1 \text{ percent per year })$$

- The equity risk premium of $E(\tilde{r}) - r_f = 0.064$ (6.4 percent) is not only positive, it is huge.

- The implication of the theory is that all investors, even the most risk averse, should have some money invested in the stock market.

Risk Aversion and Portfolio Allocation

- As an example, suppose $u(Y) = \ln(Y)$
- Recall that for this utility function, $u'(Y) = 1/Y$
- Then assume that stock returns can either be good or bad:

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1 - \pi \end{cases}$$

- $r_G > r_f > r_B$ defines the "good" and "bad" states and

$$\pi r_G + (1 - \pi) r_B > r_f$$

so that $E(\tilde{r}) > r_f$ and the investor will choose $a^* > 0$

- What if the assumption $r_G > r_f > r_B$ does not hold?

Risk Aversion and Portfolio Allocation

- What if the assumption $r_G > r_f > r_B$ does not hold?

- If $r_G > r_B > r_f$, then risky asset dominates state-by-state the risk-free bond $\implies$ trivial solution: investor should put all wealth in risky asset!

- If $r_f > r_G > r_B$, then the risky asset is dominated state-by-state by the the risk-free bond $\implies$ trivial solution: investor should put all wealth in the bond!

- To make the problem non-trivial, we always assume $r_G > r_f > r_B$.

Risk Aversion and Portfolio Allocation

- Given that $\tilde{r} = (r_G, r_B, \pi)$, investor's final wealth is

$$\widetilde{Y}_1 = \begin{cases} Y_0 (1 + r_f) + a (r_G - r_f) & \text{with prob. } \pi, \\ Y_0 (1 + r_f) + a (r_B - r_f) & \text{with prob. } 1 - \pi \end{cases}$$

- Since $u(Y) = \log(Y)$, and given $\widetilde{Y}_1$ above, the problem

$$\max_a E\left[\left(\widetilde{Y}_1\right)\right]$$

specializes to

$$\begin{aligned} \max_a \quad & \pi \ln \left[Y_0 (1 + r_f) + a (r_G - r_f) \right] \\ & + (1 - \pi) \ln \left[Y_0 (1 + r_f) + a (r_B - r_f) \right] \end{aligned}$$

Risk Aversion and Portfolio Allocation

- The problem

$$\begin{aligned} \max_a \quad & \pi \ln \left[Y_0 \left(1 + r_f\right) + a \left(r_G - r_f\right) \right] \\ & + (1 - \pi) \ln \left[Y_0 \left(1 + r_f\right) + a \left(r_B - r_f\right) \right] \end{aligned}$$

has first-order condition

$$\frac{\pi \left(r_G - r_f\right)}{Y_0 \left(1 + r_f\right) + a^* \left(r_G - r_f\right)} + \frac{(1 - \pi) \left(r_B - r_f\right)}{Y_0 \left(1 + r_f\right) + a^* \left(r_B - r_f\right)} = 0$$

- Let's do some algebra and solve for a^*

Risk Aversion and Portfolio Allocation

$$\frac{\pi\left(r_G - r_f\right)}{Y_0\left(1 + r_f\right) + a^*\left(r_G - r_f\right)} + \frac{\left(1 - \pi\right)\left(r_B - r_f\right)}{Y_0\left(1 + r_f\right) + a^*\left(r_B - r_f\right)} = 0$$

$$\begin{aligned} & \pi\left(r_G - r_f\right)\left[Y_0\left(1 + r_f\right) + a^*\left(r_B - r_f\right)\right] \\ = \; & \left(1 - \pi\right)\left(r_f - r_B\right)\left[Y_0\left(1 + r_f\right) + a^*\left(r_G - r_f\right)\right] \end{aligned}$$

$$\begin{aligned} & a^*\left(r_G - r_f\right)\left(r_f - r_B\right) \\ = \; & Y_0\left(1 + r_f\right)\left[\pi\left(r_G - r_f\right) + \left(1 - \pi\right)\left(r_B - r_f\right)\right] \end{aligned}$$

Risk Aversion and Portfolio Allocation

$$
\begin{aligned}
& a^*\left(r_G - r_f\right)\left(r_f - r_B\right) \\
= & Y_0\left(1 + r_f\right)\left[\pi\left(r_G - r_f\right) + (1-\pi)\left(r_B - r_f\right)\right]
\end{aligned}
$$

implies

$$
\frac{a^*}{Y_0} = \frac{\left(1 + r_f\right)\left[\pi\left(r_G - r_f\right) + (1-\pi)\left(r_f - r_B\right)\right]}{\left(r_G - r_f\right)\left(r_f - r_B\right)}
$$

which is positive, since $r_G > r_f > r_B$ and

$$
E(\tilde{r}) - r_f = \pi\left(r_G - r_f\right) + (1-\pi)\left(r_B - r_f\right) > 0
$$

Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = \frac{(1 + r_f)\left[\pi(r_G - r_f) + (1 - \pi)(r_B - r_f)\right]}{(r_G - r_f)(r_f - r_B)}$$

In this case, a^* :

- Rises proportionally with Y_0.
- Increases as $E(\tilde{r}) - r_f$ rises.
- Falls as r_G and r_B move father away from r_f, holding $E(\tilde{r})$ constant; that is, in response to a mean preserving spread.

Risk Aversion and Portfolio Allocation

$$\frac{a^*}{Y_0} = \frac{(1 + r_f)\left[\pi\left(r_G - r_f\right) + (1 - \pi)\left(r_B - r_f\right)\right]}{\left(r_G - r_f\right)\left(r_f - r_B\right)}$$

r_f	r_G	r_B	π	$E(\tilde{r})$	a^*/Y_0
0.05	0.40	-0.20	0.50	0.10	0.60
0.05	**0.30**	-**0.10**	0.50	0.10	1.40
0.05	0.40	-0.20	**0.75**	**0.25**	2.40

The fraction of initial wealth allocated to stocks rises when stocks become less risky (second row) or pay higher expected returns (third row)

Risk Aversion and Portfolio Allocation

- Before moving on, return to the general problem

$$\max_a E\left\{u\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\right\}$$

but assume now that the investor is risk-neutral, with

$$u(Y)=\alpha Y+\beta$$

and $\alpha>0$, so that more wealth is preferred to less.

Risk Aversion and Portfolio Allocation

- The risk-neutral investor solves

$$\begin{aligned}
&\max_a E\left\{u\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\right\} \\
=&\max_a E\left\{\alpha\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]+\beta\right\} \\
=&\max_a \alpha\left\{Y_0\left(1+r_f\right)+a\left[E(\tilde{r})-r_f\right]\right\}+\beta
\end{aligned}$$

- So long as $E(\tilde{r})-r_f>0$, the risk-neutral investor will choose a^* to be as large as possible, borrowing as much as he or she is allowed to in order to buy more stocks on margin.

Outline

Portfolios, Risk Aversion, and Wealth

- The previous examples call out for a more detailed analysis of how optimal portfolio allocation decisions, summarized by the value of a^* that solves

$$\max_a E\left\{u\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\right\}$$

are influenced by the investor's degree of risk aversion and his or her level of wealth.

Portfolios, Risk Aversion, and Wealth

The following result was proven by **Kenneth Arrow** in "*The Theory of Risk Aversion*," (1971)

Theorem Consider two investors, $i = 1$ and $i = 2$, and suppose that for all wealth levels $Y > 0$, $R_A^1(Y) > R_A^2(Y)$, where $R_A^i(Y)$ is investor i 's coefficient of absolute risk aversion. Then $a_1^*(Y) < a_2^*(Y)$, where $a_i^*(Y)$ is amount allocated by investor i to stocks when he or she has initial wealth Y.

Portfolios, Risk Aversion, and Wealth

Recall that the coefficients of absolute and relative risk aversion are

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} \text{ and } R_R(Y) = -\frac{Yu''(Y)}{u'(Y)}.$$

Thus

$$R_A^1(Y) > R_A^2(Y) \text{ or } -\frac{u_1''(Y)}{u_1'(Y)} > -\frac{u_2''(Y)}{u_2'(Y)}$$

implies

$$-\frac{Yu_1''(Y)}{u_1'(Y)} > -\frac{Yu_2''(Y)}{u_2'(Y)} \text{ or } R_R^1(Y) > R_R^2(Y)$$

Portfolios, Risk Aversion, and Wealth

Therefore Arrow's result applies equally well to **relative** risk aversion:

Theorem Consider two investors, $i = 1$ and $i = 2$, and suppose that for all wealth levels $Y > 0$, $R_R^1(Y) > R_R^2(Y)$, where $R_R^i(Y)$ is investor i's coefficient of relative risk aversion. Then $a_1^*(Y) < a_2^*(Y)$, where $a_i^*(Y)$ is amount allocated by investor i to stocks when he or she has initial wealth Y.

Portfolios, Risk Aversion, and Wealth

- Let's test Arrow's proposition out, by generalizing our previous example with logarithmic utility to the case where

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

with $\gamma > 0$.

- For this Bernoulli utility function, the **coefficient of relative risk aversion is constant and equal to** γ. If $\gamma = 1$ we are back to the case with log utility.

Portfolios, Risk Aversion, and Wealth

- Hence, in this extended example,

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma} \text{ implies } u'(Y) = Y^{-\gamma} = \frac{1}{Y^{\gamma}}.$$

- Stock returns can either be good or bad

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1-\pi \end{cases}$$

where $r_G > r_f > r_B$ defines the "good" and "bad" states and

$$\pi r_G + (1-\pi) r_B > r_f,$$

so that $E(\tilde{r}) > r_f$ and the investor will choose $a^* > 0$.

Portfolios, Risk Aversion, and Wealth

- With CRRA (constant relative risk aversion) utility and two states for $\tilde{r}$, the problem

$$\max_a E \left\{ u \left[Y_0 \left(1 + r_f \right) + a \left(\tilde{r} - r_f \right) \right] \right\}$$

specializes to

$$\begin{aligned} \max_a \quad & \pi \left\{ \frac{\left[Y_0 \left(1 + r_f \right) + a \left(r_G - r_f \right) \right]^{1-\gamma} - 1}{1-\gamma} \right\} \\ & + (1-\pi) \left\{ \frac{\left[Y_0 \left(1 + r_f \right) + a \left(r_B - r_f \right) \right]^{1-\gamma} - 1}{1-\gamma} \right\} \end{aligned}$$

Portfolios, Risk Aversion, and Wealth

- The problem

$$\max_{a} \quad \pi \left\{ \frac{\left[Y_0 \left(1 + r_f\right) + a \left(r_G - r_f\right)\right]^{1-\gamma} - 1}{1 - \gamma} \right\} + (1 - \pi) \left\{ \frac{\left[Y_0 \left(1 + r_f\right) + a \left(r_B - r_f\right)\right]^{1-\gamma} - 1}{1 - \gamma} \right\}$$

has first-order condition

$$\frac{\pi \left(r_G - r_f\right)}{\left[Y_0 \left(1 + r_f\right) + a^* \left(r_G - r_f\right)\right]^{\gamma}} + \frac{(1 - \pi)\left(r_B - r_f\right)}{\left[Y_0 \left(1 + r_f\right) + a^* \left(r_B - r_f\right)\right]^{\gamma}} = 0$$

- We have to solve the equation above for a^*!

Portfolios, Risk Aversion, and Wealth

- Solving

$$\frac{\pi\left(r_G - r_f\right)}{\left[Y_0\left(1+r_f\right) + a^*\left(r_G - r_f\right)\right]^{\gamma}} + \frac{\left(1-\pi\right)\left(r_B - r_f\right)}{\left[Y_0\left(1+r_f\right) + a^*\left(r_B - r_f\right)\right]^{\gamma}} = 0$$

we obtain Derivation

$$\frac{a^*}{Y_0} = \frac{\left(1+r_f\right)\left\{\left[\pi\left(r_G - r_f\right)\right]^{1/\gamma} - \left[\left(1-\pi\right)\left(r_f - r_B\right)\right]^{1/\gamma}\right\}}{\left(r_G - r_f\right)\left[\left(1-\pi\right)\left(r_f - r_B\right)\right]^{1/\gamma} + \left(r_f - r_B\right)\left[\pi\left(r_G - r_f\right)\right]^{1/\gamma}}$$

Portfolios, Risk Aversion, and Wealth

- Comparative statics exercise: we vary γ and compute $\frac{a^*}{Y_0}$ for many different values of γ, using the equation just derived

$$\frac{a^*}{Y_0} = \frac{(1+r_f)\left\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}\right\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

γ	r_f	r_G	r_B	π	$E(\tilde{r})$	a^*/Y_0
0.5	0.05	0.40	-0.20	0.50	0.10	1.20
1	0.05	0.40	-0.20	0.50	0.10	0.60
2	0.05	0.40	-0.20	0.50	0.10	0.30
3	0.05	0.40	-0.20	0.50	0.10	0.20
5	0.05	0.40	-0.20	0.50	0.10	0.12
10	0.05	0.40	-0.20	0.50	0.10	0.06

Portfolios, Risk Aversion, and Wealth

γ	r_f	r_G	r_B	π	$E(\tilde{r})$	a^*/Y_0
0.5	0.05	0.40	-0.20	0.50	0.10	1.20
1	0.05	0.40	-0.20	0.50	0.10	0.60
2	0.05	0.40	-0.20	0.50	0.10	0.30
3	0.05	0.40	-0.20	0.50	0.10	0.20
5	0.05	0.40	-0.20	0.50	0.10	0.12
10	0.05	0.40	-0.20	0.50	0.10	0.06

- Consistent with Arrow's theorem, higher coefficients of relative risk aversion are associated with smaller values of a^*

$$\uparrow \gamma = R_R(Y) \quad \Longrightarrow \quad \downarrow a^*$$

Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1+r_f)\left\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}\right\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

- Note that **with constant relative risk aversion, a^* rises proportionally with wealth**.

- Two additional theorems, also proven by Arrow, tell us more about the relationship between a^* and wealth.

Portfolios, Risk Aversion, and Wealth

Theorem Let $a^*(Y_0)$ be the solution to

$$\max_a E\left\{u\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\right\}$$

If $u(Y)$ is such that

$$\text{(a) } R_A'(Y) < 0 \text{ then } \frac{da^*(Y_0)}{dY_0} > 0$$

$$\text{(b) } R_A'(Y) = 0 \text{ then } \frac{da^*(Y_0)}{dY_0} = 0$$

$$\text{(c) } R_A'(Y) > 0 \text{ then } \frac{da^*(Y_0)}{dY_0} < 0$$

Portfolios, Risk Aversion, and Wealth

- Part (a)

$$R_A'(Y) < 0 \text{ then } \frac{da^*(Y_0)}{dY_0} > 0$$

describes the "normal" case where absolute risk aversion falls as wealth rises.

- This case is called **DARA**: decreasing absolute risk aversion

- In this case, wealthier individuals allocate more wealth to stocks.

Portfolios, Risk Aversion, and Wealth

- Part (b)

$$R'_A(Y) = 0 \text{ then } \frac{da^*(Y_0)}{dY_0} = 0$$

means that investors with constant absolute risk aversion (**CARA**)

$$u(Y) = -\frac{1}{\nu}e^{-\nu Y}$$

allocate a constant amount of wealth to stocks.

Portfolios, Risk Aversion, and Wealth

- Part (c)

$$R'_A(Y) > 0 \text{ then } \frac{da^*(Y_0)}{dY_0} < 0$$

  describes the case where absolute risk aversion rises (**IARA**, increasing absolute risk aversion) as wealth rises.

- The implication that wealthier individuals allocate less wealth to stocks makes this case seem less plausible.

Portfolios, Risk Aversion, and Wealth

Theorem Let $a^*(Y_0)$ be the solution to

$$\max_a E\left\{u\left[Y_0(1+r_f)+a(\tilde{r}-r_f)\right]\right\}.$$

If $u(Y)$ is such that

$$\text{(a) } R_A'(Y) < 0 \text{ then } \frac{da^*(Y_0)}{dY_0} > 0$$

$$\text{(b) } R_A'(Y) = 0 \text{ then } \frac{da^*(Y_0)}{dY_0} = 0$$

$$\text{(c) } R_A'(Y) > 0 \text{ then } \frac{da^*(Y_0)}{dY_0} < 0$$

This result relates changes in absolute risk aversion to the absolute amount of wealth allocated to stocks.

Portfolios, Risk Aversion, and Wealth

- Consistent with our results with CRRA utility, the next result relates changes in relative risk aversion to changes in the proportion of wealth allocated to stocks.

- Define the elasticity of the function $a^*(Y_0)$ as

$$\eta = \frac{d \ln a^*(Y_0)}{d \ln Y_0} = \frac{da^*(Y_0)}{dY_0} \frac{Y_0}{a^*(Y_0)}$$

- The elasticity measures the percentage change in a^* brought about by a percentage-point change in Y_0.

Portfolios, Risk Aversion, and Wealth

Theorem Let $a^*(Y_0)$ be the solution to

$$\max_a E\left\{u\left[Y_0\left(1+r_f\right)+a\left(\tilde{r}-r_f\right)\right]\right\}$$

If $u(Y)$ is such that

(a) $R_R'(Y) < 0$ (decreasing relative risk aversion) then $\eta > 1$
(b) $R_R'(Y) = 0$ (constant relative risk aversion) then $\eta = 1$
(c) $R_R'(Y) > 0$ (increasing relative risk aversion) then $\eta < 1$

- The theorem confirms what we know about CRRA utility: it implies that a^* rises proportionally with Y_0. In other words, a^*/Y_0 is constant.

Portfolios, Risk Aversion, and Wealth

- With CRRA utility:

$$\frac{a^*}{Y_0} = K$$

where

$$K = \frac{(1 + r_f)\left\{[\pi (r_G - r_f)]^{1/\gamma} - \left[[(1 - \pi)(r_f - r_B)]^{1/\gamma}\right]\right\}}{(r_G - r_f)[(1 - \pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi (r_G - r_f)]^{1/\gamma}}$$

- Hence

$$\ln (a^* (Y_0)) = \ln(K) + \ln (Y_0)$$

and

$$\eta = \frac{d \ln a^* (Y_0)}{d \ln Y_0} = 1$$

Exercise

- Let's check your understanding with a simple exercise
- Consider two investors, $i = 1$ and $i = 2$, each of whom divides up his or her initial wealth Y_0^i into an amount a_i allocated to risky stocks and an amount $Y_0^i - a_i$ allocated to risk-free bonds.
- Each investor has Bernoulli utility function $u(Y) = \log(Y)$
- Answer these questions:

  1. Write down the portfolio allocation problem solved by each investor $i = 1$ and $i = 2$

Exercise, cont'd

2. Suppose that investor $i = 1$ has initial wealth $Y_0^1 = 100$ and investor $i = 2$ has initial wealth $Y_0^2 = 1000$. Let a_i^* be the absolute dollar amount that investor $i = 1, 2$ allocates to stocks. Will a_1^* be larger than, smaller than, or equal to a_2^*? *Hint*: No calculations are needed, just apply Arrow's theorems.
3. Continue to assume that $Y_0^1 = 100$ and $Y_0^2 = 1000$. Will $w_1^* = a_1^*/Y_0^1$, the share of wealth that investor $i = 1$ allocates to stocks, be larger than, smaller than, or the same as $w_2^* = a_2^*/Y_0^2$, the share of wealth that investor $i = 2$ allocates to stocks? *Hint*: Again, you don't have to actually find the numerical values of w_1^* and w_2^*

4. Suppose now that both investors have the same amount of initial wealth, so that $Y_0^1 = Y_0^2 = 100$, but that instead of having logarithmic Bernoulli utility functions, investor $i = 1$ has Bernoulli utility function

$$u_1\left(Y_1^1\right) = \frac{\left(Y_1^1\right)^{-2} - 1}{-2}$$

while investor $i = 2$ has Bernoulli utility function

$$u_2\left(Y_1^2\right) = \frac{\left(Y_1^2\right)^{-4} - 1}{-4}$$

In this case, will a_1^*, the absolute dollar amount that investor $i = 1$ allocates to stocks, be larger than, smaller than, or the same as a_2^*, the absolute dollar amount that investor $i = 2$ allocates to stocks? *Hint*: Again, you don't have to actually find the numerical values of a_1^* and a_2^*

Outline

Risk Aversion and Saving Behavior

- So far, we've assumed that investors only receive utility from the terminal value of their wealth

- We have asked how they should split their initial wealth - accumulated, presumably, through past saving - across risky and riskless assets $\implies$ maximize the expected utility from terminal wealth.

- Now, let's take the possibly random return on the investor's portfolio of assets as given, and ask **how they should optimally determine savings under conditions of uncertainty.**

Risk Aversion and Saving Behavior

- Suppose there are two periods, $t = 0$ and $t = 1$, and let

$$\begin{aligned} Y_0 &= \text{initial wealth} \\ s &= \text{amount saved in period } t = 0 \\ c_0 &= Y_0 - s = \text{ amount consumed in period } t = 0 \\ \tilde{R} &= 1 + \tilde{r} = \text{ random, gross return on savings} \\ \tilde{c}_1 &= s\tilde{R} = \text{ amount consumed in period } t = 1 \end{aligned}$$

- Suppose also that the investor has vN-M expected utility over consumption during periods $t = 0$ and $t = 1$ given by

$$u(c_0) + \beta E[u(\tilde{c}_1)] = u(Y_0 - s) + \beta E[u(s\tilde{R})],$$

where the discount factor β is a measure of patience.

Risk Aversion and Saving Behavior

- The solution to the investor's saving problem

$$\max_{s} u\left(Y_0 - s\right) + \beta E[u(s\tilde{R})]$$

is described by the first-order condition (FOC)

$$-u'\left(Y_0 - s^*\right) + \beta E\left[u'\left(s^*\tilde{R}\right)\tilde{R}\right] = 0$$

- The FOC above can be rewritten as

$$u'\left(Y_0 - s^*\right) = \beta E\left[u'\left(s^*\tilde{R}\right)\tilde{R}\right]$$

This is known as **Euler equation**

Risk Aversion and Saving Behavior

$$\boxed{u'\left(Y_0 - s^*\right) = \beta E\left[u'\left(s^*\tilde{R}\right)\tilde{R}\right]}$$

- How does optimal saving s^* respond to an increase in risk, in the form of a mean preserving spread in the distribution of $\tilde{R}$?
- Intuitively, one might expect there to be *two offsetting effects*:

1. The riskier return will make saving less attractive and thereby reduce s^*.
2. The riskier return might lead to "precautionary saving" in order to cushion period $t = 1$ consumption against the possibility of a bad output and thereby increase s^*.

Risk Aversion and Saving Behavior

$$u'(Y_0 - s^*) = \beta E\left[u'\left(s^*\tilde{R}\right)\tilde{R}\right] \qquad \text{(Euler)}$$

- To see which of these two effects dominates, define

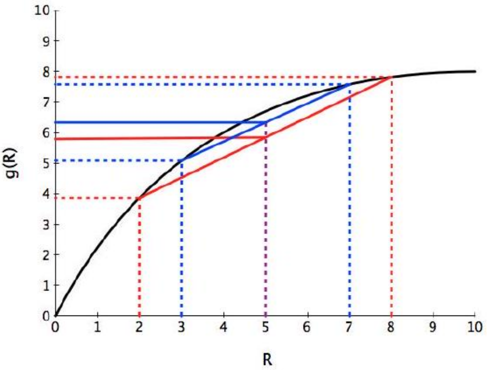
$$g(\tilde{R}) = u'\left(s^*\tilde{R}\right)\tilde{R}$$

  so that the right-hand side of Euler equation becomes

$$\beta E[g(\tilde{R})]$$

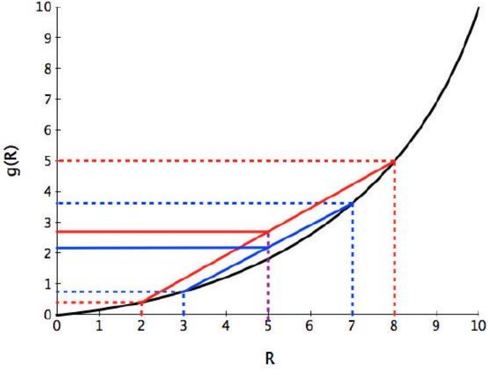
- **Jensen's inequality** $\Longrightarrow$ after a mean preserving spread in the distribution of $\tilde{R}$, this expectation will fall if g is concave and rise if g is convex.

Risk Aversion and Saving Behavior



When g is **concave**, a mean preserving spread in the distribution of $\tilde{R}$ will decrease $E[g(\tilde{R})]$.

Risk Aversion and Saving Behavior



When g is **convex**, a mean preserving spread in the distribution of $\tilde{R}$ will increase $E[g(\tilde{R})]$.

Risk Aversion and Saving Behavior

- The definition

$$g(\tilde{R}) = u'\left(s^*\tilde{R}\right)\tilde{R}$$

  suggests that the concavity or convexity of g will depend on the sign of the third derivative of u.

- Indeed, the function g is convex if and only if $g'' > 0$

- The product and chain rules for differentiation imply

$$g'(\tilde{R}) = u''\left(s^*\tilde{R}\right)s^*\tilde{R} + u'\left(s^*\tilde{R}\right)$$

$$g''(\tilde{R}) = u'''\left(s^*\tilde{R}\right)(s^*)^2\,\tilde{R} + u''\left(s^*\tilde{R}\right)s + u''\left(s^*\tilde{R}\right)s$$

Risk Aversion and Saving Behavior

- The equation

$$\begin{aligned} g''(\tilde{R}) &= u'''\left(s^*\tilde{R}\right)(s^*)^2\,\tilde{R} + u''\left(s^*\tilde{R}\right)s + u''\left(s^*\tilde{R}\right)s \\ &= u'''\left(s^*\tilde{R}\right)(s^*)^2\,\tilde{R} + 2u''\left(s^*\tilde{R}\right)s \end{aligned}$$

  implies that $g''(\tilde{R})$ has the same sign as

$$u'''\left(s^*\tilde{R}\right)s\tilde{R} + 2u''\left(s^*\tilde{R}\right)$$

- Then $g(\tilde{R})$ is convex if and only if

$$u'''\left(s^*\tilde{R}\right)s\tilde{R} + 2u''\left(s^*\tilde{R}\right) > 0$$

Risk Aversion and Saving Behavior

- To understand precautionary saving behavior, the concept of **prudence** is defined by **Miles Kimball**, "*Precautionary Saving in the Small and in the Large*," Econometrica Vol. 58 (January 1990): pp.53-73.

- Whereas risk aversion is summarized by the second derivative of the Bernoulli utility function u, **prudence is summarized by the third derivative of** u.

Risk Aversion and Saving Behavior

- Kimball defines the coefficient of absolute **prudence** as

$$P_A(Y) = -\frac{u'''(Y)}{u''(Y)}$$

and the coefficient of relative **prudence** as

$$P_R(Y) = -\frac{Yu'''(Y)}{u''(Y)}$$

thereby extending the analogous measures of absolute and relative risk aversion.

Risk Aversion and Saving Behavior

- Recall that $g''(\tilde{R})$ has the same sign as

$$u'''\left(s^*\tilde{R}\right)s\tilde{R} + 2u''\left(s^*\tilde{R}\right)$$

- But this can be rewritten as

$$u'''(Y)Y + 2u''(Y) = u''(Y)\left[\frac{u'''(Y)Y}{u''(Y)} + 2\right] = u''(Y)\left[2 - P_R(Y)\right]$$

- Therefore, since $u''(Y) < 0$,

$$g''(\tilde{R}) \text{ is positive if } 2 < P_R(Y)$$
$$g''(\tilde{R}) \text{ is negative if } 2 > P_R(Y)$$

Risk Aversion and Saving Behavior

- Hence, if $2 < P_R(Y)$, then $g''(\tilde{R}) > 0$. Since g is convex, a mean preserving spread in the distribution of $\tilde{R}$ increases the right hand side of the optimality condition

$$u'\left(Y_0 - s^*\right) = \beta E\left[u'\left(s^*\tilde{R}\right)\tilde{R}\right]$$

and s^* must increase to maintain the equality.

- **The precautionary saving effect dominates if the coefficient of relative prudence exceeds 2**

Risk Aversion and Saving Behavior

- To apply these results, let's calculate the coefficient of relative prudence implied by the CRRA utility function

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

with $\gamma > 0$

- Since $u'(Y) = Y^{-\gamma}$,

$$u''(Y) = -\gamma Y^{-\gamma-1} \text{ and } u'''(Y) = \gamma(\gamma+1)Y^{-\gamma-2}$$

imply

$$P_R(Y) = -\frac{Yu'''(Y)}{u''(Y)} = \frac{Y\gamma(\gamma+1)Y^{-\gamma-2}}{\gamma Y^{-\gamma-1}} = \gamma + 1$$

Risk Aversion and Saving Behavior

- Hence, the CRRA utility function

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

implies both a constant coefficient of relative risk aversion equal to γ and a constant coefficient of relative prudence equal to $\gamma + 1$.

- If $\gamma > 1$, saving rises in response to a mean preserving spread in the distribution of $\tilde{R}$. When $\gamma < 1$, saving falls. In the special case $\gamma = 1$ of logarithmic utility, saving is unaffected.

Outline

Separating Risk and Time Preferences

- Our first set of results focused on the optimal choice of a, the amount of wealth to allocate to a risky asset.

- Our second set of results focused on the optimal choice of s, the amount of saving to carry from period $t = 0$ to period $t = 1$.

- Now let's combine the two problems to consider the **simultaneous choices of a and s.**

Separating Risk and Time Preferences

- Suppose again that there are two periods, $t = 0$ and $t = 1$, and let

$$\begin{aligned}
Y_0 &= \text{initial wealth} \\
s &= \text{amount saved in period } t = 0 \\
c_0 &= Y_0 - s = \text{ amount consumed in period } t = 0 \\
a &= \text{amount allocated to stocks in period } t = 0 \\
s - a &= \text{amount allocated to the riskless asset in period } t = 0 \\
\tilde{r} &= \text{random return on stocks} \\
r_f &= \text{return on riskless asset} \\
\tilde{c}_1 &= \text{amount consumed in period } t = 1
\end{aligned}$$

- Then

$$\tilde{c}_1 = (1 + r_f)(s - a) + a(1 + \tilde{r}) = (1 + r_f)\, s + a(\tilde{r} - r)$$

Separating Risk and Time Preferences

- If the investor has vN-M expected utility, his or her problem can be stated as

$$\max_{s,a} u(c_0) + \beta E[u(\widetilde{c}_1)]$$

subject to

$$c_0 + s = Y_0,$$

$$\widetilde{c}_1 = s(1 + r_f) + a(\widetilde{r} - r_f)$$

- Substituting the constraints in the objective function, we get the equivalent problem

$$\max_{s,a} u(Y_0 - s) + \beta E\{u[s(1 + r_f) + a(\tilde{r} - r)]\}$$

Separating Risk and Time Preferences

$$\max_{s,a} u\left(Y_0 - s\right) + \beta E\left\{u\left[s\left(1 + r_f\right) + a(\tilde{r} - r)\right]\right\}$$

- The first-order condition for s is

$$u'\left(Y_0 - s^*\right) = \beta\left(1 + r_f\right) E\left\{u'\left[s^*\left(1 + r_f\right) + a^*(\tilde{r} - r)\right]\right\}$$

- The first-order condition for a is

$$\beta E\left\{u'\left[s^*\left(1 + r_f\right) + a^*(\tilde{r} - r)\right]\left(\tilde{r} - r_f\right)\right\} = 0$$

Separating Risk and Time Preferences

- The first-order conditions

$$u'(Y_0 - s^*) = \beta(1 + r_f)E\left\{u'\left[s^*(1 + r_f) + a^*(\tilde{r} - r)\right]\right\}$$
$$\beta E\left\{u'\left[s^*(1 + r_f) + a^*(\tilde{r} - r)\right](\tilde{r} - r_f)\right\} = 0$$

  form a system of two equations in the two unknowns a^* and s^*, which can be solved numerically using a computer.

- The model can be enriched further by considering additional periods $t = 0, 1, 2, \ldots, T$ and introducing labor income.

Separating Risk and Time Preferences

- Note, however, that the first-order condition for a

$$\beta E\left\{u'\left[s^*\left(1+r_f\right)+a^*(\tilde{r}-r)\right]\left(\tilde{r}-r_f\right)\right\}=0$$

takes the same form as in the simpler problem without saving:

$$E\left\{u'\left[Y_0\left(1+r_f\right)+a^*\left(\tilde{r}-r_f\right)\right]\left(\tilde{r}-r_f\right)\right\}=0$$

- Hence, some of our previous results carry over to the more general case.

- With CRRA utility, for example, a^* will change proportionally with s^*, to maintain an optimal fraction of saving allocated to the risky asset.

Separating Risk and Time Preferences

- As a final exercise, let's return to the optimal saving problem

$$\max_s u\left(Y_0 - s\right) + \beta E[u(s\tilde{R})]$$

- Simplify by eliminating randomness from the return $\tilde{R}$ and by assuming from the start that the utility function takes the CRRA form:

$$\max_s \frac{(Y_0 - s)^{1-\gamma} - 1}{1-\gamma} + \beta \left[\frac{(sR)^{1-\gamma} - 1}{1-\gamma}\right]$$

Separating Risk and Time Preferences

$$\max_s \frac{(Y_0 - s)^{1-\gamma} - 1}{1-\gamma} + \beta \left[\frac{(sR)^{1-\gamma} - 1}{1-\gamma} \right]$$

- The first-order condition for the optimal choice of s is

$$(Y_0 - s)^{-\gamma} = \beta (sR)^{-\gamma} R$$

- Recalling that $c_0 = Y_0 - s$ and $c_1 = sR$,

$$c_0^{-\gamma} = \beta R c_1^{-\gamma}$$

Separating Risk and Time Preferences

- Rearranging the Euler equation yields

$$c_0^{-\gamma} = \beta R c_1^{-\gamma}$$
$$(c_1/c_0)^{\gamma} = \beta R$$
$$c_1/c_0 = (\beta R)^{1/\gamma}$$
$$\ln(c_1/c_0) = (1/\gamma)\ln(\beta) + (1/\gamma)\ln(R)$$

- This last expression reveals that with this preference specification, γ measures the constant coefficient of relative risk aversion, but

$$\frac{1}{\gamma} = \frac{d\ln(c_1/c_0)}{d\ln(R)}$$

measures the constant elasticity of intertemporal substitution.

Separating Risk and Time Preferences

- Although the link between aversion to risk (γ) and willingness to substitute consumption intertemporally ($1/\gamma$) is particularly clear in the CRRA case, it holds more generally...

- ...since both features of preferences are reflected in the concavity of the Bernoulli utility function in the vN-M expected utility framework.

Separating Risk and Time Preferences

- Empirical evidence: this link between risk aversion and intertemporal substitution is too restrictive to describe optimal saving and investment behavior

- A more general preference specification is proposed by **Larry Epstein** and **Stanley Zin**, "*Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework*," Econometrica Vol. 57 (July 1989): 00.937-969.

Separating Risk and Time Preferences

- Epstein and Zin work consider a multi-period framework
- Here for simplicity we focus on a **two-period** version
- Their proposed utility function over consumption c_0 at $t = 0$ and consumption $\tilde{c}_1$, possibly dependent on random asset returns, at $t = 1$, is

$$U(c_0, \tilde{c}_1) = \left\{(1-\beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta\left[\left(E\left(\tilde{c}_1^{1-\alpha}\right)\right)^{\frac{1}{1-\alpha}}\right]^{\frac{\sigma-1}{\sigma}}\right\}^{\frac{\sigma}{\sigma-1}}$$

Separating Risk and Time Preferences

$$U(c_0, \tilde{c}_1) = \left\{ (1-\beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[\left(E\left(\tilde{c}_1^{1-\alpha} \right) \right)^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

- Note, first, that if there is no uncertainty, so that $\tilde{c}_1 = c_1$ and $E(\tilde{c}_1)^{1-\alpha} = c_1^{1-\alpha}$, then this utility function implies

$$U(c_0, c_1) = \left\{ (1-\beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[\left(c_1^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

$$= \left\{ (1-\beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta c_1^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

Separating Risk and Time Preferences

- Without uncertainty,

$$U(c_0, c_1) = \left\{(1-\beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta c_1^{\frac{\sigma-1}{\sigma}}\right\}^{\frac{\sigma}{\sigma-1}}$$

Define

$$V(c_0, c_1) = [U(c_0, c_1)]^{\frac{\sigma-1}{\sigma}} = (1-\beta)c_0^{\frac{\sigma-1}{\sigma}} + \beta c_1^{\frac{\sigma-1}{\sigma}}$$

and note that

$$\frac{\sigma-1}{\sigma} = 1 - \frac{1}{\sigma}$$

to see that under certainty, the Epstein-Zin utility function implies an elasticity of intertemporal substitution equal to σ.

Separating Risk and Time Preferences

$$U(c_0, \tilde{c}_1) = \left\{ (1-\beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[\left(E\left(\tilde{c}_1^{1-\alpha} \right) \right)^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

- On the other hand, under uncertainty, once period $t = 1$ arrives, the investor cares about

$$E\left(\tilde{c}_1^{1-\alpha} \right)$$

so α is like the coefficient of relative risk aversion. Hence, the Epstein-Zin utility function allows the coefficient of relative risk aversion α to differ from the inverse of the elasticity of intertemporal substitution σ.

Separating Risk and Time Preferences

Note that under uncertainty, when $\alpha = 1/\sigma$,

$$1 - \alpha = \frac{\sigma - 1}{\sigma}$$

and

$$\begin{aligned}
U(c_0, \tilde{c}_1) &= \left\{ (1-\beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[\left(E\left(\tilde{c}_1^{1-\alpha} \right) \right)^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}} \\
&= \left\{ (1-\beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta E\left(\tilde{c}_1^{\frac{\sigma-1}{\sigma}} \right) \right\}^{\frac{\sigma}{\sigma-1}} \\
&= \left\{ (1-\beta) c_0^{1-\alpha} + \beta E\left(\tilde{c}_1^{1-\alpha} \right) \right\}^{\frac{1}{1-\alpha}}
\end{aligned}$$

Separating Risk and Time Preferences

Under uncertainty, when $\alpha = 1/\sigma$,

$$U(c_0, \tilde{c}_1) = \left\{(1-\beta)c_0^{1-\alpha} + \beta E\left(\tilde{c}_1^{1-\alpha}\right)\right\}^{\frac{1}{1-\alpha}}$$

Define

$$V(c_0, \tilde{c}_1) = \left[U(c_0, \tilde{c}_1)\right]^{1-\alpha} = (1-\beta)c_0^{1-\alpha} + \beta E\left(\tilde{c}_1^{1-\alpha}\right)$$

to see that in this case, the Epstein-Zin specification collapses to the standard CRRA case, where α measures the coefficient of relative risk aversion and $1/\alpha$ measures the elasticity of intertemporal substitution.

Separating Risk and Time Preferences

Finally, note that in the general Epstein-Zin formulation

$$U(c_0, \tilde{c}_1) = \left\{ (1-\beta) c_0^{\frac{\sigma-1}{\sigma}} + \beta \left[\left(E\left(\tilde{c}_1^{1-\alpha} \right) \right)^{\frac{1}{1-\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

The expectation $E\left(\tilde{c}_1^{1-\alpha}\right)$ gets raised to the power

$$\left(\frac{1}{1-\alpha}\right)\left(\frac{\sigma-1}{\sigma}\right)$$

Unless $\alpha = 1/\sigma$, so that this product equals one, the probabilities of different states at $t = 1$ will enter this utility function nonlinearly: the Epstein-Zin nonexpected utility function is a special case of those considered earlier by Kreps and Porteus.

Separating Risk and Time Preferences

- Hence, Epstein and Zin show that the coefficient of relative risk aversion and the elasticity of intertemporal substitution can be disentangled, but only at the cost of departing from the vN-M expected utility framework.

- Alternatively, we can think of Epstein and Zin's study as giving us another reason to be interested in nonexpected utility: besides describing preferences over early versus late resolution of uncertainty, it also allows risk and time preferences to be separated.

Appendix

Portfolio Choice with CRRA Utility

$$\frac{\pi\left(r_G - r_f\right)}{\left[Y_0\left(1+r_f\right) + a^*\left(r_G - r_f\right)\right]^{\gamma}} + \frac{\left(1-\pi\right)\left(r_B - r_f\right)}{\left[Y_0\left(1+r_f\right) + a^*\left(r_B - r_f\right)\right]^{\gamma}} = 0$$

$$\pi\left(r_G - r_f\right)\left[Y_0\left(1+r_f\right) + a^*\left(r_B - r_f\right)\right]^{\gamma}$$
$$= \left(1-\pi\right)\left(r_f - r_B\right)\left[Y_0\left(1+r_f\right) + a^*\left(r_G - r_f\right)\right]^{\gamma}$$

$$\left[\pi\left(r_G - r_f\right)\right)^{1/\gamma}\left[Y_0\left(1+r_f\right) + a^*\left(r_B - r_f\right)\right]$$
$$= \left[\left(1-\pi\right)\left(r_f - r_B\right)\right]^{1/\gamma}\left[Y_0\left(1+r_f\right) + a^*\left(r_G - r_f\right)\right]$$

Portfolio Choice with CRRA Utility

$$[\pi (r_G - r_f)]^{1/\gamma} [Y_0 (1 + r_f) + a^* (r_B - r_f)]$$
$$= [(1 - \pi) (r_f - r_B)]^{1/\gamma} [Y_0 (1 + r_f) + a^* (r_G - r_f)]$$

$$\begin{aligned} Y_0 (1 + r_f) [\pi (r_G - r_f)]^{1/\gamma} + a^* (r_B - r_f) [\pi (r_G - r_f)]^{1/\gamma} \\ = Y_0 (1 + r_f) [(1 - \pi) (r_f - r_B)]^{1/\gamma} \\ + a^* (r_G - r_f) [(1 - \pi) (r_f - r_B)]^{1/\gamma} \end{aligned}$$

$$Y_0(1 + r_f) \left\{ [\pi (r_G - r_f)]^{1/\gamma} - [(1 - \pi) (r_f - r_B)]^{1/\gamma} \right\}$$
$$= a^* \left\{ (r_G - r_f) [(1 - \pi) (r_f - r_B)]^{1/\gamma} + (r_f - r_B) [\pi (r_G - r_f)]^{1/\gamma} \right\}$$

Back