HW Solution

CS/ECE 374: Algorithms & Models of Computation, Spring 2019 Version: **1.0**

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(100 PTS.) A recurrence.
Consider the recurrence

$$T(n) = \begin{cases} T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n & n \geq 6 \\ 1 & n < 6. \end{cases}$$

Prove by induction that $T(n) = O(n)$.
(An easier proof follows from using the techniques described in section 3 of these notes on recurrences.)

3 Solution:

1. We will prove by induction on n that $T(n) = O(n)$ which is equivalent to $T(n) \leq c * n$ for some constant c. Let $f(n) = c * n$.

2. Base case: $n < 6$. By definition of T, $T(n) = 1$ for $n < 6$. For $n \geq 1$, $f(n) \geq 1$, since $f(n)$ is an increasing function of n and $f(1) = 1$. Therefore $T(n) \leq f(n)$ for $n < 6$.

3. Inductive hypothesis: let $k \geq 6$ be an integer. Assume $T(n) \leq c * n$ holds for $n \leq k$.

4. Inductive step: We will prove $T(n) \leq c * n$ holds for $n = k + 1$. By definition of $T(n) = T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n$, the floor operators $\lfloor n/3 \rfloor$, $\lfloor n/4 \rfloor$, $\lfloor n/5 \rfloor$, $\lfloor n/6 \rfloor$ are smaller than n when $n \geq 6$. By applying inductive hypothesis, and asserting a $c \geq 1$ we have

   $T(n/3) \leq \lfloor n/3 \rfloor * c$,

   $T(n/4) \leq \lfloor n/4 \rfloor * c$,

   $T(n/5) \leq \lfloor n/5 \rfloor * c$, and

   $T(n/6) \leq \lfloor n/3 \rfloor * c$. Therefore,

   $T(n) \leq \lfloor n/3 \rfloor * c + \lfloor n/4 \rfloor * c + \lfloor n/5 \rfloor * c + \lfloor n/6 \rfloor * c + n$.

   Because the function $f(n) = c * n$ is an increasing of n given that $c \geq 1$, and $\lfloor n \rfloor \leq n$ for all n, thus we have $\lfloor n/\{3, 4, 5, 6\} \rfloor * c + n \leq n/\{3, 4, 5, 6\} * c + n$.

   Thus giving us, $T(n) \leq n/3 * c + n/4 * c + n/5 * c + n/6 * c + n$.

   We can simplify this into $T(n) \leq (c(1/3 + 1/4 + 1/5 + 1/6) + 1)n$. The coefficient of n can be written as a single constant therefore giving us the expression $c * n$. This brings us back to our original statement of $T(n) \leq c * n$, which proves that $T(n) = O(n)$.