Prove: P - Given an undirected graph G, with n vertices $\{v_0, v_1, \ldots, v_n\}$, using the greedy coloring algorithm there is no edge uv s.t. $f(u) = f(v)$ where $f(v_i)$ denotes the coloring of vertex v_i.

Proof by contradiction: If we assume P is false ($\neg P$) and reach 2 contradictory assertions Q and $\neg Q$, then the assumption $\neg P$ cannot be correct, and therefore P must be true.

Assume: There is an edge uv in G such that $f(u) = f(v)$. ($\neg P$)

<u>Assertion</u>: Q = in the i^{th} iteration, the algorithm assigns v_i the smallest color (i.e. positive integer) K, s.t. none of its neighbors that are already colored have color K.

<u>Assertion</u>: S = Since there is an edge uv s.t. $f(u) = f(v)$, u and v (neighbors) have been colored with the same k.

<u>Assertion</u>: $\neg Q$ = Since S is true given the assumption $\neg P$, Q must be false, as a vertex u or v has been colored the same as one of its already colored neighbors. $\therefore \neg Q$

<u>Contradiction</u>: Since Q and $\neg Q$ are both true under the assumption $\neg P$, $\neg P$ must be false because it produces a contradiction. Therefore, P must be true.

HW Solution

CS/ECE 374: Algorithms & Models of Computation, Spring 2019 Version: **1.0**

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(100 PTS.) Greedy coloring

Given an undirected graph G with n vertices, the greedy coloring algorithm order the vertices of G in an arbitrary order $v_1, \ldots, v_n$. Initially all the vertices are not colored. In the ith iteration, the algorithm assigns v_i the smallest color (i.e., positive integer) k such that none of its neighbors that are already colored have color k. Let $f(v_i)$ denote the assigned color to v_i.

1 (30 PTS.) Prove that the above algorithm computes a valid coloring of the graph (i.e., there are is no edge uv in G such that $f(u) = f(v)$).

Solution:

See first page (handwritten).

2 (30 PTS.) Prove that if a vertex v is colored by color k, then there is a simple path in the graph $u_1, u_2, \ldots, u_k = v$, such that for $i = 1, \ldots, k$, we have $f(u_i) = i$ (and $u_i u_{i+1} \in \mathsf{E}(\mathsf{G})$ for all $i = 1, \ldots, k-1$).

Solution:

Prove that if a vertex v is colored by color k, then there is a simple path in the graph $u_1, u_2, \ldots, u_k = v$, such that for $i = 1, \ldots, k$, we have $f(u_i) = i$ (and $u_i u_{i+1} \in \mathsf{E}(\mathsf{G})$ for all $i = 1, \ldots, k-1$).

1. We will prove by induction on K.

2. Base case: K = 1, then there is a simple path $u_1, ..., u_k = u_1 = v$ S.T. for i = k we have $f(u_i) = i$ because $f(u_1) = 1$.

3. Inductive hypothesis: let $k \geq 1$ be a positive integer. Assume there exists a path in the graph $u_1, ..., u_k$ such that for $i = 1, ..., k$ we have $f(u_i) = i$, and that this holds for $i \leq k$.

4. Inductive step: we will prove the inductive hypothesis holds for $i = k + 1$. In the i^{th} iteration where $(i = k + 1) \geq 2$, the algorithm assigns v_i the smallest color k s.t. none of its neighbors that are already colored have color k. In the i^{th} iteration, the vertex u_{k+1} is assigned the smallest color, $k + 1$, therefore this shows that $f(u_{k+1}) = k + 1$. Because of the algorithm's criteria, we also know that the neighbor set of u_{k+1} must include vertices of all colors $1, ..., k$. This means a vertex u_k where $f(u_k) = k$ is a neighbor of u_{k+1}, and being a neighbor, there is a path between u_k and u_{k+1}. Using the assumption of the inductive hypothesis, we know there is a valid path to u_k: $u_1, ..., u_k$ such that for $i = 1, ..., k$ we have $f(u_i) = i$. Since there is also a path from u_k to u_{k+1}, we can concatenate these paths, and therefore there exists a path in G $u_1, ..., u_k, u_{k+1}$ such that for $i = 1, ..., k, k + 1$, we have $f(u_i) = i$. Because this path exists between consecutive vertices in V(G) from $u_1, ..., u_{i+1}$, then for every $i = 1, ..., k + 1$ there exists an edge $u_i u_{i+1} \in \mathsf{E}(\mathsf{G})$.

3 (40 PTS.) Prove that G either have a simple path of length $\lfloor \sqrt{n} \rfloor$, G contains an independent set of size $\lfloor \sqrt{n} \rfloor$. A set of vertices $X \subseteq \mathsf{V}(\mathsf{G})$ is ***independent*** if no two vertices $x, y \in X$ form an edge in G.

Solution:

We will prove by contradiction that graph G either has a simple path of $\lfloor\sqrt{n}\rfloor$ or that G has an independent set of size $\lfloor\sqrt{n}\rfloor$. Let P be the condition that graph G has a simple path of $\lfloor\sqrt{n}\rfloor$. Let Q be the condition that G has an independent set of size $\lfloor\sqrt{n}\rfloor$. Simplifying this, we will prove that graph G must satisfy P || Q.

Proof by contradiction: Assume G has neither condition true ($\neg P$ and $\neg Q$)

Assertion: if $\neg P$, then G may have at most $\lfloor\sqrt{n}\rfloor$ colors.

In our proof for 1b, we proved that if a vertex in G exists with color k, G has a simple path of at least length $k-1$. Using the contrapositive of this claim, we can say that if G does not have a simple path of length $k-1$ (or more specifically, $\lfloor\sqrt{n}\rfloor$); then it has no vertex colored with k (i.e, at most $\lfloor\sqrt{n}\rfloor$ colors in G).

We can define an independent set within G as all vertices sharing a single color, as those vertices cannot share an edge. This satisfies the definition of a set of vertices $X \subseteq \mathsf{V}(\mathsf{G})$ being ***independent*** if no two vertices $x, y \in X$ form an edge in G.

If G has n vertices and at most $\lfloor\sqrt{n}\rfloor$ colors, then the largest independent set in G contains at least $n/\lfloor\sqrt{n}\rfloor$ vertices.

It logically follows that:

$n/\lfloor\sqrt{n}\rfloor \geq n/\sqrt{n} = \sqrt{n} \geq \lfloor\sqrt{n}\rfloor$

Therefore the largest independent set in G must contain more than $\lfloor\sqrt{n}\rfloor$ vertices, and Q must be true.

We've arrived at a contradiction. We've shown that Q must be true, but to get there we've assumed $\neg Q$. Therefore, P and Q may not both be false. Therefore, P or Q must be true.

In other words, graph G either has a simple path of $\lfloor\sqrt{n}\rfloor$ or G has an independent set of size $\lfloor\sqrt{n}\rfloor$.