27 (100 PTS.) Stay stable

We are given a directed graph with n vertices and m edges ($m \geq n$), where each edge e has a weight $w(e)$ (you can assume that no two edges have the same weight). For a cycle C with edge sequence $e_1 e_2 \cdots e_\ell e_1$, define the *fluctuation* of C to be

$$f(C) = |w(e_1) - w(e_2)| + |w(e_2) - w(e_3)| + \cdots + |w(e_\ell) - w(e_1)|.$$

27.A. (10 PTS.) Show that the cycle with the minimum fluctuation cannot have repeated vertices or edges, i.e., it must be a simple cycle.

27.B. (90 PTS.) Describe a polynomial-time algorithm, as fast as possible, to find the cycle with the minimum fluctuation.

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Solution:

1. Here we will show that any cycle with repeated vertices or edges contains a smaller, simple cycle, with lower fluctuation.

   Let our graph G, contain a simple cycle C. If we were to extend this cycle further by adding any more edges/vertices, we know that fluctuation sum would only increase due to the function using the absolute value of a difference. By definition of absolute value, additions can only be positive. Therefore by repeating any vertices, we only increase our fluctuation sum. Any other cycle containing C will have a fluctuation greater than C.

2. **Idea:** We went with the approach of calling Dijkstra's for every node v in the graph. Within each Dijsktra's call, we calculate the fluctuation value of $|w(e_1) - w(e_2)|$ for every edge it traverses. Instead of optimizing for minimum path weight, we now optimize for minimum fluctuation. The Dijkstras call returns the minimum fluctuation path to every node from V.

   After this, we iterate over each vertex with a backedge to v. We take the path that leads to that vertex, append the last backedge to v thus completing a cycle, and calculate the fluctuation of that whole cycle, and store it in a 'candidate' array.

   Now that we have the fluctuation values of these 'candidate' cycles containing v, we take the minimum fluctuation cycle and store it in an array.

   After we have stored all such cycles for each vertex v, we select and return the cycle with the minimum fluctuation from our stored array.

   Psuedocode: See image 1.

   Analysis: n calls of a modified Djikstra's for each node gives us a polynomial run time of $O(n(m + nlogn))$.
