Math 3101: Homework II Solutions

Krantz: Ch. 2 $\#9, 14, 20, 22, 28, 32, 37, 43$

9. Imitate the proof of Pythagoras's theorem to show that the number 5 does not have a rational square root.

   Proof. Suppose the contrary. Then there exist $p, q \in \mathbb{Z}$ such that $q \neq 0$, p and q have no common divisors, and $\sqrt{5} = \frac{p}{q}$. Squaring both sides gives $5 = \frac{p^2}{q^2}$ which implies that $5q^2 = p^2$. Therefore, since 5 divides $5q^2$ it must divide p^2. Furthermore, because 5 is prime and divides p^2, it must divide p. Thus, there is some integer k such that $p = 5k$. Substituting this into $5q^2 = p^2$, we obtain $5q^2 = 25k^2$. Hence, $q^2 = 5k^2$ implying that 5 divides q^2 which further implies that 5 divides q. We have reached a contradiction as we assumed that p and q have no common divisors, but just showed they are both multiplies for 5. Therefore, the number 5 does not have a rational square root, and this completes the proof. $\square$

14. Prove that if the product of two integers is odd, then both of them must be odd.

    Proof. We will prove this by contrapositive. That is, if we have an even integer $m = 2r$ and an odd integer $n = 2s + 1$, then their product is even. Observe:

    $$m \cdot n = 2r(2s + 1) = 2(2rs + r).$$

    Hence, we have written $m \cdot n$ as a product of 2 and the integer $2rs + s$ proving that it is even. $\square$

In each of Exercises 20 and 22, either prove that the statement is true or provide a counter example.

20. The difference of two perfect squares is never a prime number.

    Proof. False. Consider the perfect squares $2^2 = 4$ and $3^2 = 9$. We see that

    $$3^2 - 2^2 = 9 - 4 = 5,$$

    a prime number, disproving the claim. $\square$

22. For x a positive real number, we have $1 + x^2 < (1 + x)^2$.

    Proof. True. Suppose that $x \in \mathbb{R}$ is such that $x > 0$. Then we certainly have that $0 < 2x$. Adding $1 + x^2$ to both sides of the inequality gives:

    $$1 + x^2 < 2x + 1 + x^2 = x^2 + 2x + 1 = (1 + x)^2.$$

    Thus, if x is a positive real number, we have $1 + x^2 < (1 + x)^2$. $\square$

28. Prove by induction that the sum of angles interior to a convex polygon with k sides is $(k - 2) \cdot 180°$ (begin with $k = 3$ and you may assume that the result is known for triangles).

    Proof. Our statement $P(k)$ is that the interior angles of a convex k-gon sum to $(k - 2) \cdot 180°$. We assume that the base case $k = 3$ is true as stated above. Now, assume that the statement is true for all n-gons where $n \leq k$. Now, we want to show that $P(k + 1)$ also holds. Suppose we have a convex $(k + 1)$-gon. Call it T and name its vertices $v_1, v_2, \ldots, v_k, v_{k+1}$ where v_i is adjacent to v_{i+1} for $i = 1$ to $i = k$ and v_{k+1} is adjacent to v_1. Because T is convex, we can draw an edge between v_1 andv_3 that is entirely contained in T. Then we are left with T divided into a triangle and a k-gon whose interior angles make up the interior angles of T. By our inductive hypothesis, the interior angles of the triangle sum to $180°$ and the interior angles of the k-gon sum to $(k - 2) \cdot 180°$. Hence, the interior angles of T sum to the following:

    $$180° + (k - 2) \cdot 180° = ((k + 1) - 2) \cdot 180°.$$

    This completes the proof.

$\square$

32. Fix a number $q > 1$. Use induction on the *positive* integer n to prove the following formula of Ramanujan:

$$1 + \frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)} + \cdots + \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

Proof. Note that q is fixed and that we are inducting on n where $P(n)$ is the statement of the equality above. Let us prove the base case first: suppose that $n = 1$. Then we have the following:

$$1 + \frac{q^1}{1-q^1} = \frac{1-q}{1-q} + \frac{q}{1-q} = \frac{1-q+q}{1-q} = \frac{1}{1-q}.$$

Thus, the base case holds. Now, assume that $P(k)$ holds. Then we have:

$$1 + \frac{q}{1-q} + \cdots + \frac{q^k}{(1-q)\cdots(1-q^k)} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$$

Adding $\frac{q^{k+1}}{(1-q)\cdots(1-q^k)(1-q^{k+1})}$ to both sides, we obtain the following:

$$\left(1 + \frac{q}{1-q} + \cdots + \frac{q^k}{(1-q)\cdots(1-q^k)}\right) + \frac{q^{k+1}}{(1-q)\cdots(1-q^k)(1-q^{k+1})}$$

$$= \left(\frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}\right) + \frac{q^{k+1}}{(1-q)\cdots(1-q^k)(1-q^{k+1})}$$

$$= \left(\frac{1}{(1-q)(1-q^2)\cdots(1-q^k)} \cdot \frac{1-q^{k+1}}{1-q^{k+1}}\right) + \frac{q^{k+1}}{(1-q)\cdots(1-q^k)(1-q^{k+1})}$$

$$= \frac{1-q^{k+1}+q^{k+1}}{(1-q)\cdots(1-q^k)(1-q^{k+1})}$$

$$= \frac{1}{(1-q)\cdots(1-q^k)(1-q^{k+1})}$$

Thus, $P(k+1)$ holds provided $P(k)$ holds. $\square$

37. You write 27 letters to 27 different people. Then you address the 27 envelopes. You close your eyes and stuff one letter into each envelope. What is the probability that just one letter is in the wrong envelope?

Proof. We are assuming every letter goes into an envelope. Therefore, if one letter is in the wrong envelope, at least two letters are in the wrong envelopes. (Convince this to yourself. I explained this in class. If you still don't understand, we can discuss it more.) Therefore, the probability that exactly one letter is in the wrong envelope zero. $\square$

43. Use induction to prove the identity

$$\frac{2^2}{1\cdot 3}\cdot\frac{3^2}{2\cdot 4}\cdots\frac{n^2}{(n-1)\cdot(n+1)}=\frac{2n}{n+1}.$$

for $n\in\mathbb{N}$ where $n>1$.

Proof. Note that the statement $P(n)$ is stated as the equality above. In our base case, $P(2)$, we have $\frac{2^2}{(2-1)(2+1)}=\frac{4}{3}=\frac{2\cdot 2}{2+1}$, so it holds. Now, assume $P(k)$ holds. That is, suppose the following equality is true.

$$\frac{2^2}{1\cdot 3}\cdot\frac{3^2}{2\cdot 4}\cdots\frac{k^2}{(k-1)\cdot(k+1)}=\frac{2k}{k+1}.$$

Multiplying both sides by $\frac{(k+1)^2}{((k+1)-1)((k+1)+1)}$, we obtain:

$$\begin{aligned}
\frac{2^2}{1\cdot 3}\cdot\frac{3^2}{2\cdot 4}\cdots\frac{k^2}{(k-1)\cdot(k+1)}\cdot\frac{(k+1)^2}{((k+1)-1)((k+1)+1)} &= \frac{2k}{k+1}\cdot\frac{(k+1)^2}{((k+1)-1)((k+1)+1)}\\
\frac{2^2}{1\cdot 3}\cdot\frac{3^2}{2\cdot 4}\cdots\frac{k^2}{(k-1)\cdot(k+1)}\cdot\frac{(k+1)^2}{k(k+2)} &= \frac{2k}{k+1}\cdot\frac{(k+1)^2}{k(k+2)}\\
&= \frac{2(k+1)}{k+2}\cdot\frac{k(k+1)}{k(k+1)}\\
&= \frac{2(k+1)}{(k+1)+1}.
\end{aligned}$$

Thus, $P(k)$ implies $P(k+1)$. $\square$