Please show **all** your work! Answers without supporting work will not be given credit. Write answers in spaces provided. You have 1 hour and 30 minutes to complete this exam.

Name:____

1. **(10 pts.)** Determine if $\mathbf{P} \Leftrightarrow \mathbf{Q}$ and $\sim \mathbf{P} \Leftrightarrow \sim \mathbf{Q}$ are logically equivalent. Use a truth table or other means to explain your conclusion.

P	Q	$\sim P$	$\sim Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$Q \Leftrightarrow P$	$\neg P \Rightarrow \neg Q$	$\neg Q \Rightarrow \neg P$	$\neg P \Leftrightarrow \neg Q$
T	T	F	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T	F	F
F	T	T	F	T	F	F	F	T	F
F	F	T	T	T	T	T	T	T	T

Because the columns for $Q \Leftrightarrow P$ and $\neg P \Leftrightarrow \neg Q$ are ~~equi~~ identical, the statements are logically equivalent.

2. **(10 pts.)** Determine if $\mathbf{P} \Rightarrow \mathbf{Q}$ and $\mathbf{P} \vee \sim \mathbf{Q}$ are logically equivalent. Use a truth table or other means to explain your conclusion.

P	Q	$P \Rightarrow Q$	$\neg Q$	$P \vee \neg Q$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Because the columns for $P \Rightarrow Q$ and $P \vee \neg Q$ differ, the statements are not logically equivalent.

3. **(15 pts.)** Write the contrapositive, converse, and negation of the following statement:

If x is a real number, then $x > 0$ of $x \leq 0$.

Contrapositive:

If $x \leq 0$ and $x > 0$, then x is not a real number.

or (If $x \not> 0$ and $x \not\leq 0$, then x is not a real number.)

Converse:

If $x > 0$ or $x \leq 0$, then x is a real number.

Negation:

x is a real number, $x > 0$, and $x \leq 0$.

4. **(5 pts.)** Suppose we have the following statements:

A = **I believe everything I read.**
B = **All fish have eyelids.**

Express the statement: "If I do not believe some of what I read, then some fish do not have eyelids," using the letter **A** and **B** and the connectives $\vee, \wedge, \sim, \Rightarrow, \Leftrightarrow$.

If $\neg A$, then $\neg B$

$\neg A \Rightarrow \neg B$

Cont.

5. **(5 pts.)** Suppose we have the following statements.

P	=	**All politicians are honest.**
Q	=	**I don't have two brain cells to rub together..**
R	=	**The pie is in the sky.**

Translate the following statement into English: $\sim \mathbf{P} \Leftrightarrow (\mathbf{R} \Rightarrow \sim \mathbf{Q})$.

There exists a dishonest politician if and only if the pie in the sky implies that I have two brain calls to rub together.

6. **(10 pts.)** Prove that if $\sqrt{x} \in \mathbb{Q}$, then $x \in \mathbb{Q}$.

Proof.

If $\sqrt{x} \in \mathbb{Q}$, then there exist $p, q \in \mathbb{Z}, q \neq 0$ such that $\sqrt{x} = \frac{p}{q}$. Squaring both sides gives $x = \frac{p^2}{q^2}$. Because $p^2, q^2 \in \mathbb{Z}$ and $q^2 \neq 0$, $x \in \mathbb{Q}$ by definition.

$\square$

Cont.

7. **(15 pts.)** Prove that if $x \in \mathbb{Z}$, then $x^2 + x$ is even. Do this by cases. (You may prove using some other method for partial credit.)

Proof.

First suppose $x = 2k$ is even.

Then $x^2 + x = (2k)^2 + 2k$

$= 4k^2 + 2k$

$= 2(2k^2 + k)$.

Hence $x^2 + x$ is even.

Now suppose $x = 2k+1$ is odd.

Then

$x^2 + x = (2k+1)^2 + (2k+1)$

$= 4k^2 + 4k + 1 + 2k + 1$

$= 4k^2 + 6k + 2$

$= 2(2k^2 + 3k + 1)$.

Hence $x^2 + x$ is even.

$\square$

Cont.

8. **(15 pts.)** Prove by induction that the sum of the first n odd integers is n^2 [redacted]. That is, show the following:

$$\sum_{i=1}^{n}(2i-1) = n^2$$ [redacted]

Proof.

We proceed by induction. Clearly

$$\sum_{i=1}^{1} 2i-1 = 2(1)-1 = 1 = 1^2.$$

Now assume that

$$\sum_{i=1}^{k} 2i-1 = k^2.$$

Adding the next odd number $2k+1$ gives:

$$2k+1 + \sum_{i=1}^{k} 2i-1 = k^2+(2k+1)$$

$$[2(k+1)-1] + \sum_{i=1}^{k} 2i-1 = k^2+2k+1$$

$$\sum_{i=1}^{k+1} 2i-1 = (k+1)^2$$

completing the proof. □

Cont.

9. **(15 pts.)** Prove that $\sqrt{2}$ is irrational.

Proof.

Spz not. Then there exist $p, q \in \mathbb{Q}$ such that $q \neq 0$, p and q have no common divisors, and $\sqrt{2} = \frac{p}{q}$. Squaring gives:

$$2 = \frac{p^2}{q^2};$$

So,

$$2q^2 = p^2$$

Therefore, p^2 is even implying p is even. Hence $p = 2k$ for some $k \in \mathbb{Z}$. Then

$$2q^2 = (2k)^2 = 4k^2,$$

So

$$q^2 = 2k^2$$

implying q is even. This contradicts the fact that p and q had no common divisors. □

The End.