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Singular Solutions of Rolling Balls Model a Topological View

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The cardinal notions of number, place, and combination...three intersecting but distinct spheres of thought to which all mathematical ideas admit of being referred.

— James Joseph Sylvester

Contents

Abstract (Italian)

Il "Rolling Balls Model", il modello che descrive il rotolamento senza torsioni né scivolamenti di una sfera sulla superficie di un'altra sfera, trova la sua più naturale formulazione nel linguaggio della Teoria del Controllo Ottimale e più precisamente nella sua formulazione moderna data dalla Geometria sub-Riemanniana. Una struttura sub-Riemanniana è costituita da una varietà differenziabile M (spazio delle configurazioni del sistema) e da una mappa di controllo $L : W \to M$ che identifica un fibrato vettoriale W su M con un sottofibrato di TM, definendo cosi sulla varietà M una distribuzione di sottospazi tangenti. I cammini ammissibili, che rappresentano le soluzioni del problema, sono quindi particolari curve localmente Lipschitziane in M il cui campo di velocità (nei punti in cui essa è definita) appartiene a tale distribuzione vettoriale.

Considerata una generica varietà sub-Riemanniana (W, L) e fissato un punto $p \in M$, lo spazio Ω_p delle curve ammissibili rispetto a (W, L) uscenti dal punto p ammette una struttura di varietà di Hilbert sullo spazio funzionale $L^2([0, 1], \mathbb{R}^k)$. Sulla varietà Ω_p è quindi definita in modo naturale una mappa, detta *endpoint map*

$$\mathrm{end} : \Omega_p \to M$$

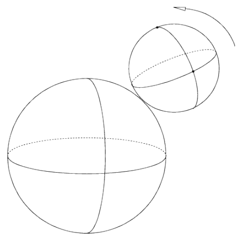
$$\mathrm{end}(\gamma) = \gamma(1)$$

che associa ad ognuna di queste curve ammissibili il punto finale, che si dimostra essere differenziabile rispetto alla struttura di varietà di Hilbert di Ω_p. Le soluzioni singolari sono per definizione i punti critici di tale mappa, che si dimostrano essere esattamente le proiezioni su M degli estremali abnormali definiti dal Principio del Massimo di Pontryagin su T^*M (vedi [4]). L'importanza di tali soluzioni è legata al fatto che esse distinguono le strutture sub-Riemanniane da quelle Riemmaniane: nel caso in cui la distribuzione su M abbia rango massimo, essa definisce infatti una struttura riemanniana su M per cui si dimostra però la non esistenza di tale tipo di soluzioni.

Nel caso specifico del rotolamento reciproco di due sfere di raggi rispettivamente r ed R, lo spazio delle configurazioni è una varietà differenziabile M di dimensione 5 ed in particolare un fibrato principale

$$SO(2, \mathbb{R}) \to M \to S^2 \times S^2$$

su cui è definita una distribuzione D_ρ di rango due. Tale distribuzione e dunque il sistema di equazioni differenziali (ODE) associato, dipende dal rapporto ρ dei raggi delle due sfere ed è in particolare integrabile se e soltanto se tale rapporto è unitario.



Le simmetrie del sistema (ovvero della distribuzione associata al problema), che per un generico valore del rapporto dei raggi sono rappresentate dal gruppo $SO(3) \times SO(3)$ (ovvero il prodotto dei gruppi di simmetria delle due sfere), si allargano nel caso (e solo nel caso) di rapporto dei raggi pari a $1:3$ agli automorfismi associati al gruppo di Lie eccezionale G_2 (in particolare della sua forma reale non compatta, detta forma split) ottenibile dalla classificazione dei gruppi di Lie semisemplici. Inoltre tale gruppo agisce, sempre per rapporto $1:3$, anche sulla varietà delle soluzioni singolari. La proprietà di avere tale gruppo come gruppo di simmetria è una più generale proprietà dei problemi di rotolamento di due corpi l'uno sull'altro[1], ma assume un'importanza particolare nel caso in cui tali corpi siano due sfere, in quanto in questo specifico caso lo spazio delle configurazioni (più precisamente il suo rivestimento universale $\mathbf{M}$) può essere descritto in termini dell'algebra degli ottonioni in forma split $\mathbb{O}_s$, il cui gruppo di automorfismi è costituito proprio dalla forma split del gruppo di Lie G_2. La peculiarità di tale descrizione è di rendere esplicita la condizione sul rapporto dei raggi, spiegando cosi perché il valore $1:3$ sia necessario affinchè il sistema e lo spazio delle curve singolari associato ammettano tale gruppo come gruppo di simmetria.

Le proiezioni sulle due sfere delle soluzioni singolari in M associate al problema si dimostrano essere, nel moto effettivo, le traiettorie associate a coppie di geodetiche sulle rispettive sfere. Nella parte originale della tesi si dimostra come tale proprietà permetta di descrivere tali spazi di soluzioni singolari N_ρ, per valori razionali del rapporto dei raggi, in modo geometrico come varietà quoziente rispetto ad un'azione del gruppo $SO(2)$ (dipendente dal rapporto dei raggi ρ) sul prodotto dei fibrati tangenti normali alle due sfere. Al variare di tale rapporto (razionale) rimane quindi definita una famiglia di varietà differenziabili N_ρ di dimensione 5 per cui, cosi come per la varietà delle configurazioni del sistema, è possibile dare una descrizione in termini dell'algebra degli ottonioni $\mathbb{O}_s$. Nello specifico ognuna di esse è diffeomorfa globalmente al quoziente di $S^3 \times S^3 \subset \mathbb{O}_s$ rispetto ad una rispettiva azione $SO_\rho(2)$ da cui si conclude che, nel caso particolare del rapporto dei raggi unitario, tale varietà è diffeomorfa (globalmente) proprio al rivestimento $\mathbf{M}$ della varietà delle configurazioni M. Si utilizza quindi tale descrizione per calcolarne alcuni degli invarianti topologici principali per ogni valore (razionale) del rapporto, quali i gruppi di omotopia, di omologia e di coomologia di De Rham, che si dimostrano però essere identici e dunque indipendenti da tale rapporto, aprendo cosi ad interessanti quesiti sulla struttura topologica di tali varietà: il gruppo di Lie G_2 può agisce infatti su tale spazio di soluzioni se e soltanto se tale rapporto è pari a 3, suggerendo tali varietà siano in generale non omeomorfe. Si mostra quindi come, nel caso di rapporto intero (positivo) k, sia possibile definire una famiglia di rivestimenti ramificati

$$p_k : \mathbf{M} \to \mathbf{N}_k$$

[1] La condizione di avere come gruppo di simmetria il G_2 è legata ai valori del rapporto delle curvature Gaussiane dei due corpi e si riduce, nel caso specifico in cui tali corpi siano due sfere, alla condizione sul rapporto dei loro raggi.

di grado k^2 ad ognuno dei quali è in particolare associata una famiglia a due parametri di embedding (al variare dei parametri in S^2) dello spazio lenticolare $L_k := S^3/\mathbb{Z}_k$ nella varietà $\mathbf{N}_k$.

Nel primo capitolo si introduce alla classificazione dei gruppi di Lie semisemplici da cui si deducono l'esistenza delle diverse forme del gruppo di Lie G_2 e le sue principali proprietà. Si introduce poi alle algebre di divisione ed in particolare allo spazio dei quaternioni descrivendone le proprietà geometriche ed agli ottonioni e la loro forma split, per poi mostrare come il gruppo G_2 costituisca il gruppo degli automorfismi di quest'ultimi. Nel secondo capitolo si riassumono i principali risultati riguardanti la Geometria sub-Riemmaniana ed i suoi legami con i sistemi Hamitloniani, per poi introdurre la nozione di soluzione singolare. Si focalizza quindi lo studio sul caso specifico del rolling balls problem ed in quest'ottica e se ne mostrano i legami con le altre strutture algebriche citate: la descrizione ottonionica mediante i quaternioni ed i suoi legami con il gruppo di simmetria dato dal gruppo di Lie G_2. Si conclude quindi la tesi presentando i risultati originali riguardanti lo studio di tali spazi di soluzioni singolari.

Abstract

The "Rolling Balls Model", the model describing a pair of spheres rolling one on another without slipping or twisting is an Optima Control Theory problem or rather of its modern formulation, sub-Riemannian geometry. A sub-Riemannian structure is given by a differential manifold M (the configuration space of the system) and a control map $L : W \to M$ that identifies a vector bundle W on M with a vector sub-bundle of TM, defining on the manifold M a distribution of tangent subspaces. The admissible paths, the solutions of the problem are particular local Lipschitz curves in M which velocity fields (in the points on which defined) belong to such distribution. Consider a sub-Riemannian manifold (W, L), fixed a point $p \in M$, the space Ω_p of admissible curves with respect to (W, L) outgoing from p has a structure of smooth Hilbert manifold on the functional space $L^2([0, 1], \mathbb{R}^k)$. On the manifold Ω_p is then defined a map, called *endpoint map*

$$\mathrm{end} : \Omega_p \to M$$

$$\mathrm{end}(\gamma) = \gamma(1)$$

which is differentiable with respect to the Hilbert manifold structure of Ω_p. Singular solutions are then by definition the critical points of such map that can be proved being the projection on M of the abnormal extremals defined on T^*M by Pontryagin's Maximum Principle (see [4]). This class of solutions distinguishes sub-Riemannin structures from Riemman ones: if the rank of the distribution on M is maximal, the distribution defines a Riemannian structure on M which, however, has no singular solutions.

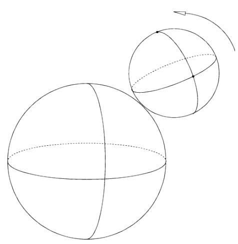
For the system of two spheres of radius respectively r and R rolling one on another, the configuration space is a 5-dimensional manifold M and in particular a principal bundle

$$SO(2, \mathbb{R}) \to M \to S^2 \times S^2$$

on which is defined a rank 2 distribution D_ρ. This distribution and then the related system of differential equation (ODE), depends on the ratio ρ of the sphere's rays and it is in particular integrable if and only if this ratio is one.

The symmetries of the system, that for general values of the ratio are represented by the group $SO(3) \times SO(3)$ (the product of the symmetry groups of the two spheres), in the case (and only in the case) of ratio equal to $1 : 3$ increase to the exceptional Lie group G_2 (its real, non-compact form) obtained by the semisimple Lie groups' classification. Moreover for the same value of the ratio such group acts also, on the singular solution space. The properties to have

such group as symmetry group is a more general property of rolling problems [2] but it takes a particular importance for rolling sphere since the configuration space (or rather its universal cover $\mathbf{M}$) can be described by using the split form of octonions algebra $\mathbb{O}_s$, which automorphism group is given by the split form of the real Lie group G_2. This description makes explicit the condition on the ratio, explaining why the specific value $1:3$ is necessary to have a G_2-simmetry.



The projections of the singular solutions in M on the two spheres are given by pairs of geodesics on their respective spheres. In the original part of this dissertation we then show how such property allows us to describe such spaces of singular curves N_ρ (for rational values of the ratio) as quotient manifolds with respect to a $SO(2)$-action depending on the ratio ρ, defined on the product of the normal fiber bundles on the two spheres. It is then defined a family of 5-dimensional differential manifolds N_ρ for which it is possible to give a split-octonionic description. In particular any of such manifold is globally diffeomorphic to the quotient of $S^3 \times S^3 \subset \mathbb{O}_s$ with respect to a $SO_\rho(2)$-action that in particular, for ratio equal to one, gives the covering $\mathbf{M}$ of the configuration space. It is then possible to use such description to compute some topological invariants of these manifolds. It turns out that Homotopy, Homology and de Rham Cohomology groups are independent on this ratio. Then the fact that the G_2 group could act on these manifold if and only if the ratio equals 3, it suggests these manifolds could be in general not homeomorphic. For integer values of the ratio it is moreover possible to define a family of branched covering

$$p_k : \mathbf{M} \to \mathbf{N}_k$$

of degree k^2, each of which is associated with a family of two parameters of embedding (varying the parameters in S^2) of the lens spaces $L_k := S^3/\mathbb{Z}_k$ in the manifold $\mathbf{N}_k$.

In the first chapter we introduce to semisimple Lie groups' classification from which the different forms for the Lie group G_2 and their properties follow. Then we introduce division algebras and in particular the quaternion space and its geometrical properties and octonions and their split form, in order to show that G_2 is their autmorphism group. In the second chapter we summarize the principal results about sub-Riemannian geometry and its relations with Hamiltonians systems, in order to introduce singular solutions. We then focus on rolling balls problem showing its relations with the other algebric structures: the octonionic description using quaternions and its relation with the symmetry group G_2. Finally, in the last part of the dissertation we present the original results about these singular solution spaces.

[2]The condition to have symmetry group G_2 is related to the values of the Gaussian curvatures ratio of the two manifolds and becomes, for the case of spheres, a condition on the rays.

Chapter 1

The G_2 group and Octonions

Lie groups describe differentiable transformations of geometrical spaces depending on some complex (real) parameters, that give a structure of finite dimensional complex (real) differentiable manifold. The left (right) invariant vector fields defined on these groups, i.e. the vector fields invariant respect to the left (right) group multiplication, represent infinitesimal transformations related to the group and allow us to define an additional structure of Lie algebra on the tangent space at the identity, which describes the whole group. For a group acting on a manifold, these structures also connect the symmetries of the manifold induced by the group to the vector fields defined on it and are then a fundamental tool in the study of solutions of differential relations.

In this chapter we introduce the classification of a fundamental class of Lie groups, the so-called *semisimple* Lie groups. As we'll see, at first we can narrow down to the complex case, from which it is possible to derive the real one. The importance of such class of groups is related to the fact that it includes the so called *classical Lie groups*; the term "classical group", coined by Hermann Weyl at the beginning of the last century, is not related to mathematical properties but to the historical importance of these groups in mathematics and physics. These Lie groups are the special linear groups [1] $SL(n,\mathbb{R})$, $SL(n,\mathbb{C})$, $SL(n,\mathbb{H})$

G		$\mathfrak{g}$		dim
$\mathrm{GL}(n,\mathbb{R})$	/	$\mathfrak{gl}(n,\mathbb{R})$	/	n^2
$\mathrm{SL}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$	$\det A = 1$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathrm{Tr}\, A = 0$	$n^2 - 1$
$\mathrm{Sp}(n,\mathbb{R}) \subseteq \mathrm{GL}(2n,\mathbb{R})$	$A^T JA = J$	$\mathfrak{sp}(n,\mathbb{R})$	$JA + A^T J = 0$	$n(2n+1)$
$\mathrm{O}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$	$AA^T = I$	$\mathfrak{o}(n,\mathbb{R})$	$A + A^T = 0$	$n(n-1)/2$
$\mathrm{SO}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$	$\det A = 1, AA^T = I$	$\mathfrak{so}(n,\mathbb{R})$	$A + A^T = 0$	$n(n-1)/2$
$\mathrm{U}(n) \subseteq \mathrm{GL}(n,\mathbb{C})$	$AA^\dagger = I$	$\mathfrak{u}(n)$	$A + A^\dagger = 0$	n^2
$\mathrm{SU}(n) \subseteq \mathrm{GL}(n,\mathbb{C})$	$\det A = 1, AA^\dagger = I$	$\mathfrak{su}(n)$	$\mathrm{Tr}\, A = 0, A + A^\dagger = 0$	$n^2 - 1$

[1] Here the symbol $\mathbb{H}$ denote the set of quaternions.

G		$\mathfrak{g}$		$\dim_{\mathbb{C}}$
$\mathrm{GL}(n,\mathbb{C})$	/	$\mathfrak{gl}(n,\mathbb{C})$	/	n^2
$\mathrm{SL}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$	$\det A = 1$	$\mathfrak{sl}(n,\mathbb{C})$	$\mathrm{Tr}\, A = 0$	$n^2 - 1$
$\mathrm{O}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$	$AA^T = I$	$\mathfrak{o}(n,\mathbb{C})$	$A + A^T = 0$	$n(n-1)/2$
$\mathrm{SO}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$	$\det A = 1, AA^T = I$	$\mathfrak{so}(n,\mathbb{C})$	$\mathrm{Tr}\, A = 0, A + A^T = 0$	$n(n-1)/2$

Figure 1.1: Real and Complex Lie Groups and Algebras.

together with the special[2] automorphism groups of symmetric or skew-symmetric bilinear forms and Hermitian or skew-Hermitian sesquilinear forms defined on real, complex and quaternionic finite-dimensional vector spaces.

Semisimple complex Lie groups' classification is based on the classification of the possible Lie algebras associated. From such classification it turns out that there exist some "exceptional Lie groups" G_2, F_4, E_6, E_7, E_8, associated to five addictional complex Lie algebras, each of which has two different real forms. In particular we analyze the properties of one of them, the group G_2 and its relations with the algebric structure of octonions for which, as we'll see, one of the real form of the group G_2 is the automorphism group. This analysis is justified by the fact that in next chapter we'll see how this group is related to the symmetry group of the rolling bodies problem, the main topic of this dissertation.

1.1 Lie Groups and Lie Algebras

Consider a complex (real) Lie group G of dimension n. Recall that a generic Lie Algebra $\mathfrak{g}$ is a vector space with an antisymmetric bilinear form $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

$$[[A,B],C] + [[C,A],B] + [[B,C],A] = 0$$

The Lie algebra $\mathfrak{g}$ of G is the vector space of left (right) - invariant vector fields, with the bilinear form given by the vector fields commutator, that is a finite dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ isomorphic to the tangent space at the identity of the group. A complex (real) Lie algebra can be specified by a set of *generators* $\{E^k\}$ i.e. a basis of $\mathfrak{g}$ as vector space, with commutation relations

$$[E^k, E^l] = \sum_{l=1}^{n} c_m^{kl} E^m \quad , \quad c_m^{kl} \in \mathbb{C}, \mathbb{R}$$

where the constants c_m^{kl}, called *structure constants* of the algebra, satisfy the antisymmetric property $c_m^{kl} = -c_m^{lk}$. An example of complex Lie algebra is given by the complex Lie group $SL(2,\mathbb{C}) \subset GL(2,\mathbb{C})$. This group is the group of complex 2×2 matrices

$$M = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

with determinant 1. It's easy to show that 1 is regular value for the map

$$\det : SL(2,\mathbb{C}) \to \mathbb{C}$$

[2]The subgroups with determinant equal to 1.

and then $SL(2,\mathbb{C})$ is actually a Lie subgroup of $GL(2,\mathbb{C})$. Its Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is the set

$$\mathfrak{sl}(2,\mathbb{C}) = \big\{A \in \mathbb{C}^{2\times 2} \mid \mathrm{Tr}(A) = 0\big\}$$

and a possible set of generators is given by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Instead, an example of real Lie group is the 3-dimensional group $SU(2)$ i.e. the set of (complex) matrices of the type

$$M = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2} \qquad M^\dagger = M^{-1}, \quad \det(M) = 1$$

Notice that this group, also if defined by complex matrices, does not admit a structure of complex Lie group. The subalgebra associated is the real Lie algebra

$$\mathfrak{su}(2) = \big\{A \in \mathbb{C}^{2\times 2} \mid A = A^\dagger,\ \mathrm{Tr}(A) = 0\big\}$$

while a set of possible generators is given by

$$J_1 = i\frac{\sigma_1}{2} = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \qquad J_2 = i\frac{\sigma_2}{2} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad J_3 = i\frac{\sigma_3}{2} = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices (notice that all the structure constants are real). Anyway, if we consider the generators

$$J_\pm := J_1 \pm iJ_2$$

and the vector space generated over $\mathbb{C}$ (i.e. the complexification of the algebra), we get a complex Lie algebra, that is $\mathfrak{sl}(2,\mathbb{C})$.

1.1.1 The Adjoint Representation

To study Lie groups' properties it is natural to think about them as groups of linear transformations of some vector space. This notion is related to the representation one. Consider a finite dimensional complex vector space $V_\mathbb{C}$ and the automorphisms group associated

$$\mathrm{Aut}(V_\mathbb{C}) = \big\{L : V_\mathbb{C} \to V_\mathbb{C} \mid L \text{ isomorphism}\big\}$$

Fixed a basis of $V_\mathbb{C}$, we have the identification $\mathrm{Aut}(V_\mathbb{C}) \cong GL(n,\mathbb{C})$ therefore $\mathrm{Aut}(V_\mathbb{C})$ has a structure of Lie group.

Definition 1.
Let G be a complex Lie group and $V_\mathbb{C}$ a finite dimensional vector space. We call *representation* of G on $V_\mathbb{C}$ a Lie groups homomorphism

$$T : G \to \mathrm{Aut}(V_\mathbb{C})$$

$$g \to T_g$$

i.e. such that for all $g \in G$ the associated map T_g is smooth and

$$T_{g_1 g_2} = T_{g_1} \circ T_{g_2} \qquad , \qquad T_{g^{-1}} = T_g^{-1}$$

We now show how for a given Lie group it is possible to define an intrinsic representation as the automorphisms group of its Lie algebra. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Consider a left invariant vector field $A \in \mathfrak{g}$ and its integral curve through the identity γ^A. This solution defines a map, called *exponential map*

$$\exp : \mathfrak{g} \to G$$
$$\exp(A) := \gamma^A(1)$$

Since A is smooth, by Cauchy Theorem also exp is smooth and then by the uniqueness of the solution of the ODE it follows that for each $t \in \mathbb{R}$ it satisfies

$$\gamma^{tA}(s) = \gamma^A(ts)$$

then in particular

$$\gamma^{tA}(1) = \gamma^A(t)$$

Moreover it's clear that the set $\{\exp(tA)\}_{t\in\mathbb{R}}$ is exactly the one parameter group of diffeomorphisms generated by the field A. Consider now a Lie groups homomorphism $\phi : G \to H$. It induces a linear map

$$d\phi_e : T_eG \to T_eH$$

We denote by ϕ_* the associated map

$$\phi_* : \mathfrak{g} \to \mathfrak{h}$$

between the Lie algebras of the groups induced by the identifications $T_eG \cong \mathfrak{g}$ and $T_eH \cong \mathfrak{h}$.

Proposition 1.
Let $\phi : G \to H$ be a Lie groups homomorphism; the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\phi"'] & H \end{tikzcd}$$

Proof.
Given $A \in \mathfrak{g}$, let γ_A be it's integral curve that is

$$\gamma_A(0) = e \qquad \frac{d}{dt}(\gamma_A)_{|t=0} = A$$

Then

$$\phi_*(A) = \frac{d}{dt}(\phi(\gamma_A(t))_{|t=0}$$
$$\exp \circ \phi_*(A) = \gamma_{\phi_*(A)}(1)$$

while

$$\phi \circ \exp(A) = \phi(\gamma_A(1))$$

but $\gamma_{\phi_*(A)}$ and $\phi(\gamma_A)$ solve the same Cauchy problem, then the thesis.

□

Consider now a smooth manifold M and an action of G on M

$$\mu : G \times M \to M$$

and assume there exists a fixed point $p \in M$ for the map $\mu_g : M \to M$ defined by $\mu_g(p) := gp$, for each $g \in G$. Then its differential

$$d\mu_g : T_pM \to T_pM$$

is an automorphism of T_pM and is therefore well defined a map $G \to \mathrm{Aut}(T_pM)$ by

$$g \to d\mu_g$$

Since it's differentiable respect to g and satisfies

$$d\mu_g(X) = \frac{d}{dt}\mu(g, \gamma(t))_{|t=0}$$

it's a smooth map. Moreover it satisfies

$$d\mu_{gh} = d\big(\mu_g \circ d\mu_h\big) = d\mu_g \circ d\mu_h$$

then it is a representation of G on T_pM. Clearly the condition of a point p to be a fixed point for each map μ_g is a strong request for a generic manifold M. On the other hand it is not if we consider a natural action of G on itself. We define the *adjoint action* of G on itself by the map

$$a : G \times G \to G$$

$$a(g, h) := ghg^{-1}$$

The identity $e \in G$ is a fixed point for a_g for each element $g \in G$. From the identification $\mathfrak{g} \cong T_eG$, denoted by

$$\mathrm{Ad}_g := (a_g)_* : \mathfrak{g} \to \mathfrak{g}$$

we can define a representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ by

$$g \to \mathrm{Ad}_g$$

called *adjoint representation* of G on $\mathfrak{g}$. The group $\mathrm{Aut}(\mathfrak{g}) \cong GL(n, \mathbb{R})$ is also a Lie group and its Lie algebra is $\mathrm{End}(\mathfrak{g})$ with the commutator of matrices. Then since the map

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$$

is a group homomorphism, denoted by $\mathrm{ad} := (\mathrm{Ad})_*$ it's differential at the identity as map between the Lie algebra $\mathfrak{g}$ of G and $\mathrm{End}(\mathfrak{g})$ of $\mathrm{Aut}(\mathfrak{g})$, we get a map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

$$A \to \mathrm{ad}_A$$

Notice that since B is a left-invariant vector field

$$\mathrm{ad}_A(B) = \left[\frac{d}{dt}\mathrm{Ad}_{\exp(tA)}B\right]_{t=0} = \left[\frac{d}{dt}(R_{\exp(-tA)*})(L_{\exp(tA)})_*B\right]_{t=0} =$$

$$= \left[\frac{d}{dt}(R_{\exp(-tA)*})B\right]_{t=0} = \mathcal{L}_A(B) = [A, B]$$

Moreover ad defines an action of $\mathfrak{g}$ on $\mathfrak{g}$

$$\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}$$

$$\mathrm{ad}_A(B) = [A, B]$$

for all $B \in \mathfrak{g}$. In the same way we have done for groups, we can define the notion of representation also for Lie algebras.

Definition 2.
Let $\mathfrak{g}$ be a complex (real) Lie algebra and V be complex (real) a vector space. A *Lie algebra representation* of $\mathfrak{g}$ on V is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to End(V)$$

i.e. a linear map $A \to \pi_A$ satisfying

$$\pi_{[A,B]} = \pi_A \pi_B - \pi_B \pi_A$$

The adjoint action of $\mathfrak{g}$ on itself induces a representation

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

$$A \to \mathrm{ad}_A$$

called *adjoint representation* of $\mathfrak{g}$ on $\mathfrak{g}$. It is effective a Lie algebra representation since for each $C \in \mathfrak{g}$ by the Jacoby identity

$$\begin{aligned} \mathrm{ad}_{[A,B]}(C) = [[A, B], C] = [A, [B, C]] - [B, [A, C]] = \\ = \mathrm{ad}_A \mathrm{ad}_B(C) - \mathrm{ad}_B \mathrm{ad}_A(C) \end{aligned}$$

Finally, for the Proposition the diagram

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & End(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"'] & End(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd}$$

commutes. Lie group's structure is strongly related to its Lie algebra one and then the adjoint representation becomes a fundamental stuff in Lie groups' classification.

1.1.2 From Groups to Algebras

Consider a Lie group G and its Lie algebra $\mathfrak{g}$. There is not correspondence between general subgroups of G and subalgebras of $\mathfrak{g}$. For example, $\mathbb{Q}$ is a proper subgroup of $\mathbb{R}$ which is not even a manifold; anyway if we consider only Lie subgroups the following holds.

Theorem 1.
Let $\mathfrak{a}$ be finite dimensional complex (real) Lie algebra. Then there exists a Lie group G such that its Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{a}$. Moreover, given a Lie group G with Lie algebra $\mathfrak{g}$, to any Lie subalgebra of $\mathfrak{g}$ is associated to a Lie subgroup of G.

Let G be a Lie group with closed subgroup $H \subset G$ and let $\pi : G \to G/H$ be the projection onto the coset space. If H is normal in G it is possible to prove (see *Cartan Theorem*) that it defines a topological group G/H admitting a structure of Lie group such that the projection π is a Lie group homomorphism and an open map. Remark that the correspondence established by the Theorem is not bijective, then to a Lie algebra could be associated more that one Lie group. To solve this asymmetry, we have to take into account also discrete subgroups of G.

Definition 3.
Let G be a Lie group and $Z \subset G$ be a subgroup. We call Z *discrete subgroup* if there exists an open cover of G for which every open subset contains exactly one element of Z.

In other words, a subgroup $Z \subset G$ is discrete if the subspace topology of Z in G is the discrete topology. As example the set of integers $\mathbb{Z}$, form a discrete subgroup of the reals numbers $\mathbb{R}$. Consider now the orthogonal group $O(n, \mathbb{C})$. It has two connected components: the one containing the identity is the special orthogonal group $SO(n, \mathbb{C})$ while the remaining component consists in the reflections of $\mathbb{R}^n$ (that don't form a Lie group): it's related to a discrete subgroup of $O(n, \mathbb{C})$. This is a more general property of Lie groups.

Proposition 2.
Let G be a Lie group and $G_e \subseteq G$ be his connected component which contains the identity. Then G_e is a normal Lie subgroup of G and the quotient G/G_e is a discrete subgroup.

Remark that a connected Lie group G is generated by a neighborhood of its identity. Let U be a neighborhood of $e \in G$; if necessary replacing U with $U \cap U^{-1}$ where

$$U^{-1} := \left\{ g^{-1} \mid g \in U \right\}$$

we can suppose that $U = U^{-1}$. Consider the set

$$S := \left\{ g_1 \cdots g_n \mid g_i \in U, n \in \mathbb{N} \right\}$$

It is non empty and since $gU \subset S$ is an open neighborhood of g (notice that $e \in U$) for all $g \in S$, it is also open. On the other hand it is also closed. Indeed consider gU for $g \notin S$ and suppose $gU \cap S \neq \emptyset$; thus $gu \in S$ for some $u \in S$ and then $g = (gu)u^{-1} \in S$, giving a contradiction. Since G is connected, must be $S = G$. We call *analytic subgroup* a connected Lie subgroup H of G, then we say that a connected group G is *simple* if it has no proper, nontrivial analytic normal subgroups [3]. The above example describes how any analytic subgroup has an associated subalgebra. It turns out that this association is bijective.

Corollary 1.
Let $\mathfrak{g}$ be a Lie algebra of a Lie group G. Every subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra of a unique analytic subgroup H of G.

Consider as example the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ i.e. the algebra $gl(n, \mathbb{C})$ of the $n \times n$ complex matrices with bilinear form given by the commutator. The

[3] This contrasts with the definition of simple abstract groups, since a simple Lie group may still have normal subgroups (but no normal Lie subgroups).

Theorem states that to each subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ it is possible to associate a Lie group. These groups are fundamental since if we consider a representation of a group or a Lie algebra on a vector space, we obtain linear operators that fixed a basis are represented by matrices. Recall that if G and H are two Lie groups and $\phi_1, \phi_2 : G \to H$ are Lie groups morphisms, G is connected and $\phi_* = \phi'_*$ then also $\phi \equiv \phi'$, that is each morphism

$$\phi : G \to H$$

is determined by its induced Lie algebra map. We then end that to classify Lie groups we can suppose (within certain limits) the groups are connected. The natural question is now: is this group unique? The answer is still negative also for connected Lie groups. Consider as example $\mathbb{R}^n$ and $\mathbb{T}^n$; they are both connected and have isomorphic Lie algebras with an isomorphism given by the projection, but they are clearly not isomorphic. To explain this asymmetry, we nedd to investigate the other topological properties of Lie groups. Remark that $\mathbb{R}^n$ is simply connected and $\mathbb{T}^n$ is the quotient of $\mathbb{R}^n$ by the action of the discrete subgroup $\mathbb{Z}^n$, that is the fundamental group of $\mathbb{T}^n$.

Proposition 3.
Let G be a connected Lie group and $H \subset G$ a normal subgroup. Then the projection $\pi : G \to G/H$ defines a covering space if and only if H is discrete.

Proof.
i) First, if π is a covering map, then $H = \pi^{-1}(1)$ is clearly discrete.
ii) Conversely, if H is discrete there exists a neighborhood $U \subset G$ of the identity $e \in G$ such that $U \cap H = e$. In particular we can choose U to be small enough so that $U \cap hU \neq \emptyset$ for any $h \in H$. Then the restriction of π to one of the sets hU is an homeomorphism onto its image. Finally if we consider the coset $gH \in G/H$ the preimage of the neighborhood $(gH)\pi(U)$ consists of the union

$$\bigcup_{h \in H} ghU$$

□

Let G be a Lie group. We define the *center* of G as the set

$$Z(G) := \{g \in G \mid gh = hg \ \forall h \in G\}$$

then a subgroup $H \subset G$ is called *central* if $H \subset Z(G)$. Notice that $Z(G)$ is also an abelian normal subgroup of G.

Proposition 4.
Any discrete normal subgroup H of a connected Lie group G is central.

These two Proposition show that if a Lie group G can be realized as the quotient of a group H by a discrete central subgroup, since the projection is a smooth covering map and thus a diffeomorphism in a neighborhood of the identity, it must induces a Lie algebras isomorphism that is

$$\mathfrak{g}_G \cong \mathfrak{g}_{G/H}$$

Conversely, consider an algebra $\mathfrak{g}$ associated to a group G. Are all the groups with same algebra $\mathfrak{g}$, quotients of G by some discrete central subgroups?

Proposition 5.
Let G and H be two connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and

$$\Phi : G \to H$$

a Lie groups homomorphism such that the induced map

$$\Phi^* : \mathfrak{g} \to \mathfrak{h}$$

is an isomorphism. Then $\Phi : G \to H$ is a covering map.

Proof.
a) Since $\Phi_* = d_e\Phi$, by the Inverse Function Theorem there exists a neighborhood $U \subset G$ of the identity diffeomorphic to a neighborhood V of the identity in H. By translating V by elements of H, the connectedness of H implies Φ is surjective.
b) Consider the same neighborhoods of the identities U and V. Since $e_G \in U$ and $e_H \in V$, for all $g \in G$ and $h \in H$, gU and hV are neighborhood of g and h respectively. Then if we consider $h \in H$ and $g \in G$ such that $\Phi(g) = h$, must be $\Phi(gU) = \Phi(hV)$.

□

From the previous Theorems it follows that if we know the connected Lie group associated to a Lie algebra, computing the center of this group and taking the quotient respect to central subgroups, the resulting groups are all the other Lie groups with the same Lie algebra. An example of that is given by the covering

$$SU(2) \to SO(3)$$

where $SU(2)$ is simply connected, defined by the quotient respect to the discrete subgroups $\mathbb{Z}_2 \subset SU(2)$. This example is developed in more details in Section 2.3. The previous Propositions and examples suggest could be useful to investigate the relationship between Lie groups and their universal covering spaces, which is closely related to quotients of Lie groups.

Theorem 2.
Let G be a Lie group and $p : \tilde{G} \to G$ be its universal cover. There exists a discrete central subgroup H of $\tilde{G}$ such that $G = \tilde{G}/H$ and the projection $\pi : \tilde{G}/H \to G$ is a covering map.

Proof.
We shall sketch a proof.
i) Denote by $m : G \times G \to G$ the multiplication and by $\Phi : \tilde{G} \times \tilde{G} \to G$ the composition $m \circ (p, p)$. By the simple-connectedness of $\tilde{G}$

$$1 = \phi_*(\pi_1(\tilde{G} \times \tilde{G}, \tilde{1} \times \tilde{1})) \subset p_*(\pi_1(\tilde{G}, \tilde{1}))$$

It is possible to prove that this defines a unique continuous map

$$\tilde{\Phi} : \tilde{G} \times \tilde{G} \to \tilde{G}$$

satisfying $\Phi = p \circ \tilde{\Phi}$ and $\tilde{\Phi}(\tilde{1}, \tilde{1}) = \tilde{1}$ that induces a group multiplication on $\tilde{G}$ for which p is a group homomorphism. With charts given by the covering map, $\tilde{G}$ has a manifold structure in such a way that the covering map is smooth, which

implies that multiplication and inversion in $\tilde{G}$ are smooth.
(ii) The kernel H of the covering map p is a discrete closed normal subgroup of G, then by Proposition H is a central subgroup. By the standard Isomorphism Theorem we then obtain $G = \tilde{G}/H$. □

Thus the class of groups with a fixed Lie algebra contains at least one connected and simply-connected representative. Next Lemma allows to show a Fundamental Theorem that establishes its uniqueness.

Lemma 1.
Let G, H be Lie groups with G simply-connected and H connected, $\lambda : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebras morphism. There exists a morphism of Lie groups

$$\Phi : G \to H$$

such that $d\Phi = \lambda$.

Proof.
Let $\mathfrak{s}$ denote the graph of λ in the Lie algebra of $G \times H$. This subalgebra corresponds to an analytic subgroup S of $G \times H$. Let

$$p_G : G \times H \to G \qquad p_H : G \times H \to H$$

be the projections, which are both Lie group homomorphisms. Since dp_G is a Lie algebras isomorphism, by Proposition p_G is a covering map. Then since G is also is simply-connected, p_G must be an isomorphism. We obtain a map $\Phi : G \to H$ defined by

$$\Phi = p_H \circ p_G^{-1}$$

satisfying the statement.

□

Theorem 3 (Fundamental Theorem of Lie groups)**.**
Let $\mathfrak{g}$ be a finite dimensional Lie algebra. There exists a unique (up to isomorphism) connected and simply connected Lie group G with $\mathfrak{g}$ as its Lie algebra. If G' is another connected Lie group with same Lie algebra, it is of the form $G/Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is some discrete central subgroup of G.

Proof.
i) Suppose G and H are connected and simply-connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. By Lemma an isomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$ induces a map $\phi : G \to H$. The inverse $\Psi : \mathfrak{h} \to \mathfrak{g}$ also induces a map $\psi : H \to G$. Since G and H are simply connected, these maps are unique and then ϕ and ψ must be double-sided inverses.
ii) It follows from i) and the previous Theorems.

□

In summary, complex (real) Lie algebras coincide bijectively with simply-connected complex (real) Lie groups and any sub-group with a given Lie algebra can be realized as quotient of its (unique) simply-connected representative by a discrete central subgroup. Instead, given any discrete normal subgroup of a Lie group the quotient group is a Lie group and the quotient map is a covering homomorphism; in particular two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic. So, to classify complex and real Lie group (the connected and simply connected ones) we can classify complex and real Lie algebras.

$$\begin{tikzcd}
 & & \text{Lie group} \arrow[dl] \arrow[dr] & & & \\
\big(& \text{Connected} \arrow[ur, dotted, bend left, "\textit{discrete sub.}"] \arrow[dr, dotted] & + & \text{S.Connected} \arrow[ul, dotted, bend right, "\textit{univ. covering}"'] \arrow[dl, dotted] & \big) \arrow[r, dotted, leftrightarrow] & \text{Lie Algebra} \\
 & & \text{Simple Group} & & &
\end{tikzcd}$$

We finally introduce the notion of *maximal torus* of a Lie group G that, as we'll see, is related to a fundamental tool in Lie algebra Classification Theory. Let G be a compact Lie group. A *torus* of G is a compact, connected, abelian Lie subgroup of G. A *maximal torus* is a torus which is maximal among such subgroups (respect to the inclusion). Notice that by dimensional considerations it follows that every torus is contained in a maximal torus. The dimension of a maximal torus in G is called the *rank* of G. The previous definitions are motivated by the following Proposition.

Proposition 6.
Let G be an abelian connected Lie group of dimension n. Then it is isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$.

Proof.
We first prove that $\exp : \mathfrak{g} \to G$ is an homomorphism. The moltiplication $m : G \times G \to G$ is an homomorphism and it's differential in (e, e) is given by

$$m_* : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

$$(X, Y) \to X + Y$$

then since by Proposition we have $\exp(X + Y) = \exp(X)\exp(Y)$, exp is an homomorphism. Since G is connected and exp is a local diffeomorphism in $0 \in \mathfrak{g}$, its image contains a set of generators of G and is then a subgroup of G. Now, exp is a local diffeomorphism then its kernel is a discrete subgroup of G, thus we have an isomorphism

$$\mathfrak{g}/\ker(\exp) \to G$$

Since $\ker(\exp)$ is generated by k linear indipendent vectors in $\mathfrak{g} \cong \mathbb{R}^n$ and each discrete subgroup of $\mathbb{R}^n$ is a lattice, we conclude

$$G \cong \mathfrak{g}/\ker(\exp) \cong \mathbb{T}^k \times \mathbb{R}^{n-k}$$

$\square$

Corollary 2.
Each connected, abelian compact Lie group G of dimension n is diffeomorphic to $\mathbb{T}^n$.

From the Corollary it follows that each torus of a Lie group G is isomorphic to the standard torus $\mathbb{T}^k$ for some $k > 0$. Consider as example the unitary group $U(n)$. It has maximal torus the subgroup of diagonal matrices

$$T = \left\{ \mathrm{diag}\left(e^{i\theta_1}, \dots e^{i\theta_2}, \dots, e^{i\theta_n}\right) \mid \theta_k \in \mathbb{R} \right\}$$

Clearly T is isomorphic to the product of n circles, so the unitary group $U(n)$ has rank n. A maximal torus in the special unitary group $SU(n)$ is instead of the form $T \cap SU(n)$ that is an $n-1$ torus, hence $SU(n)$ has rank $n-1$.

1.1.3 Simple and Semisimple Lie Algebras

We now introduce complex Lie algebras' classification. The first idea is to decompose any Lie algebra in an almost abelian and not abelian part and thus reduce the study to the second one. To do that we first need the definition of ideal of a Lie algebra. As for rings, this notion is necessary to define the quotient algebras but in this case the internal product operation has represented (in some sense) by the commutator. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}, \mathfrak{h}$ subalgebras of $\mathfrak{g}$. We define the set

$$[\mathfrak{a}, \mathfrak{h}] := \{[A, B] \mid A \in \mathfrak{a}, B \in \mathfrak{h}\}$$

Definition 4.
Let $\mathfrak{g}$ be a Lie algebra on $\mathbb{C}$, $\mathbb{R}$ and $\mathfrak{h} \subset \mathfrak{g}$ a linear subspace. We call $\mathfrak{h}$ *subalgebra* of $\mathfrak{g}$ if it is closed under Lie brakets i.e. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ while *ideal* of $\mathfrak{g}$ if it is invariant under Lie brakets i.e. $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

We first summarize some basic properties of ideals and subalgebras.

Proposition 7.
Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, $\mathbb{R}$. Then
i) If $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are ideals, also $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are ideals.
ii) If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal and $\mathfrak{b} \subset \mathfrak{g}$ is a subalgebra, also $\mathfrak{a} + \mathfrak{b}$ is a subalgebra.

Moreover if $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebras homomorphism it is possible to prove that $\ker \phi$ is an ideal of $\mathfrak{g}_1$, $\mathrm{Im}(\phi)$ is a subalgebra of $\mathfrak{g}_2$ and then

$$\mathrm{Im}(\phi) \cong \mathfrak{g}_1 / \ker(\phi)$$

At Lie groups level, we have the following.

Proposition 8.
Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ an analytic subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Then H is normal in G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

Abelian Lie algebras are not particularly interesting since all Lie brackets are zero; moreover a connected Lie group associated to an abelian Lie algebra is abelian, hence it's of the form explain above. So, we can just study the "almost abelian" subspaces of a Lie algebra.

Definition 5.
Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, $\mathbb{R}$. Consider the subspaces of $\mathfrak{g}$

$$\mathfrak{g}^{(0)} = \mathfrak{g} \qquad \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \qquad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$$

We define the *derived series*

$$\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots \supseteq \mathfrak{g}^{(n)}$$

A Lie algebra $\mathfrak{g}$ is called *solvable* if $\mathfrak{g}^{(n)} = 0$ for some $n > 0$. It is possible to show that any subalgebra of a solvable Lie algebra is solvable and if $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are solvable ideals then also $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are solvable ideals. Notice that $\mathfrak{g}^{(n)}$ is an ideal for $n > 1$ and if $\mathfrak{g}$ is a solvable Lie algebra i.e. $0 = \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$ then $\mathfrak{g}^{(n)}$ is an abelian subalgebra of $\mathfrak{g}$. Then, clearly

$$\{\text{abelian}\} \subsetneq \{\text{solvable}\}$$

Consider now a maximal (respect to dimension) solvable ideal of $\mathfrak{g}$ and denote it with $\mathbf{rad}(\mathfrak{g})$. If $\mathfrak{a}$ is another solvable ideal of $\mathfrak{g}$, then $\mathfrak{a}+\mathbf{rad}(\mathfrak{g})$ is again a solvable ideal. Since $\mathbf{rad}(\mathfrak{g})$ is of maximal dimension, must be $\mathfrak{a} + \mathbf{rad}(\mathfrak{g}) = \mathbf{rad}(\mathfrak{g})$ and $\mathfrak{a} \subset \mathbf{rad}(\mathfrak{g})$. Then if there are two distinct maximal dimensional solvable ideals of $\mathfrak{g}$, by above explanation the sum is actually equal to both ideals, hence the maximal one is unique and contains any solvable ideal of $\mathfrak{g}$. We call it *radical* of $\mathfrak{g}$, that we denote by $\mathbf{rad}(\mathfrak{g})$. Then $\mathfrak{g}$ is called semisimple if it doesn't contain any non zero solvable ideal, that is $\mathbf{rad}(\mathfrak{g}) = 0$.

Proposition 9.
A finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if either of the following two conditions holds:

i) Any solvable ideal of $\mathfrak{g}$ is zero.

ii) Any abelian ideal of $\mathfrak{g}$ is zero.

Proof.
i) The first condition is equivalent to the definition of semisimplicity.
ii)
a) Suppose $\mathfrak{g}$ contains a non-zero solvable ideal $\mathfrak{a}$. For some k, we have

$$\mathfrak{a} \supset \mathfrak{a}^{(1)} \supset \mathfrak{a}^{(2)} \supset \cdots \supsetneq \mathfrak{a}^{(k)} = 0$$

hence $\mathfrak{a}^{(k-1)}$ is a nonzero ideal and it's abelian since $[\mathfrak{a}^{(k-1)}, \mathfrak{a}^{(k-1)}] = \mathfrak{a}^{(k)} = 0$, since all the $\mathfrak{a}^{(i)}$ are ideals of $\mathfrak{g}$.
b) Abelian ideals are solvable, so the other direction.

□

The previous Proposition suggests how to decompose a Lie algebra on its almost abelian part (the radical) and the non abelian one, that is the semisimple one. This is the established in the following Levi's theorem.

Theorem 4 (Levi)**.**
Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}, \mathbb{R}$. Then $\mathfrak{g}$ admits a Levi decomposition, that is the direct sum

$$\mathfrak{g} = \mathbf{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$$

where $\mathfrak{g}_{ss}$ is a semisimple subalgebra of $\mathfrak{g}$.

Proof.
Let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ be the quotient homomorphism, $\mathfrak{h}$ be a solvable ideal in $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ and $\mathfrak{a} = \pi^{-1}(\mathfrak{h}) \subset \mathfrak{g}$. Then $\pi(\mathfrak{a}) = \mathfrak{h}$ is solvable and $\ker(\pi_{|\mathfrak{a}})$ is solvable, being in $\mathrm{rad}(\mathfrak{g})$. So $\mathfrak{a}$ is solvable, hence $\mathfrak{a} \subset \mathrm{Rad}(\mathfrak{g})$ and $\mathfrak{h} = 0$. Therefore $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.

□

Notice that the decomposition allows to reduce the problem to semisimple Lie algebra. Moreover remark that Lie algebras with a proper ideal, i.e. an ideal $\mathfrak{h} \subsetneq \mathfrak{g}$, can be understood as extensions of the quotient $\mathfrak{g}/\mathfrak{h}$ by $\mathfrak{h}$. This justifies the study of not abelian Lie algebras with no proper ideals, the so called *simple* Lie algebras.

Proposition 10.
A connected Lie group is simple (as Lie group) if and only if its Lie algebra is simple.

Proof.
It follows from the fact that a Lie subgroup is normal if and only if its Lie algebra is an ideal.

□

Remark that since ideals correspond to normal Lie subgroups and simple Lie algebras correspond to simple Lie groups, then simple algebras represent the fundamental blocks in Lie algebras' classification. We'll then first decompose any semisimple Lie algebra in simple Lie algebras, that is the goal of this Section. Consider the adjoint representation of $\mathfrak{g}$ on itself, i.e. the map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

We define the *center* of a Lie algebra the set

$$Z(\mathfrak{g}) = \{A \in \mathfrak{g} | [A, \mathfrak{g}] = 0\}$$

that is the set of vectors commuting with each element of the algebra i.e. the set of elements for which the linear operator ad_A induced by the adjoint representation is the null operator. Notice that $\ker(\mathrm{ad}) = Z(\mathfrak{g})$ and then the center is an ideal, in particular abelian. Thus, since $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$ is an homomorphism and $\mathfrak{g}/\ker(\mathrm{ad}) \cong \mathrm{Im}(\mathrm{ad})$ it induces an embedding

$$\mathfrak{g}/Z(\mathfrak{g}) \to \mathfrak{gl}_n(\mathbb{C})$$

This is a simple case of the more general one described by Ado's Theorem.

Theorem 5 (Ado)**.**
Any finite dimensional complex or real Lie algebra embeds respectively in $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{R})$ for some natural $n > 0$.

It is also possible to prove that, for a connected Lie group G with Lie algebra $\mathfrak{g}$, the Lie algebra associated to its subgroup $Z(G)$ is exactly[4] $Z(\mathfrak{g})$. To prove the semisimple Lie algebra's decomposition in simple ones, we need to investigate

[4]Generally (without the connectedness assumption) the Lie algebra of $Z(G)$ is contained in $Z(\mathfrak{g})$.

the properties of the linear operators defined on the algebras. Consider a Lie algebra $\mathfrak{g}$ and an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $H_1, \dots H_m$ be a basis of $\mathfrak{h}$. Since $\mathfrak{h}$ is abelian all H_i commute and thus since $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a Lie algebra homomorphism, also

$$[\mathrm{ad}_{H_i}, \mathrm{ad}_{H_j}] = 0$$

then the maps ad_{H_i} are simultaneously diagonalizable. Suppose the operators ad_{H_i} admit a common eigenvector E and eigenvalues λ_{H_i} for each H_i. If

$$H = \sum_{i}^{k} c_i H_i$$

is a generic element in $\mathfrak{h}$, denoted by λ_H the eigenvalue for H i.e such that $\mathrm{ad}_H(E) = \lambda_H E$ we have

$$\mathrm{ad}_H(E) = [H, E] = \sum_{i=1}^{m} c_i [H_i, E] = \sum_{i=1}^{m} c_i \lambda_{H_i} E = \left(\sum_{i=1}^{m} c_i \lambda_{H_i} \right) E$$

then

$$\lambda_H = \sum_{i=1}^{m} c_i \lambda_{H_i}$$

This relation tell us that we can think about the eigenvalue λ_H associated to H as an element of the dual space $\mathfrak{h}^*$ i.e. as the map

$$\lambda : \mathfrak{h} \to \mathbb{C}$$

$$\lambda(H) = \sum_{i=1}^{m} \lambda(H_i) c_i$$

where $c_1, \dots, c_m$ are the coordinates of H respect to the basis $H_1, \dots, H_m$. Notice that since the line $\mathbb{C}A$ spanned by an element $A \in \mathfrak{g}$ is always an abelian subalgebra, we can always consider the eigenvalue as a dual form on $\mathbb{C}A$. Consider now $A \in \mathfrak{g}$ and $\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}$. We define the *weight space* of A associated to $\lambda \in \mathfrak{g}^*$

$$V_\lambda^A := \left\{ B \in \mathfrak{g} \mid \mathrm{ad}_A(B) = \lambda_A B \right\}$$

We can generalize this definition for a set of operators.

Definition 6.
Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ ($\mathbb{R}$) and $\mathfrak{g}^*$ its dual vector space. Let

$$\pi : \mathfrak{g} \to \mathrm{End}(V_{\mathbb{C}})$$

be a representation of $\mathfrak{g}$ on $V_{\mathbb{C}}$ ($V_{\mathbb{R}}$). Given $\lambda \in \mathfrak{g}^*$ we define the *weight space* of $\mathfrak{g}$ associated to λ

$$V_\lambda^{\mathfrak{g}} = \{ B \in V \mid \pi_A(B) = \lambda_A B \quad , \ \forall A \in \mathfrak{g} \}$$

and if $V_\lambda^{\mathfrak{g}} \neq 0$, we call λ *weight* for π.

If we consider in particular the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ we have

$$\mathfrak{g}_\lambda^{\mathfrak{g}} = \{ B \in \mathfrak{g} \mid [A, B] = \lambda_A B \quad , \ \forall A \in \mathfrak{g} \}$$

Lemma 2 (Lie).
Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Let

$$\pi : \mathfrak{g} \to End(V_{\mathbb{C}}) \qquad \pi_{|_{\mathfrak{h}}} : \mathfrak{h} \to End(V_{\mathbb{C}})$$

be representations of $\mathfrak{g}$ and $\mathfrak{h}$ on $V_{\mathbb{C}}$. Then each weightspace $V_\lambda^{\mathfrak{h}}$ for the restricted representation $\pi_{|_{\mathfrak{h}}}$ is invariant under $\mathfrak{g}$ i.e $\pi_A(\ V_\lambda^{\mathfrak{h}}) \subset V_\lambda^{\mathfrak{h}}$ for all $A \in \mathfrak{g}$.

Proof.
We have to show that if $B \in V_\lambda^{\mathfrak{h}}$ then $\pi_A(B) \in V_\lambda^{\mathfrak{h}}$ for all $A \in \mathfrak{g}$. But this is verified if and only if

$$\pi_{A_2}\pi_{A_1}(B) = \lambda(A_2)\pi_{A_1}B \quad \forall A_2 \in \mathfrak{h},\ \forall A_1 \in \mathfrak{g}$$

now,

$$\pi_{A_2}\pi_{A_1}B = [\pi_{A_2}, \pi_{A_1}]B + \pi_{A_1}\pi_{A_2}B = \pi_{[A_1,A_2]}B + \pi_{A_1}\lambda(A_2)B$$

Since $\mathfrak{h}$ is an ideal, $[A_1, A_2] \in \mathfrak{h}$ then

$$\pi_{A_2}\pi_{A_1}B = \lambda_{[A_1,A_2]}B + \pi_{A_1}\lambda_{A_2}B$$

Now we have to show that $\lambda_{[A_1,A_2]} = 0$ for all $A_2 \in \mathfrak{h}$ with $V_\lambda^{\mathfrak{h}} \neq 0$. Let $A_1 \in \mathfrak{g}$ and $B \neq 0$ be an element in $V_\lambda^{\mathfrak{h}} \neq \{0\}$. Consider the subspaces

$$W_m = \text{span}\,\langle B, \pi_{A_1}B, \pi_{A_1}^2 B, \ldots, \pi_{A_1}^m B\rangle \qquad \forall m \geq 0 \quad, W_{-1} = \{0\}$$

Since V is finite dimensional there exists an integer n that is the maximal integer for which all the generators of W_n are linearly independent. Then we have $W_n = W_{n+1} = \ldots$ hence

$$\pi_{A_1}W_n \subset W_n$$

Consider the increasing sequence of subspaces

$$W_{-1} = \{0\} \subset W_0 = \mathbb{C}B \subset \cdots \subset W_n$$

We claim that $\forall m \geq 0$, W_m is invariant under $\pi(\mathfrak{h})$ and furthermore

$$\forall A_2 \in \mathfrak{h} \qquad \pi_{A_2}\pi_{A_1}^m B - \lambda_{A_2}\pi_{A_1}^m B \in W_{m-1}$$

We prove the equation on m. The case $m = 0$ is true since $B \in V_\lambda^{\mathfrak{h}}$. Suppose we have proved the assumption for $m-1$; then

$$\pi_{A_2}\pi_{A_1}^m B - \lambda_{A_2}\pi_{A_1}^m B \in W_{m-1} = [\pi_{A_2}, \pi_{A_1}]\pi_{A_1}^{m-1}B + \pi_{A_1}\pi_{A_2}\pi_{A_1}^{m-1}B - \lambda_{A_2}\pi_{A_1}^m B =$$

$$= [\pi_{A_2}, \pi_{A_1}]\pi_{A_1}^{m-1}B + \pi_{A_1}\pi_{A_2}\pi_{A_1}^{m-1}B - \pi_{A_1}\lambda_{A_2}\pi_{A_1}^{m-1}B$$

thus by induction

$$w = \pi_{A_2}\pi_{A_1}^{m-1}v - \lambda_{A_2}\pi_{A_1}^{m-1}B \in W_{m-2}$$

and $\pi_{A_1}w \in W_{m-1}$ by construction of W_i. Moreover $\mathfrak{h}$ is an ideal so that $[\pi_{A_2}, \pi_{A_1}] \in \pi(\mathfrak{h})$ and by inductive hypothesis

$$[\pi_{A_2}, \pi_{A_1}]\pi_{A_1}^{m-1}B \in W_{m-1}$$

thus

$$\pi_{A_2}\pi_{A_1}^m B - \lambda_{A_2}\pi_{A_1}^m B \in W_{m-1}$$

cause it is a sum of elements in W_{m-1}, then we have the induction. Now, we know that W_n is invariant both for π_{A_1} and for π_{A_2}, for all $A_2 \in \mathfrak{h}$. In particular the equation shows that for all $A_2 \in \mathfrak{h}$ the operator π_{A_2} acts on W_n as an upper triangular matrix on the basis $B, \pi_{A_1}B, \dots, \pi_{A_1}^n B$

$$\begin{pmatrix} \lambda_{A_2} & * & \cdots & \cdots & * \\ 0 & \ddots & * & & \vdots \\ 0 & 0 & \lambda_{A_2} & * & \vdots \\ 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 0 & \lambda_{A_2} \end{pmatrix}$$

then

$$\mathrm{tr}_{W_n}\big([\pi_{A_2}, \pi_{A_1}]\big) = 0 = \mathrm{tr}\big(\pi_{[A_1,A_2]})\big) = n\lambda_{([A_2,A_1])}$$

which implies that $\lambda_{([A_2,A_1])} = 0$.

$\square$

The natural question is now: given a Lie algebra $\mathfrak{g}$, does it admit weight?

Theorem 6 (Lie).
Let $\mathfrak{g}$ be complex solvable Lie algebra and $\pi : \mathfrak{g} \to End(V_{\mathbb{C}})$ a representation of $\mathfrak{g}$ on a finite dimensional vector space $V_{\mathbb{C}} \neq 0$. There exists a weight $\lambda \in \mathfrak{g}^*$ for π, that is $V_\lambda^{\mathfrak{g}} \neq \{0\}$.

Proof.
The following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\pi"] \arrow[d, "\phi"] & \mathrm{End}(V) \\ \pi(\mathfrak{g}) \arrow[ur, "i"'] & \end{tikzcd}$$

then since $\mathfrak{g}$ is solvable and $\pi(\mathfrak{g})^{(n)} = \pi(\mathfrak{g}^{(n)})$, also $\pi(\mathfrak{g})$ is solvable. We prove the Theorem by induction on $dim(\mathfrak{g}) = m$. If $dim(\mathfrak{g}) = 0$ is trivial, then consider $m > 0$. Since $\mathfrak{g}$ is solvable, of positive dimension, $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$ properly and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, any subspace is automatcally an ideal. Consider a subspace of codimension one in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Its inverse image $\mathfrak{h}$ by the projection on the quotient is an ideal of codimension one in $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$. Thus we have the decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathbb{C}A$$

for some $A \in \mathfrak{g}$. Now, $dim(\mathfrak{h}) = m - 1$ and is solvable (it's an ideal in a solvable algebra). Hence by inductive hypothesis there exists $\lambda \in \mathfrak{h}^*$ such that the weight space $V_\lambda^{\mathfrak{h}} \neq \{0\}$. By Lie's Lemma, the subspace $V_\lambda^{\mathfrak{h}}$ is invariant under the action of $\pi(\mathfrak{g})$ then in particular $\pi_A\big(V_\lambda^{\mathfrak{h}}\big) \subset V_\lambda^{\mathfrak{h}}$, thus there exists $B \in V_\lambda^{\mathfrak{h}}$, $B \neq 0$ such that

$$\pi_A(B) = cA$$

for some $c \in \mathbb{C}$. We define a linear functional $\lambda' \in \mathfrak{g}^*$ on $\mathfrak{g}$ by

$$\lambda'(H + \mu A) := \lambda(H) + \mu c \quad \forall H \in \mathfrak{h} \quad , c \in \mathbb{C}$$

By construction B is in $V^{\mathfrak{g}}_{\lambda'}$ then in particular $V^{\mathfrak{g}}_{\lambda'} \neq \{0\}$.

□

Corollary 3.
i) Let $\pi : \mathfrak{g} \to End(V_{\mathbb{C}})$ be a representation of a solvable Lie algebra $\mathfrak{g}$. There exists a basis of $V_{\mathbb{C}}$ for which the matrices of $\pi(\mathfrak{g})$ are upper triangular
ii) Let $V_{\mathbb{C}}$ be a finite dimensional complex vector space. Any solvable subalgebra $\mathfrak{g} \subset \mathfrak{gl}_V(\mathbb{C})$ is contained in the subalgebra of upper triangular matrices for some basis of $V_{\mathbb{C}}$.

We are now ready to prove the semisimple Lie algebras decomposition. To do that, we need to introduce a bilinear form on the algebra. Let $\mathfrak{g}$ be a Lie algebra and $\pi : \mathfrak{g} \to \mathrm{End}(V_{\mathbb{C}})$ a representation. We call *trace form* the bilinear form on $\mathfrak{g}$ defined by

$$\langle A, B\rangle_V = \mathrm{tr}(\pi_A \pi_B)$$

The trace form is

- symmetric: $\langle A, B\rangle_V = \langle B, A\rangle_V$

- invariant: $\langle [A, B], C\rangle_V = \langle A, [B, C]\rangle_V$.

If $\mathfrak{g}$ is a finite dimensional Lie algebra, the trace form of the adjoint representation

$$k(A, B) = \mathrm{tr}((\mathrm{ad}_A)(\mathrm{ad}_B))$$

is called *Killing form* on $\mathfrak{g}$.

Lemma 3.
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$ or $\mathbb{R}$ and $\langle\cdot, \cdot\rangle$ be a symmetric invariant bilinear form on $\mathfrak{g}$. If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal

1. *The complement $\mathfrak{h}^{\perp}$ is also an ideal.*

2. *If $\langle\cdot, \cdot\rangle_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ is a direct sum of Lie algebras.*

Proof.
i) If $B \in \mathfrak{a}^{\perp}$ then $\langle B, \mathfrak{a}\rangle = 0$. Consider $C \in \mathfrak{g}$; since the form is invariant and $\mathfrak{a}$ is an ideal, for each $D \in \mathfrak{a}$ we have

$$\langle [B, C], D\rangle = \langle B, [C, D]\rangle = 0$$

Hence $\mathfrak{a}^{\perp}$ is an ideal.
ii) Since $\mathfrak{a}$ and also $\mathfrak{a}^{\perp}$ are ideals, they are both subalgebra then by the classical decomposition for vector spaces we obtain the thesis.

□

We report the following Lemma that we don't proof.

Lemma 4.
Let $\mathfrak{g} \subset \mathfrak{gl}_{\mathbb{C}}(V)$ be a subalgebra such that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Then there exists $A \in \mathfrak{g}$ such that $\langle A, A \rangle_V \neq 0$.

We can prove the following characterization of solvable Lie algebras.

Theorem 7 (Cartan's criterion)**.**
Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}_{\mathbb{C}}(V)$. The following are equivalent

i) $\langle \mathfrak{g}, [\mathfrak{g}, \mathfrak{g}] \rangle = 0$

ii) $\langle A, A \rangle_V = 0$ *for all* $A \in [\mathfrak{g}, \mathfrak{g}]$

iii) $\mathfrak{g}$ *is solvable*

Proof.
$i) \Rightarrow (ii)$: Obvious.
$iii) \Rightarrow i)$: By Lie's Theorem, in some basis of V all matrices of $\mathfrak{g}$ are upper triangular and thus $[\mathfrak{g}, \mathfrak{g}]$ is strictly upper triangular. Then π_{AB} is strictly upper triangular and $\langle A, B \rangle_V = 0$ if $A \in \mathfrak{g}, B \in [\mathfrak{g}, \mathfrak{g}]$.
$ii) \Rightarrow iii)$: Suppose not. Then there exists k such that the derived series of $\mathfrak{g}$ stabilizes i.e.

$$[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^{(k)}$$

with $\mathfrak{g}^{(k)} \neq 0$. Then $\langle A, A \rangle_V = 0$ for $A \in \mathfrak{g}^{(k)}$ and $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^{(k)}$ thus by the previous Lemma we get a contradiction.

□

Corollary 4.
A finite dimensional complex Lie algebra $\mathfrak{g}$ is solvable if and only if

$$k(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$$

Proof.
Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$. Its kernel is $Z(\mathfrak{g})$. So $\mathfrak{g}$ is solvable iff $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}_{\mathfrak{g}}$ is a solvable. But by Cartan's criterion, $\mathrm{ad}(\mathfrak{g})$ is solvable iff $k(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

□

The previous Theorem allows to prove the following characterization of semisimple Lie algebras in terms of the Killing form.

Theorem 8.
Let $\mathfrak{g}$ be a complex Lie algebra. The Killing form on $\mathfrak{g}$ is non-degenerate if and only if $\mathfrak{g}$ is semisimple. Moreover, if $\mathfrak{g}$ is semisimple and $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, the restriction

$$k_{|\mathfrak{h} \times \mathfrak{h}} \mathfrak{h} \times \mathfrak{h} \to \mathbb{F}$$

is also non-degenerate and coincides with the Killing form of $\mathfrak{h}$.

Proof.
a) Suppose k is non-degenerate on $\mathfrak{g}$ but $\mathfrak{g}$ is not semisimple; there exists an abelian ideal $\mathfrak{a} \subset \mathfrak{g}$. Then if $B \in \mathfrak{g}$ and $A \in \mathfrak{a}$

$$(\mathrm{ad}_B)(\mathrm{ad}_A)C = [B, [A, C]] \in \mathfrak{a}$$

forall $C \in \mathfrak{g}$ and 0 forall $C \in \mathfrak{a}$. It follows that in the basis $E_1, ..., E_k$ of $\mathfrak{a}$, $E_1, ..., E_k, E_{k+1}, \ldots, E_n$ basis of $\mathfrak{g}$, the matrix of $(\mathrm{ad}_B)(\mathrm{ad}_A)$ is of the form

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

But the trace of this matrix is 0, so $k(B, A) = 0$ and then $k(\mathfrak{a}, \mathfrak{g}) = 0$, contradicting non-degeneracy of k.

b) Conversely, let $\mathfrak{g}$ be semisimple, $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. If $k_{\mathfrak{a}\times\mathfrak{a}}$ is degenerate from the Lemma $\mathfrak{a} \cap \mathfrak{a}^\perp \neq 0$, hence $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$ such that $k(\mathfrak{b}, \mathfrak{b}) = 0$. By considering the adjoint representation of $\mathfrak{b}$ on $\mathfrak{g}$ and applying the Cartan criterion we conclude that $\mathfrak{b}$ is solvable. Since $\mathfrak{g}$ is semisimple, we deduce that $\mathfrak{b} = 0$. Thus if $\mathfrak{g}$ is semisimple the Killing form is non-degenerate, by taking $\mathfrak{a} = \mathfrak{g}$. As for the second part, we already proved that $k_{|_{\mathfrak{a}\times\mathfrak{a}}}$ is non-degenerate, so $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. By lemma it is a direct sum of ideals then $[\mathfrak{a}, \mathfrak{a}^\perp] = 0$ and k for $\mathfrak{a}$ equals k of $\mathfrak{g}$ restricted.

□

We can then proof the goal of this section.

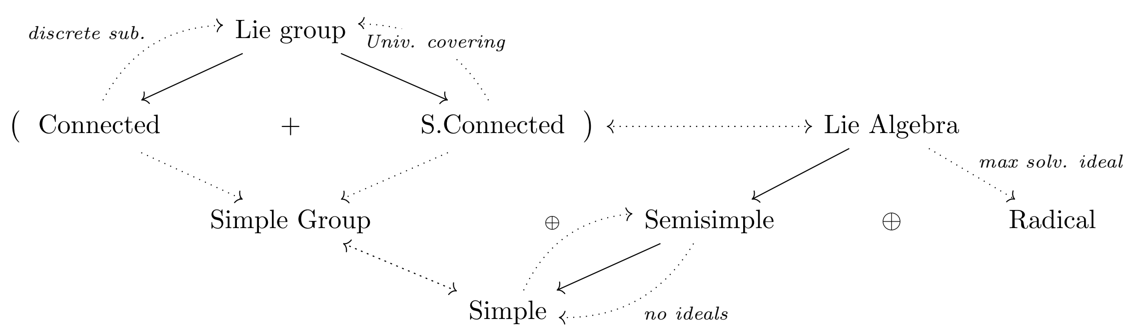
Theorem 9.
Any semisimple, finite-dimensional Lie algebra $\mathfrak{g}$ *over* $\mathbb{C}$ *or* $\mathbb{R}$ *is a direct sum*

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_s^i$$

of simple Lie algebras $\mathfrak{g}_s^i$.

Proof.
If $\mathfrak{g}$ is semisimple but not simple and $\mathfrak{a}$ is an ideal, by the Theorem the Killing form restricted to $\mathfrak{a}$ is non-degenerate, hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, where $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are also semisimple. After finitely many steps $\mathfrak{g}$ can then be decomposed into simple algebras. □

We summarize what we have done in the following.



1.2 Classification of Semisimple Lie Algebras

An important idea behind the classification is that, if $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$, it has no nontrivial abelian ideals, hence $Z(\mathfrak{g}) = 0$. Then a semisimple Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra of linear transformations

$$\{\mathrm{ad}_A : A \in \mathfrak{g}\}$$

so the structure of a Lie algebra $\mathfrak{g}$ is related to the structure of this set of operators. To classify these operators is natural to analyze their eigenvectors and eigenvalues. Moreover, since we are interested in the properties of the whole algebra, we have to consider all the diagonalizable elements together; in other words, we need to study the bigger set of elements of the algebra that are simultaneously diagonalizable. Recall that generally an endomorphism is not diagonalizable. Anyway, in the complex field, it is possible to represent any linear operator in Jordan form. We recall the results valid for a generic vector space.

Definition 7.
Let T be a linear operator on a vector space $V_{\mathbb{C}}$ and $\lambda \in \mathbb{C}$. We define the *generalized eigenspace* of T with eigenvalue λ as the set

$$V_\lambda^n = \{v \in V \mid (T - \lambda I)^n v = 0\}$$

Let T be a linear operator on a finite dimensional complex vector space $V_{\mathbb{C}}$ of dimension n with eigenvalues $\lambda_1, \dots \lambda_k$ and multiplicities $n_1, \dots n_k$. The Jordan's Theorem establishes that in some basis $e_1, e_2, \dots, e_n$ the matrix A_T associated to the operator T assumes the *Jordan normal form*

$$A_T = \begin{pmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_n} \end{pmatrix}$$

where each J_{λ_i} is a $n_i \times n_i$ matrix with values λ_i on the diagonal, 0 or 1 in each entry just above the diagonal and 0 everywhere else. Each of the generalized eigenspaces

$$V_{\lambda_1} = \mathrm{span}\left\langle e_1, e_2, \dots, e_{n_1} \right\rangle, V_{\lambda_2} = \mathrm{span}\left\langle e_{n_1+1}, ..., e_{n_1+n_2} \right\rangle, \dots$$

related to the restricted operators J_{λ_i} are T-invariant and the generalized eigenspace decomposition

$$V = \bigoplus_{i=1}^{k} V_{\lambda_k}^{n_k}$$

where $\dim(V_{\lambda_i}) = n_i$ holds. Moreover since we can decompose the Jordan blocks as $J_{\lambda_i} = \lambda_i I_{n_i} + N_i$ where N_i is nilpotent and since $J_{\lambda_i} = A_{T|_{V_{\lambda_i}}}$, also the *Jordan decomposition*

$$T = D + N$$

where D is the diagonalizable and N is the nilpotent component of T holds. Moreover

$$V = \bigoplus_{i=0}^{s} V_{\lambda_i}$$

with $TV_{\lambda_i} \subset V_{\lambda_i}$ and $D_{|V_{\lambda_i}} = \lambda_i I$, hence $DN = ND$. Note that

$$V_\lambda \subset V_\lambda^2 \subset \dots \subset V_\lambda^n \subset \dots$$

then the eigenspace of T with eigenvalue λ is a subspace of V_λ; finally remark that T is a nilpotent operator if and only if $V = V_0$, that is has only eigenvalue the zero.

We can do the same thing for the semisimple Lie algebra: we want to decompose a generic operator ad_A where $A \in \mathfrak{g}$ in the Jordan form. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and π its representation on a finite-dimensional vector space $V_{\mathbb{C}}$. We define the *generalized weight space* of A in V attached to λ

$$V_\lambda^A = \left\{ v \in V | (\pi_A - \lambda I)^n v = 0 \ \text{ for some } n \in \mathbf{N} \right\}$$

From the above considerations fixed $A \in \mathfrak{g}$ the generalized eigenspace decomposition

$$V_{\mathbb{C}} = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda^A$$

follows. In particular for the adjoint representation we have

$$\mathfrak{g}_\alpha^A = \left\{ B \in \mathfrak{g} | (\mathrm{ad}_A - \alpha I)^n B = 0 \ \text{ for some } n \in \mathbf{N} \right\}$$

and the decompostion

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{C}} \mathfrak{g}_\alpha^A$$

We call *abstract Jordan decomposition* of an element $A \in \mathfrak{g}$ a decomposition of the form

$$A = A_d + A_n$$

where ad_{A_n} and ad_{A_d} are respectively a nilpotent and a diagonalizable operator, and $[A_n, A_d] = 0$.

Theorem 10.
Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Then any $A \in \mathfrak{g}$ admits a unique Jordan decomposition.

The Jordan decomposition into generalized eigenspaces is related to a singular operator. It is possible to generalize the idea and to give the definition of generalized eigenspaces for a Lie algebra. Let $\mathfrak{g}$ be a Lie algebra and $\pi : \mathfrak{g} \to V$ a representation, $\lambda \in \mathfrak{g}^*$. We set

$$V_\lambda^{\mathfrak{g}} = \{ v \in V \mid (\pi_A - \lambda_A I)^n v = 0 \mid \text{some } n > 0 \text{ (dependent of } A) \ , \ \forall A \in \mathfrak{g} \}$$

called *generalized weight space* of $\mathfrak{g}$ in V attached to λ. We now need to investigate how the Jordan form of an operator is reflected in the subalgebras of $\mathfrak{g}$; in particular it suggests to study sets of nilpotent operators and diagonalizable operators.

1.2.1 Nilpotent Subalgebras

We first investigate nilpotent operators. Consider an element $A \in \mathfrak{g}$; the power of the operator associated by the adjoint map is for each $B \in \mathfrak{g}$

$$\mathrm{ad}_A^n(B) = [A, [A, [A, \cdots, [A, B]] \cdots]$$

then is natural the following.

Definition 8.
Let $\mathfrak{g}$ be a Lie algebra on $\mathbb{C}$, $\mathbb{R}$. Consider the subspaces of $\mathfrak{g}$

$$\mathfrak{g}^1 = \mathfrak{g}, \ \ldots \ , \ \mathfrak{g}^{n+1} := [\mathfrak{g}, \mathfrak{g}^n]$$

We define the *lower central series* of $\mathfrak{g}$ the descending chain

$$\mathfrak{g} \supseteq \mathfrak{g}^2 \supseteq \mathfrak{g}^3 \supseteq \cdots \supseteq \mathfrak{g}^n$$

A Lie algebra $\mathfrak{g}$ is then called *nilpotent* if $\mathfrak{g}^n = 0$ for some $n > 0$. Notice that $\mathfrak{g}^n$ is an ideal. Moreover and it is possible to prove that any subalgebra of a nilpotent Lie algebra is nilpotent. Remark also that for each $n > 0$, $\mathfrak{g}^{(n)} \subset \mathfrak{g}^n$ then we have the inclusion

$$\{\text{abelian}\} \subsetneq \{\text{nilpotent}\} \subsetneq \{\text{solvable}\}$$

Theorem 11.
Let $\mathfrak{g}$ be a Lie algebra.
(i) If $\mathfrak{g}$ is a nonzero nilpotent Lie algebra, $Z(\mathfrak{g})$ is nonzero.
(ii) If $\mathfrak{g}$ is a finite-dimensional Lie algebra such that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, $\mathfrak{g}$ is nilpotent.

Proof.
(i) Let $n > 0$ be the minimal integer such that $\mathfrak{g}^n = 0$. Since $\mathfrak{g} \neq 0$ we have that must be $n \geq 2$ then $\mathfrak{g}^{n-1} \neq 0$ and

$$[\mathfrak{g}, \mathfrak{g}^{n-1}] = \mathfrak{g}^n = 0$$

so $\mathfrak{g}^{n-1} \subset Z(\mathfrak{g})$, thus $Z(\mathfrak{g}) \neq \{0\}$.
(ii) Suppose $\mathfrak{g} = \mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, i.e., $\mathfrak{g}^n = 0$ for some n. This implies that $\mathfrak{g}^n \subset Z(\mathfrak{g})$, but then

$$\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \subset [\mathfrak{g}, Z(\mathfrak{g})] = 0$$

□

We now want to prove that if an algebra $\mathfrak{g}$ is nilpotent, the induced maps ad_A are nilpotent for each $A \in \mathfrak{g}$. We first report the following Lemma.

Lemma 5.
Let A be a nilpotent operator on a vector space V.
(a) There exists a non-zero $v \in V$ such that $Av = 0$.
(b) ad_A is a nilpotent operator in gl_V .

Remark the following: given a representation of $\pi : \mathfrak{g} \to \mathrm{End}(V_{\mathbb{C}})$ of a Lie algebra $\mathfrak{g}$ in V, if $W \subset V$ is a subspace invariant with respect to all operators π_A for $A \in \mathfrak{g}$, we have the subrepresentation

$$\pi_W : \mathfrak{g} \to \mathrm{End}(W)$$

$$A \to \pi_{A|_W}$$

of $\mathfrak{g}$ on W. Denoted by V/W the quotient space, the subrepresentation induces a *factor representation*

$$\pi_{V/W} : \mathfrak{g} \to \mathrm{End}(V/W)$$

of $\mathfrak{g}$ on the quotient V/U defined by

$$A \to \pi_{A|_{V/U}}$$

The previous considerations can be used to prove a fundamental Theorem that establish a general property of nilpotent matrices on a vector space.

Theorem 12 (Engel)**.**
Let $V_{\mathbb{C}}$ be a non-zero complex vector space and $\mathfrak{g} \subset \mathfrak{gl}_{\mathbb{C}}(V)$ be a finite dimensional subalgebra which consists of nilpotent operators. There exists a non-zero vector $v \in V$ such that $Av = 0$ for all $A \in \mathfrak{g}$.

Proof.
By induction on $dim(\mathfrak{g}) = n$.
a) For $n = 1$:
We have that $\mathfrak{g} = \mathbb{C}A$ for $A \in \mathfrak{gl}_{\mathbb{C}}(V)$ and then by Lemma, Engel's Theorem holds.
b) For $n \geq 2$, assuming $n - 1$:
Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal proper subalgebra. Since $\mathbb{C}A$ is always a subalgebra, we have that $dim(\mathfrak{h}) \geq 1$. Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Since $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, its restriction

$$\mathrm{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$$

is an invariant subspace for the representation. Therefore, we may consider the factor representation

$$\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$$

for the quotient $\mathfrak{g}/\mathfrak{h}$. Then $\pi(\mathfrak{h}) \subset \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ and

$$dim(\pi(\mathfrak{h})) \leq dim(\mathfrak{h}) \leq dim(\mathfrak{g})$$

Remark that by Lemma $\pi(\mathfrak{h})$ consists of nilpotent operators. Applying the inductive assumption there exists a non-zero vector $A_1 \in \mathfrak{g}/\mathfrak{h}$ such that

$$\pi_H(A_1) = 0 \quad \forall H \in \mathfrak{h}$$

Let $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the projection on the factor and let $A \in \mathfrak{g}$ such that $p(A) = A_1$. We get that $[\mathfrak{h}, A] \subset \mathfrak{h}$ and since $A_1 \neq 0, A \notin \mathfrak{h}$. Hence

$$\mathfrak{h} \oplus \mathbb{C}A$$

is a subalgebra of $\mathfrak{g}$. This subalgebra is larger than $\mathfrak{h}$, but $\mathfrak{h}$ was a maximal proper subalgebra of $\mathfrak{g}$, which implies $\mathfrak{h} \oplus \mathbb{C}A = \mathfrak{g}$. Then have to be $\dim(\mathfrak{h}) < n$.

Come back to the representation $\pi : \mathfrak{g} \to \mathrm{End}(V)$. By inductive assumption, there exists a non-zero vector $v \in V$ such that $Av = 0$ for all $A \in \mathfrak{h}$. Let V_0 denote the space

$$V_0 = \left\{v \in V \mid \mathrm{ad}_H(v) = 0 \ \ \forall H \in \mathfrak{h}\right\}$$

of all vectors $v \in V$ satisfying $Av = 0$. If $v \in V_0$ for all $H \in \mathfrak{h}$

$$\pi_H(Av) = [\pi_H, A] + A\mathrm{ad}_H(v) = 0 + 0 = 0$$

then $AV_0 \subset V_0$. By the Lemma there exists a non-zero vector $v \in V_0$ such that $Av = 0$. Therefore on v all the operators associated to elements of $\mathfrak{h}$ and A take the value zero, hence for all elements in $\mathfrak{g}$.

□

Corollary 5.
Let $\pi : \mathfrak{g} \to End(V)$ be a representation of a Lie algebra $\mathfrak{g}$ in a finite dimensional vector space V such that π_A is a nilpotent operator for all $A \in \mathfrak{g}$. There exists a basis of V on which all operators π_A, $A \in \mathfrak{g}$ are strictly upper triangular matrices.

Proof.
By Engel's Theorem, there exists a non-zero vector e_1 such that $\pi_A(e_1) = 0$ for all $A \in \mathfrak{g}$. Since $\mathbb{C}e_1$ is an invariant subspace, we consider the factor representation of $\mathfrak{g}$ in $V/\mathbb{C}e_1$. Apply the inductive assumption to get the basis $\tilde{e}_1, \ldots, \tilde{e}_n$ of $V/\mathbb{C}e_1$ in which all matrices of $\pi_{|V}\left(\mathbb{C}e_1\right)$ are strictly upper triangular. Taking $e_1, \ldots, e_n$ preimages of $\tilde{e}_1, \ldots, \tilde{e}_n$, in the basis $e_1, \ldots, e_n$ of V all matrices $\pi(\mathfrak{g})$ are strictly upper triangular. □

We now give a characterization of nilpotent Lie algebras in terms of the operators associated, that is the goal of this section.

Theorem 13 (Characterization of Nilpotent Lie Algebras)**.**
Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ is nilpotent if and only if for each $A \in \mathfrak{g}$ there exists $n > 0$ such that $(ad_A)^n = 0$. One may always take $n = dim\ \mathfrak{g}$

Proof.
If $\mathfrak{g}$ is nilpotent then $\mathfrak{g}^{n+1} = 0$ for some n and in particular $\mathrm{ad}_A^n(B) = 0$ for all $A, B \in \mathfrak{g}$. Conversely, the adjoint representation gives an injective homomorphism $\mathfrak{g}/Z(\mathfrak{g}) \to \mathfrak{gl}_\mathfrak{g}$ which image consists by assumption of nilpotent operators. So by Engel's Theorem, $\mathfrak{g}/Z(\mathfrak{g})$ consists of strictly upper triangular matrices in the same basis. Therefore $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent and hence $\mathfrak{g}$ is nilpotent as well. □

Theorem 14.
Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and $\pi : \mathfrak{g} \to End(V_\mathbb{C})$ be a representation over the vector space $V_\mathbb{C}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a nilpotent subalgebra. Then the following decomposition holds

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^\mathfrak{h} \qquad\qquad \pi(\mathfrak{g}_\alpha^\mathfrak{h})V_\lambda^\mathfrak{h} \subset V_{\lambda+\alpha}^\mathfrak{h}$$

Proof.
i) We first prove the decomposition by induction on the dimension of the eigenspaces.
a) If for each $A \in \mathfrak{h}$, π_A has only one eigenvalue then V is a generalized eigenspace $V^A_{\lambda_A}$ for every $A \in \mathfrak{h}$ thus we have only to prove the linearity of λ. Since $\mathfrak{h}$ is nilpotent is also solvable, then for Lie's Theorem there exists a weight λ' and a related nonzero weight space $V^{\mathfrak{h}}_{\lambda'}$ thus λ'_A must be eigenvalue of π_A and π_A acts on $V^{\mathfrak{h}}_{\lambda'}$ so $\lambda' = \lambda$. Therefore λ is linear and so V is the generalized weight space $V^{\mathfrak{h}}_{\lambda}$.
b) Suppose now for some $A_0 \in \mathfrak{h}$ the map π_{A_0} has at least two distinct eigenvalues. Since $\mathfrak{h}$ is nilpotent for all $A \in \mathfrak{h}$ the associated operator ad_A is a nilpotent operator on $\mathfrak{h}$, thus $\mathfrak{h} \subset \mathfrak{g}^A_0$. Then by the Theorem for any $A \in \mathfrak{h}$

$$\pi(\mathfrak{h})V^A_\lambda \subset V^A_\lambda$$

Since $\mathbb{C}$ is algebrically closed, V can be written as direct sum of the generalized spaces of A_0. Since each $V^{A_0}_\lambda$ is invariant under the action of $\mathfrak{h}$ is defined a representation

$$\pi|_{V^{A_0}_\lambda} : \mathfrak{h} \to \mathrm{End}(V^{A_0}_\lambda)$$

of $\mathfrak{h}$ on $V^{A_0}_\lambda$. Since $dim(V^{A_0}_\lambda) < dim(V)$ we may apply induction on $dim\ V$, obtaining the equality.
ii) We finally prove the inclusion.
Suppose $\alpha, \lambda \in \mathfrak{h}^*$ and $B \in \mathfrak{g}^{\mathfrak{h}}_\alpha$; then $B \in \mathfrak{g}^A_{\alpha_A}$ for all $A \in \mathfrak{h}$ thus by Theorem

$$\pi_B V^A_{\lambda_A} \subset V^A_{\lambda_A + \alpha_A}$$

for all $A \in \mathfrak{h}$. Since by definition

$$V^{\mathfrak{h}}_\lambda = \bigcap_{A \in \mathfrak{h}} V^A_{\lambda_A}$$

we have that

$$v \in \bigcap_{A \in \mathfrak{h}} V^A_{\lambda_A} \qquad \Rightarrow \qquad \pi_B v \in \bigcap_{A \in \mathfrak{h}} V^A_{\lambda_A + \alpha_A}$$

□

If we consider in particular the adjoint representation and a nilpotent subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we have the decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^{\mathfrak{h}}_\alpha \qquad\qquad [\mathfrak{g}^{\mathfrak{h}}_\alpha, \mathfrak{g}^{\mathfrak{h}}_\beta] \subset \mathfrak{g}^{\mathfrak{h}}_{\alpha+\beta}$$

Where

$$\mathfrak{g}^{\mathfrak{h}}_\alpha = \{B \in \mathfrak{g} \mid (\mathrm{ad}_A - \alpha_A I)^n B = 0 \mid \text{some } n > 0 \ , \ \forall A \in \mathfrak{h}\}$$

1.2.2 Cartan Subalgebra Decomposition

We now want to investigate the other component of the Jordan decomposition, that are the diagonalizable operators. Remark that if an operator is diagonalizable on some subspace, it leaves such subspace invariant.

Definition 9.
Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We call *normalizer* of $\mathfrak{h}$ the set

$$N_{\mathfrak{g}}(\mathfrak{h}) := \{A \in \mathfrak{g} \mid [A, \mathfrak{h}] \subset \mathfrak{h}\}$$

Notice that $N_{\mathfrak{g}}(\mathfrak{h})$ is a subalgebra and it's the set of elements in $\mathfrak{h}$ such that $\mathrm{Im}(\mathrm{ad}_A) \subset \mathfrak{h}$, then the restriction

$$\mathrm{ad}_{A|_{\mathfrak{h}}} : \mathfrak{h} \to \mathfrak{h}$$

has value on $\mathfrak{h}$.

Lemma 6.
Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra such that $\mathfrak{h} \neq \mathfrak{g}$. Then

$$\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h})$$

Proof.
Consider the central series $\mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots \supset \mathfrak{g}^n = 0$ where

$$\mathfrak{g}^1 = \mathfrak{g} \qquad \mathfrak{g}^{n+1} := [\mathfrak{g}, \mathfrak{g}^n]$$

Take j to be the maximal possible positive integer such that $\mathfrak{g}^j \subsetneq \mathfrak{h}$; it must be $1 < j < n$, hence

$$[\mathfrak{g}^j, \mathfrak{h}] \subset \mathfrak{g}^{j+1} \subset \mathfrak{h}$$

by the choice made on j, thus $\mathfrak{g}^j \subset N_{\mathfrak{g}}(\mathfrak{h})$ which is not a subspace of $\mathfrak{h}$ and then we have that $\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h})$.

□

We call a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ *Cartan subalgebra* if

$$N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$$

and we denote it with $\mathfrak{h}_c$. Cartan subalgebras are fundamental components of a Lie algebra and encode its whole structure.

Corollary 6.
Any Cartan subalgebra of $\mathfrak{g}$ is a maximal [5] *nilpotent subalgebra*

Proof.
It follows directly from Lemma and the definition of Cartan subalgebras. □

The following Proposition establishes a class of such algebras and allows to understand their nature.

Proposition 11.
Let $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ be a subalgebra containing a diagonal matrix $A = diag(a_1, ..., a_n)$ with distinct a_i and $\mathfrak{h}$ be the subspace of all diagonal matrices in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra.

[5] That is, it is not contained in another proper subalgebra of $\mathfrak{g}$.

Consider now a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. In the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ we have the decomposition

$$\mathfrak{g} = \bigoplus_{\alpha\in\mathbb{C}} \mathfrak{g}^A_\alpha$$

where

$$\mathfrak{g}^A_\alpha = \{B \in \mathfrak{g} \mid (\mathrm{ad}_A - \alpha I)^m B = 0\}$$

for some $m > 0$. Let n be the dimension of $\mathfrak{g}$ and consider the characteristic polynomial of an endomorphism ad_A for some $A \in \mathfrak{g}$

$$\det{}_{\mathfrak{g}}(\mathrm{ad}_A - \lambda I_n) = (-\lambda)^n + c_{n-1}(-\lambda)^{n-1} + \cdots + \det(\mathrm{ad}_A)$$

It is a polynomial of degree n and since

$$\mathrm{ad}_A(A) = [A, A] = 0$$

must be $\det(\mathrm{ad}_A) = 0$ then the constant term in the polynomial vanishes. Let r be the smallest positive integer such that $c_r(A)$ is not the zero polynomial on $\mathfrak{g}$; then $1 \le r \le n$. An element $A \in \mathfrak{g}$ is called *regular* if $c_r(A) \neq 0$.

Theorem 15 (E.Cartan)**.**
Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra, $A \in \mathfrak{g}$ a regular element (which exists since $\mathbb{C}$ is infinite) and

$$\mathfrak{g} = \bigoplus_{\alpha\in\mathbb{C}} \mathfrak{g}^A_\alpha$$

the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to ad_A. Then $\pi(\mathfrak{g}^A_\alpha) \subset V^A_\lambda \subset V^A_{\lambda+\alpha}$ and $\mathfrak{h}_c = \mathfrak{g}^A_0$ is a Cartan subalgebra.

We now prove the most important theorem of Cartan algebras, establishing the crucial properties that allow to classify semisimple Lie algebras.

Theorem 16 (Cartan Subalgebras)**.**
Let $\mathfrak{g}$ a semisimple Lie algebra and $\mathfrak{h}_c$ be a Cartan subalgebra. Then
i) $\mathfrak{h}_c$ is abelian.
ii) For each $H \in \mathfrak{h}_c$ the map $ad_H : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable.

Proof.
i) Since $\mathfrak{h}_c$ is nilpotent, it is solvable and $\mathrm{ad}(\mathfrak{h}_c) \subset \mathrm{End}(\mathfrak{g})$ is solvable as subalgebra of transformations of $\mathfrak{g}$ then by Lie's Theorem it's triangular in some basis. Recall that for any three triangular matrices A, B, C we have

$$\mathrm{tr}(ABC) = \mathrm{tr}(BAC)$$

Therefore

$$\mathrm{tr}\big(\mathrm{ad}_{[H_1,H_2]}\mathrm{ad}_H\big)\big) = 0$$

for $H_1, H_2, H \in \mathfrak{h}_c$. If α is a generalized non zero weight and $E_\alpha \in \mathfrak{g}_\alpha$, $H \in \mathfrak{h}$ then $\mathrm{ad}_H \mathrm{ad}_{E_\alpha}$ carries $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$. Since

$$\mathfrak{g} = \bigoplus_{\alpha\in\mathbb{C}} \mathfrak{g}_\alpha$$

we get

$$\operatorname{tr}\Big(\mathrm{ad}_H \mathrm{ad}_{E_\alpha}\Big) = 0$$

then for $H = [H_1, H_2]$ we have $k([H_1, H_2], E_\alpha) = 0$ (where k is the killing form). The first relation implies that

$$k\big([H_1, H_2], B\big) = 0$$

for all $B \in \mathfrak{g}$. Since k is non degenerate for $\mathfrak{g}$ semisimple we conclude that must be $[H_1, H_2] = 0$.

ii) Consider $H \in \mathfrak{h}_c$. By the previous Theorem H admits an abstract Jordan decomposition of the form $H = H_d + H_n$, where $H_d, H_n \in \mathfrak{g}$ are such that ad_{H_d} is diagonalizable, ad_{H_n} is nilpotent and $[H_d, H_n] = 0$. Then we have $[H, \mathfrak{h}_c] = 0$ and hence for all $H' \in \mathfrak{h}_c$

$$0 = \mathrm{ad}_{[H', H]} = [\mathrm{ad}_{H'}, \mathrm{ad}_H]$$

It's easy to see that also $[\mathrm{ad}_{H'}, (\mathrm{ad}_H)_d] = 0$, then

$$0 = [\mathrm{ad}_{H'}, (\mathrm{ad}_H)_d] = [\mathrm{ad}_{H'}, \mathrm{ad}_{H_d}] = \mathrm{ad}_{[H', H_d]}$$

Since $Z(\mathfrak{g}) = 0$ this implies $[H', H_d] = 0$ for $H' \in \mathfrak{h}_c$. But since $\mathfrak{h}_c$ is a maximal nilpotent subalgebra, we conclude that $H_d \in \mathfrak{h}_c$. It remains to be shown that $H_n = 0$. Note that

$$H_n = H - H_d \in \mathfrak{h}_c$$

Since $\mathrm{ad}(\mathfrak{h}_c) \subset \mathrm{End}(\mathfrak{g})$ is solvable, by Lie's Theorem there exists a basis of $\mathfrak{gl}_{\mathbb{C}}(\mathfrak{g})$ such that the elements of $\mathrm{ad}(\mathfrak{h}_c)$ are upper triangular. Since ad_{H_n} is nilpotent, have to be in particular strictly upper triangular. Therefore

$$\operatorname{tr}((\mathrm{ad}_{H'})(\mathrm{ad}_{H_n})) = 0$$

for all $H' \in \mathfrak{h}_c$ that is

$$k(H, H_n) = 0$$

Since $\mathfrak{g}$ is semisimple, also

$$k_{|_{\mathfrak{h}_c \times \mathfrak{h}_c}} : \mathfrak{h}_c \times \mathfrak{h}_c \to \mathbb{C}$$

is nondegenerate, so must be $H_n = 0$. This implies that $H = H_d$ that means H is diagonalizable.

□

Moreover, given a complex Lie group G with complex Lie algebra $\mathfrak{g}$ and cartan subalgebra $\mathfrak{h}_c$, it is possible to prove that $\mathfrak{h}_c$ is the Lie algebra associated to the maximal torus subgroup[6] of G. This subgroup is the so called *Cartan subgroup* of G.

So, a Cartan subalgebra is defined as a maximal subalgebra $\mathfrak{h}$ such that ad_H is diagonalizable for all $H \in \mathfrak{h}$. Consider now a finite dimensional complex Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}_c \subset \mathfrak{g}$. We can decompose $\mathfrak{g} = \mathfrak{h}_c \oplus W$

[6]This holds for finite dimensional complex Lie algebras.

for some subspace $W \subset \mathfrak{g}$. Fixed an element $A \in \mathfrak{h}_c$, consider the associated operator

$$\mathrm{ad}_A : \mathfrak{h}_c \oplus W \to \mathfrak{h}_c \oplus W$$

From definition of Cartan subalgebra the restriction on $\mathfrak{h}_c$ has image in $\mathfrak{h}_c$ i.e.

$$\mathrm{ad}_{A|_{\mathfrak{h}_c}} : \mathfrak{h}_c \to \mathfrak{h}_c$$

and since $\mathfrak{h}_c$ is nilpotent this restriction is a nilpotent operator, then $\mathfrak{h}_c$ corrisponds to the zero eigenvalue. Since $\mathfrak{h}_c$ is abelian, on the remaining part of $\mathfrak{g}$ the map is diagonalizable with non-zero eigenvalues. This fact holds for all the operator ad_A for an element $A \in \mathfrak{h}_c$ then since all of them commute, the decomposition holds for the intere cartan subalgebra $\mathfrak{h}_c$. These considerations are formally statement in the following Theorem.

Proposition 12.
Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and let $\mathfrak{h}_c \subset \mathfrak{g}$ be a Cartan subalgebra. Consider the generalized weight space decomposition with respect to $\mathfrak{h}_c$:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}_c^*} \mathfrak{g}_\alpha$$

Then $\mathfrak{g}_0 = \mathfrak{h}_c$.

Proof.
i) The inclusion $\mathfrak{h}_c \subset \mathfrak{g}_0$ follows from the previous discussion.
ii) Let's prove the equality.
Suppose $\mathfrak{h}_c \subsetneq \mathfrak{g}_0$. By definition of $\mathfrak{g}$, for all elements $H \in \mathfrak{h}_c$

$$\mathrm{ad}_{H|_{\mathfrak{g}_0}} : \mathfrak{g}_0 \to \mathfrak{g}$$

is a nilpotent operator. Hence

$$\left(\mathrm{ad}_{\mathfrak{g}_0/\mathfrak{h}_c}\right)_H : \mathfrak{g}_0 \to \mathrm{End}\Big(\mathfrak{g}_0/\mathfrak{h}_c\Big)$$

is a nilpotent operator for all $H \in \mathfrak{h}_c$. Therefore, by Engle's Theorem, there exists a non-zero $B_1 \in \mathfrak{g}_0, B_1 \notin \mathfrak{h}_c$ on which each operator of the type $\mathrm{ad}_{H|_{\mathfrak{g}_0/\mathfrak{h}_c}}$ has value zero. Taking a pre-image $B \in \mathfrak{g}$ of B_1 , this means that $[B, \mathfrak{h}_c] \subset \mathfrak{h}_c$ i.e.

$$\mathfrak{h}_c \neq N_{\mathfrak{g}}(\mathfrak{h}_c)$$

which contradicts the fact that $\mathfrak{h}_c$ is a Cartan subalgebra. □

Finally, the following Theorem ensures this decomposition is in some sense unique and then really encodes the Lie algebra's structure.

Theorem 17 (Chevalley)**.**
Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. Any Cartan subalgebra $\mathfrak{h}_c$ in $\mathfrak{g}$ is of the form $\mathfrak{g}_0^A$ for some regular element $A \in \mathfrak{g}$. Moreover all such subalgebras $\mathfrak{g}_0^A$ are isomorphic.

1.2.3 Root Systems and Dynkin Diagrams

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and $\mathfrak{h}_c \subset \mathfrak{g}$ be a Cartan subalgebra, from Chevalley Theorem unique up to an isomorphism. If $\mathfrak{g}$ is semisimple, the operator corresponding to each element in $\mathfrak{h}_c$ is a diagonalizable operator then the generalized eigenspaces become simply eigenspaces, i.e.

$$\mathfrak{g}_\alpha = \{A \in \mathfrak{g} \mid [B, A] = \alpha(B)A \quad , \quad \forall B \in \mathfrak{h}_c\}$$

We call *root space* $\Delta \subset \mathfrak{h}_c^*$ the set of all $\alpha \in \mathfrak{h}_c^*$ such that $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, each $\alpha \in \Delta$ *root* and $\mathfrak{g}_\alpha$ the corresponding *root space.* From the Theorems of previous section the decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h}_c \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

follows. This decomposition completely describes the structure of finite dimensional (complex and real) semisimple Lie algebras and allows to classify them, analyzing the relations between the different roots in the root space Δ.

Theorem 18.
Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}_c \subset \mathfrak{g}$ be a Cartan subalgebra. Let k be the Killing form on $\mathfrak{g}$.
a) If $\alpha + \beta \neq 0$, then $k(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.
b) The form $k_{|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}}$ is nondegenerate. Consequently $k_{|_{\mathfrak{h}_c \times \mathfrak{h}_c}}$ is nondegenerate.

Proof.
a) Consider $A \in \mathfrak{g}_\alpha$ and $B \in \mathfrak{g}_\beta$ with $\alpha + \beta \neq 0$. Since $\mathfrak{g}$ is finite dimensional, for some for $n > 0$

$$\big((\mathrm{ad}_A)(\mathrm{ad}_B)\big)^n \mathfrak{g}_\lambda \subset \mathfrak{g}_{\lambda + n\alpha + n\beta} = 0$$

thus we can choose n sufficiently large so that the operator $\big((\mathrm{ad}_A)(\mathrm{ad}_B)\big)^n$ is zero on each summand of the decomposition, therefore $(\mathrm{ad}_A)(\mathrm{ad}_B)$ is nilpotent. It follows that the eigenvalues of $(\mathrm{ad}_A)(\mathrm{ad}_B)$ are zero, implying that

$$k(A, B) = \mathrm{tr}_{\mathfrak{g}}\big((\mathrm{ad}_A)(\mathrm{ad}_B)\big) = 0$$

b) By semisimplicity, we know that k is nondegenerate on $\mathfrak{g}$. By part (a), we have $k(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha \neq -\beta$, so necessarily $k_{|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}}$ have to be nondegenerate.

□

To relate different roots in Δ we introduce a bilinear form on $\mathfrak{h}_c^*$ extending the Killing form on $\mathfrak{h}_c$. Remark that we have a canonical linear map $\nu : \mathfrak{h}_c \to \mathfrak{h}_c^*$

$$B \to k(B, \cdot) \in \mathfrak{h}_c^*$$

Since $k_{|_{\mathfrak{h}_c \times \mathfrak{h}_c}}$ is nondegenerate, ν is injective and then determines a vector space isomorphism inducing a bilinear form on $\mathfrak{h}_c^*$

$$k^* : \mathfrak{h}_c^* \times \mathfrak{h}_c^* \to \mathbb{C}$$

defined by

$$k^*(\lambda_1, \lambda_2) = k(\nu^{-1}(\lambda_1), \nu^{-1}(\lambda_2))$$

that now denote by $\langle \alpha, \beta \rangle := k^*(\alpha, \beta)$. It turns out that $\langle \alpha, \beta \rangle \in \mathbb{Q}$ for all $\alpha, \beta \in \Delta$ and $\langle \alpha, \alpha \rangle > 0$. If we consider the vector space

$$V := \sum_{\alpha \in \Delta} \mathbb{R}\alpha$$

this bilinear form defines on V a scalar product and then V can be seen as an Euclidean vector space containing Δ on which we can compute angles and lengths of the roots.

Definition 10.
Let $\Delta \subset V$ be a finite subset in a finite dimensional vector space V over $\mathbb{R}$ with a positive definite bilinear form $\langle \cdot, \cdot \rangle$. We call Δ *root system* if satisfy:
1) Δ span all the space V and $0 \notin \Delta$.
2) If $\alpha \in \Delta$ then $\mathbb{R}\alpha \cap \Delta = \pm\alpha$.
3) Given two roots $\alpha, \beta \in \Delta$ then

$$C_{\alpha\beta} := \frac{2 \langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$$

is an integer.
4) For each $\alpha \in \Delta$, the reflection σ_α on the hyperplane $(\mathbb{R}\alpha)^\perp$ defined for all $\beta \in \Delta$ by

$$\sigma_\alpha(\beta) := \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

leaves Δ invariant, i.e. $\sigma_\alpha(\beta) \in \Delta$ for all $\alpha, \beta \in \Delta$.

It is possible to prove that the roots space Δ associated to the decomposition respect to a Cartan subalgebra $\mathfrak{h}_c$ satisfies the conditions of a root system. This reduces the difficult algebrical problem of classifying complex simple Lie algebras to the easier geometrical problem of classifying root systems. Moreover the definition of root system is so restrictive that there exist only a few possibilities for them. Let Δ and Δ' be root systems; a vector space isomorphism $\Phi : V \to V'$ such that $\Phi(\Delta) = \Delta'$ is called *root systems isomorphism* if

$$\langle \Phi(\alpha), \Phi(\beta)) \rangle' = c \langle \alpha, \beta \rangle$$

for all $\alpha, \beta \in \Delta$ where c is a positive constant, independent of α and β. In particular, replacing $\langle \cdot, \cdot \rangle$ by $c \langle \cdot, \cdot \rangle$, where $c > 0$, we get, by definition, an isomorphic root system. Moreover it is possible to prove that a root system is a *lattice* in the Euclidean space V i.e it is a discrete subgroup $(Q, +)$ of V , which spans V over $\mathbb{R}$. Consider now a vector $v_p \in V$ such that $\langle v_p, \alpha \rangle \neq 0$ for any $\alpha \in \Delta$. This choice allows to define *positive root* (resp. negative root) a root α such that $\langle v_p, \alpha \rangle > 0$ (resp. $\langle v_p, \alpha \rangle < 0$). Then if Δ_+ and Δ_- denote the set of positive and negative roots respectively, from condition 2) of the definition of root space it follows that $\Delta_- = -\Delta_+$ and so the decomposition

$$\Delta = \Delta_+ \cup \Delta_-$$

called *polarization* of Δ. The informations contained in the whole root system, are in some sense redundant; to uniquely identify a root system, we have to consider the so called simple root. A positive root is called *simple root* if it is not the sum of two positive roots.

Proposition 13.
Let $\Delta \subset V$ be a root system and $S = \{\alpha_1, \dots, \alpha_r\}$ be a set of simple roots due to a polarization v_p. Then S is a basis of V and every $\alpha \in \Delta$ is of the form

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i$$

where $n_i \in \mathbb{Z}$ and n_i is non positive if $\alpha_i \in \Delta_-$ and non negative if $\alpha_i \in \Delta_+$.

We would like to reduce the classification of root systems Δ to simple root systems S. To do that we have to show that any choice of polarization v_p gives equivalent simple root systems and that any root system Δ can be uniquely reconstructed from its simple root system S. These two conditions are established in the next Proposition.

Proposition 14.
There is a bijection between root systems and simple root systems. Every root system Δ has a unique (up to an orthogonal stransformation) simple root system S. Conversely we can uniquely reconstruct Δ from a simple root system S.

Type	$\mathfrak{g}$	V	Δ
$A_n, \quad n \geq 1$	$\mathfrak{sl}(2n{+}1, \mathbb{C})$	$\Big\{ \sum_{i=1}^{n+1} a_i e_i \mid a_i \in \mathbb{R}, \sum_{i}^{n+1} a_i = 0 \Big\}$	$\Big\{ e_i - e_j \mid 1 \leq i,\ j \leq n+1 \Big\}$
$B_n, \quad n \geq 1$	$\mathfrak{so}(2n{+}1, \mathbb{C})$	$\Big\{ \sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{R} \Big\}$	$\Big\{ \pm e_i \pm e_j, \pm e_i \mid 1 \leq i,\ j \leq n,\ i \neq j \Big\}$
$C_n, \quad n \geq 1$	$\mathfrak{sp}(2n, \mathbb{C})$	$\Big\{ \sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{R} \Big\}$	$\Big\{ \pm e_i \pm e_j, \pm 2e_i \mid 1 \leq i,\ j \leq n,\ i \neq j \Big\}$
$D_n, \quad n \geq 2$	$\mathfrak{so}(2n, \mathbb{C})$	$\Big\{ \sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{R} \Big\}$	$\Big\{ \pm e_i \pm e_j \mid 1 \leq i,\ j \leq n,\ i \neq j \Big\}$

Table 1.1: Classical Simple Groups.

Furthermore, we know that for a root system in a n-dimensional space, the simple root system has exactly n linearly independent elements. We call a root system Δ *irreducible* if it is not a disjoint union of two orthogonal root systems. In the previous table we report the roots systems associated to some simple classical groups.[7] So, since to every root system corresponds a semisimple Lie algebra, it is important to know all the root systems. In addiction to the four *classical root systems* A_n, B_n, C_n and D_n, it turns out there exist other five exceptional roots systems G_2, F_4, E_6, E_7, E_8.

$n_{\alpha\beta}$	$n_{\beta\alpha}$	$\lvert\alpha\rvert/\lvert\beta\rvert$	φ [rad]	φ [°]
0	0	/	$\pi/2$	90
-1	-1	1	$2\pi/3$	120
1	1	1	$\pi/3$	60
-2	-1	$\sqrt{2}$	$3\pi/4$	135
2	1	$\sqrt{2}$	$\pi/4$	45
-3	-1	$\sqrt{3}$	$\pi/6$	30
3	1	$\sqrt{3}$	$5\pi/6$	150

Figure 1.2: From Lijubiana File.

Suppose we have a simple roots system S. Since taking an orthogonal transformation we still obtain an equivalent root system, the relevant informations contained in such a system are not the position or the length of the roots. The important properties are their relative length to each other and the angle between them. Since for simple roots $\alpha, \beta \in S$ the inequality $\langle \alpha, \beta \rangle \leq 0$ holds, the angle between simple roots is ≥ 90. We have the four familiar possibilities, reported in the previous table.

We can also present these informations in a matrix. We define the matrix

$$C_{\alpha\beta} = 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$$

conserved via root system isomorphisms. This matrix is the so called *Cartan matrix* associated to the root system. By the previous results it follows that $C_{ii} = 2$ for all $i \in \{1, \ldots, n\}$. Moreover, since the scalar product of simple roots $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$, the non-diagonal entries in the Cartan matrix are not positive, that is $C_{ij} \leq 0$ for $i \neq j$. It is also possible to present the information encoded in the Cartan matrix in a graphical way, via the so called *Dynkin diagrams*.

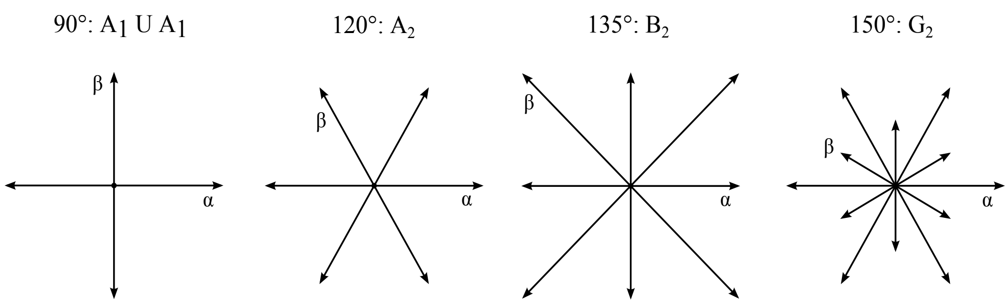


Figure 1.3: Rank two root systems.

We now define these diagrams by the rules necessary to draw them.

[7] In the table, $\langle e_i, e_j \rangle = \delta_{ij}$.

Definition 11.
Suppose $S \subset \Delta$ is a simple root system. The *Dynkin diagram* of S is a graph constructed by the following prescription:
i) For each $\alpha_i \in S$ we construct a vertex.
ii) For each pairs of roots α_i, α_j we construct a vertex we draw a connection, depending by the angle α between them:

- If $\varphi = \frac{\pi}{2}$ the verticies are not connected;
- If $\varphi = \frac{2\pi}{3}$ the verticies have a single edge;
- If $\varphi = \frac{3\pi}{4}$ the verticies have a double edge;
- If $\varphi = \frac{5\pi}{6}$ the verticies have a triple edge;

iii) For double and triple edges connecting two roots, we direct them towards the shorter root (we draw an arrow pointing to the shorter root).

We are expecting that general root systems can be decompose in irreducible ones. This is what the following Proposition establish.

Proposition 15.
i) Every root system Δ can be decompose in a finite disjoint union

$$\Delta = \bigsqcup \Delta_i$$

where Δ_i are irreducible root systems and $\Delta_i \perp \Delta_j$ if $i \neq j$ where

$$S = \bigsqcup S_i$$

is the simple root system of Δ (under some polarization), and $S_i = \Delta_i \cap S$ is the simple root system of Δ_i (under the same polarization).
ii) If S_i are simple root systems, $S_i \perp S_j$ for $i \neq j$, $S = \bigsqcup S_i$ and Δ_i are the root systems generated by the simple systems S_i, then the root system of S is

$$\Delta = \bigsqcup \Delta_i$$

Next Theorem shows that the previous Proposition is in some sense the analogous for root systems of semisimple Lie algebra's decomposition in simple ones and then allows to classify semisimple Lie algebras from generic root systems.

Theorem 19.
Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\mathfrak{h}_c \subset \mathfrak{g}$ a Cartan subalgebra. The following are equivalent:
1. The Lie algebra $\mathfrak{g}$ is simple.
2. The root system Δ associated to the decomposition induced by any Cartan subalgebra $\mathfrak{h}_c$ is irreducible.
3. The Dynkin diagram associated to the decomposition induced by $\mathfrak{h}_c$ is connected.

Finally, Dynkin diagrams encode all the structure of a root system.

Theorem 20 (Classification of Dynkin diagrams)**.**
i) Let Δ and Δ' be two root systems, constructed from the same Dynkin diagram. Then Δ and Δ' are isomorphic.
ii) Let Δ be an irreducible system. Then its Dynkin diagram is isomorphic to a diagram from the list in figure, which index in the label is always equal to the number of simple roots. On the other hand, each of the diagrams in figure is realized for some irreducible root system Δ.

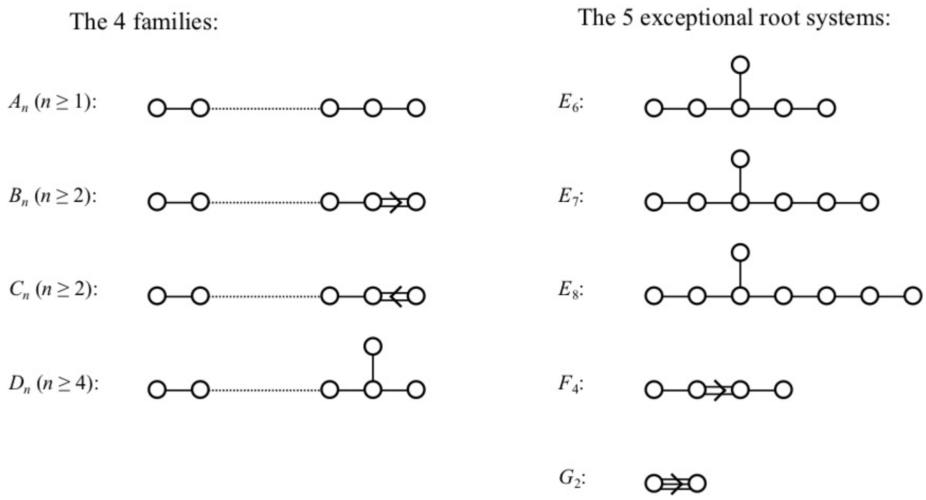
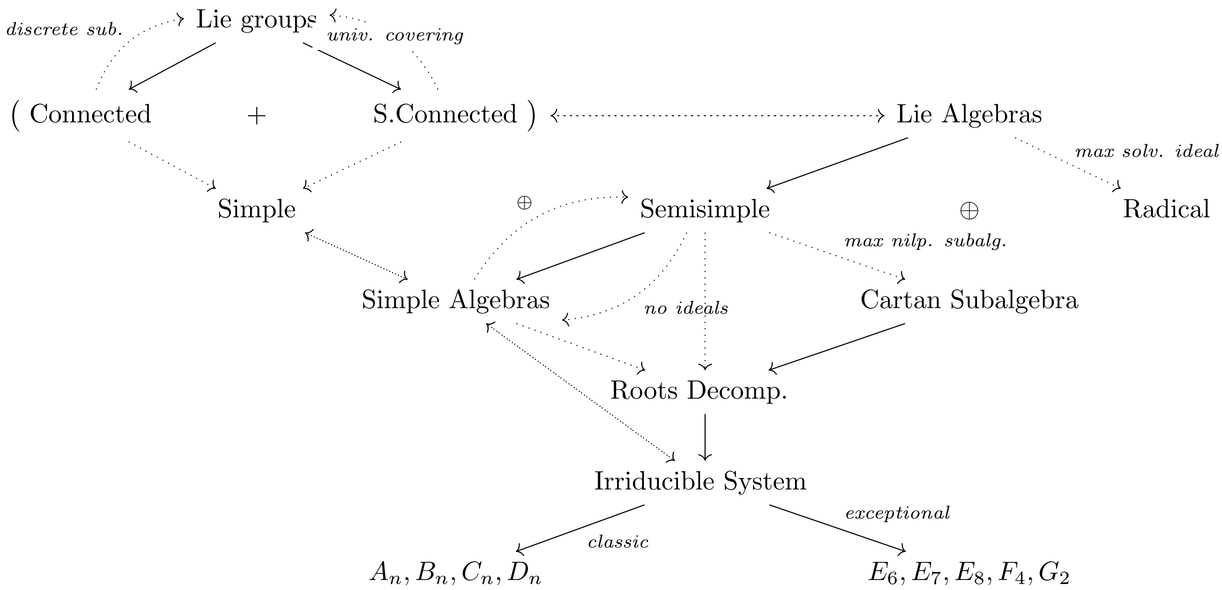


Figure 1.4: Possible Dynkin Diagrams for irreducible systems.

We conclude that any of such exceptional systems corresponds to a simple complex Lie algebra and then to a connected and simply connected complex Lie group. We now finally complete our scheme, that explicitly shows the process of classification of simple Lie groups and Lie algebras we have done.



1.2.4 The Exceptional Groups G_2

As we have seen, any irreducible root system identifies a particular finite dimensional complex simple Lie algebra. The G_2 root system identifies the Lie algebra of the connected, simply connected simple complex Lie group G_2. Consider the euclidean space $V_{G_2} = V_{A_2}$ and

$$\Delta = \left\{ (a_1, a_2, a_3) | a_1 + a_2 + a_3 = 0 \mid a_i \in \mathbb{Z}, \quad \sum_i a_i^2 \in \{2, 6\} \right\}$$

Let's determine the elements of Δ explicitly. In order for an element of Δ to have square sum 2, exactly two of its coordinates a_i must be equal to ± 1, so we have the only possibilities

$$\pm(1, -1, 0), \quad \pm(1, 0, -1), \quad \pm(0, 1, -1)$$

Instead, in order to have square sum 6, exactly one of the coordinates must be ± 2 while the others two must both be ∓ 1, then we get

$$\pm(2, -1, -1), \quad \pm(-1, 2, -1), \quad \pm(1, 1, -2)$$

Moreover, since the vector space spanned by the roots has dimension equal to 2, the system has rank 2. Two simple roots are the vectors

$$\alpha_1 = (1, -2, 1) \quad \alpha_2 = (-1, 1, 0)$$

while positive roots are

$$\Delta_+ = \left\{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2 \right\}$$

Computing the quantities $\langle \alpha, \beta \rangle, \langle \alpha, \alpha \rangle$ and following the definition, it turns out that G_2 is effective a root system. To see that G_2 is also irreducible, note that no two of the shorter roots are perpendicular and each of the longer ones is not perpendicular to one of the shorter ones.

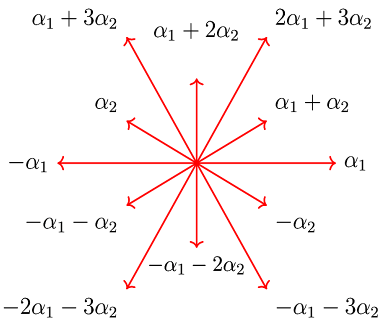


Figure 1.5: G_2 Roots System.

We can easy compute the Cartan matrix of G_2 obtaining

$$C_{G_2} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

The group G_2 has also two real form, the real compact one and the real split one associated respectively to the real and split real form of the complex Lie algebra associated to the G_2 diagram, which complexification gives the complex one. These two real forms are respectively the automorphism group of octonions and split-octonions, that we'll introduce in next section. We now see how to obtain these two real algebras and more general the real forms of any complex simple Lie algebra. The basic idea is based on the fact that given a complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, if we find a set of generators $\{J^k\}$ i.e. a basis of $\mathfrak{g}_{\mathbb{C}}$ with commutation relations

$$[J^k, J^l] = \sum_{l=1}^{n} c_m^{kl} J^m$$

such that all the constant structure c_m^{kl} are *real*, then the vector space spanned by real linear combinations of $J_1, \ldots, J_n$ is an real Lie algebra. For simple Lie algebras, a convenient choice of basis is the so called *Cartan-Weyl basis*. We first find a maximal set of commuting (Hermitian) generators H_i, $i = 1, \ldots, r$ for the Cartan subalgebra $\mathfrak{h}_c$, then since the generators of the Cartan subalgebra can be simultaneously diagonalized, we choose the remaining generators to be combinations E^j of the J^k' s that are common eigenvectors, i.e. such that

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

Proposition 16.
For the Cartan-Weyl basis, the following commutation relations hold

$$[H_i, H_j] = 0, \qquad [H_i, E_\alpha] = \alpha_i E_\alpha, \qquad [E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta} & \textit{if} \quad \alpha + \beta \in \Delta \\ \frac{2\alpha_i H^i}{|\alpha|^2} & \textit{if} \quad \alpha = -\beta \\ 0 & \textit{otherwise.} \end{cases}$$

Proof.
Let us evaluate the restriction of the Killing form to $\mathfrak{h}_c$. We have $\mathrm{ad}_{H_i}(H_j) = 0$ and $\mathrm{ad}_{H_i}(E_\alpha) = \alpha_i E_\alpha$. Therefore, in this basis, ad_{H_i} is a diagonal matrix with entries α_i on E_α. We have that

$$\mathrm{Tr}\big(\mathrm{ad}_{H_i} \mathrm{ad}_{H_j}\big) = \sum_\alpha \alpha_i \alpha_j$$

Rescaling k of a positive quantity we can assume that $k(H_i, H_j) = \delta_{ij}$. The Jacobi identity implies

$$[H_i, [E_\alpha, E_\beta]] = (\alpha_i + \beta_i)[E_\alpha, E_\beta]$$

then:
(i) If $\alpha+\beta \in \Delta$: the commutator $[E_\alpha, E_\beta]$ must be proportional to $E_{\alpha+\beta}$ with a proportionality non-vanishing constants $N_{\alpha,\beta} \neq 0$.
(ii) If $0 \neq \alpha+\beta \notin \Delta$: then $[E_\alpha, E_\beta] = 0$.
(iii) If $\beta = -\alpha$, then $[E_\alpha, E_{-\alpha}]$ commutes with all H_i and thus must belong to the Cartan subalgebra. Since

$$k([E_\alpha, E_{-\alpha}], H_i) = k(E_{-\alpha}, [H_i, E_\alpha]) = \alpha_i k(E_\alpha, E_{-\alpha})$$

it follows that $[E_\alpha, E_{-\alpha}] = \alpha_i H^i k(E_\alpha, E_{-\alpha})$. We can then rescale the generators such that

$$k(E_\alpha, E_{-\alpha}) = \frac{2}{|\alpha|^2}$$

□

The real Lie algebra $\mathfrak{g}^s_{\mathbb{R}}$ obtained considering real linear combinations of Cartan-Weyl basis $\{H_i, E_\alpha\}$ is the so called *split real form* of $\mathfrak{g}_{\mathbb{C}}$. In particular the split forms associated to classical groups are the real algebras

$$A_r \to \mathfrak{sl}(r+1,\mathbb{R}) \qquad B_r \to \mathfrak{so}(r+1,r)$$

$$C_r \to \mathfrak{sp}(2r,\mathbb{R}) \qquad D_r \to \mathfrak{so}(r,r)$$

Clearly, also the exceptional algebras E_6, E_7, E_7, F_4 and G_2 have split forms. The Cartan-Weyl basis is not the only set of real[8] generators. Consider the basis

$$iH^i \qquad J_1^\alpha = \frac{E^\alpha + E^{-\alpha}}{i\sqrt{2}} \qquad J_2^\alpha = \frac{E^\alpha - E^{-\alpha}}{\sqrt{2}}$$

and the real algebra $\mathfrak{g}_{\mathbb{R}}$ of its real linear combinations. With some computations it turns out that, again, the structure constants are real. Since

$$k(H_i, H_j) = \delta_{ij}, \qquad k(H_i, E_\alpha) = k(E_\alpha, E_\beta) = 0, \quad \beta \neq -\alpha$$

and

$$k(E_\alpha, E_{-\alpha}) = \frac{2}{|\alpha|^2}$$

in the basis above the Killing form is diagonal and negative-definite. It is possible to prove that last property guarantees that the corresponding Lie group is compact and $\mathfrak{g}_{\mathbb{R}}$ is therefore called the *compact real form* of $\mathfrak{g}_{\mathbb{C}}$. In the next sections we still investigate the real forms of the group G_2, in respect of two algebric structures for which these two real forms are the automorphisms group, the spaces of octonions and split octonions.

1.3 Quaternions and Space Rotations

Before introducing to octonions and their algebric structure, it is useful to investigate another easier space with similar algebric properties, the quaternions. The space $\mathbb{H}$ of quaternions can be considered the natural four dimensional

[8] With "real" we mean that the constant structures of this set of generators are real.

extension of complex numbers. It is the four dimensional real vector space with basis the elements $1, i, k, j$ that is

$$\mathbb{H} = \{a + bi + cj + dk \ \mid a, b, c, d \in \mathbb{R}\}$$

on which is defined a multiplication i.e. a bilinear map

$$\mathbb{H} \times \mathbb{H} \to \mathbb{H}$$

such that $1 \cdot w = w \cdot 1 = w$, for all $w \in \mathbb{H}$ and t

$$ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j$$

while

$$i^2 = k^2 = j^2 = -1$$

that makes quaternions a non-commutative ring over $\mathbb{R}$. We now show that the space of quaternions is finally a non commutative field, defining for each element an inverse. We can decompose the space $\mathbb{H}$ as the direct sum [9]

$$\mathbb{H} = \mathbb{R} \oplus \mathrm{Im}(\mathbb{H}) = \langle 1 \rangle \oplus \langle i, j, k \rangle$$

where $\mathrm{Im}(\mathbb{H}) := \langle i, j, k \rangle$ is by definition the set of *imaginary quaternions*. Since it follows that any $w \in \mathbb{H}$ can be represented as a sum $w = a + u$ for $a \in \mathbb{R}$, $u \in \mathrm{Im}(\mathbb{H})$ it is natural to define the *conjugate* of a quaternion w

$$\overline{w} = a - u = a - bi - ck - dk$$

With simple computations we get that

$$\overline{w_1 w_2} = \overline{w}_2 \overline{w}_1 \qquad Re(w_1 \overline{w}_2) = Re(\overline{w}_2 w_1) = \langle w_1, w_2 \rangle$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^4$. Remark that in particular, unlike the complex numbers, the conjugation is for quaternions anticommutative. On $\mathbb{H}$ we can also consider the euclidean norm

$$|w| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

induced by the scalar product

$$\langle u, v \rangle := a^2 + b^2 + c^2 + d^2$$

that moreover satisfies

$$|w_1 w_2| = |w_1||w_2| \qquad w\overline{w} = |w|^2$$

From the last property it follows that the inverse of a quaternion w is given by

$$w^{-1} = \frac{\overline{w}}{|w|^2}$$

then $\mathbb{H}$ with multiplication has also a structure of four dimensional real Lie group. It is possible to define quaternions in other (equivalent) forms, sometimes useful to explain some geometrical and algebrical properties.

[9]Here the brakets denote the vector space spanned by the elements.

i) *Complex form*: Given a quaternion $w \in \mathbb{H}$ we can rewrite it with simple algebric transformation as

$$w = (a + ib) + (c + id)j$$

Then denoted by $(a+ib, c+id) = (z_1, z_2) \in \mathbb{C}^2$ we have that $w = z_1 + z_2 j$ from which the identification $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}j$ follows. As we'll see in the last part of the thesis, complex form allows to define (under some restriction) functions on quaternions built from functions on $\mathbb{C}^2$.

ii) *Matrix form*: Quaternions can be also defined to be the set of 2×2 complex matrices of the type

$$w = \begin{bmatrix} z_1 & -z_2 \\ \overline{z}_2 & \overline{z}_1 \end{bmatrix}$$

for $z_1, z_2 \in \mathbb{C}$, with the standard sum and product of matrixes. The imaginary units can then be identified with the matrices

$$1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad i := \begin{bmatrix} i & -0 \\ 0 & -i \end{bmatrix} \quad j := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad k := \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

Quaternions represent a central tool to describe rotations of euclidean space $\mathbb{R}^3$. Indeed given $u, v \in \mathrm{Im}(\mathbb{H})$

$$u = bi + cj + dk \qquad v = b'i + c'j + d'k$$

we can define the wedge product $u \wedge v \in \mathrm{Im}(\mathbb{H})$ between u and v by

$$u \wedge v = (cd' - c'd)i - (bd' - db')j + (bc' - cb')k$$

that is the standard wedge product in $\mathbb{R}^3$, if we think about the elements i, j, k as a clockwise ordered triad of unit vectors in $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$. Clearly

$$i \wedge j = k \quad j \wedge k = i \quad k \wedge i = j$$

With some algebric computations it is possible to prove the following (beautiful) geometrical relation between quaternion product, scalar product and wedge product of elements in $\mathrm{Im}(\mathbb{H})$

$$uv = -\langle u, v \rangle + u \wedge v$$

These geometrical properties of quaternions are encoded in their group structure. Indeed if we consider the three dimensional sphere

$$S^3 = \{w \in \mathbb{H} \mid |w| = 1\}$$

in the matrix description of $\mathbb{H}$ as subgroup of $GL(2, \mathbb{C})$ for each matrix $U \in \mathbb{H}$, denoted by w_U the associated quaternion in the standard description, we have that

$$\det(U) = |w_U|^2$$

that suggests an identification between $SU(2)$ and $S^3 \subset \mathbb{H}$. More formally.

Proposition 17.
The map $SU(2) \to S^3 \subset \mathbb{H}$ *defined for* $U \in SU(2)$ *of the type*

$$U = \begin{bmatrix} z_1 & -z_2 \\ \overline{z}_2 & \overline{z}_1 \end{bmatrix}$$

by $U \to z_1 + jz_2$ *is a group isomomorphism.*

Proof.
The correspondence is clearly bijective. To see it is an homomorphism of groups, we compute

$$\begin{bmatrix} z_1 & -z_2 \\ \overline{z}_2 & \overline{z}_1 \end{bmatrix} \begin{bmatrix} z_1' & -z_2' \\ \overline{z}_2' & \overline{z}_1' \end{bmatrix} = \begin{bmatrix} z_1 z_1' - \overline{z}_2' z_2 & -z_1 z_2' - z_2 \overline{z}_1' \\ \overline{z}_2 z_1' + \overline{z}_1 \overline{z}_2' & -\overline{z}_2 z_2' + \overline{z}_1 \overline{z}_1' \end{bmatrix}$$

Remark that for $a, b \in \mathbb{R}$ we have $j(a + bi) = (a - bi)j$ thus

$$\begin{aligned}(z_1 + j\overline{z}_2)(z_1' + jz_2') &= z_1 z_1' + z_1 j z_2' + j z_2 z_1' + j z_1 j z_1' = \\ &= z_1 z_1' - \overline{z}_1 z_1' + j(\overline{z}_1 z_2' + z_2 z_1')\end{aligned}$$

hence the thesis.

□

Moreover, since $\mathbb{H}$ is a Lie group, it is defined the adjoint action of $\mathbb{H}$ on itself i.e. the action $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$

$$(w, q) \to w^{-1} q w$$

Theorem 21.
Let $w \in \mathbb{H}$ *be a quaternion and consider the adjoint action of* $\mathbb{H}$ *on* $\mathbb{H}$. *Denoted by* $\sigma_w : \mathbb{H} \to \mathbb{H}$

$$\sigma_w(q) = w^{-1} q w$$

the function associated to q, *if* $u \in Im(\mathbb{H})$ *is imaginary and* $w \notin \mathbb{R}$ *the map* $\sigma_w(u)$ *is a rotation of an angle* θ *around the vector*

$$n_w = \frac{Im(w)}{|Im(w)|}$$

in the direction $u \wedge Im(w)$.

Remark that $\sigma_{w_2} \circ \sigma_{w_1} = \sigma_{w_1 w_2}$ then if $w_1, w_2 \in S^3 \subset \mathrm{Im}(\mathbb{H})$ we have that

$$\sigma_{w_1} = \sigma_{w_2}$$

if and only if $w_1 = \pm w_2$. The left implication is clear, while if $\sigma_{w_1} = \sigma_{\mp w_2}$ then $w_1^{-1} q w_1 = w_2^{-1} q w_2$ for each $q \in \mathbb{H}$ that is

$$w_2 w_1^{-1} q w_1 w_2^{-1} = q$$

thus $\sigma_{w_1} \circ \sigma_{\mp w_2} = \sigma_1$, from which

$$\sigma_{w_1} = \sigma_{\mp w_2}^{-1} = \sigma_{\pm w_2}$$

follows. Then the map $S^3 \to SO(3)$

$$w \to \sigma_w$$

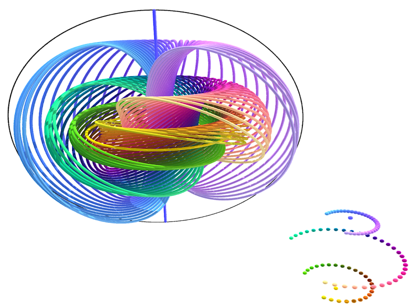
is a differentiable $2:1$ map. Moreover, since each element in $SO(3)$ is a rotation around some axis, the map is also surjective and then defines a double covering

$$SU(2) \cong S^3 \to SO(3)$$

Finally, on the quotient that identifies w and $-w$ the map becomes a diffeomorphism, hence $SO(3) \cong \mathbb{P}^3(\mathbb{R})$. Remark that since S^3 is simple connected, the covering has degree 2 and the only group of two elements is $\mathbb{Z}_2$, it follows that the fundamental group of the Lie group $SO(3)$ is $\mathbb{Z}_2$. Consider now

$$S^2 = \{u \in \mathrm{Im}(\mathbb{H}) \mid |u| = 1\}$$

The action of $SO(3)$ on S^2 is transitive, then fixed the nord pole $i \in S^2$ we can reach any other point by rotations in $SO(3)$.



The composition of this action with the diffeomorphism above defines a map

$$\mathfrak{h} : S^3 \to S^2$$

$$\mathfrak{h}(w) = \overline{w} i w$$

called *quaternionic Hopf fibration* that gives to S^3 a structure of fibre bundle, with fiber through $w \in S^3$

$$S^1_w = \{e^{i\theta} w \mid \theta \in [0, 2\pi)\}$$

diffeomorphic to the circle S^1. Hopf fibration explains the relation between the sphere S^2 and the group S^3 as double covering of the rotation group $SO(3)$ and it's a fundamental tool to investigate the topology of spheres.

1.4 Composition Algebras and Octonions

Quaternions and octonions are examples of a more general algebric construction, called *Composition Algebra.* We now give the basic notions of such algebras useful to explain the algebric properties of octonions and another composition algebra related, the split octonions . Recall that given a vector space V over a field K a *quadratic form* on V is a map

$$N : V \to K$$

such that $N(\lambda x) = \lambda^2 N(x)$ and the map $\langle \cdot, \cdot \rangle : V \times V \to K$

$$\langle x, y \rangle := N(x + y) - N(x) - N(y)$$

is bilinear. Then $\langle \cdot, \cdot \rangle$ is called *inner product* associated to the quadratic one. Notice that from the definition it follows that $\langle \cdot, \cdot \rangle$ is symmetric but could be in general degenerate. Indeed

$$\langle x, x \rangle = N(x + x) - N(x) - N(x) = N(2x) - 2N(x) = 2N(x)$$

so

$$N(x) = \frac{1}{2} \langle x, x \rangle$$

We can then distinguish two kinds of quadratic forms. A quadratic form N is called *isotropic* if there exists a vector $x \neq 0$ such that $N(x) = 0$ while *anisotropic* if $N(x) \neq 0$ for all $x \in V$.

Definition 12.
A composition algebra C over a field K is a not necessarily associative algebra over K equipped with a quadratic form

$$N : C \to K$$

such that

$$N(xy) = N(x)N(y)$$

Then N is called *norm* and the associate bilinear form, *inner product.*

Let $e \in C$ be the identity element. Since by definition $N(ex) = N(e)N(x)$ it follows that $N(e) = 1$. Moreover

$$N(xy_1 + xy_2) = N(x(y_1 + y_2)) = N(x)(N(y_1) + N(x)N(y_2) + \langle y_1, y_2 \rangle N(x)$$

but also

$$N(xy_1 + xy_2) = N(xy_1) + N(xy_2) = N(x)N(y_1) + N(x)N(y_2) + \langle xy_1, xy_2 \rangle$$

then

$$\langle xy_1, xy_2 \rangle = N(x) \langle y_1, y_2 \rangle$$

that becomes for $y = y_1 + y_2$

$$\langle x_1, x_2 \rangle \langle y_1, y_2 \rangle = \langle x_1 y_1, x_2 y_2 \rangle + \langle x_1 y_2, x_2 y_1 \rangle$$

We can now define the *conjugation* in a composition algebra C by

$$\overline{x} := \langle x, e \rangle e - x$$

that is minus the reflection in $e^{\perp}$. We now summarize the principal algebraic properties of these functions and their relations with the quadratic form.

Proposition 18.
Let (C, N) be a composition algebra and $x, y, e \in C$, where e is the unitary element. Then the following elementary properties hold
i) For the conjugation

$$\overline{xy} = \overline{y}\ \overline{x}\ , \qquad \overline{\overline{x}} = x\ , \qquad \overline{x + y} = \overline{x} + \overline{y}$$

ii) For the quadratic form

$$N(\overline{x}) = N(x) \qquad x\overline{x} = \overline{x}x = N(x)e$$

iii) For the inner product

$$\langle \overline{x}, \overline{y} \rangle = \langle x, y \rangle$$

We report also the following fundamental identities that allows to recover (in some sense) the associativity in the algebra, called *Moufang identities.*

Theorem 22 (Moufang identities)**.**
In any composition algebra (C, N)*, for each* $x, y, z \in C$

$$(zx)(yz) = z((xy)z)$$
$$z(x(zy)) = (z(xz)y)$$
$$x(z(yz)) = ((xz)y)z)$$

Remark that the Proposition implies that an element x has an inverse if and only if $N(x) \neq 0$ and in this case

$$x^{-1} = N(x)^{-1}\overline{x}$$

It is then natural to distinguish two cases, the case of N isotropic and the one with N anisotropic, i.e. $N(x) \neq 0$ for all $x \in C$. In the first case, C is called *split composition algebra*, in the second one C is called *composition division algebra.* Composition algebras are completely classified. The classification falls outside the goal of this thesis, then we simply report the result for completeness.

Theorem 23.
The possible dimensions of a composition algebra are 1 *(if char*$(K) \neq 2$*),* $2, 4, 8$*. Moreover in each of the dimension* $2, 4, 8$ *there is, up to isomorphism, exactly one split composition algebra and these are the only composition algebras containing zero divisors.*

1.4.1 Octonions and Split Octonions

As quaternions can be thought as the extension of the complex numbers, octonions can be thought as the extension of quaternions. If in the quaternion space we have lost the commutative of the product compared to complex numbers, in the octonions one we loose also the associativity.

		e_j							
	e_ie_j	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_i	e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
	e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
	e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
	e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
	e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
	e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Figure 1.6: Multiplication table of Octonions

As for the quaternions, the octonions could be defined as an eight dimensional vector space spanned by the basis

$$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$$

with a multiplication defined on the elements of the basis by the rules in the table. Anyway, a more useful way to define them is following the *Cayley-Dickson construction*[10]. Consider the space of quaternions $\mathbb{H}$ in the complex representation $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ and let

$$w_1 = z_1 + jz_2 \qquad w_2 = \eta_1 + j\eta_2$$

be two quaternions. Their product is given by

$$w_1 w_2 = (z_1\eta_1 - \overline{\eta}_2 z_2, \eta_2 z_1 + z_2 \overline{\eta}_1)$$

The same holds if we decompose $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$. The idea is then to define octonions in the same way, i.e. as the direct product $\mathbb{O} = \mathbb{H} \times \mathbb{H}$ with the multiplication defined by

$$(w_1, lw_2)(q_1, lq_2) := (w_1 q_1 - \overline{q}_2 w_2, q_2 w_1 + w_2 \overline{q}_1)$$

With this definition, the previous basis becomes the basis

$$(1,0), (i,0), (j,0), (k,0), (0,1), (0,i), (0,j), (0,k)$$

In analogy we use the notation $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l$ and then we define the product

$$(w_1 + lw_2)(q_1 + lq_2) := w_1 q_1 - \overline{q}_2 w_2 + l(q_2 w_1 + w_2 \overline{q}_1)$$

Instead the quadratic form is defined on an element $x = w_1 + lw_2 \in \mathbb{O}$ by

$$N(x) := |w_1|^2 + |w_2|^2$$

and it's then positive defined. Notice that since $x\overline{x} = \overline{x}x = N(x) = |x|^2$, the form is anisotropic hence the inverse of an element $x \in \mathbb{O}$ is alway defined and it's given by

$$x^{-1} = \frac{\overline{x}}{|x|^2}$$

while the *conjugate* of $x \in \mathbb{O}$ is

$$\overline{w_1 + lw_2} := \overline{w}_1 - lw_2$$

We can then define the real and imaginary components respectively

$$\mathrm{Re}(x) := \frac{x + \overline{x}}{2} = Re(w_1) \qquad \mathrm{Im}(x) := \mathrm{Re}(x) - \overline{x}$$

It follows that for each $x \in \mathbb{O}$ we have $x = 2\mathrm{Re}(x) - \overline{x}$ and then by the definition of imaginary component the decomposition

$$x = \mathrm{Re}(x) + \mathrm{Im}(x)$$

holds. Instead, as for quaternions, the *cross product* of $u, v \in \mathrm{Im}(\mathbb{O})$ is defined by

$$u \wedge v := \frac{1}{2}(uv - vu)$$

We summarize the principal algebric properties of octonions.

[10]This construction is a more general construction for division algebras.

Proposition 19.
Consider $x, y \in \mathbb{O}$. Then

$$\overline{xy} = \overline{y}\ \overline{x} \qquad , \ |xy| = |x||y|$$

while given $u, v \in Im(\mathbb{O})$

- $uv + vu = -2\langle u, v\rangle$
- $uv = -\langle u, v\rangle + u \wedge v$
- $|uv|^2 = |\langle u, v\rangle|^2 + |u \wedge v|^2$

Moreover, denoted by $w = u \wedge v$

$$\langle w, u\rangle = \langle w, v\rangle = 0$$

We finally report a fundamental relation that we'll often use in next section.

Proposition 20.
For each $x, y, z \in \mathbb{O}$, whenever two of them coincide

$$x(yz) = (xy)z$$

Split Octonions are the (only) eight dimensional split division algebra. They are defined by the same set of octonions i.e. the set

$$\mathbb{O} := \{w_1 + lw_2 \mid w_i \in \mathbb{H}\}$$

but in this case the product is defined by the rule

$$(w_1 + lw_2)(q_1 + lq_2) := (w_1 q_1 + q_2 \overline{w_2}) + l(\overline{w_1} q_2 + q_1 w_2)$$

We denote the *split octonions* (the octonions with such product) with $\mathbb{O}_s$. Let now $x = w_1 + lw_2$ be in $\mathbb{O}_s$. The quadratic form is defined by

$$N_s(x) := |w_1|^2 - |w_2|^2$$

Then if $x = w_1 + lw_2 \in \mathbb{O}_s$, with some computations easy follows that the conjugate of an element is given by

$$\overline{x} := \overline{w_1} - lw_2$$

and then we effective find that $N(x) := \overline{x}x$. Moreover the set of the zero divisors of the algebra $(\mathbb{O}_s, N_s)$ define the cone $K := N_s^{-1}(0)$.

1.4.2 G_2 as Automorphism Group

Let C be an algebra. The set $\mathrm{Aut}(C)$ of the *automorphisms* of C is the set of the linear isomorphisms $T : C \to C$ that preserve the algebra product i.e. satisfying

$$T(xy) = T(x)T(y)$$

The automorphisms of an algebra clearly form a group and in particular for quaternions, octonions and split octonions form a Lie group. Consider as example the division algebra of quaternions. The condition for a linear operator to be automorpism means that $T(1) = T(1)^2$, then must be $T(1) = 1$ while

$$T(i)T(j) = T(k) \qquad T(j)T(k) = T(i) \qquad T(k)T(i) = T(j)$$

These relations imply that the automorphisms must preserve the orientation of $\mathrm{Im}(\mathbb{H})$. Taking the norm of these relations and solving the algebric system associated, it follows that must be

$$|T(i)| = |T(j)| = |T(k)| = 1$$

Hence, since the image of 1 is fixed to be 1 for each $T \in \mathrm{Aut}(\mathbb{H})$, we conclude that $\mathrm{Aut}(\mathbb{H}) \cong SO(3)$. The goal of this section is to establish the automorphism group of octonions. Analogous considerations could be done to this end.

Theorem 24.
Let $T \in Aut(\mathbb{O})$ be an automorphism. Then $T : Im(\mathbb{O}) \to Im(\mathbb{O})$ and conseguently $\forall x \in \mathbb{O}$

$$T(\overline{x}) = \overline{T(x)} \ , \qquad |T(x)| = |x|$$

so $T \in SO(7)$ and then $Aut(\mathbb{O}) \subset SO(7)$.

Proof.
a) Similarly for quaternions

$$T(1) = T(1)T(1)$$

then must be $T(1) = 1$, so T finally acts only on $\mathrm{Im}(\mathbb{O})$. Remark that given $x \in \mathbb{O}$, it's square x^2 is real if and only if either x is real or $x \in \mathrm{Im}(\mathbb{O})$. Now, given $u \in \mathrm{Im}(\mathbb{O})$

$$T(u)^2 = T(u^2) = -|u|^2 T(1) = -|u|^2$$

is real, so either $T(u) \in \mathrm{Im}(\mathbb{O})$ or $T(u) = a$ is real. Suppose $T(u) \in \mathrm{Im}(\mathbb{O})$. We have $T(a^{-1}u) = 1$, so $a^{-1}u = 1$ that is $u = a$, contradicting the hypothesis that $u \in \mathrm{Im}(\mathbb{O})$. Then $T(u)$ have to be real, hence $T(\overline{x}) = \overline{T(x)}$.
b) For each $x \in \mathbb{O}$ we have

$$|T(x)|^2 = T(x)\overline{T(x)} = T(x)T(\overline{x}) = T(x\overline{x}) = |x|^2$$

then the thesis follows.

□

Anyway, the automorphism group is not the whole $SO(7)$. Recall that the dimension of $SO(n)$ is $\frac{n(n-1)}{2}$, so the dimension of $SO(7)$ is 21. Since any automorphism of $\mathbb{O}$ preserve the norm, must map i somewhere on

$$S^6 \subset \mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$$

then there are six degrees of freedom in where i can go. Next, j can be map somewhere on the 6-sphere but orthogonal to i, corresponding to an additional 5 degrees of freedom. Fixed the values of these two elements, the value of k is defined since $T(k) = T(i)T(j)$. Finally l is mapped on the 6-sphere so that it is orthogonal to each $i, j, k,$, corresponding to $6 - 3 = 3$ additional degrees of freedom. From the same argument of k, also the values of the remain basis il, jl, kl are defined, then finally there are $6 + 5 + 3 = 14$ choices we can make and then

$$\dim\big(\mathrm{Aut}(\mathbb{O})\big) = 14$$

As we'll see, the automorphism group is one of the real form of the G_2 group. To prove that, we need first to investigate some particular subalgebras of $\mathbb{O}$.

Proposition 21.
Consider $u_1 \in Im(\mathbb{O})$ and $u_2 \in Im(\mathbb{O})$ orthogonal to u_1, let $u_3 := u_1 \wedge u_2$ be their wedge product. Then

$$W := span\langle 1, u_2, u_2, u_3\rangle$$

is a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

Proof.
If $u_1 \in Im(\mathbb{O})$ is such that $|u_1| = 1$ then $u_1^2 = -1$ and we have the subalgebra of $\mathbb{O}$

$$\mathrm{Span}\langle 1, u_1\rangle \cong \mathbb{C}$$

Consider now $u_2 \in \mathrm{Im}(\mathbb{O})$ such that $|u_2| = 1$ and $\langle u_1, u_2\rangle = 0$ and set $u_3 = u_1 \wedge u_2$. Since u_1 and u_2 are orthogonal, $u_3 = u_1u_2$ and also $|u_3| = |u_1||u_2| = 1$ since

$$u_2^2 = -1 \qquad u_2u_1 = -u_1u_2 = -u_3$$

Note that

$$\mathrm{Re}(u_3) = \mathrm{Re}(u_1u_2) = -\langle u_1, u_2\rangle = 0$$

and so $1 = u_3\overline{u}_3 = -u_3^2$. Furthermore, since $x(yz) = (xy)z$ whenever any two of $x, y, z \in \mathbb{O}$ coincide, we get

$$\begin{aligned} u_1u_3 &= u_1(u_1u_2) = (u_1u_1)u_2 = -u_2 \\ u_3u_2 &= (u_1u_2)u_2 = u_1(u_2u_2) = -u_1 \end{aligned}$$

Hence, by Proposition

$$\langle u_3, u_1\rangle = \langle u_3, u_2\rangle = 0$$

and we have

$$u_3u_1 = -u_1u_3 \qquad u_2u_3 = -u_3u_2$$

then each of such choice of u_1 and u_2 defines a subalgebra of $\mathbb{O}$,

$$\mathrm{Span}\langle 1, u_1, u_2, u_3\rangle \cong \mathbb{H}$$

$\square$

Proposition 22.
Given any two elements $x, y \in \mathbb{O}$, the algebra generated by $1, x$ and y is isomorphic to either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. In particular, it is associative.

Proof.
Consider $V = \mathrm{Span}\langle 1, x_1, x_2\rangle$. If $\dim(V) = 1$, then $A \cong \mathbb{R}$. If $\dim(V) = 2$, previous arguments gives $A \cong \mathbb{C}$. If $\dim(V) = 3$ then $\mathrm{Im}(x_1)$ and $\mathrm{Im}(x_2)$ are linearly independent. We can pick orthonormal elements u_1 and u_2 in their span; then if A is the algebra generated by $1, u_1$, and u_2, the previous analysis gives $A \cong \mathbb{H}$.

$\square$

We now investigate how the subalgebra $A = \mathrm{Span}\langle 1, u_1, u_2, u_3\rangle$ is related with its orthogonal complement $A^\perp$. Consider $v_0 \in A^\perp, |v_0| = 1$. Note that $v_0 \in \mathrm{Im}(\mathbb{O})$. We set

$$v_l = u_lv_0, \qquad 1 \leq l \leq 3$$

Then $\mathrm{Re}(v_l) = -\langle u_l, v_0\rangle = 0$, so $v_l \in \mathrm{Im}(\mathbb{O})$. Defining for $x \in \mathbb{O}$ the $\mathbb{R}$-linear maps

$$L_x, R_x : \mathbb{O} \to \mathbb{O}$$

$$L_x y = xy, \qquad R_x y = yx$$

we can prove the following

Proposition 23.
The set $\{1, u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$ defined as above is an orthonormal basis of $\mathbb{O}$.

Proof.
i) We first prove that the set $\{v_0, v_1, v_2, v_3\}$ is an orthonormal set in $\mathbb{O}$.
Remark that if $|x| = 1$ for each $y \in \mathbb{O}$ we have

$$|L_x y| = |R_x y| = |y|$$

Hence L_x and R_x are orthogonal transformations. Since the unit sphere in $\mathbb{O}$ is connected, $\det(L_x) = 1$ and $\det(R_x) = 1$ for such x, so $|x| = 1$ then $L_x, R_x \in SO(\mathbb{O})$. Since

$$v_0 = R_{v_0}1, \qquad v_l = R_{v_0}u_l \quad 1 \le l \le 3$$

also $R_{v_0} \in SO(\mathbb{O})$ and we have the thesis.

ii) We now prove that $v_l \perp u_m$ for $1 \le l \le 3$ and $1 \le m \le 3$.
Since $L_{u_l} \in SO(\mathbb{O})$

$$\langle v_l, u_m\rangle = \langle u_l v_0, u_m\rangle = \langle u_l(u_l v_0), u_l u_m\rangle$$

$$= \langle (u_l u_l)v_0, u_l u_m\rangle = -\langle v_0, u_l u_m\rangle = 0$$

where the third identity follows since $x(yz) = (xy)z$ whenever any two of $x, y, z \in \mathbb{O}$ coincide. Then it follows also that

$$W^\perp = \mathrm{Span}\,\langle v_0, v_1, v_2, v_3\rangle$$

and consequently $\{1, u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$ is an orthonormal basis of $\mathbb{O}$.

$\square$

The results above imply that

$$R_v : W \to W^\perp$$

and then $W \cong W^\perp$ for each $v \in W^\perp$. Consequently if $x \in W$ and $y \in W^\perp$ then $xy \in W^\perp$. Moreover if $x \in \mathrm{Im}(\mathbb{O})$ then $xy = -yx$, thus we deduce that if $x \in W, y \in W^\perp$ also $yx \in W^\perp$. Furthermore, since if $|x| = 1$ then $L_x, R_x \in SO(\mathbb{O})$ and we have that if $x \in W^\perp$

$$L_x, R_x : W^\perp \to W$$

hence

$$x, y \in W^\perp \Rightarrow xy \in W$$

We have an important result about the correspondence between the standard basis and the basis

$$\{1, u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$$

of $\mathbb{O}$. Indeed, with some computation it is possible to show that the basis

$$\{u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$$

has the same multiplication table as $\{e_1, e_2, e_3, f_0, f_1, f_2, f_3\}$; this is proved the following.

Theorem 25.
Let $u_l, v_l \in Im(\mathbb{O})$ as above. Then the orthogonal transformation $T : \mathbb{O} \to \mathbb{O}$ defined by

$$T(1) = 1 \qquad T(e_l) = u_l \qquad T(f_l) = v_l$$

is an automorphism of $\mathbb{O}$.

Let $u, v, w \in \mathrm{Im}(\mathbb{O})$. If $\{u, v, w, uv\} \subset \mathrm{Im}(\mathbb{O})$ is an orthonormal set, we call *Cayley triangle.* Notice that since $\langle u, v\rangle = 0$ we have

$$uv = u \wedge v$$

We report some properties of these triangles.

Proposition 24.
Assume $\{u, v, w\}$ is a Cayley triangle. Then

$$v(uw) = -(vu)w \qquad \text{and} \qquad \langle uv, uw\rangle = 0$$

so $\{u, v, uw\}$ is a Cayley triangle, and

$$(uv)(wu) = vw$$

Proof.
i) We first show that $v(uw) = -(vu)w$:
The hypotheses imply

$$vu = -uv, \quad vw = -wv, \quad uw = -wu, \quad (vu)w = -w(vu)$$

so

$$\begin{aligned} v(uw) + (vu)w &= -\, v(wu) - w(vu) \\ &= (v^2 + w^2)u - (v + w)(vu + wu) \\ &= (v + w)2u - (v + w)((v + w)u) \\ &= 0 \end{aligned}$$

ii) We now show that $\langle uv, uw\rangle = 0$:
Since $L_u \in SO(\mathbb{O})$ we have

$$\langle uv, uw\rangle = \langle L_u v, L_u w\rangle = \langle u, w\rangle = 0$$

Thus $\{u, v, uw\}$ is a Cayley triangle. Applying *i)* to this Cayley triangle

$$\begin{aligned} (vu)(uw) &= -v(u(uw)) = \\ &= -v(u^2 w) \\ &= vw \end{aligned}$$

yielding the thesis.

□

Next Theorem shows how these triangles are related to the automorphism group of octonions and allows to determine some topological properties of this group.

Theorem 26.
The formulas
$$u_1 = Te_1, \qquad u_2 = Te_2, \qquad v_0 = Tf_0$$
provide a one to ne corrispondence between Aut($\mathbb{O}$) and the set of ordered orthonormal triples $(u_1, u_2, v_0) \in$ Im($\mathbb{O}$) such that v_0 is also orthogonal to $u_1 \wedge u_2$, that is set of Cayley triangles in Im($\mathbb{O}$).

Proof.
Consider $T \in \mathrm{Aut}(\mathbb{O})$ and define u_1, u_2, and v_0 by
$$u_1 = Te_1 \qquad u_2 = Te_2 \qquad v_0 = Tf_0$$
By the first Theorem of this section
$$T(\overline{x}) = \overline{T(x)}\ , \qquad |T(x)| = |x|$$
then these are orthonormal elements of Im($\mathbb{O}$). Moreover $W = T(H)$, spanned by $1, u_1, u_2$ and $u_1u_2 = u_1 \wedge u_2$, is a subalgebra of $\mathbb{O}$ and $v_0 \in W^{\perp}$. These observations, together with previous Theorem, yield the thesis.

□

Corollary 7.
The automorphism group of octonions Aut($\mathbb{O}$) is compact and connected.

Let $\mathcal{L}(\mathbb{O})$ be the set of linear maps from $\mathbb{O}$ to $\mathbb{O}$ and consider $A \in \mathcal{L}(\mathbb{O})$. We can define a one-parameter linear map
$$T(t) = e^{tA} \quad , \quad A \in \mathcal{L}(\mathbb{O})$$
where e^{tA} is the matrix exponential. This map is an automorphisms if satisfy
$$T(t)(xy) = (T(t)x)(T(t)y) \qquad x, y \in \mathbb{O}\ , t \in \mathbb{R}$$
If we differentiate this identity, we obtain the condition for A
$$A(xy) = (Ax)y + x(Ay) \qquad x, y \in \mathbb{O}$$
Then we define the set of *derivations* of $\mathbb{O}$
$$Der(\mathbb{O}) = \left\{ A \in \mathcal{L}(\mathbb{O}) \mid A(xy) = (Ax)y + x(Ay) \quad \forall x, y \in \mathbb{O} \right\}$$
The following Proposition explains the algebric structure of the derivations and their relations with the automorphisms group of octonions.

Theorem 27.
The set of derivations Der($\mathbb{O}$) has a structure of Lie algebra with the matrix commutator. Moreover, given $A \in \mathcal{L}(\mathbb{O})$ its exponential e^{tA} is an automorphism of $\mathbb{O}$ for all $t \in \mathbb{R}$ if and only if $A \in$ Der($\mathbb{O}$).

Proof.
i) From the defining property is clear that $\mathrm{Der}(\mathbb{O})$ is a linear space. Consider now $A, B \in \mathrm{Der}(\mathbb{O})$. For all $x, y \in \mathbb{O}$ we have

$$\begin{aligned} AB(xy) &= A((Bx)y) + A(x(By)) \\ &= (ABx)y + (Bx)(Ay) + (Ax)(By) + x(ABy) \end{aligned}$$

and similarly

$$BA(xy) = (BAx)y + (Ax)(By) + (Bx)(Ay) + x(BAy)$$

Hence

$$[A, B](xy) = ([A, B]x)y + x([A, B]y)$$

ii) The first implication follows from the definition of the algebra. For the converse, suppose A as in the statement, $x, y \in \mathbb{O}$ and set

$$X(t) = (e^{tA}x)(e^{tA}y)$$

Differentiating it we obtain (using the definition of elements in $\mathrm{Der}(\mathbb{O})$)

$$\begin{aligned} \frac{dX(t)}{dt} &= (Ae^{tA}x)(e^{tA}y) + (e^{tA}x)(Ae^{tA}y) \\ &= A\big((e^{tA}x)(e^{tA}y)\big) = AX(t) \end{aligned}$$

Since $X(0) = xy$ it follows by the standard uniqueness argument of ODE that

$$X(t) = e^{tA}(xy)$$

hence the thesis.

$\square$

Moreover it is possible to prove the following.

Theorem 28.
If $A_j \in Der(\mathbb{O})$ are mutually commuting, for $j = 1, \ldots, d$ and if $\{A_j\}_j$ are linearly independent in $\mathcal{L}(\mathbb{O})$, then $d \leq 2$.

It follows that $\mathrm{Aut}(\mathbb{O})$ has rank 2.

Proposition 25.
Let G be a compact Lie group of rank 2. If its Lie algebra $\mathfrak{g}$ has a non-trivial ideal $\mathfrak{h}$, then $dim(G) \leq 6$.

Proof.
Give $\mathfrak{g}$ an ad-invariant inner product. If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, then $\mathrm{ad}(\mathfrak{g})$ preserves both $\mathfrak{h}$ and $\mathfrak{h}^\perp$, so $\mathfrak{h}^\perp$ is also an ideal and each $X \in \mathfrak{h}$ commutes with each $Y \in \mathfrak{h}^\perp$. Now if $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are both nonzero,

$$\mathrm{Rank}(\mathfrak{g}) = 2 \quad \Rightarrow \quad \mathrm{Rank}(\mathfrak{h}) = \mathrm{Rank}(\mathfrak{h}^\perp) = 1$$

But, as well known,

$$\mathrm{Rank}(\mathfrak{h}) = 1 \quad \Rightarrow \quad \dim(\mathfrak{h}) = 1 \;\text{ or }\; 3$$

so we conclude that $\dim(G) \leq 6$.

$\square$

Corollary 8.
The Lie group Aut($\mathbb{O}$) *is a simple Lie group.*

Proof.
We have proved that Aut($\mathbb{O}$) has rank 2. Since also dim(Aut($\mathbb{O}$)) = 14 from the previous Proposition it follows that it has no non-trivial ideals i.e. have to be simple.

$\square$

We finally establish the fundamental group of Aut($\mathbb{O}$).

Proposition 26.
The group Aut($\mathbb{O}$) *has trivial center, that is, is simply connected.*

Proof.
Consider T_0 in the center of Aut($\mathbb{O}$). Then T_0 is defined by a one-parameter subgroup e^{tA} for some $A \in \mathrm{Der}(\mathbb{O})$. In general it is possible to prove (is a linear algebra computation) that if $A \in \mathrm{Der}(\mathbb{O})$ then

$$\ker(A) \cap \mathrm{Im}(\mathbb{O}) \neq 0$$

Then since

$$e^{tA} = I + tA + \frac{t^2A^2}{2} + \dots$$

there exists $u \in S^7 \subset \mathrm{Im}(\mathbb{O})$ fixed under the action of e^{tA}, hence fixed by T_0. Since T_0 is an element of the center, for each $T \in \mathrm{Aut}(\mathbb{O})$ we have that

$$TT_0T^{-1} = T_0$$

Then T_0 also fixes Tu. Since Aut($\mathbb{O}$) acts transitively on the unit sphere $S^7 \subset \mathrm{Im}(\mathbb{O})$, T_0 must fix each point of Im($\mathbb{O}$), so have to be $T_0 = Id$. $\square$

We can summarize all the results of this section in the following Theorem, based on the Lie groups' classification results.

Theorem 29.
The automorphism group of octonions is isomorphic to the compact real form of the simple simply connected Lie groups G_2.

Instead, it is possible to prove that the automorphism group of split octonions is isomorphic to the non compact real form of the connected Lie groups G_2. In this case, its fundamental groups is not trivial, but equal to $\mathbb{Z}_2$; anyway, we can always consider its universal cover; we denote it with G_2^s. The automorphism group preserve the quadratic form N_s and in particular it's cone K. Consider the action of $\mathbb{R}_+$, $\mathbb{R}_+ \times \mathrm{Im}(\mathbb{H}) \times \mathbb{H} \to \mathbb{H}$

$$\lambda \times w \to \lambda w$$

We call *spherization* of the cone K the quotient $\mathbf{K} := K/\mathbb{R}_+$. Notice that it's simply $\mathbf{K} \cong S^2 \times S^3$; the automorphism group acts then transitively also in this quotient. If instead we consider the Cartan subgroup H_c associated, it is possible to define a linear action of it on the split octonions. In particular this action can be identified with the action of the plane in $\mathbb{R}^3$ defined by

$$\left\{(t_1, t_2, t_3) \in \mathbb{R}^3 \;\; | \; t_1 + t_2 + t_3 = 0\right\}$$

in the following way. Consider the matrices blocks

$$\mathrm{ch}(t) := \begin{bmatrix} \cosh(t_1) & 0 & 0 \\ 0 & \cosh(t_2) & 0 \\ 0 & 0 & \cosh(t_3) \end{bmatrix}$$

The action is defined by the matrix product on $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ of

$$A_t = \begin{bmatrix} \mathrm{ch}(t) & 0 & \mathrm{ch}(t) \\ 0 & 1 & 0 \\ \mathrm{ch}(t) & 0 & \mathrm{ch}(t) \end{bmatrix}$$

In the next chapter we'll see how the group G_2 is related to the symmetries of a Control Theory problem, that is the Rolling Balls model, one of the first physical application of this group of transformations.

Chapter 2

Sub-Riemannian Geometry

Sub-Riemannian Geometry represent the modern formulation of Optimal Control Theory and it be thought as a generalization of Riemannian Geometry. In this setting rolling bodies problem is one of the early examples of study of such theory and its key features. In the first part of the Chapter we summarize the fundamental mathematical objects used in the theory and show the principal proprieties of such constructions and their relations with Hamiltonians systems, in order to introduce a particular class of solution for sub Riemannian problems, the so-called singular solutions. Finally, we give a new a geometrical description of the spaces of such curves for rolling spheres problem and we investigate their principal topological properties.

2.1 Sub-Riemannian Geometry

For a dynamical system defined on a configuration space M each solution is determined for the initial state q_0 by the evolution law induced by the flow Φ_t that is the system differential equations

$$\dot{\gamma}(t) = X(\gamma(t))$$

To control the dynamic we can more generally consider a family of dynamical systems

$$\dot{\gamma} = X_u(\gamma(t))$$

where $u \in U$ is a generic parameter in the set of parameters U. A such system is called *control system*, while the variable u *control parameter*. Consider the problem to drive a car from a point to another; it can be formulated as a control problem in which the solution, if exists, is a curve that join two points (the initial and final configuration of the car) in the states space of the problem. Generally for such problems becomes interesting require that the curve of configurations minimizes also a cost, represented by a functional on the possible controls

$$J(u) = \int_0^{t_1} \varphi(q_u(t), u(t))dt)$$

where $\varphi : M \times U \to \mathbb{R}$. More formally, consider a distribution of subspaces of the tangent space of a manifold and the ODE problem associated to the control condition. Suppose that X_u is a smooth vector field on M for any fixed u, $X_u(q)$ is continuous as map in both the variables and the same it's derivate respect to q. If the controls $u \subset \mathbb{R}^m$ are measurable locally bounded maps with values in the space of the parameters i.e. maps $t \to u(t) \in U$, by the classical Theorem of ODE for any $q \in M$ the Cauchy problem

$$\dot{\gamma} = X_u(q)$$

has a unique solution. We can then formally define the *optimal control problem* as the problem to minimize the functional J among all admissible controls $u = u(t)$ for which the corresponding solution $q_u(t)$ of the Cauchy problem satisfies the boundary condition $q_u(t_1) = q_1$ i.e the problem

$$\begin{cases} \dot{q}(t) = f_u(q(t)) \\ \gamma(0) = q, \quad \gamma(t_1) = p \\ J(u) \to min \end{cases}$$

Definition 13.
Let M be a smooth manifold and $\mathcal{F} \subset \mathrm{Vect}(M)$ a family of (smooth) vector fields. We define the *Lie Algebra generated by the family* $\mathcal{F}$

$$\mathrm{Lie}(\mathcal{F}) := \mathrm{span}\{[X_1, \ldots, [X_{j-1}, X_j]\ X_i \in \mathcal{F}\}$$

i.e. the smallest sub algebra of $\mathrm{Vect}(M)$ containing $\mathcal{F}$. We say that $\mathcal{F}$ is *bracket generating* if

$$T_pM = \mathrm{Lie}_p(\mathcal{F}) = \{X_p, X \in \mathrm{Lie}(\mathcal{F})\}$$

We then set

$$\mathrm{Lie}^k(\mathcal{F}) := \mathrm{span}\{[X_1, \ldots, [X_{i-1}, X_i]],\ X_i \in \mathcal{F},\ i \leq k\}$$

And we say that the family has step k in $p \in M$ if $\mathrm{Lie}^k_p(\mathcal{F}) = T_pM$ (in general the step should depend of the point).

Definition 14.
Let M be a smooth manifold. A *sub-Riemannian structure* for M is a pair (W, L) where:
i) $W \to M$ is an euclidean bundle on M.
ii) The map $L : W \to TM$ is a linear-morphism[1] of vector bundles i.e the following diagram commutes

$$\begin{tikzcd} W \arrow[r, "L"] \arrow[dr, "\pi_W"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

iii) The set of horizontal vector fields

$$D := \big\{Lv \mid v : M \to W \text{ smooth section}\big\}$$

is a bracket-generating family of vectors fields for M.

[1]The map is linear between the fibers.

We call *sub-Riemaniann manifold* a smooth manifold M (suppose always connected) with a sub-Riemannian structure (W, L). [2] In particular, if the bundle W is trivial, we say the structure *free.* A sub-Riemannian manifold has associated the *distribution* D on M defined at each point by

$$D_p := L(W_p) \subset T_pM$$

Notice that if m is the rank of W as vector bundle and n is the dimension of M then for each $p \in M$ the rank r of the distribution D is $r \leq \min\{m, n\}$ (the map L in a fixed point could not be injective) but if $v_1, \ldots, v_m \in W_p$ form an orthonormal base of W_p and L is injective then its image by L is an orthonormal basis of D_p. If (p, v) is an element of W_p we denote

$$L_pv := L(p, v)$$

that is a vector in T_pM. Moreover the set of the horizontal vector fields D as subset of the module $\chi(M)$ (on the ring $C^\infty(M)$) has a structure of finite generated sub-module. Before to give the definition of control, recall that a function $f : M \to \mathbb{R}$ is *essentially bounded* [3] if it is measurable and has a bounded rappresentative in $L^\infty(M)$, that is it's measurable and differs from a bounded function only on a measure zero set.

Definition 15.
We define a Lipschitz curve $\gamma : [0, T] \to M$ to be *admissible* or *horizontal* for a sub-Riemannian structure (W, L) if there exists [4] an essentially bounded *control* function

$$u : [0, T] \to W_{\gamma(t)}$$

such that

$$\dot{\gamma}(t) = L_{\gamma(t)}u(t)$$

for each time $t \in [0, T]$.

We say that u is *a control of* γ and denote the set of admissible path by

$$\Omega([a, b]) = \big\{\gamma : [a, b] \to M \ admissible,\ Lipschitz\big\}$$

and to simplify the notation $\Omega := \Omega([0, 1])$.

Notice that in general given (W, M) and (W', M) could be $D_q = D'_q$ but $D \neq D'$; this means that also if the vector space spanned at each point by the distribution are the same, the admissible curves could be different. Then the property to have velocity $\dot{\gamma} \in D$ is not enough for a curve to be admissible but admissible condition also depends on the map L that describes the identification between W and the subspace. If we require the solutions to be Lipschitz then

[2] Remark that if the map L is surjective, we get an Riemannian Manifold.

[3] An example is given by the functions

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

The function f is essentially bounded and g is its rappresentative in $L^\infty(M)$,

[4] Notice that it is possible to find different controls for the same trajectory, then it is usually not unique.

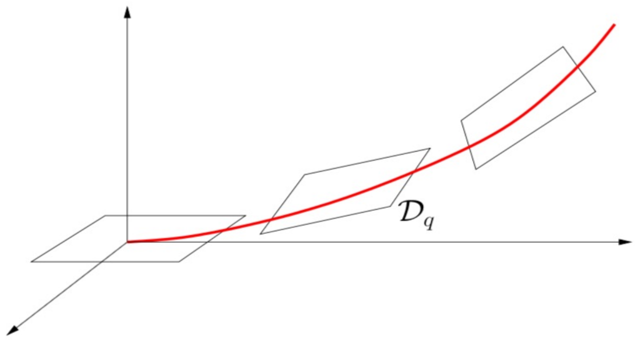


Figure 2.1: A horizontal trajectory for a distribution.

they are rectifiable and almost everywhere differentiable curves. Let $p \in M$ be a point and $U \times \mathbb{R}^m$ a local trivialization of W where U is an neighborhood of p. Since L is linear on the fiber, denoted by $\{\nu_i\}_i$ the image of a basis of W by L, for $(p, v) \in U \times \mathbb{R}^m$ it takes the form

$$L_p v = \sum_i^m v^i \nu_i(p)$$

then a Lipschitz curve γ is admissible if there exists a function $u = (u_1, \dots u_m) \in L^\infty([0,T], \mathbb{R}^m)$ such that

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) \nu_i(\gamma(t))$$

and then by the Theorem of existence and unicity of solutions, for each condition $p \in M$ and $u \in L^\infty([0,T], \mathbb{R}^m)$ in a neighborhood in U there exists an admissible curve γ such that u is the control associated with γ and $\gamma(0) = p$.

Definition 16.
We define the *sub-Riemannian norm* of $v \in D_p$ as the norm of the minimum (respect to euclidean one) control u that makes a curve with velocity $L_p v$ admissible in p i.e.

$$\|v\|_s := \min_{v = L_p u} \|u\|$$

Since L_p is linear with respect to u, the minimum is always a unique point and it's then well defined. It is easy to prove that it's effective a norme; moreover since an admissible curve γ is differentiable at almost every point, it is well define almost everywhere in $[0,T]$ the unique control function $t \to u^*(t)$ associated to γ realizing the minimum. Denote for a generic $g : \mathbb{R}^n \to \mathbb{R}$ the set of its minimum points

$$\arg\min f := \{x_m \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n \; f(x_m) \leq f(y)\}$$

We can give the definition of minimal control.

Definition 17.
Let γ be an admissible curve and consider the set (depending on time)

$$C_{\gamma(t)} = \{u : [0,T] \to W_p \mid \dot{\gamma}(t) = (Lu)_{\gamma(t)}\}$$

We define in every differentiability point of an admissible curve $\gamma \in \Omega([0,T])$ the *minimal control* u^* associated by

$$u^*(t) := \arg\min_{u \in C_{\gamma(t)}} |u|$$

The minimal control u^* is then pointwise defined for $t \in [0,T]$ as the minimal vector (in norme) that has image equal to the velocity of γ in the tangent space of the image of the point by L. The natural question is then which is its regularity. In general, the minimal control may not be continuous; if we consider as example the free sub-Riemannian structure in $\mathbb{R}^2 \times \mathbb{R}^2$

$$L(x, y, v_1, v_2) = (x, y, u_1, u_2 x)$$

and the admissible curve $\gamma : [-1,1] \to \mathbb{R}^2$ defined by

$$\gamma(t) = (t, t^2)$$

its minimal control u^* satisfies the condition $(u_1^*, u_2^*) = (1,2)$ for $t \neq 0$ while $(u_1^*(0), u_2^*(0)) = (1,0)$ hence it can not be continuos. It is possible to prove that the minimal control u^* associated to an admissible curve $\gamma : [0,T] \to M$ is a measurable and essentially bounded map on $[0,T]$.

Definition 18.
Let (W, L) and (W', L') be two sub-Riemannian structures for a smooth manifold M; we call them:

i) *Equivalent as distributions* if there exists an Euclidean bundle V and two surjective vector bundle morphisms $f : V \to W$ and $f' : V \to W'$ such that the following diagram commutes

$$\begin{tikzcd} & W \arrow[dr, "L"] & \\ V \arrow[ur, "f"] \arrow[dr, "f'"'] & & TM \\ & W' \arrow[ur, "L'"'] & \end{tikzcd}$$

ii) *Equivalent as sub-Riemannian structures* if $i)$ is satisfied and the maps f, f' are compatible with the scalar product, i.e.

$$|w| = \min\{|v| \mid f(v) = w\} \qquad |w'| = \min\{|v| \mid f'(v) = w'\}$$

for all $w \in W, w' \in W'$

A surface M in $\mathbb{R}^3$ with the induced metric can not be generally regard as a sub- Riemannian structure of rank 2 but always as a free one of rank 3 considering $W = M \times \mathbb{R}^3$ and defining $L : W \to TM$ by

$$L_q u := \pi_q^\perp(u)$$

where $\pi_q^\perp : \mathbb{R}^3 \to T_pM$ is the orthogonal projection on the tangent space of the surface. In this case the generating family is the family of fields defined at each point q by

$$\left\{\pi_q^\perp \partial_x \ , \ \pi_q^\perp \partial_y \ , \ \pi_q^\perp \partial_z\right\}$$

This is a more general property of sub-Riemannian structures.

Proposition 27.
Every sub-Riemannian structure (W, L) on M is equivalent to a sub-Riemannian structure (W', L') where W' is a trivial bundle.

Moreover two sub-Riemannian structures are equivalent as distributions if and only if the two modules of horizontal vector fields D and D' coincide; that's the answers of the previous observation about equivalence of sub Riemannian structures and distributions associated. As consequence of the proposition we can assume that there exists a generating family of vector fields $\nu_1, \dots \nu_m,\ m \leq n$ globally defined on M such that every admissible curve γ for the sub- Riemannian structure can be represented as

$$\dot{\gamma}(t) = \sum_{i=1}^{m} u_i(t)\nu_i(\gamma(t))$$

2.1.1 Optimal Control

Similarly to Riemannian Geometry we can define a notion of length also for sub-Riemannian structures.

Definition 19.
Let $\gamma \in \Omega([0, T])$ be an admissible curve. We define the *sub- Riemannian length* of γ by

$$l_s(\gamma) := \int_0^T \|\dot{\gamma}(t)\|_s \, dt$$

where $\|\cdot\|_s$ is the sub-Riemannian norm.

Since for $v \in D_p$ the sub-Riemannian norm is defined by $\|v\|_s := \min_{v=L_p u} |u|$ and the minimal control u^* minimizes at each point the norm, the length is simply

$$l_s(\gamma) = \int_0^T \|u^*(t)\| \, dt$$

then for the Lemma the length of a curve is well defined (u^* is measurable) and each admissible curve has finite length.

Lemma 7.
The sub-Riemannian length l_s of an admissible curve $\gamma \in \Omega([0, T])$ is invariant by Lipschitz reparameterization. Moreover if a curve has length $l_s > 0$ then it's a Lipschitz reparameterization of an arc-length parametrized admissible one.

Proof.

i) Let $\gamma : [0, T] \to M$ be an admissible curve and $\varphi : [0, T'] \to [0, T]$ a Lipschitz reparemeterization i.e. a Lipschitz and monotone surjective map. Consider the reparametrized curve

$$\tilde{\gamma} : [0, T'] \to M$$
$$\tilde{\gamma} := \gamma \circ \varphi$$

Then $\tilde{\gamma}$ is composition of Lipschitz functions, hence Lipschitz. Moreover $\tilde{\gamma}$ is admissible since by the linearity of L has minimal control $(u^* \circ \varphi)\dot{\varphi} \in L^\infty$ where u^* is the minimal control of γ. Using the change of variable $t = \varphi(\tau)$ we get

$$l_s(\tilde{\gamma}) = \int_0^T \left\|\dot{\tilde{\gamma}}(\tau)\right\|_s d\tau = \int_0^T \|u^*(\varphi(\tau))\| \, |\dot{\varphi}(\tau)| d\tau = \int_0^T \|u^*(\tau)\| \, d\tau = \int_0^T \|\dot{\gamma}(\tau)\|_s \, d\tau = l_s(\gamma)$$

ii) Let $\gamma : [0,T] \to M$ be an admissible curve with $l_s(\gamma) > 0$ and minimal control u^*. Consider $\varphi : [0,T] \to [0, l_s(\gamma)]$

$$\varphi(t) := \int_0^t |u^*(\tau)| d\tau$$

For the integral properties it's Lipschitz and monotone. Then if $\varphi(t_2) = \varphi(t_2)$ from the monotonicity $\gamma(t_1) = \gamma(t_2)$. Hence we are allowed to define the curve $\alpha : [0, l(\gamma)] \to M$ by

$$\alpha(s) := \gamma(t) \quad \text{ if } s = \varphi(t) \text{ for some } t \in [0,T]$$

i.e. $\gamma = \alpha \circ \varphi$. We now show that α is Lipschitz. Fixed $K \subset M$ compact such that $\gamma([0,T]) \subset K$ and set

$$C := \max_{x \in K} \left(\sum_{i=1}^m |\nu_i(x)|^2 \right)^{\frac{1}{2}}$$

then

$$|\gamma(t_1) - \gamma(t_0)| \leq \int_{t_0}^{t_1} \sum_{i=1}^m |u_i^* \nu_i(\gamma(t))| \leq$$

$$\leq \int_{t_0}^{t_1} \sqrt{\sum_{i=1}^m |u_i^*(t)|^2} \sqrt{\sum_{i=1}^m |\nu_i(\gamma(t))|^2} dt \leq C \int_{t_0}^{t_1} |u^*(t)| dt$$

If $s_0 = \varphi(t_0)$ and $s_1 = \varphi(t_1)$

$$|\alpha(s_1) - \alpha(s_0)| = |\gamma(t_1) - \gamma(t_0)| \leq C \int_{t_0}^{t_1} |u^*(\tau)| d\tau = C|s_1 - s_0|$$

which proves that α is Lipschitz and in particular $\dot{\alpha}(s)$ exists for a.e. $s \in [0, l_s(\gamma)]$. If we now define for each s such that $s = \varphi(t)$, $\dot{\varphi}(t)$ exists and $\dot{\varphi} \neq 0$ the control

$$v(s) := \frac{u^*(t)}{\dot{\varphi}(t)} = \frac{u^*(t)}{|u^*(t)|}$$

then the control v is defined for a.e. $s \in [0, l_s(\gamma)]$. Moreover by construction $|v(s)| = 1$ for a.e. $s \in [0, l_s(\gamma)]$ hence v is the minimal control associated to α.

□

In particular it is possible to prove that the length of a curve admissible for two equivalent sub-Riemannian structures is the same, hence we can still reduce the problem to the free one. Since the structure is free, in case using the classical Gram-Schimdt procedure, we can assume that ν_i are image of an orthonormal frame on the fiber and then the length of the curve reduced to

$$l_s(\gamma) = \int_0^T \sqrt{\sum_{i=0}^m u_i^*(t)^2 dt}$$

where $u^*(t)$ is the minimal control. As said in the introduction to the chapter, a fundamental goal in Optimal Control Theory is to find the shortest admissible curve joining two points. If we set

$$\Omega_q^p([0,T]) = \big\{\gamma \in \Omega([0,T]) \mid \gamma(0) = q, \quad \gamma(T) = p\big\}$$

given an admissible curve $\gamma \in \Omega_q^p([0,T])$ we say that γ is *length-minimizer* if it minimizes the functional

$$l_s : \Omega_q^p([0,T]) \to \mathbb{R}^+$$

i.e. minimizes the length on the admissible curves in $\Omega_q^p([0,T])$. It is generally well known from Riemannian Geometry that such curves could not exist [5] and they existence is determined by the topology of the manifold M. To relate the two notions, in the same way as can be done for Riemannian manifolds, we introduce a structure of metric space on M.

Definition 20.
Let M be a sub-Riemannian manifold and $q, p \in M$, we define the sub-Riemannan distance

$$d_s(q,p) = \inf\{l(\gamma) \ |\gamma \in \Omega_q^p \ \}$$

The fundamental topological properties of this distance are summarized in the following Theorem, that we don't prove.

Theorem 30.
Let M be a sub-Riemannian Manifold. Then
i) (M, d_s) is a metric space .
ii)The topology induced by (M, d_s) is equivalent to the manifold topology.
iii)The map $d_s : M \times M \to \mathbb{R}$ is continuous.
iv) The metric space (M, d) is locally compact, i.e. for each point $p \in M$ there exists an $\varepsilon > 0$ such that $B_r(p)$ is compact, for each $r < \varepsilon$.

One of the main consequences of this result is that, thanks to the bracket-generating condition, for every $q_0, q_1 \in M$ there always exists an admissible curve that joining them, hence $d(q_0, q_1) < \infty$. We now summarize the ideas behind the proof of existence of minimizing functions. We can obtain the minimizing curve as (uniformly) limit point of a sequence of curves, which length approximates the distance between q_0 and q_1. To do that is first necessary to show that the functional l_s is semicontinuous and that the metric space is locally compact, hence next Theorem follows.

Theorem 31 (Existence of Minimizers)**.**
Let M be a sub-Riemannian manifold and $q_0 \in M$. There exists $\varepsilon > 0$ such that for every $q_1 \in B_\varepsilon(q_0)$ there exists a minimizing curve joining q_0 and q_1. In particular if M is also complete the minimizing curve exists for each pair of point $q_0, q_1 \in M$.

Let γ be an admissible curve. We define the *energy functional* E on the set of Lipschitz curves in M by

$$E(\gamma) = \frac{1}{2}\int_0^T \|\dot{\gamma}(t)\|_s^2 \, dt$$

[5] An example is given by the manifold $\mathbb{R}^2 \backslash \{0\}$ with the Euclidean norm.

Since the definition depends on the parametrization, if we don't fix the final time T, is possible to show that the infimum of E on the curves joing two fixed point is zero (if you imagine to walk on the curve slowly your kinetic energy is low, but in a longer time you still arrive to the end point).

Lemma 8.
An admissible curve $\gamma \in \Omega_q^p([0,T])$ for fixed time T is minimizing of the length on $\Omega_q^p([0,T])$ and has constant speed if and only if it is minimizing of J in $\Omega_q^p([0,T])$.

Proof.
By Cauchy -Schwartz inequality

$$l(\gamma)^2 = \left(\int_0^T \|\gamma(t)\|_s \cdot 1 dt \right)^2 \leq \int_0^T \|\gamma(t)\|_s^2 \, dt \int_0^T 1^2 dt = 2E(\gamma)T$$

and the equality holds if and only if $\|\gamma(t)\|_s = 1$ □

Summarizing what we have done, if $\gamma \in \Omega_q^p([0,T])$ is an admissible curve which is length minimizing, parametrized by constant speed and u^* its minimal control the following equations hold

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i^*(t)\nu_i(\gamma(t)) \qquad l(\gamma) = \int_0^T |u^*(t)| dt = d(\gamma(0), \gamma(T))$$

Consider now the flow $\Phi_t^{-1} : M \to M$ associated to the solution γ and it's pull back map $(\Phi_t)^* : T^*M \to T^*M$. We define *dual-flow* by

$$(\Phi_t^{-1})^* : \mathbb{R} \times T^*M \to T^*M$$

Fixed a covector $\lambda_0 \in T^*M$ is then defined a curve $\lambda : [0,T] \to T^*M$ with $\pi(\lambda(t)) = \gamma(t)$ usually called *extremal.*

		Curves
Manifold	$p \in M \xrightarrow[X]{\Phi_t} M$	$\gamma : [0,1] \to M$
Co-Bundle	$\lambda_0 \in T_p^*M \xrightarrow[X^*]{(\Phi_t^{-1})^*} T_{\Phi_t(p)}^*M$	$\lambda : [0,1] \to T^*M$

Definition 21.
Consider $\gamma \in \Omega([0,T])$ and $u^* \in L^\infty([0,T], \mathbb{R}^m)$ its minimal control. Let $\lambda(t)$ be an extremal curve associated to γ with initial co-velocity $\lambda_0 \in T_{\gamma(0)}^*M \backslash \{0\}$. We call $\lambda(t)$

i) *normal extremal* if it satisfies the condition

$$u_i^*(t) = \langle \lambda(t), \nu_i(\gamma(t)) \rangle \qquad \forall i = 1, \dots, m$$

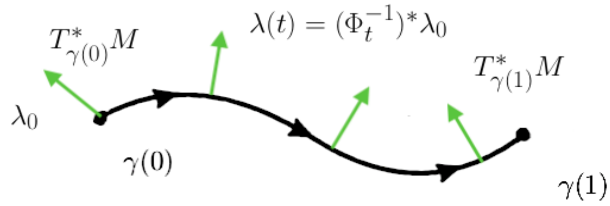


Figure 2.2: The black curve represent the flow Φ_t, the green covectors the dual flow in the cotangent bundle.

ii) *abnormal extremal* if it satisfies the condition

$$\langle \lambda(t), \nu_i(\gamma(t)) \rangle = 0 \qquad \forall i = 1, \dots, m$$

Moreover, if an admissible curve $\gamma : [0,1] \to M$ has a normal or abnormal extremal we call it *normal trajectory* and *abnormal trajectory* respectively. Notice that though the two conditions for the extremals are exclusive, the definitions for the trajectory are not. Indeed the same trajectory $\gamma : [0,1] \to M$ could has both a normal or abnormal extremal defined for a different choice of the initial covector λ_0. The next fundamental Theorem shows how each admissible curve has at least an extremal of one of these types.

Theorem 32 (Pontryagin Extremals)**.**
Let $\gamma \in \Omega^p_q([0,T])$ be an admissible curve which is length minimizing, parametrized by constant speed. Let u^* be its minimal control where $|u^*(t)|$ is constant a.e. on $[0,T]$ and denote with Φ_t the flow of the nonautonomous vector field

$$Lu^* = \sum_{i=1}^{k} u_i^*(t)\nu_i$$

Then there exists a covector $\lambda_0 \in T^*_{\gamma(0)}M$ such that the co-curve $\lambda(t) \in T^*M_{\gamma(t)}$

$$\lambda(t) := (\Phi_t^{-1})^*\lambda_0$$

is a normal or an abnormal extremal.

Proof.
By the Lemma we know that γ is minimizing of E on $\Omega^p_q([0,T])$. We define the functional on $L^\infty([0,T], \mathbb{R}^m)$

$$J(u) := \frac{1}{2}\int_0^T |u(t)|^2 dt$$

Then the minimal control u^* of γ minimize the functional J i.e.

$$J(u^*) \le J(u)$$

for all $u \in L^\infty([0,T],\mathbb{R}^m)$ associated to trajectories in $\Omega_q^p([0,T])$. Consider a variation $u = u^* + v$ of u^* and the associated trajectory $q(t)$ solution of

$$\dot{q}(t) = L_{q(t)}u(t) \quad q(0) = q_0$$

Φ_t the local flow associated to u^* and γ the optimal admissible curve. We introduce the curve

$$x(t) = \Phi_t^{-1}(q(t))$$

obtained applying the inverse of the flow of the optimal control to the solution associated with the new control u. Then the ODE satisfied by $x(t)$ is obtained by differentiating the equation $q(t) = \Phi_t(x(t))$

$$\dot{q}(t) = L_{q(t)}u^*(t) + (\Phi_t)_*(\dot{x}(t)) = L_{\Phi_t(x(t))}u^*(t) + (\Phi_t)_*(\dot{x}(t))$$

since $\dot{q}(t) = L_{q(t)}u(t) = L_{\Phi_t(x(t))}u(t)$ we get

$$(\Phi_t)_*(\dot{x}(t)) = L_{\Phi_t(x(t))}u(t) - L_{\Phi_t(x(t))}u^*(t)$$

and then inverting $(\Phi_t)_*$

$$\dot{x}(t) = (\Phi_t^{-1})_*\big(L_{\Phi_t(x(t))}(u(t) - u^*(t))\big) =$$

$$= (\Phi_t^{-1})_*\big(L_{x(t)}\big(u(t) - u^*(t)\big)\big) = (\Phi_t^{-1})_*\big(L_{x(t)}v(t)\big)$$

then defining the nonautonomous vector field $X_{v(t)} := \big(\Phi_t^{-1}\big)_* Lv(t)$ the Cauchy problem becomes

$$\dot{x}(t) = X_{v(t)}(t,(x(t)) \qquad x(0) = q_0$$

This vector field is linear respect to v. Fixed the control v, we denote by $x(t; u^* + sv)$ the solution associated to this problem starting from q_0 and $J(u^* + sv)$ the cost functional; we define the map $\mathbb{R} \to \mathbb{R} \times M$ for $s \in \mathbb{R}$

$$s \to \begin{pmatrix} J(u^* + sv) \\ x(T; u^* + sv) \end{pmatrix}$$

The Lemma (that follows the Theorem) implies that there exists $\lambda_0 \in T^*_{q_0}M$ such that for all $v \in L^\infty([0,T],\mathbb{R}^m)$

$$\langle \lambda_0, \left(\frac{\partial J(u^* + sv)}{\partial s}, \frac{\partial_x(T, u^* + sv)}{\partial s} \right) \rangle = 0$$

then for each $v \in L^\infty([0,T],\mathbb{R}^m)$ one of the following identities is satisfied

$$i) \quad \frac{\partial J(u^* + sv)}{\partial s}_{|s=0} = \langle \lambda_0, \frac{\partial_x(T, u^* + sv)}{\partial s}_{|s=0} \rangle$$

$$ii) \quad \langle \lambda_0, \frac{\partial_x(T, u^* + sv)}{\partial s}_{|s=0} \rangle = 0$$

Let's show that

$$i) \quad \frac{\partial J(u^* + sv)}{\partial s}_{|s=0} = \int_0^T \sum_{i=1}^m u_i^*(t)v_i(t)dt$$

$$ii) \quad \langle \lambda_0 \frac{\partial_x(T, u^* + sv)}{\partial s}_{|s=0} \rangle = \int_0^T V^t_{v(t)}(q_0)dt = \int_0^T \sum_{i=1}^m \big((\Phi_t^{-1})_*\nu_i\big)(q_0)v_i(t)dt$$

The first relation comes from the definition of J

$$J(u^* + sv) = \frac{1}{2}\int_0^T |u^* + sv|^2 dt$$

indeed

$$\frac{\partial J(u^* + sv)}{\partial s}_{|s=0} = \int_0^T \sum_{i=1}^m v_i(u_i^* + sv_i)dt_{\,|s=0} = \int_0^T \sum_{i=1}^m v_i u_i^* dt$$

Instead for the second one, considering the ODE in coordinates

$$\dot{x}(t) = X_{v(t)}^t(x(t)) \qquad x(0) = q_0$$

we find that

$$x(T; u^* + sv) = q_0 + s\int_0^T X_{v(t)}^t(x(t; u^* + sv))dt$$

since the vector field is defined by $X_{v(t)}^t := \left(\Phi_t^{-1}\right)_* Lv(t)$ deriving

$$\frac{\partial x(T; u^* + sv)}{\partial s}_{|s=0} = \int_0^T X_{v(t)}^t(x(t; u^*+sv))dt + s\frac{\partial}{\partial s}\int_0^T X_{v(t)}^t(x(t; u^*+sv))dt \quad _{|s=0} =$$

$$= \int_0^T X_{v(t)}^t(x(t; u^*))dt = \int_0^T \left(\Phi_t^{-1}\right)_* Lv(t)(x(t; u^*))dt$$

We now show that $i)$ is equivalent to normal condition; similarly $ii)$ is equivalent to abnormal condition. We have

$$\int_0^T \sum_{i=1}^m u_i^*(t)v_i(t)dt = \frac{\partial J(u^* + sv)}{\partial s}_{|s=0} = \langle \lambda_0, \frac{\partial_x(T, u^* + sv)}{\partial s}_{|s=0} \rangle =$$

$$= \langle \lambda_0, \int_0^T \sum_{i=1}^m \left((\Phi_t^{-1})_* \nu_i\right)(q_0)v_i(t)dt \rangle = \int_0^T \sum_{i=1}^m \langle \lambda(t), \nu_i(\gamma(t)\rangle\, v_i(t)dt$$

$\square$

In the previous Theorem we have used the following Lemma.

Lemma 9.
There exists $\lambda^* \in (\mathbb{R} \oplus T_qM)^*$ with $\lambda^* \neq 0$ such that for all $v \in L^\infty([0, T], \mathbb{R}^m)$

$$\langle \lambda^*, \left(\frac{\partial J(u^* + sv)}{\partial s}, \frac{\partial_x(T, u^* + sv)}{\partial s}\right)\rangle = 0$$

It is also possible to show that the minimal control associated to the normal extremals is smooth. We conclude this section remark that in the special case of Riemannian Geometry, since L is surjective on the fiber, we can find m vector fields $\nu_1, \ldots \nu_m$ on M such that

$$T_qM = \operatorname{span}\{\nu_1, \ldots \nu_m\}$$

then the abnormal condition implies that at each point

$$\langle \lambda(t), \nu_i(\gamma(t))\rangle = 0 \qquad \forall i = 1, \ldots, m$$

hence $\lambda \equiv 0$ a.e..

2.1.2 Hamiltonian Formulation

In Pontryagin Extremals Theorem has been proved that the dual flow

$$(\Phi_t^{-1})^* : T^*M \to T^*M$$

defines for each initial covector λ_0 a co-curve $\lambda(t) \in T^*_{\gamma(t)}M$. We now investigate the relations between this kind of co-curve and the natural geometric structure of the tange bundle, its symplectic structure; in particular we'll show that it is possible to define an Hamiltonian system on T^*M which solutions are the admissible curves and that allows to characterize normal and abnormal extremals. To do that we first need to analyze the relations between vector fields on M and functions on T^*M. In particular we now show how a vector field on T^*M is completely characterized by its action on functions that are affine on the fibers.

The pull-back of the projection $\pi : T^*M \to M$ induces a one-to-one correspondence from the set $C^\infty(M)$ of smooth functions on M with the set $C^\infty(T^*M)$ of functions that are constant on the fiber. In a similar way we can think about a vector field $\mathrm{Vect}(M) = \Gamma(TM)$ on M as a function on T^*M with values in $\mathbb{R}$

$$f_X : T^*M \to \mathbb{R}$$

$$f_X(p, \lambda_p) := \langle X_p, \lambda_p \rangle$$

In particular defining on the set $C^\infty_{\mathrm{lin}}(T^*M)$ of the functions in $C^\infty(T^*M)$ that are linear on the fibers a product for an element of $C^\infty(M)$ by

$$\alpha \cdot f_X := (\pi^*\alpha) f_X = f_{\alpha X}$$

it get a structure of module while the map

$$\mathrm{Vect}(M) \to C^\infty_{\mathrm{lin}}(T^*M)$$

$$X \to f_X$$

defines an isomorphism of modules. Denoted by $\alpha \in C^\infty_{cst}(T^*M)$ the functions in $C^\infty(T^*M)$ that are constant on the fiber, we then call a function $f \in C^\infty(T^*M)$ *affine on the fiber* if there exist two functions $\alpha \in C^\infty_{cst}(T^*M)$ and $f_X \in C^\infty_{\mathrm{lin}}(T^*M)$ such that

$$f(\lambda) = \alpha(q) + \langle \lambda, X(q) \rangle \qquad q = \pi(\lambda)$$

and we denote by $C^\infty_{\mathrm{aff}}(T^*M)$ the set of such functions. Consider $V \in \mathrm{Vect}(T^*M)$ and $f \in C^\infty TM$. Then $V(f) = \langle d_\lambda f, V \rangle$ depends on the differential of f at each point. Hence for each $\lambda \in T^*M$, to compute the vector field $V(f)$ one can replace the function f with any affine function with same differential in λ. The vector field associated to the flow

$$\left(\Phi_t^{-X}\right)^* : T^*M \to T^*M$$

is an autonomous vector field X^* on T^*M that by definition satisfies the relation

$$(\Phi_t^{-X})^* = \Phi_t^{X^*}$$

for all time t.

Proposition 28.
Let $X^* \in$ Vect(T^*M) be the dual vector field related to the dual flow associated to the vector field $X \in$ Vect(M) on M. Then for each function $f \in C^\infty_{Aff}(T^*M)$

$$X^*(\beta + f_X) = X\beta + f_{[X,Y]}$$

Proof.
For every $\lambda \in T^*M$ we have

$$\frac{d}{dt}\beta \circ \pi\big((\Phi_t^{-X})^*\lambda\big)|_{t=0} = \frac{d}{dt}\beta\big(\Phi_t^X(q)\big)|_{t=0} = (X\beta)(q)$$

If we now consider a function $f_Y(\lambda) = \langle \lambda, Y(q)\rangle$ linear on the fiber we have

$$\frac{d}{dt} f_Y\big((\Phi_t^{-X})^*\lambda\big)|_{t=0} = \frac{d}{dt}\left\langle (\Phi_t^{-X})^*\lambda, Y\big(\Phi_t^X(q)\big)\right\rangle|_{t=0} =$$

$$= \frac{d}{dt}\left\langle \lambda, \big((\Phi_t^{-X})_* Y\big)(q)\right\rangle|_{t=0} = \langle \lambda, [X,Y](q)\rangle = f_{[X,Y]}(\lambda)$$

thus by linearity on the affine functions

$$X^*(\beta + f_Y) = X\beta + f_{[X,Y]}$$

□

Let $f_X, f_Y \in C^\infty(T^*M)$ be the linear fiber functions associated to $X, Y \in$ Vect(M). Can be proved that the Poisson braket satisfies

$$\{f_X, f_Y\} := f_{[X,Y]}$$

where $f_{[X,Y]}$ is the function in $C^\infty_{lin}(T^*M)$ associated to the vector field $[X,Y]$. Finally recall that the Hamiltonian vector field associated to a function f is the vector field X_f defined on each $g \in C^\infty(M)$ by

$$X_f(g) := \{f, g\}$$

Let's see how we can translate the Pontryagin Extremals Theorem using the symplectic language.

Proposition 29.
Let $X \in$ Vect(M) be a complete vector field with flow Φ_t^X and consider the Hamiltonian

$$h_X(\lambda) := \langle \lambda, X_p\rangle$$

where $p = \pi(\lambda)$. The dual flow on T^*M defined by $(\Phi_t^{-1})^*$ is generated by the vector field X_{h_X} i.e.

$$X^* = X_{h_X}$$

Proof.
To prove that $X^* = X_h$ it is sufficient to prove that their action on affine functions is the same. By definition of Hamiltonian vector field and from the previous proposition

$$X_{h_X}(\alpha) = \{h_X, \alpha\} = X\alpha = X^*(\alpha)$$
$$X_{h_X}(f_Y) = \{h_X, f_Y\} = f_{[X,Y]} = X^*(f_Y)$$

hence the action is the same of X^* and then $X^* = X_h$.

□

More generally, for a nonautonomous vector field.

Proposition 30.
Let $X_1, \dots, X_m$ be vector fields and $X[t] = \sum_i^m u_i(t) X^i$ be a nonautonomous vector field with $u_i \in L^\infty([0,T], \mathbb{R}^m)$. Consider the Hamiltonian

$$h_{X[t]}(\lambda) = \langle \lambda, X[t](q) \rangle$$

The dual flow $(\Phi_t^{-1})^*$ is the generated by the nonautonomous Hamiltonian vector field on T^*M associated to $h_{X[t]}$ i.e

$$X^*[t] = X_{h_{X[t]}}$$

Proof. (sketch)
From the autonomous case we have first to prove that the flow generated by $X^*[t]$ and $(\Phi_t^{-1})^*$ for

$$X[t] = \sum_{i=1}^m u_i(t) X^i$$

where u piecewise constant, are the same. Then if we prove the continuity of both the flows respect to u in the L^1 topology, since one can approximate any $L^\infty([0,T], \mathbb{R}^m)$ control by piecewise constant ones, the required identity follows for any nonautonomous vector field of the form $X[t]$.

□

Let (W, L) be a sub-Riemannian structure on a manifold M with generating family $\{\nu_1, \dots, \nu_m\}$. We call *partial Hamiltonians* the fiberwise linear functions on T^*M

$$h_i : T^*M \to \mathbb{R}$$

$$h_i(\lambda) := \langle \lambda, \nu_i(q) \rangle \quad i = 1, \dots, m$$

Theorem 33 (Hamiltonian Characterization)**.**
Let $\gamma \in \Omega_q^p([0,T])$ be a length minimizing, parametrized by constant speed and u^* be the corresponding minimal control. There exists a Lipschitz curve $\lambda(t) \in T^*_{\gamma(t)}M$ solution of the nonautonomous Hamiltonian problem defined by the Hamiltonian

$$H_u(\lambda, t) = \sum_i u_i^*(t) h_i(\lambda)$$

that is such that

$$\dot{\lambda}(t) = \sum_{i=1}^m u_i^*(t) X_{h_i}(\lambda(t))$$

And one of the following conditions is satisfied:

i) $h_i(\lambda(t)) = u_i^*(t) \quad i = 1, \dots, m, \quad \forall t$
ii) $h_i(\lambda(t)) = 0 \quad i = 1, \dots, m, \quad \forall t$

i.e. $\lambda(t)$ is a normal or an abnormal extremal.

Proof.

For the Pontryagin Extremals Theorem there exists an initial covector $\lambda_0 \in T^*_{\gamma(0)}M$ such that the co-curves $\lambda(t) \in T^*M_{\gamma(t)}$ i.e. the curve

$$\lambda(t) := (\Phi_t^{-1})^* \lambda_0$$

associated to the vector field $X^*[t]$ on T^*M with initial co-velocity λ_0 is a normal or an abnormal extremal. Remark that

$$p \in M \xrightarrow[X]{\Phi_t} M \qquad\qquad \gamma : [0,1] \to M$$

$$\lambda_0 \in T^*_p M \xrightarrow[X^*]{(\Phi_t^{-1})^*} T^*_{\Phi_t(p)} M \qquad\qquad \lambda : [0,1] \to T^*M$$

Since the flow Φ_t is generated by the non autonomous vector field

$$X[t] = Lu^* = \sum_{i=1}^{k} u_i^*(t)\nu_i$$

the dual flow is generated by the vector field $X^*[t]$ that for the proposition is the Hamiltonian vector filed $X^*[t] = X_H[t]$ where

$$H_u(\lambda, t) := \langle \lambda, X[t](q) \rangle = \sum_i u_i^*(t) \langle \lambda, \nu_i \rangle = \sum_i u_i^*(t) h_i(\lambda)$$

$\square$

The Hamiltonian system defined in the Theorem, which solutions are the extremals, is in general nonautonomous and depends on the trajectory by the presence of the control u^*. We now see that how normal extremals are characterized to be solutions of a *smooth autonomous* Hamiltonian system.

Definition 22.
Let (M, W, L) be a sub-Riemannian manifold. We call *sub-Riemannian Hamiltonian* the function $H : T^*M \to \mathbb{R}$ defined by

$$H_u(\lambda) := \max_{u \in W_p} \left(\langle \lambda, L_p u \rangle - \frac{1}{2}|u|^2 \right), \quad p = \pi(\lambda)$$

It is possible to show that two equivalent sub-Riemannian structures (W, L) and (W', L') on M define the same Hamiltonian. Next Proposition allows us to express this function in an explicit form.

Proposition 31.
For every generating family $\{\nu_1, \dots, \nu_m\}$ of the structure, the sub-Riemannian Hamiltonian can be expressed as

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^{m} \langle \lambda, \nu_i(p) \rangle$$

where $\lambda \in T^*_p M$ and $p = \pi(\lambda)$ and it's then smooth and quadratic on the fibers.

Proof.
In terms of the family $\{\nu_1, \dots, \nu_m\}$ the Hamiltonian can be expressed as

$$H(\lambda) := \max_{u \in W_p} \left(\sum_i u_i \langle \lambda, \nu_i(p) \rangle - \frac{1}{2} \sum_i u_i^2 \right), \quad p = \pi(\lambda)$$

By differentiating the function

$$f(u) = \sum_i u_i \langle \lambda, \nu_i(p) \rangle - \frac{1}{2} \sum_i u_i^2$$

we get the maximum for $u_i = \langle \lambda, \nu_i(p) \rangle$ then replacing it in the definition of H we get the statement form. From that is clear that this map is also smooth and quadratic on the fiber.

□

Remark that since abnormal extremals have the property that

$$\langle \lambda(t), \nu_i(\gamma(t)) \rangle = 0 \qquad \forall i = 1, \dots, m$$

from the definition of H we conclude that abnormal extremals are contained in the level set $H^{-1}(0)$. Anyway, as we show in the next Theorem, *the solutions of the sub-Riemannian Hamiltonian system are exactly the normal extremals.*

Theorem 34.
A Lipschitz curve $\lambda : [0, T] \to T^*M$ is a normal extremal if and only if it is a solution for the Hamiltonian system

$$\dot{\lambda}(t) = X_H(\lambda(t))$$

Moreover given a normal extremal the corresponding normal extremal trajectory $\gamma(t) = \pi(\lambda(t))$ is smooth and has constant speed satisfying

$$\frac{1}{2} \|\dot{\gamma}(t)\|^2 = H(\lambda(t)) \qquad \forall t \in [0, T]$$

Proof.
i) Let $\{\nu_1, \dots, \nu_m\}$ be a generating family and $h_i(\lambda) = \langle \lambda, \nu_i(p) \rangle$. Since $X_{h_i^2} = 2h_i X_{h_i}$ it follows that

$$X_H = \frac{1}{2} X_{\sum_i h_i^2} = \sum_i h_i X_{h_i}$$

Let $\lambda(t)$ be a normal extremal. In particular $h_i(\lambda(t)) = u_i^*(t)$ and from the normal condition we get

$$X_H(\lambda(t)) = \sum_i h_i\big(\lambda(t)\big) X_{h_i}(\lambda(t)) = \sum_i u_i^*(t) X_{h_i}(\lambda(t)) = \dot{\lambda}(t)$$

ii) Let now $\lambda(t)$ be the solution of $\dot{\lambda}(t) = X_H(\lambda(t))$. Then

$$\dot{\lambda}(t) = \sum_i h_i\big(\lambda(t)\big) X_{h_i}(\lambda(t))$$

instead if $\gamma(t) = \pi(\lambda(t))$ one has

$$\dot{\gamma}(t) = \sum_i h_i(\lambda(t))\nu_i(\gamma(t)) = \sum_i \langle \lambda(t), \nu_i(\gamma(t))\rangle \, \nu_i(\gamma(t))$$

since $\nu_i = \pi_* X_{h_i}$. Hence $\overline{u}_i(t) := \langle \lambda(t), \nu_i(\gamma(t))\rangle$ defines a control for γ that is also minimal. Finally notice that

$$\frac{1}{2}\|\dot{\gamma}(t)\|^2 = \frac{1}{2}\sum_i \overline{u}_i(t)^2 = \frac{1}{2}\sum_i \langle \lambda(t), \nu_i(\gamma(t))\rangle^1 = H(\lambda(t))$$

□

Corollary 9.
A normal extremal trajectory is parameterized by constant speed and it is arc length parameterized if and only if its extremal lift is contained in the level set $H^{-1}(1/2)$.

Proof.
Since H is constant along $\lambda(t)$, also $\|\dot{\gamma}(t)\|$ is constant. Moreover, $\|\dot{\gamma}(t)\| = 1$ if and only if $H(\lambda(t)) = \frac{1}{2}$. □

Let's see how the Corollary can be used to compute some Riemannian properties of a surface in $\mathbb{R}^3$. Let M be a 2-dim manifold with a standard Riemannian metric and let p_1, p_2 be two points in M. Consider the problem to find the geodetics of the metric joining p_1 and p_2 i.e. the curves from p_1 to p_2 minimizing the Riemannian length l. If we also require these solutions to be arc-length parametrized, considered a frame e_1, e_2 with structure constant $c_1, c_2 \in C^\infty(M)$ from the PMP it follows that setting the partial hamiltonians $h_i(\lambda) := \langle \lambda, e_i\rangle$ and

$$H = \frac{1}{2}\left(h_1^2 + h_2^2\right)$$

the arc-length parametrized normal trajectories $x(t)$ on M are projections of trajectories of the normal Hamiltonian vector field X_H:

$$\begin{cases} \lambda(t) = \pi \circ \Phi_t^{X_H}(\lambda_0) \\ \lambda_0 \in \{H = \frac{1}{2}\} \end{cases}$$

The level surface $\{H = \frac{1}{2}\}$ is the spherical co-bundle N^*M over M with fiber

$$\{h_1^2 + h_2^2 = 1\} \cap T_p^*M \cong S^1$$

Moreover since $\{h_1, h_2\}(\lambda) = \langle \lambda, [e_1 e_2]\rangle$ from $[e_1, e_2] = c_1 e_1 + c_2 e_2$ we obtain that

$$\{h_1, h_2\} = c_1 h_1 + c_2 h_2$$

By definition of normal extremal $u_i(t) = h_i(\lambda(t))$ and then the equation on the base manifold becomes

$$\dot{x}(t) = h_1 e_1 + h_2 e_2$$

Parametrized N^*M by coordinates $h_1 = \cos(\theta)$ and $h_2 = \sin(\theta)$ since $\dot{h}_1 = \{H, h_1\}$ and $\dot{h}_2 = \{H, h_2\}$ we get the system

$$\begin{cases} \dot{x}(t) = h_1 e_1 + h_2 e_2 \\ \dot{h}_1 = -\{h_1, h_2\} h_2 \\ \dot{h}_2 = \{h_1, h_2\} h_1 \end{cases}$$

that in these coordinates becomes

$$\begin{cases} \dot{x} = \cos(\theta) e_1(x) + \sin(\theta) e_2(x) \\ \dot{\theta} = \cos(\theta) c_1(x) + \sin(\theta) c_2(x) \end{cases}$$

If μ_1, μ_2 is the frame dual to e_1, e_2, the Levi-Civita connection on M can be expressed by some coefficients $a_1, a_2 \in C^\infty(M)$ by

$$\omega = d\theta + a_1 \mu_1 + a_2 \mu_2$$

Since the Hamiltonian vector field in these coordinates takes the form

$$X_H = \cos(\theta) e_1 + \sin(\theta) e_2 + \big(c_1 \cos(\theta) + c_2 \sin(\theta)\big) \frac{\partial}{\partial \theta}$$

the normal trajectories are projections of integral curves of X_H and moreover

$$\omega(X_H) = 0 \quad \Rightarrow \quad a_1 = -c_1,\ a_2 = -c_2 \quad \Rightarrow \quad \omega = d\theta - c_1 \mu_1 - c_2 \mu_2$$

Applying ω to an curve on N^*M project on γ and hence satisfing

$$\dot{\lambda} = \cos(\theta) e_2 + \sin(\theta) e_2 + \dot{\theta} \frac{\partial}{\partial \theta}$$

we finally find the geodesic curvature $k_g(\gamma) = \dot{\theta} - c_1 \cos(\theta) - c_2 \sin(\theta)$

Proposition 32.
The Gaussian curvature k of the Riemannian structure on M defined in a local orthonormal frame e_1, e_2 is

$$k = e_1(c_2) - e_2(c_1) - c_1^2 - c_2^2$$

where $c_1, c_2 \in C^\infty(M)$ are the constant of the frame.

Proof.
If $dV = \mu_1 \wedge \mu_2$ is the Riemannian volume form, since $d\omega = -k dV$ in this frame

$$dc_i = e_1(c_i) \mu_1 + e_2(c_2) \mu_2$$

while

$$d\mu_i = d\mu_i(e_1, e_2) \mu_1 \wedge \mu_2 = -c_i \mu_1 \wedge \mu_2$$

then

$$d\omega = d(-c_1 \mu_1 - c_2 \mu_2) = -dc_1 \wedge \mu_1 - dc_2 \wedge \mu_2 - c_1 d\mu_1 - c_2 d\mu_2 =$$

$$= -\big(e_1(c_2) - e_2(c_1) - c_1^2 - c_2^2\big) \mu_1 \wedge \mu_2$$

□

2.2 Singular Solutions

Consider a sub-Riemannian manifold (M, W). Fixed a point p we can define the space of the admissible curves respect to (W, L) based at p

$$\Omega_p := \bigcup_{q \in M} \Omega_p^q$$

i.e. the set of all admissible curves $\gamma : [0, 1] \to M$ flowing out from $\gamma(0) = p$. To each of these curves we can associate the end point $q = \gamma(1)$. This defines a map

$$end : \Omega_p \to M$$
$$end(\gamma) = \gamma(1)$$

called *endpoint map.* Let's see how it is possible to define a differential structure on Ω_p such that Ω_p becomes an *Hilbert manifold* and the endpoint map a differentiable function on it.

Definition 23.
An *Hilbert manifold* X is a separable metrizable topological space such that every point admit a neighborhood homeomorphic to an open subset of an Hilbert space H. In particular a submanifold of X is a subset $Y \subset X$ such that for every point $y \in Y$ there is an open neighborhood V of $y \in X$ and a homeomorphism $\psi : V \to W$ to an open subset $W \subset H$ such that $\psi(V \cap Y) = W \cap U$ for a closed linear subspace U of H.

The basic idea of this construction is to consider the vector control function $u = (u_1, \dots, u_m) \in L^2_k = L^2([0, 1], \mathbb{R}^k)$ as local coordinates for the admissible paths. Indeed considered global orthonormal frame of complete vector fields $\nu_1, \dots \nu_m$ of D each admissible curve has velocity field satisfying

$$\dot{\gamma}(t) = \sum_i^m u_i(t)\nu_i(\gamma(t))$$

A technical issue: since in general these paths might self intersect, we have to taking in account also the time, that is represent the vector field respect to a time-dependent frame $\nu_i = \nu_i(t, x) \in W_x$. With this tweak we can expand

$$\dot{\gamma} = \sum u_i(t)\nu_i(t, \gamma(t))$$

In order to obtain a one-to-one correspondence it is then necessary to prove that the coordinates maps

$$\psi : \Omega_p \to L^2_k$$
$$\psi(\gamma) = u$$

are actually invertible, condition guaranteed by the well posedness of the differential problem

$$\begin{cases} \dot{\gamma}(t) = \sum_i u_i(t)\nu_i(t, \gamma(t)) \\ \gamma(0) = p \end{cases}$$

since it implies that for each fixed vector $u \in L^2([0,1],\mathbb{R}^k)$ and $q \in M$ there exists a unique trajectory γ satisfying the above problem. This is established in the following.

Lemma 10.
Let (t_0, x) be in the domain of the frame X and $u \in L^2(I,\mathbb{R}^k)$. There exists a $t_1 > t_0$, $t_1 \in I$ such that the differential equation above with initial condition $\gamma(t_0) = p$ admits a solution γ on $0 \leq t \leq t_1$ and the solution is unique within the space of absolutely continuous curves defined on $[t_0, t_1]$.

Using the Lemma is then possible to prove the following Theorem, that summarize the previous arguments.

Theorem 35.
The space $\Omega \cap L^2$ is an Hilbert manifold with the charts defined as before with values in $L^2([0,1],\mathbb{R}^k)$. Moreover,
i) The inclusion $i : \Omega \cap L^2 \to C^0(I, M)$ is continuous.
ii) The inclusion $i : \Omega \cap L^2 \to C^0(I, M)$ is continuous with respect to the weak topology on $\Omega \cap L^2$ induced bu the weak topology on L^2.

Moreover, it is possible to prove that in these coordinates the end point map assume the form

$$end : L^2([0,1],\mathbb{R}^k) \to M$$

$$u \to \gamma_u(1)$$

and it's also differentiable. We call the critical and regular points of the endpoint map respectively *singular curves* and *regular curves*. To investigate these two kind of curves, we have to compute the differential of the endpoint map first. Let $\gamma_p \in \Omega$ be a curve from the point $p \in M$ and $(u_1, \dots u_n) \in L^2(I,\mathbb{R}^k)$ its coordinates on Ω_p. The curve is described in term of the flow by $\Phi_t(u) = \gamma(t)$ while the endpoint map by $end(\gamma) = \Phi_1(u)$. Thus if $v \in L^2(I,\mathbb{R}^k)$ is an arbitrary direction the differential of *end*

$$d(end)_\gamma : L^2([0,1],\mathbb{R}^k) \to T_{\gamma(1)}M$$

is given by

$$d(end)_\gamma(v) = \frac{\partial \Phi_1(u + \varepsilon v)}{\partial \varepsilon}_{|\varepsilon=0}$$

In the next Lemma we explicitly compute this differential.

Lemma 11.
For each $p \in M$, the differential of the endpoint map $end : \Omega_p \to M$ is given by

$$d(end)_\gamma(v) = \sum_i^m d\Phi_1 \int_0^1 v_i(t) d\Phi_t^{-1}\big(\nu_i(t)\big)(t) dt$$

Proof.
Let $\Phi_t(u + \varepsilon v)$ be the curve corresponding to $u + \varepsilon v$ and

$$\partial_\varepsilon \Phi_t(u + \varepsilon v) := \frac{\partial \Phi_t(u + \varepsilon v)}{\partial \varepsilon}$$

Denoted by $v \cdot \nu = \sum_i^m v_i \nu_i$, since $\Phi_t(u+\varepsilon v)$ is a solution it satisfies

$$\frac{\partial \Phi_t(u+\varepsilon v)}{\partial t} = \big(u(t)+\varepsilon v(t)\big) \cdot \nu(\Phi_t(u+\varepsilon v))$$

Thus since the partial derivatives to respect to t and ε commute, fixed a frame[6]

$$\frac{d\partial_\varepsilon \Phi_t(u+\varepsilon v)}{dt}|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\frac{\partial \Phi_t(u+\varepsilon v)}{\partial t}|_{\varepsilon=0} =$$

$$= (v(t) \cdot \nu(\Phi_t(u+\varepsilon v))) + \big(u(t)+\varepsilon v(t)\big)\big) \cdot \frac{\partial \nu}{\partial x} \partial_\varepsilon \Phi_t(u+\varepsilon v)|_{\varepsilon=0}$$

Then if we denote by $W(t) := \partial_\varepsilon \Phi_t(u+\varepsilon v)|_{\varepsilon=0}$ we get the equation

$$\frac{dW(t)}{dt} = (v(t)\cdot \nu)|_{\gamma(t)} + u(t) \cdot \frac{\partial \nu}{\partial x} W(t)$$

that is an inhomogeneous linear differential equation for $W(t)$, which can be solved by variation of the parameters method. To simplify the notation denote

$$\begin{cases} j(t) := (v(t)\cdot\nu)|_{\gamma(t)} \\ A(t) := u(t) \cdot \frac{\partial X}{\partial x} \end{cases}$$

The vector field W satisfies the inhomogeneous linear differential equation

$$\frac{dW}{dt} = j(t) + A(t)W(t)$$

Assuming that $W(t) = \Psi(t)w(t)$, we get that

$$w(t)\frac{d\Psi(t)}{dt} + \Psi(t)\frac{dw(t)}{dt} = j(t) + A(t)\Psi(t)w(t)$$

If we look for the fundamental matrix solution Ψ solving the homogeneous equation

$$\frac{d\Psi}{dt} = A(t)\Psi(t)\ , \quad \Psi(0) = Id$$

and replacing this relation in the upper equation we obtain the condition

$$\Psi(t)\frac{dw(t)}{dt} = j(t)$$

Then since

$$w(t) = \int_0^1 \Psi(s)^{-1} j(s) ds$$

the solution is

$$W(t) = \Psi(t)\bigg(\int_0^1 \Psi(s)^{-1} j(s) ds\bigg)$$

with initial value $W(0) = 0$. Let's compute $\Psi(t)$, showing that it's equal to $d\Phi_t(q_0)$. By definition it solves the linear homogeneous equation

$$\frac{d\Psi(t)}{dt} = u(t) \cdot \frac{\partial \nu}{\partial x}\Psi(t)$$

[6] Necessary to compute the term $\frac{\partial X}{\partial x}$.

While the flow Φ_t is defined by the differential equation

$$\frac{d\Phi_t(x)}{dt} = u(t) \cdot \nu(t, \Phi_t(x))$$

with initial condition $\Phi_0(x) = x$. If we consider a variation $x = q_0 + \varepsilon\Psi(0)$ and the system

$$\frac{d\Phi_t(q_0 + \varepsilon\Psi(0))}{dt} = u(t) \cdot \nu(t, \Phi_t(q_0 + \varepsilon\Psi(0)))$$

The differential with respect to ε of this differential equation at $\varepsilon = 0$ is

$$\frac{d}{d\varepsilon}\frac{d\Phi_t(q_0 + \varepsilon\Psi(0))}{dt}|_{\varepsilon=0} = \frac{d}{d\varepsilon}u(t) \cdot \nu(t, \Phi_t(q_0 + \varepsilon\Psi(0)))|_{\varepsilon=0}$$

Switching the derivatives we obtain

$$\frac{d}{dt}d\Phi_t(q_0, \Psi(0)) = u(t) \cdot \frac{\partial\nu}{\partial x}d\Phi_t(q_0, \Psi(0))$$

then since $d\Phi_0 = Id$, both $\Psi(t)$ and $d\Phi_t$ must satisfy the same ODE with same initial conditions and then must be $\Psi(t) = d\Phi_t(q_0)$. Replacing it in the general form of the solution, we get

$$d(end)_\gamma(v) = d\Phi_1 \int_0^1 d\Phi_t^{-1}\big(v \cdot \nu\big)(t)dt$$

Then since

$$(v \cdot \nu)(t) = \sum_i^m v_i(t)\nu_i(\gamma(t))$$

for the linearity of $d\Phi_t^{-1}$ at each time t

$$d(end)_\gamma(v) = d\Phi_1 \int_0^1 d\Phi_t^{-1}\big(\sum_i^m v_i(t)\nu_i(t)\big)(t)dt = \sum_i^m d\Phi_1 \int_0^1 v_i(t)d\Phi_t^{-1}\big(\nu_i(t)\big)(t)dt$$

$\square$

We now need to compute also the transpose of this differential i.e the map

$$d(\text{end})^*_\gamma : T^*_{\gamma(1)}M \to L^2(I, \mathbb{R}^k)^*$$

$$\big(d(\text{end})^*_\gamma\lambda\big)(v) := \langle\lambda, d(end)_\gamma(v)\rangle$$

Let $\lambda_1 \in T^*_{\gamma(1)}M$ be a covector at the endpoint $\text{end}(\gamma) = \gamma(1)$ of the horizontal curve γ and $\lambda_0 := d\Phi_1^*(\lambda_1)$. The dual braket is given by

$$\langle\lambda_1, d(end)_\gamma(v)\rangle = \langle\lambda_0, \int_0^1 d\Phi_t^{-1}(v \cdot \nu)(t)dt\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between vectors and covectors. Considered

$$\lambda(t) := \big(d\Phi_t^{-1})^*(\lambda_0)$$

with $\lambda(0) = \lambda_0$ and defined

$$\lambda_1 := \lambda(1) = \big(d\Phi_1^{-1}\big)^*(\lambda_0)$$

Since

$$d(end)_\gamma(v) = d\Phi_1 \int_0^1 d\Phi_t^{-1}\big(v \cdot \nu\big)(t)dt$$

we see that

$$\langle \lambda_1, d(end)_\gamma(v)\rangle = \langle \big(d\Phi_1^{-1}\big)^*(\lambda_0), d\Phi_1 \int_0^1 d\Phi_t^{-1}\big(v \cdot \nu\big)(t)dt\rangle =$$
$$= \int_0^1 \langle \lambda(t), (v \cdot \nu)(t)\rangle \, dt$$

As we saw in the previous chapter the curve is the solution for the Hamiltonian system with initial conditions λ_0 and Hamiltonian H_u defined by

$$H_u(\lambda, t) = \sum_{i=1}^k u_i(t) h_i(\lambda) \qquad h_i(\lambda) = \langle \lambda, \nu_i(q)\rangle$$

Remark that there is a technical problem: the functions u_i need not to be smooth, rather only L^2. This lack of smoothness is solvable requiring some reworking of existence-uniqueness theory for solutions to time-dependent Hamiltonians equations that are only in L^2 in the time variable t. Assuming them, we have

$$\langle \lambda(t), (v \cdot \nu)(t)\rangle = \sum_{i=1}^k v^i(t) \, \langle \lambda(t), \nu_i(q(t))\rangle \,\big) = \sum_{i=1}^k v^i(t) h_i(\lambda(t))$$

Hence replacing it in the equation it becomes

$$\langle \lambda_1, d(end)_\gamma(v)\rangle = \int_0^1 \langle \lambda(t), (v \cdot \nu)(t)\rangle \, dt = \sum_{i=1}^k \int_0^1 v^i(t) h_i(\lambda(t)) dt$$

This shows that

$$d(\mathrm{end})^*_\gamma \lambda = \big(h_1(\lambda(t)), \ldots, h_k(\lambda(t))\big)$$

This expression relates the singular curves with the Hamiltonian formulation of the Pontryagin Extremals Theorem.

Theorem 36.
The singular curves are exactly the projections of abnormal extremals. More formally, let $\gamma \in \Omega_p([0,1])$ be a curve corresponding to the control $u(t) \in L^2([0,1],\mathbb{R}^k)$ via the frame $\nu_1, \dots \nu_m$ of D. Then γ is singular if and only if there exists a solution $\lambda(t)$ for the Hamiltonian system

$$\begin{cases} \dot{\lambda} = X_{H_u} \\ \lambda(0) = \lambda_0 \end{cases}$$

with time-dependent Hamiltonian

$$H_u(\lambda, t) = \sum u_i(t) h_i(\lambda)$$

and initial conditions $p \in M$, $\lambda_0 \in T_p^*M, \lambda_0 \neq 0$ which in addiction satisfies $\lambda(t) \in D^\perp$ for all time t.

Proof.
Suppose that a curve γ is singular. The image of $d(\text{end})_\gamma$ is a proper subspace of $T_{\gamma(1)}M$, consequently there exists a nonzero covector $\lambda \in T^*_{\gamma(1)}M$ that annihilates this subspace i.e.

$$\langle \lambda, d(\text{end}_\gamma(v)) \rangle = 0$$

for all $v \in L^2(I, \mathbb{R}^k)$. Since v is arbitrary must be $h_i(\lambda(t)) = 0$, $i = 1, \dots, k$. The vector fields ν_i frame D and $h_i(\lambda(t)) = \langle \lambda(t), \nu_i(\gamma(t)) \rangle$ so that the vanishing of the h_i is equivalent to the assertion that $\lambda(t)$ annihilates W, thus $\lambda(t)$ is an abnormal extremal. On the other hand, if λ is an abnormal extremal and $\gamma = \pi(\lambda)$, since $h_i(\lambda(t)) = 0$, $i = 1, \dots, k$ must be $d(end)^*_\gamma \equiv 0$

□

Since Riemannian structures have no abnormal extremals, the next Corollary follows.

Corollary 10.
A Riemannian structure (as sub-Riemannian structure) has no singular curves.

2.3 Rolling Bodies Model

We now investigate a specific problem of Control Theory. We can describe the motion of two solid bodies in $\mathbb{R}^3$ rolling one over the another without slipping or twisting by a pair of two dimensional differential manifolds (with some more general assumptions) related by an identification of the tangent spaces at the contact points.

Let Q and $\hat{Q}$ be the two connected and orientable 2-dimensional manifolds, equipped with a Riemannian structure and p, q a pair of contact points, respectively on Q and $\hat{Q}$. Since during the motion the two tangent spaces are coincident in $\mathbb{R}^n$, fixed two frames e_1, e_2 and $\hat{e}_1, \hat{e}_2$ respectively on Q and $\hat{Q}$ in two neighborhoods of the points, at each time t the mutual position of these two basis is identified by an isometry

$$A : T_pQ \to T_q\hat{Q}$$

that send one bases on the other. Since the orientation of the manifolds (and then of the frames) doesn't change, the isometry is orientation preserving, hence the configuration space of the problem is given by the set

$$M = \left\{ A : T_pQ \to T_q\hat{Q} \mid p \in Q,\ q \in \hat{Q},\ A \in SO(2) \right\}$$

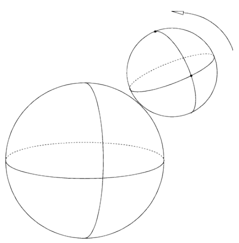
Respect to these frames at each contact pairs in some pair of neighborhoods of p and p' these isometries are parameterized by some angle $\theta \in [0, 2\pi]$ such that

$$\begin{aligned} A_\theta e_1 &= \cos(\theta)\hat{e}_1 + \sin(\theta)\hat{e}_2 \\ A_\theta e_2 &= -\sin(\theta)\hat{e}_1 + \cos(\theta)\hat{e}_2 \end{aligned} \tag{2.1}$$

Then if $(x_1, x_2), (\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2$ are local coordinates for p and q in these neighborhoods, we can define a structure of differentiable manifold on M parameterizing each of these isometries $A_\theta : T_pM \to T_qM$ with coordinates $(x_1, x_2, \hat{x}_1, \hat{x}_2, \theta)$ for $\theta \in S^1$. It follows that the configuration space M has a structure of smooth principal bundle

$$SO(2) \to M \to Q \times \hat{Q}$$

Let $A(t)$ be a trajectory of the system in M and denote by $x(t) = \pi(A(t))$ and $\hat{x}(t) = \hat{\pi}(A(t))$ the two curves respectively on Q and $\hat{Q}$. We impose the condition of



- Absence of *slipping*: the velocity fields $\dot{x}(t)$ and $\dot{\hat{x}}(t)$ are related via the isometry by

$$A_t\dot{x}(t) = \dot{\hat{x}}(t)$$

- Absence of *twisting*: the solution $A(t)$ transforms parallel vector fields along the solution $x(t)$ in parallel vector fields along the solution $\hat{x}(t)$. More formally if X is a vector field on Q such that $\nabla_{\dot{x}(t)}X = 0$ and $\hat{X} := A_tX$ then[7]

$$\nabla_{\dot{\hat{x}}}\hat{X} = 0$$

2.3.1 A Sub-Riemannian Problem

Let's show that the two conditions explicitly formulated describe a sub-Riemannian problem. If we decompose the velocities of the contact curves respect to the previous frame as

$$\begin{aligned} \dot{x} &= a_1 e_1(x) + a_2 e_2(x) \\ \dot{\hat{x}} &= b_1 \hat{e}_1(x) + b_2 \hat{e}_2(x) \end{aligned} \tag{2.2}$$

replacing in $A_t\dot{x} = \dot{\hat{x}}$ the equations

$$A_\theta e_1 = \cos(\theta)\hat{e}_1 + \sin(\theta)\hat{e}_2 \qquad A_\theta e_2 = -\sin(\theta)\hat{e}_1 + \cos(\theta)\hat{e}_2$$

[7]Here we denote both the connections (on Q and $\hat{Q}$) with ∇, since we consider the one induced by the euclidean structure of $\mathbb{R}^3$ in both the cases.

we obtain that the no slipping condition reads

$$b_1 = a_1 \cos(\theta) - a_2 \sin(\theta) \qquad b_2 = a_1 \sin(\theta) + a_2 \cos(\theta)$$

Let's explain the no-twisting condition. Let $c_1, c_2, \hat{c}_1, \hat{c}_2$ be the structure constants of the frames i.e. the elements of $C^\infty(M)$ such that

$$[e_2, e_2] = c_1 e_1 + c_2 e_2$$

$$[\hat{e}_1, \hat{e}_2] = \hat{c}_1 \hat{e}_1 + \hat{c}_2 \hat{e}_2$$

Consider the Riemannian/sub-Riemannian problem to find the geodetics for Q; let H be the Hamiltonian associated and

$$E_{\frac{1}{2}} := H^{-1}(0)$$

It is possible to prove the following.

Proposition 33.
Let be $Y_H =: [\partial_\theta, X_H]$. *For each* $\lambda \in T^*Q$

$$T_\lambda E_{\frac{1}{2}} = span\Big(X_H, \partial_\theta, Y_H\Big)$$

Moreover in the frame $e_1, e_2, \hat{e}_1, \hat{e}_2$

$$X_H = h_1\big(e_1 + c_1 \partial_\theta\big) + h_2\big(e_2 + c_2 \partial_\theta\big)$$

$$Y_H = -h_2\big(e_1 + c_1 \partial_\theta\big) + h_1\big(e_2 + c_2 \partial_\theta\big)$$

Since Levi-Civita connection is the unique connection on the bundle $NM \to M$ such that the hamiltonian vector field X_H is horizontal (i.e. parallel along geodetics) and the horizontal lifts of vector fields on M commute with the vector field ∂_θ that determines the element of length in NM, that is $\big(\Phi_s^{\partial_\theta}\big)_* D = D$, we have that

$$D = \text{span}\Big(\big(\Phi_s^{\partial_\theta}\big)_* X_H \mid s \in \mathbb{R}\Big)$$

Explicitly

$$\big(\Phi_s^{\partial_\theta}\big)_* X_H = h_1(\theta - s)\Big(e_1 + c_1 \partial_\theta\Big) + h_2(\theta - s)\Big(e_2 + c_2 \partial_\theta\Big)$$

then finally

$$D = \text{span}\big(X_H, Y_H\big)$$

The connection 1-form $\omega \in \Lambda^1 E_{\frac{1}{2}}$ associated to D i.e. such that $D = \ker \omega$ reads

$$\omega = c_1 \eta_1 + c_2 \eta_2 - d\theta$$

where η_1, η_2 is the dual co-frame associated to e_1, e_2. Let $\tilde{A} : T_p^* M \to T_q^* \hat{M}$ be the map induced by the identification $T^*M \cong TM$. Then

$$\tilde{A}\eta_1 = \cos(\theta)\hat{\eta}_1 + \sin(\theta)\hat{\eta}_2$$

$$\tilde{A}\eta_2 = -\sin(\theta)\hat{\eta}_1 + \cos(\theta)\hat{\eta}_2$$

Proposition 34.
Let $c_1, c_2 \in C^\infty(Q)$ and $\hat{c}_1, \hat{c}_2 \in C^\infty(\hat{Q})$ be the structure constant of the frames. A curve A_t in M is an admissible motion of the rolling bodies if and only if it is a trajectory of the control system

$$\dot{A} = u_1 X_1 + u_2 X_2 \quad , \; A \in M, u_1, u_2 \in \mathbb{R}$$

where

$$X_1 = e_1 + \cos(\theta)\hat{e}_1 + \sin(\theta)\hat{e}_2 + \big(- c_1 + \hat{c}_1 \cos(\theta) + \hat{c}_2 \sin(\theta)\big)\partial_\theta$$

$$X_2 = e_2 - \sin(\theta)\hat{e}_1 + \cos(\theta)\hat{e}_2 + \big(- c_2 - \hat{c}_1 \sin(\theta) + \hat{c}_2 \cos(\theta)\big)\partial_\theta$$

Proof.
The nontwisting condition means that if $\lambda(t) = (x(t), \varphi(t)) \in E_{\frac{1}{2}}$ is a parallel covector field along a curve $x(t)$ on M then $\hat{\lambda}(t) = \tilde{A}_t\lambda(t) \in \tilde{E}_{\frac{1}{2}}$ is a parallel covector field along $\hat{x}(t)$. Then if the isometry A_t rotates the tangent space of an angle $\theta(t)$, $\tilde{A}$ rotates the cotangent space with the same angle,

$$\hat{\varphi}(t) = \varphi(t) + \theta(t)$$

hence

$$\dot{\theta}(t) = \dot{\hat{\varphi}}(t) - \dot{\varphi}(t)$$

From definition a covector field is parallel along $x(t)$ if and only if $\dot{\lambda} \in \ker(\Omega)$ then we get for φ and $\hat{\varphi}$ the equations

$$\begin{aligned}
\dot{\varphi} &= \langle c_1\eta_1 + c_2\eta_2, \dot{x}\rangle = c_1 a_1 + c_2 a_2 \\
\dot{\hat{\varphi}} &= \langle \hat{c}_1\hat{\eta}_1 + \hat{c}_2\hat{\eta}_2, \dot{\hat{x}}\rangle = \hat{c}_1\hat{a}_1 + \hat{c}_2\hat{a}_2 = \\
&= a_1(\hat{c}_1\cos(\theta) + \hat{c}_2\sin(\theta)) + a_2(-\hat{c}_1\sin(\theta) + \hat{c}_2\cos(\theta))
\end{aligned}$$

Replacing in the previous equation we obtain

$$\dot{\theta} = a_1(-c_1 + \hat{c}_1\cos(\theta) + \hat{c}_2\sin(\theta)) + a_2(-c_2 - \hat{c}_1\sin(\theta) + \hat{c}_2\cos(\theta))$$

Since

$$\begin{aligned}
\dot{x} &= a_1 e_1(x) + a_2 e_2(x) \\
\dot{\hat{x}} &= b_1\hat{e}_1(x) + b_2\hat{e}_2(x)
\end{aligned} \tag{2.3}$$

The velocity field of an solution $A(t) = (x(t), \hat{x}(t), \theta(t))$ in the local frame $e_1, e_2, \hat{e}_1, \hat{e}_2, \partial_\theta$ of TM reads

$$\dot{A} = a_1 e_1(x) + a_2 e_2(x) + b_1\hat{e}_1(x) + b_2\hat{e}_2(x) + \dot{\theta}\frac{\partial}{\partial\theta}$$

where

$$b_1 = a_1\cos(\theta) - a_2\sin(\theta) \qquad b_2 = a_1\sin(\theta) + a_2\cos(\theta)$$

hence simply denoting $u_1 = a_1$ and $u_2 = a_2$ we obtain the condition

$$\dot{A} = u_1 X_1 + u_2 X_2$$

□

If we consider the distribution $W = \text{span}\big(X_1, X_2\big)$, admissible motions of the rolling bodies are then trajectories of the control system

$$\dot{A} = u_1 X_1 + u_2 X_2$$

The natural question is now if this distribution is integrable or not. First of all remark that we have calculated the Gaussian curvatures of Q and $\hat{Q}$ that read

$$k = e_1(c_2) - e_2(c_1) - c_1^2 - c_2^2$$

$$\hat{k} = \hat{e}_1(\hat{c}_2) - \hat{e}_2(\hat{c}_1) - \hat{c}_1^2 - \hat{c}_2^2$$

Theorem 37.
Let k and $\hat{k}$ be the gaussian curvatures of Q and $\hat{Q}$ respectively. The distribution is integrable if and only if $k \equiv \hat{k}$.

Proof.
Computing the Lie brackets of the fields X_1 and X_2 we get

$$\begin{aligned}
X_3 &:= [X_1, X_2] = c_1 X_1 + c_2 X_2 + (\hat{k} - k)\frac{\partial}{\partial\theta} \\
X_4 &:= [X_1, X_3] = (X_1 c_1) X_1 + (X_1 c_2) X_2 + c_2 X_3 + \\
&\qquad\qquad + \big(X_1(\hat{k} - k)\big)\frac{\partial}{\partial\theta}(\hat{k} - k)\big[X_1, \frac{\partial}{\partial\theta}\big] \\
X_5 &:= [X_2, X_3] = (X_2 c_1) X_1 + (X_2 c_2) X_2 - c_1 X_3 + \\
&\qquad\qquad + \big(X_2(\hat{k} - k)\big)\frac{\partial}{\partial\theta} + (\hat{k} - k)\big)\Big[X_2, \frac{\partial}{\partial\theta}\Big]
\end{aligned}$$

Then from the first equation we obtain that D is integrable if and only if $k(p) \equiv \hat{k}(p)$ since under this condition the distribution is involutive, that is

$$[X_1, X_2] = c_1 X_1 + c_2 X_2$$

Moreover it is easy to see that if $k(p) \neq \hat{k}(p)$

$$Lie(X_1, X_2)(p) = span(X_1, X_2, X_3, X_4, X_5)(p) = T_p M$$

□

In particular considering two rolling spheres.

Corollary 11.
The system of two rolling spheres S_r^2 and S_R^2 of rays r and R respectively is integrable if and only if the ratio of the rays is equal to one.

Proof.
Let S_r^2 and S_R^2 be the two spheres. It is well known that the gaussian curvatures are respectively $k_r = 1/r^2$ and $k_R = 1/R^2$ then by the Theorem it follows that the distribution associated is integrable if and only if

$$\frac{R}{r} = 1$$

□

Finally, it is possible to prove that the projections of abnormal extremal curve $q(t)$

$$x(t) = \pi(q(t)) \qquad \hat{x}(t) = \hat{\pi}(q(t))$$

are Riemannian geodesics respectively on Q and $\hat{Q}$ i.e. that singular curves are defined by pairs of geodetics in Q and $\hat{Q}$ respectively. As we'll see this property of singular solutions allows to give a geometric construction of the space of such curves in the case of rolling spheres. Let's now investigate the symmetries of these systems. Consider a distribution D on a smooth manifold M. A diffeomorphism $\psi : M \to M$ is a *local symmetry* of D if it transforms integral curves of D in integral curves of D, that is

$$\psi \circ \phi_t = \phi_t \circ \psi$$

for each integral curve ϕ_t of D or, in an equivalent way, if

$$\psi_* D = D$$

Clearly the set of symmetries of a distribution, usually denoted by $\mathrm{sim}(M, D)$, has a group structure. Conversely if G is a Lie group that acts on a manifold M, then D is called *G-invariant* if $G \subset \mathrm{sim}(M, D)$. In the first chapter we introduced Lie groups' classification with a focus on the group G_2, which real split form G_2^s is the automorphism group of split octonions. As we anticipated, this group is related to the symmetry group of the rolling bodies problem. More formally, the following general Theorem holds.

Theorem 38 (D. An, P. Nurowski)**.**
Let (Q_1, g_1) be a Riemann surface with Gaussian curvature k, which has a Killing vector and let (Q_2, g_2) be a Riemann surface of constant Gaussian curvature λ. Consider the configuration space M of the two surfaces rolling on each other without slipping or twisting and the associate distribution D. Then in order for the distribution D to have local symmetry G_2, the curvatures must satisfy:

$$(9k - \lambda)(k - 9\lambda)\lambda = 0$$

Corollary 12.
The system of two rolling spheres S_r^2 and S_R^2 of rays r and R respectively is integrable if and only if the ratio of the rays is equal to 3 *or* $1/3$.

Proof.
Since the Gaussian curvature are respectively $k_r = 1/r^2$ and $k_R = 1/R^2$ then by the Theorem it follows that the distribution associated has G_2-simmetry if and only if

$$\frac{R}{r} = 3 \qquad \frac{R}{r} = \frac{1}{3}$$

□

The proof of this result is based on the so called *equivalent method*, developed by Henri Cartan at the beginning of the last century. This so general (as long and difficult) computation is outside the goal of this thesis, but in the next section we'll see that in the special case of rolling spheres, further considerations can be done to this end.

2.3.2 The G_2 Symmetries of Rolling Balls

Consider now the case of rolling spheres. From the last Corollary of previous section we know that the system has a G_2^s-symmetry if and only if the ratio of the rays is equal to 3 or 1/3. But is G_2^s the whole symmetry group? And which is the symmetry group in the other cases? The only known way to answer these questions is still based on the equivalent method. Anyway, in order to show that G_2^s *is contained* in the symmety group, for ratio $3, 1/3$, others approach are possible. In [4](Montgomery) is described a method using general properties of invariant distribution; we now sketch the fundamental ideas beside it. We start introducing a general approach for the symmetries problem of distributions. Let G be a Lie group acting on a manifold M. We define the *isotropy group* of G at p

$$H_p := \{g \in G \mid gp = p\}$$

If M is a differential manifold and G acts continuously on M, then H_p is a closed subgroup of G.

Theorem 39.
Let M be a smooth manifold and G a connected [8] Lie group acting smoothly on M. If the action of G on M is transitive, all isotropy groups are isomorphic. Moreover, for each $p \in M$ the map

$$f : G/H_p \to M$$

$$g \cdot H_p \to gp$$

is a diffeomorphism that induces for each $p \in M$ an identification

$$M \cong G/H_p$$

and a H_p-equivariant identification $T_pM \cong \mathfrak{g}/\mathfrak{h}_p$ where $\mathfrak{h}_p$ is the Lie algebra of H_p.

A general manifold M representable as quotient $M = G/H_p$ respect to a transitive Lie group action, is usually called *homogeneous space.* Remark that on the quotient space G/H is defined an action of G by

$$\mu : G \times G/H \to G/H$$

$$g_1(g_2H) := (g_1g_2)H$$

and under this action the identification $M \cong G/H_p$ is G- equivariant. Moreover the isotropy group H_p of a point $p \in M$ induces a group of linear transformations of the tangent vector space T_pM. Indeed it is defined a representation

$$\rho_p : H_p \to GL(T_pM)$$

$$h \to dh_p$$

of H_p on $\mathrm{Aut}(T_pM)$, called *isotropy representation* of H_p. The image of H_p via this representation, $LH_p := \rho_p(H_p)$, is the so called the *linear isotropy group* at

[8]A Lie group is second countable if and only if it has at most countably many components. In particular the the identity component is second countable.

p. Furthermore, under the Theorem's hypothesis, the isotropy representation ρ_p is now identified with the representation

$$H_p \to GL(\mathfrak{g}/\mathfrak{h}_p)$$

induced by the restriction of the adjoint representation Ad_G of G (on itself) to H_p.

Definition 24.
Let M be a manifold and D a distribution on M. Given a connected Lie group G, we call (M, D) a *G-homogeneous distribution* if G acts transitively on M and D is G-invariant.

Consider a G-homogeneous distribution (M, D). From the previous Theorem all the isotropy groups are isomorphic (as Lie groups) and then all the quotient spaces G/H_p; we can then simply denote them with H and with $\mathfrak{h} \subset \mathfrak{g}$ their lie algebra. From the identification of the Theorem it follows that for each $p \in M$ the subspace $D_p \subset T_pM$ corresponds to an invariant subspace $W \subset \mathfrak{g}/\mathfrak{h}$. We conclude that every G-homogeneous distribution corresponds to data (G, H, W) where $H \subset G$ is a closed subgroup and $W \subset \mathfrak{g}/\mathfrak{h}$ is an H-invariant subspace. In particular, the adjoint action of G defines an equivalence relation on the set of pairs (H, W) so that different choices of base points on M correspond to equivalent pairs $(H, W) \sim (H', W')$. On the other hand:

Proposition 35.
Let G be a connected Lie group, H a closed subgroup of G and $W \subset \mathfrak{g}/\mathfrak{h}$. Then (G, H, W) determines a G-homogeneous distribution (M, W).

Proof. (sketch)
Given the data (G, H, W) where $H \subset G$ is a closed subgroup and $W \subset \mathfrak{g}/\mathfrak{h}$ is an H-invariant subspace, consider the coset space $M := G/H$ and the distribution

$$D_{[e]} := W \subset \mathfrak{g}/\mathfrak{h} \cong T_{[e]}(G/H)$$

It is defined a left action (left translation) of G on M by $L : G \times M \to M$ by

$$L_a(g \cdot H) = L_a(g) \cdot H$$

We can then push $D_{[e]}$ around to all other points of M by the action of G on M defined by left translations on the right H-coset space. The distribution D obtained is by definition a G-invariant distribution $D \subset TM$.

□

Corollary 13.
The data $(\mathfrak{g}, \mathfrak{h}, W)$ determines a G-homogeneous distribution (M, D) up to a cover.

These results lead to the following.

Theorem 40.
Let (M, D) be a G-homogeneous distribution with datas (H, W) and $G_1 \subset G$ be a subgroup such that the restriction of the G action to G_1 is still transitive on M. Then the G datas (H, W) and the G_1 datas (H_1, W_1) yield diffeomorphic manifolds with distributions.

Proof.
Suppose that (M, D) is a G-homogeneous distribution with datas (H, W) and $G_1 \subset G$ is a subgroup such that the restriction of the G action to G_1 is still transitive on M. Then (M, D) is also a G_1-homogeneous distribution with associated the G_1 datas (H_1, W_1) where

$$H_1 = H \cap G_1$$

while W_1 corrisponds to W under the linear isomorphism

$$\mathfrak{g}_1/\mathfrak{h}_1 \to \mathfrak{g}/\mathfrak{h}$$

induced by the diffeomorphism

$$M = G/H = G_1/H_1$$

Then the G datas (H, W) and the G_1 datas (H_1, W_1) yield diffeomorphic manifolds with distributions.

□

The proof developed in [5] is based on an equivalent description of the configuration manifold of the problem. In particular, in this description the configuration space of the problem is the five dimensional manifold $SO(3) \times S^2$ with a 2-distribution D depending of the ratio ρ; then the proof consists on the following. Consider the covering

$$S^2 \times S^3 \to S^2 \times SO(3)$$

The elements of $SO(3) \times SO(3)$ are symmetries for the problem and the isotropy group related to this action on $S^2 \times SO(3)$ is the circle subgroup H with lie algebra $\mathfrak{h}$. This gives the algebric datas $(\mathfrak{so}(3) \times \mathfrak{so}(3), \mathfrak{h}, D)$. On the other hand in the previous chapter we have seen that G_2^s is the automprhism group of the algebra $(\mathbb{O}_s, N_s)$ of split octonions. In particular we have seen that G_2^s, preserve the quadratic form N_s and acts transitively on the spherization $\mathbf{K} \cong S^2 \times S^3$ of the cone $N_s^{-1}(0)$ that is, $\mathbf{K}$ is a G_2^s- homogeneous space. Consider for each $x \in K \setminus \{0\}$ the 3-distribution Δ defined in each point x by

$$\Delta_x = \{y \in \mathbb{R}^7 \mid xy = 0\}$$

i.e. as the set of zero divisors of x. On the quotient respect to the $\mathbb{R}_+$ action it defines a 2-distribution $\boldsymbol{\Delta} := \Delta/\mathbb{R}_+$ on $\mathbf{K}$.

Proposition 36.
The distribution $(\mathbf{K}, \Delta)$ is a G_2^s-homogeneous distribution.

Proof.
We have to show that Δ_x is G_2^s- invariant, where

$$\Delta_x = \{y \in \mathbb{R}^7 \mid xy = 0\}$$

For each $x, y \in \mathbb{O}_s$ we have $\phi(xy) = \phi(x)\phi(y)$. Since ϕ is linear, $d\phi_x = \phi$ and hence $d\phi_x(y)\phi(x) = \phi(xy) = 0$ then $d\phi_x(y) \in \Delta_{\phi(x)}$ hence the distribution is G_2^s-invariant.

□

The algebric datas related to the distribution $(\mathbf{K}, \mathbf{\Delta})$ are the datas $(\mathfrak{g}_2, \mathfrak{p}, W_1)$ where $\mathfrak{p} \subset \mathfrak{h}$ is a particular subalgebra (iperbolic subalgebra) defined by a specific choice of root vectors. The proof is then based on an application of the previous Theorem to these pairs of algebric datas, showing that $\mathfrak{so}(3) \times \mathfrak{so}(3)$ embeds in $\mathfrak{g}_2$. In this setting, the more general Theorem that we have advance at the beginning of the section becomes following.

Theorem 41 (R.Bryant)**.**
The connected component of the identity in $Aut(S^2 \times S^3, D)$ for radius ratio $1 : 3$ is isomorphic to G_2. The G_2 action on $S^2 \times S^3$ does not descend to $S^2 \times SO(3)$, but its restriction to the maximal compact $C \subset G_2$ does, covering the $SO(3) \times SO(3)$ action on $S^2 \times SO(3)$. For any other radius ratio (other then 1:1) $Sym(S^2 \times S^3, \mathbf{D})$ is isomorphic to C.

The second approach developed in [1] is based on an octonionic description of the configuration space of the problem and it's the starting point for the singular solution space construction. Consider the two spheres S^2_r and S^2_R respectively of rays r, R and let $M_{r,R}$ be the configuration manifold. We can reduce the problem to a couple of spheres of unitary rays rescaling the distribution, indeed the homothety

$$i_p : M_{R,r} \to M_{1,1}$$
$$i_p(A) = \rho A$$

transforms $M_{R,r}$ in $M_{1,1}$ and, denoted by $M := M_{1,1}$, the principal bundle in the principal bundle

$$SO(2) \to M \to S^2 \times S^2$$

where S^2 is the unit sphere. Let $\rho := \frac{R}{r}$ be the ratio of the two rays. The distribution $D^{R,r}$ on $M_{R,r}$ is rescaled by i_p and becomes the distribution D^ρ on M obtained by rescaled no-slipping condition

$$A\dot{x}(t) = \rho\dot{\hat{x}}(t)$$

Notice that if the solution in one of the spheres is arc-lenght parametrized by constant speed, the one in the other has speed ρ. Let's show how the quaternionic and octonionic descriptions of the system explains the geometrical properties of such distribution. Consider the unit tangent bundle

$$SO(2) \to NS^2 \to S^2$$

on the 2-sphere, that in quaternionic description takes form

$$NS^2 = \big\{(q, v) \in \mathbb{H}_m \times \mathbb{H}_m \mid |q| = |v| = 1,\ \langle q, v\rangle = 0\big\}$$

and the quaternionic Hopf fibration $\mathfrak{h} : S^3 \to S^2$

$$\mathfrak{h}(w) = \overline{w} i w$$

At each point of S^3 is defined a 2-dimensional distribution D_w by

$$D_w := {S^1_w}^\perp := \operatorname{span}\langle jw, kw\rangle$$

that is, as the space orthogonal to the fiber.

Proposition 37.
Let $\pi : NS^2 \to S^2$ and $\mathfrak{h} : S^3 \to S^2$ be the unit tangent bundle and the Hopf bundle respectively. The map

$$\psi : S^3 \to NS^2$$

$$\psi(w) = (\overline{w}iw, \overline{w}jw)$$

is a fiberwise map that induces a double covering $S^1_w \to S^1_{\psi(w)}$ of the fibers. Moreover:
i) The map transforms the distribution D_w in the distribution that defines the standard Levi-Civita connection for S^2.
ii) The diagram

$$\begin{tikzcd} S^3 \arrow[r, "\psi"] \arrow[d, "\mathfrak{h}_2"] & NS^2 \arrow[d, "\mathfrak{n}"] \\ S^2 \arrow[r, leftrightarrow, "Id"] & S^2 \end{tikzcd}$$

commutes.

It follows that for each pairs $w_1, w_2 \in S^3$ there exists a unique orientation preserving isometry of the fibers of $\mathfrak{h}$ i.e. a map

$$S^1_{w_1} \to S^1_{w_2}$$

that sends w_1 to w_2. Moreover pairs $(w_1, w_2), (w_1', w_2') \in S^3 \times S^3$ define the same isometry if and only if $w_1' = e^{i\theta} w_1$ and $w_2' = e^{i\theta} w_2$. Hence the coset space $\mathbf{M} := S^3 \times S^3 / \sim$ defines a double covering

$$\pi : \mathbf{M} \to M$$

Let's see the distributions.

Proposition 38.
Let $\mathbf{D}^\rho$ be the distribution defined on $\mathbf{M}$ by the pushforward of D^ρ by the double covering $\pi : \mathbf{M} \to M$. Then

$$\mathbf{D}^\rho_{\pi(w_1,w_2)} = \pi_* C_{(w_1,w_2)}$$

where C is the distribution on $S^3 \times S^3$ defined by

$$C_{(w_1,w_2)} := span\,\langle (jw_1, \rho j w_2), (kw_1, \rho k w_2)\rangle$$

In the next diagram we summarized what we've done.

$$\begin{tikzcd} \pi_* C_{(w_1,w_2)} & C_{(w_1,w_2)} \arrow[l, dotted, "\pi_*"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{D}^\rho & \mathbf{M} \arrow[d, "p"] & S^3 \times S^3 \arrow[l, "\pi"'] \arrow[r, "\psi"] \arrow[d, "\mathfrak{h}_2"] & NS^2 \times NS^2 \arrow[d, "\pi"] \\ D^\rho \arrow[u, dotted, "p^*"] & M & S^2 \times S^2 \arrow[ul, dotted, "\pi_* \mathfrak{h}_2^*"] \arrow[r, leftrightarrow, "\mathrm{Id}"] & S^2 \times S^2 \end{tikzcd}$$

We now give an explicit diffeomorphism from $\mathbf{K}$ to $\mathbf{M}$.

Theorem 42 (A.Agrachev)**.**
The map $\Phi : \mathbf{M} \to \mathbf{K}$ *defined by*

$$\Phi([w_1 + lw_2]) = [\overline{w}_1 i w_1 + l(\overline{w}_1 w_2)]$$

is a diffeomorphism and transforms the rolling distribution for ratio 3 *in* $\mathbf{\Delta}$*.*

Proof.
The inverse map $\Phi^{-1} : \mathbf{K} \to \mathbf{M}$ is defined as following. Consider $v_1 \in S^2$ with $v_1 = \mathfrak{h}(w_1)$. If $v_1 + lv_2 \in S^2 \times S^3$ then

$$\Phi^{-1}(v_1 + lv_2) := \pi(w_1 + lw_1 w_2)$$

It is differentiable and it's easy to check that composed with Φ returns the identity. Let's see the distributions. We must prove that the product

$$\Phi(x)(D_x \Phi v) = 0$$

for any $x = w_1 + lw_2 \in S^2 \in \mathbf{M}$ and $v = zjw_1 + 3l(zjw_2) \in \mathbf{D}^3$ such that $|w_1| = |w_2|$ and $z \in \mathbb{C}$. Computing explicitly in the case $|w_1| = |w_2| = 1$ we obtain

$$D_{w_1 + lw_2}\Phi(zjw_1 + l(zjw_2)) = 2\overline{w}_1 zkw_1 + 2l(\overline{w}_1 zjw_2)$$

instead for $|w_1| \neq 1$ is enough to divide the equation by the norm square. Finally if we multiply the equation by $\Phi(w_1 + lw_2)$ we find that the result is zero, from which the statement follows.

□

Corollary 14.
For $\rho = 3$ *the distribution* $(\mathbf{M}, \mathbf{D})$ *is* G_2^s*-homogeneous, that is* $G_2^s \subset Sym(\mathbf{M}, \mathbf{D})$*.*

The Theorem explains the projective structure of singular curves for $\rho = 3$. Remark that an element of $x \in \mathrm{Im}(\mathbb{O})$ is a zero divisor if and only if $x^2 = N_s(x) = 0$, that is $x \in K$. This means that given $x_0 \in S^2 \times S^3$ and a point $y \in \Delta_{x_0}$ i.e such that $x_0 y = 0$, we have

$$\text{span}\,\langle x_0, y\rangle \subset K$$

hence the singular trajectory is the projectivization of the plane span $\langle x_0, y\rangle$. Since the G_2^s action as automorphism group of split octonions is linear, it preserves the linear structure and then projective one, hence defines an action on the singular curves if and only if the ratio is equal to 3 or $1/3$.

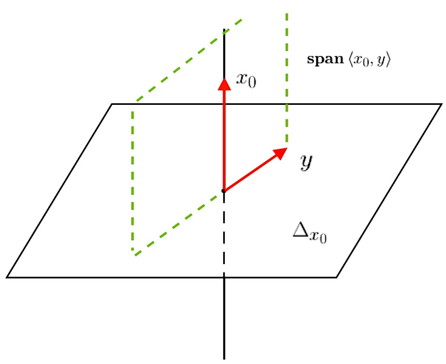
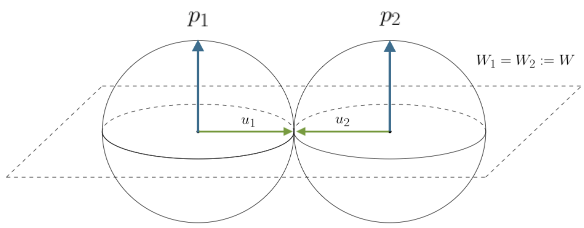


Figure 2.3: Singular solution as plane projection.

2.4 Singular Solutions Spaces

In this section we show how it is possible to describe the singular curves space of rolling spheres problem as a finite dimensional manifold. We know that a singular solution is a curve γ in M which projection on $S^2 \times S^2$ corresponds to a couple of geodetics on the two spheres. On the other hand a couple of points $(p_1, p_2) \in S^2 \times S^2$ defines two orthogonal planes $W_1 = p_1^\perp$ e $W_2 = p_2^\perp$ that identify a couple of geodetics $W_1 \cap S^2 \times W_2 \cap S^2$ on the spheres. Fixed now a couple $(u_1, u_2) \in W_1 \cap S^2 \times W_2 \cap S^2$ orthogonal to p_1 and p_2 respectively, by no-twisting condition necessary condition such that this pair of geodetics identifies a solution of the problem in M passing through (u_1, u_2) is that the geodetics are coplanar i.e. $W_1 = W_2 := W$. Notice that the points $p_1, -p_1 \in S^2$ identify the



same plane and then the four couples $(p_1, p_2), (p_1, -p_2), (-p_1, -p_2), (-p_1, p_2)$ the same couple of geodetics. Fixed an orientation of $\mathbb{R}^3$, by no slipping condition

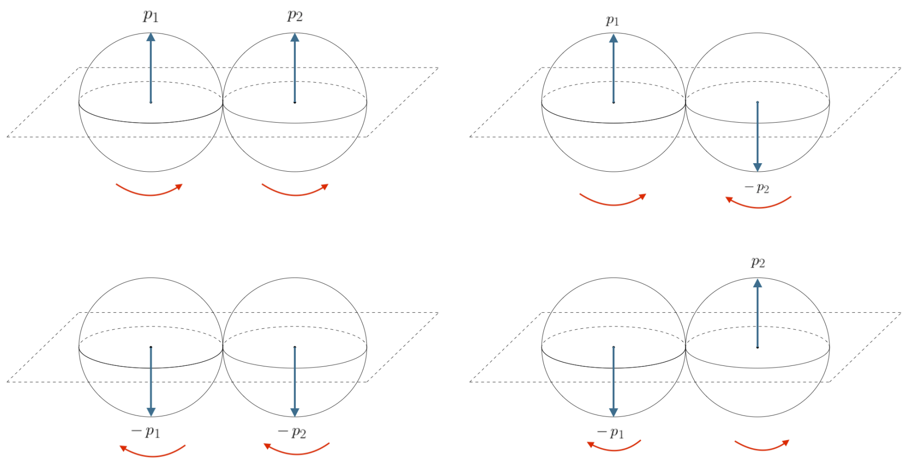


Figure 2.4: The four couple of points in $S^2 \times S^2$ giving the same couple of geodetics.

during the motion the two spheres have to roll with opposite side then only two of the configurations in figure are possible, the second and the fourth. We can then obtain the space of oriented singular solution identifying the first configuration with the second one and the third with the fourth by the action of the discrete group $\mathbb{Z}_2$. Now, for each couple of point $(u_1, u_2) \in S^2 \times S^2$ there exists at least one solution of the problem, i.e. a curve defined by D^ρ joining the two points. On the other hand by the rescaled slipping condition we know that if a curve in one of the two spheres has constant unitary speed, then the one in the other has speed ρ; then two couple of point $(u_1, u_2), (u_1', u_2') \in W \cap S^2 \times W \cap S^2$ belong to the same solution if and only if there exists an angle θ such that $(u_1', u_2') = (g_\theta u_1, g_{\rho\theta} u_2)$ where, if $g_\theta \in SO(2)$ is a rotation around the axis that defines W, $g_{\rho\theta} \in SO(2)$ is a rotation in the same direction but of an angle $\rho\theta$. Let's see how to describe it formally. The space $W \cap S^2$ orthogonal to a point q, is exactly the fiber of $q \in S^2$ in the spheric bundle NS^2

$$\pi^{-1}(q) = \{(q, v) \mid v \in \ T_qS^2, \ |v| = 1\}$$

then we define the action of $SO(2)$ on $NS^2 \times NS^2$ by

$$SO(2) \times (NS^2 \times NS^2) \to NS^2 \times NS^2$$

$$g_\theta \times (u, v)_m \to (g_\theta u, g_{\rho\theta} v)_m$$

Let's denote by $SO(2)_\rho$ the group $SO(2)$ with the action associated to the ratio ρ. We define the space

$$N_\rho := NS^2 \times NS^2 / SO(2)_\rho$$

Using quaternions it is possible to give a useful parametrization of N_ρ. The exponential of an imaginary quaternion h is defined by

$$e^{\theta h} := \cos(\theta) + h \sin(\theta)$$

while the rotation of a vector $v \in \mathbb{H}_m$ around an unitary axis $h \in \mathbb{H}_m$ of an angle θ is defined by the map

$$R_h(\theta)(x) = e^{-\frac{\theta}{2}h} v e^{\frac{\theta}{2}h}$$

Given $(q, v) \in NS^2$, since v and q are orthogonal. they anti-commute hence the rotation becomes

$$R_q(\theta)(v) = e^{-\frac{\theta}{2}q} v e^{\frac{\theta}{2}q} = R_q(\theta)(v) = e^{-\theta q} v$$

We can then parametrize the fiber of (q, v) as

$$\pi^{-1}(q) = \{(q, e^{\theta q} v_0) \mid \theta \in (0, 2\pi)\}$$

Consider now the fiber bundle product

$$\mathbb{T}^2 \to NS^2 \times NS^2 \to S^2 \times S^2$$

Given $(q, p) \in S^2 \times S^2$, denoted by $m = (q, p)$ and by $(u, v)_m$ a point of its fiber, we obtain the parametrization

$$\pi^{-1}(m) = \left\{(e^{\theta_1 q} u, e^{\theta_2 p} v)_m \mid \theta_1, \theta_2 \in [0, 2\pi)\right\}$$

It follows that the equivalent classes on the quotient N_ρ respect to the $SO(2)_\rho$ action in the quaternionic parametrization take the form

$$[u, v]_m = \left\{(e^{\theta q} u, e^{\rho\theta p} v)_m \in NS^2 \times NS^2 \mid \theta \in [0, 2\pi)\right\}$$

Recall the well known result about helices in $\mathbb{T}^2$.

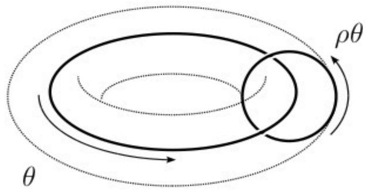


Figure 2.5: The action of $SO(2)_\rho$ on the fiber $N_qS^2 \times N_pS^2 \cong \mathbb{T}^2$

Lemma 12.
Consider the function

$$f : \mathbb{R} \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}_2$$
$$t \to [\omega_1 t, \omega_2 t]$$

and denote $\omega := \frac{\omega_1}{\omega_2}$

i) If $\omega \in \mathbb{Q}$ *the map* f *is not injective and the orbit is a closed periodic helix .*
ii) If $\omega \notin \mathbb{Q}$ *the map* f *is injective and the orbit is dense in* $\mathbb{T}^2$*.*

Using this result, we can prove the following.

Proposition 39.
The action of $SO(2)_\rho$ *on* $NS^2 \times NS^2$ *is proper and free, then the quotient* N_ρ *is a five dimensional differential manifold. In particular for ratio* $\rho \in \mathbb{Q}$ *it's a principal bundle*

$$\pi_\rho : N_\rho \to S^2 \times S^2$$

with fiber S^1*.*

Proof.
i) The action of $SO(2)_\rho$ is proper since the group $SO(2) \cong S^1$ is compact. Let's see that it's also free. Consider $\theta_0 \in SO(2)$ and $(u, v)_m$ such that

$$(e^{\theta_0 q}u, e^{\rho\theta_0 p}v)_m = (u, v)_m$$

then

$$\begin{cases} \theta_0 = 2\pi k \quad , \ k \in \mathbb{Z} \\ \rho\theta_0 = 2\pi m \quad , \ m \in \mathbb{Z} \end{cases}$$

hence $\theta_0 = 0 \mod 2\pi$ that is, it's the identity of $SO(2)$.
ii) Since the action moves only the fibers of $NS^2 \times NS^2$ that are copies of $\mathbb{T}^2$, for ratios $\rho \in \mathbb{Q}$ the fibers of N_ρ are the quotients $\mathbb{T}^2/S^1 \cong S^1$ and hence N_ρ a principal bundle

$$SO(2) \to N_\rho \to S^2 \times S^2$$

□

Recall that not all the points of N_ρ represent singular solutions. Anyway fixed $(p, q) \in S^2 \times S^2$ and (u, v) on their fibers the action, varying $\theta \in [0, 2\pi)$,

$(e^{\theta p}u, e^{\theta q}v)$ moves u and v around $\mathrm{Im}(e^{\theta p})$ and $\mathrm{Im}(e^{\theta q})$ respectively in the directions

$$u \wedge \mathrm{Im}(e^{\theta p}) \qquad v \wedge \mathrm{Im}(e^{\theta q})$$

We can define the action of $\mathbb{Z}_2$ on N_k for $\varepsilon \in \{-1, 1\}$ by

$$\mathbb{Z}_2 \times N_k \to N_k$$

$$\varepsilon \times [u, v]_m \to [u, v]_{\varepsilon m}$$

where if $m = (q, p)$ then $\varepsilon m := (q, \varepsilon p)$. Then explicitly

$$[u, v]_{\varepsilon m} = \Big\{(q, e^{\theta q}u, \varepsilon p, e^{\rho\theta\varepsilon p}v) \in NS^2 \times NS^2 \mid \theta \in [0, 2\pi)\ ,\ \varepsilon \in \{-1, 1\}\Big\}$$

This action is clary smooth, free and properly discontinuous and then the quotient $N_k/\mathbb{Z}_2$ is a five dimensional manifold, that is the effective *space of oriented singular solutions*. Since the quotient is defined by a finite discrete group, the projection $N_k \to N_k/\mathbb{Z}_2$ is a covering map then we can simply focus on N_k; as we'll see in next subsection, the space N_ρ is simply connected and it's then its universal cover. Notice that the action of $SO(2)_\rho$ is defined by a choice of one the two spheres, that is considered moving with unitary speed. We now show that the other choice is equivalent.

Proposition 40.
The diffeomorphisms $N_\rho \cong N_{1/\rho}$ holds for each $\rho > 0$.

Proof.
The two actions are defined by

$$g \cdot_1 (u, v)_m = (g_\rho u, gv)_m$$

$$g \cdot_2 (u, v)_m = (g_{1/\rho} u, gv)_m$$

Let $(u, v)_m$ be an element of $NS^2 \times NS^2$ and consider the maps

$$\varphi : SO(2) \to SO(2) \qquad \sigma : NS^2 \times NS^2 \to NS^2 \times NS^2$$

$$g_\alpha \to g_{\alpha/\rho} \qquad \sigma\big((u, v)_m\big) = (v, u)_{m'}$$

where if $m = (q, p)$ then $m' = (p, q)$. The map φ is a groups homomorphism indeed if $g_\alpha, g_\beta \in SO(2)$ are rotations of angles α e β then

$$\varphi(g_\alpha) \cdot \varphi(g_\beta) = g_{\frac{\alpha}{\rho}} \cdot g_{\frac{\beta}{\rho}} = g_{\frac{\alpha+\beta}{\rho}} = \varphi(g_\alpha \cdot g_\beta)$$

Let's show that the map is equivariant i.e

$$\sigma\big(g \cdot_2 (u, v)_m\big) = \varphi(g) \cdot_1 \sigma\big((u, v)_m\big)$$

From definitions

$$\varphi(g) \cdot_1 \sigma\big((u, v)_m\big) = g_{1/\rho} \cdot_1 (v, u)_{m'} = (gv, g_{1/\rho}u)_{m'} =$$

$$= \sigma\big((g_{1/\rho}u, gv)_m\big) = \sigma\big(g \cdot_2 (u, v)_m\big)$$

Moreover $\sigma^2 = Id$ and its then σ bijective, differentiable and equivariant, then it extends to the quotient to an diffeomorphism.

□

2.4.1 Split-Octonions Description

We now show how it is possible to describe the space N_ρ using split octonions and use this description to compute some topological invariants of such family of spaces. In the previous section we have defined the 2-covering map

$$\psi : S^3 \to NS^2$$

$$\psi(w) = (\overline{w}iw, \overline{w}jw)$$

Then if $S^3 \times S^3 \subset \hat{\mathbb{O}} \cong \mathbb{H} \times \mathbb{H}$ are two copies of the 3-spheres the map

$$\psi_d : S^3 \times S^3 \to N_s^1 \times N_s^2$$

$$\psi_d(w_1, w_2) = (\psi(w_1), \psi(w_2))$$

is still a covering map.

Lemma 13.
Consider the map $\psi : S^3 \to NS^2$ defined by $\psi(w) = (\overline{w}iw, \overline{w}jw)$ and let $w \in S^3$. If $\psi(w) = (q, e^{\theta_0 q}u)$ then

$$\psi(e^{i\alpha}w) = \big(q, e^{(\theta_0 - 2\alpha)q}u\big)$$

Proof.
Given $w \in S^3$ by definition $\psi(w) = (\overline{w}iw, \overline{w}jw)$. Denote $q := \overline{w}iw$; fixed an arbitrary element $u \in q^\perp$ there exists an angle θ_0 such that

$$\overline{w}jw = e^{\theta_0 q}u$$

then finally $\psi(w) = (q, e^{\theta_0 q}u)$. Instead if $\alpha \in [0, 2\pi]$ then

$$\psi(e^{\alpha i}w) = (\overline{w}iw, \overline{w}e^{-i\alpha}je^{i\alpha}w) = (q, \overline{w}e^{-2i\alpha}jw)$$

Since they are in the same fiber, there exists an angle φ such that $\overline{w}e^{-2\alpha i}jw = e^{\varphi q}u$. By definition $q = \overline{w}iw$ and hence since $\|w\| = 1$ and $\overline{w}w = 1$ we have

$$\overline{w}e^{-2\alpha i}jw = e^{\varphi q}u = \overline{w}e^{\varphi i}wu$$

if we multiply on the left by w

$$e^{-2\alpha i}jw = e^{\varphi i}wu$$

from which

$$\overline{w}jw = \overline{w}e^{(2\alpha+\varphi)i}wu = e^{(2\alpha+\varphi)q}u$$

By hypothesis, $\overline{w}jw = e^{\theta_0 q}u$ and then must be $\theta_0 = 2\alpha + \varphi \mod 2\pi$ that is $\varphi = \theta_0 - 2\alpha \mod 2\pi$ from which follows

$$\psi(e^{i\alpha}w) = \big(q, e^{(\theta_0 - 2\alpha)q}u\big)$$

□

Consider $S^3 \times S^3 \subset \mathbb{H} \times \mathbb{H}$ and the group action $SO(2)_\rho$ on $S^3 \times S^3$ defined by the product of $(e^{\theta i}, e^{\rho\theta i})$. Let's denote the quotient respect to this action by

$$\mathbf{N}_\rho := S^3 \times S^3/(e^{\theta i}, e^{\rho\theta i})$$

We have the following.

Proposition 41.
For each values of $\rho > 0$ *the fiberwise map* $\psi_d : S^3 \times S^3 \to NS^2 \times NS^2$ *extends to the quotients* $\mathbf{N}_\rho$ *and* N_ρ *to a map*

$$\psi_d : \mathbf{N}_\rho \to N_\rho$$

$$\psi_d([w_1, w_2]) = [\psi(w_1), \psi(w_2)]$$

that is a (global) diffeomorphism; in particular for $\rho \in \mathbb{Q}$, $\rho > 0$ *it's a diffeomorphism of principal bundles.*

Proof.
We first show that the map is well defined on the quotient. Let (w_1, w_2) and $(e^{\theta i} w_1, e^{\rho\theta i} w_2)$ be two elements of $S^3 \times S^3$ in the same class of $\mathbf{N}_\rho$. For the previous Lemma if $\psi(w_1, w_2) = (v_1, v_2)_m$ then

$$\psi(e^{\theta i} w_1, e^{\rho\theta i} w_2) = (e^{-2\theta i} v_1, e^{-2\rho\theta i} v_2)_m$$

hence for $\beta = -2\theta$ we conclude that $(e^{\beta i} v_1, e^{\rho\beta i} v_2)_m \sim (v_1, v_2)_m$ that is, the map is well defined. Now, since each of the two copies of ψ is a 2-covering with $\psi(w) = \psi(-w)$ and since in the quotient $\mathbf{N}_\rho$ we have $[w] = [-w]$ the map ψ_d became injective on $\mathbf{N}_\rho$ and then defines a global diffeomorphism from $\mathbf{N}_\rho$ and N_ρ.

□

Corollary 15.
For ratio $\rho = 1$ *the space* $\mathbf{N}_1$ *is a double covering of the configuration space of the system* M*.*

Proof.
Notice that for $\rho = 1$, i.e. in the integrable case for the distribution D_ρ, the quotient space $\mathbf{N}_\rho$ is exactly the space $\mathbf{M}$ that is a double covering of the configuration space.

□

We have a family of manifolds defined by different actions of $SO(2)$ that depend on the parameter ρ. The natural question is then if the elements of this family are topologically equivalent or not. In order to answer this question, we can use the quaternionc description to compute some of their principal topological invariants. Since for irrational values of the ratio the topology of the fibers of these manifolds is not clear, we consider only rational values.

Theorem 43.
Consider $\rho \in \mathbb{Q}$, $\rho > 0$ *and the fiber bundle* $S^1 \to S^3 \times S^3 \to \mathbf{N}_\rho$*. Let* $\pi_k(\mathbf{N}_\rho)$ *be the* k*-homotopy group. Then it doesn't depend on the value of the ratio* ρ*. In particular*

$$\pi_1(\mathbf{N}_\rho) = \{0\} \qquad \pi_2(\mathbf{N}_\rho) \cong \mathbb{Z}$$

while for $k > 2$

$$\pi_k(\mathbf{N}_\rho) \cong \pi_k(S^3) \times \pi_k(S^3)$$

Proof.
The fiber bundle

$$S^1 \to S^3 \times S^3 \to \mathbf{N}_\rho$$

induces a long exact sequence

$$\cdots \to \pi_k(S^1) \to \pi_k(S^3 \times S^3) \to \pi_k(\mathbf{N}_\rho) \to \pi_{k-1}(S^1) \to \pi_{k-1}(S^3 \times S^3) \to \pi_{k-1}(\mathbf{N}_\rho) \to \dots$$

$$\dots$$

$$\cdots \to \pi_2(S^1) \to \pi_2(S^3 \times S^3) \to \pi_2(\mathbf{N}_\rho) \to \pi_1(S^1) \to \pi_1(S^3 \times S^3) \to \pi_1(\mathbf{N}_\rho) \to \dots$$

$$\cdots \to \pi_0(S^1) \to \pi_0(S^3 \times S^3) \to \pi_0(\mathbf{N}_\rho) \to \{0\}$$

That since

$$\pi_k(S^1) = \{0\}\,, k > 1 \qquad \pi_1(S^1) = \mathbb{Z}\,, \qquad \pi_2(S^3 \times S^3) = \{0\}\,, \quad \pi_1(S^3 \times S^3) = \{0\}$$

becomes

$$\cdots \to \{0\} \to \pi_k(S^3 \times S^3) \to \pi_k(\mathbf{N}_\rho) \to \{0\} \to \pi_{k-1}(S^3 \times S^3) \to \pi_{k-1}(\mathbf{N}_\rho) \to \dots$$

$$\dots$$

$$\cdots \to \{0\} \to \{0\} \to \pi_2(\mathbf{N}_\rho) \to \mathbb{Z} \to \{0\} \to \pi_1(\mathbf{N}_\rho) \to \dots$$

$$\cdots \to \{0\} \to \{0\} \to \pi_0(\mathbf{N}_\rho) \to \{0\}$$

Then $\pi_1(\mathbf{N}_\rho) = \{0\}$ and $\pi_2(\mathbf{N}_\rho) \cong \mathbb{Z}$ while for $k > 2$

$$\pi_k(\mathbf{N}_\rho) \cong \pi_k(S^3) \times \pi_k(S^3)$$

□

Corollary 16.
For each $\rho \in \mathbb{Q}$, $\rho > 0$, the spaces $\mathbf{N}_\rho$ are simply connected and then for each of these values the space $\mathbf{N}_\rho$ is the universal cover of the oriented singular solutions space $N_\rho/\mathbb{Z}_2$.

Moreover also the following holds.

Proposition 42.
Consider $\rho \in \mathbb{Q}$, $\rho > 0$ and the fiber bundle $S^1 \to S^3 \times S^3 \to \mathbf{N}_\rho$. Let $H_k(\mathbf{N}_\rho)$ be the k-th homology group. Then it doesn't depend on the value of the ratio ρ. In particular

$$H_1(\mathbf{N}_k, \mathbb{Z}) = \{0\} \quad H_2(\mathbf{N}_k, \mathbb{Z}) = \mathbb{Z} \quad H_3(\mathbf{N}_k, \mathbb{Z}) = \mathbb{Z}$$

$$H_4(\mathbf{N}_k, \mathbb{Z}) = \{0\} \qquad H_5(\mathbf{N}_k, \mathbb{Z}) = \mathbb{Z}$$

Proof.
Since $\mathbf{N}_k$ is 1-connected by Hurewicz Theorem we know that

$$H_1(\mathbf{N}_k, \mathbb{Z}) = \pi_1(\mathbf{N}_k) = \{0\} \qquad H_2(\mathbf{N}_k, \mathbb{Z}) = \pi_2(\mathbf{N}_k) = \mathbb{Z}$$

Then since it's simply connected also $H_0(\mathbf{N}_\rho, \mathbb{Z}) \cong \mathbb{Z}$. Remark that given a topological space X and an abelian group A by universal coefficient Theorem there's a long exact sequence in omology

$$\{0\} \to H_k(X, \mathbb{Z}) \otimes A \to H_k(X, A) \to \mathrm{Tor}\big[H_{k-1}(X, \mathbb{Z}), A\big] \to \{0\}$$

For $A=\mathbb{R}$ and $k<3$, since $H_k(\mathbf{N}_\rho)$ are free groups, we have that $\mathrm{Tor}\big[H_{k-1}(\mathbf{N}_\rho),\mathbb{R}\big]=\{0\}$ and then

$$H_k(\mathbf{N}_\rho,\mathbb{R})\cong H_k(\mathbf{N}_\rho,\mathbb{Z})\otimes\mathbb{R}$$

Moreover for each abelian group A we have $\mathbb{Z}\otimes A\cong A$ then

$$H_0(\mathbf{N}_\rho,\mathbb{R})=H_0(\mathbf{N}_\rho,\mathbb{Z})\otimes\mathbb{R}=\mathbb{R}$$

$$H_1(\mathbf{N}_\rho,\mathbb{R})=H_1(\mathbf{N}_\rho,\mathbb{Z})\otimes\mathbb{R}=\{0\}$$

$$H_2(\mathbf{N}_\rho,\mathbb{R})=H_2(\mathbf{N}_\rho,\mathbb{Z})\otimes\mathbb{R}=\mathbb{R}$$

Instead since $\mathbf{N}_\rho$ is compact and orientable for the Poincarè Duality Theorem for $k\leq 5$

$$H^{n-k}_{dR}(\mathbf{N}_\rho)\cong H^k_{dR}(\mathbf{N}_\rho)\cong H_{n-k}(\mathbf{N}_\rho,\mathbb{R})$$

from which we obtain the relations

$$H^5_{dR}(\mathbf{N}_\rho)\cong\mathbb{R}\quad H^4_{dR}(\mathbf{N}_\rho)\cong\{0\}\quad H^3_{dR}(\mathbf{N}_\rho)\cong\mathbb{R}$$

$$H^2_{dR}(\mathbf{N}_\rho)\cong\mathbb{R}\quad H^1_{dR}(\mathbf{N}_\rho)\cong\{0\}\quad H^0_{dR}(\mathbf{N}_\rho)\cong\mathbb{R}$$

$\square$

Since all these invariants of the spaces $\mathbf{N}_\rho$ are independent of the ratio ρ, our question is still opened. Anyway, consider the spaces $N_3/\mathbb{Z}_2$ and $N_m/\mathbb{Z}_2$ for $m\neq 3$. As we have seen, the group $G_2^2\cong\mathrm{Aut}(\mathbb{O}_s)$ is the symmetry group of the problem if and only if $\rho=3$. This means that only for this value of the ratio for each $\phi\in G_2^s$ and for each singular solution γ we have that $\phi\circ\gamma$ is still a singular solution. This suggests the possibility to extend the action of G_2^s to $N_3/\mathbb{Z}_2$ and only to $N_3/\mathbb{Z}_2$. Notice that this prove that $N_m/\mathbb{Z}_2$ can not be homeomorphic to $N_3/\mathbb{Z}_2$, otherwise if we denote by $\Psi: N_3/\mathbb{Z}_2\to N_m/\mathbb{Z}_2$ this map and by

$$\mu_3: G_2^s\times N_3/\mathbb{Z}_2\to N_3/\mathbb{Z}_2$$

$$(\phi,[w_1,w_2]_3)\to\phi([w_1,w_2]_3)$$

the G_2^s-action, we could extend it to an action

$$\mu_m: G_2^s\times N_m/\mathbb{Z}_2\to N_m/\mathbb{Z}_2$$

defining it by

$$\mu_m\big(\phi,[w_1,w_2]_m\big):=\Psi\circ\phi\circ\Psi^{-1}([w_1,w_2]_m)$$

This is, clearly, not a proof of this fact, but motivate further studies in this direction.

2.4.2 Singular Solutions and Lens Spaces

We now show that the spaces $\mathbf{N}_\rho$ for integer value of the ratio ρ are related to another family of spaces, the *lens spaces*. This family of spaces is defined as the quotient of the odd-dimensional sphere $S^{2k-1}\subset\mathbb{C}^k$ by the action of a discrete group $\mathbb{Z}_n$. Consider $n,q_1,\ldots,q_k\in\mathbb{Z}$ such that the q_i are coprime to n. The $\mathbb{Z}_n$ action on S^{2k-1} is defined by

$$[m]\times(z_1,\ldots,z_k)\mapsto(e^{2\pi i\frac{m}{n}q_1}\cdot z_1,\ldots,e^{2\pi i\frac{m}{n}q_k}\cdot z_k)$$

Then the *lens space* $L(n;q_1,\dots q_k)$ is defined to be the quotient of S^{2k-1} by this free and properly discontinuos $\mathbb{Z}_n$-action. The fundamental group of all these lens spaces $L(n;q_1,\dots,q_k)$ is independent of the q_i, and in particular

$$\pi_1\big(L(n;q_1,\dots,q_k)\big) = \mathbb{Z}_n$$

Consider now the three dimensional case. The three dimensional lens space is usually defined to be the lens space $L(n;q) := L(n,1,q)$ on the three dimensional sphere S^3. It could be also defined starting by the solid ball with the following identification: first mark n equally spaced points on the equator of the solid ball, denote them $a_0,\dots,a_{n-1}$ then on the boundary of the ball, draw geodesic lines connecting the points to the north and south pole. Now identify spherical triangles by identifying the north pole to the south pole and the points a_i with a_{i+q} and a_{i+1} with a_{i+q+1}. The resulting space is homeomorphic to the lens space $L(n;q)$.

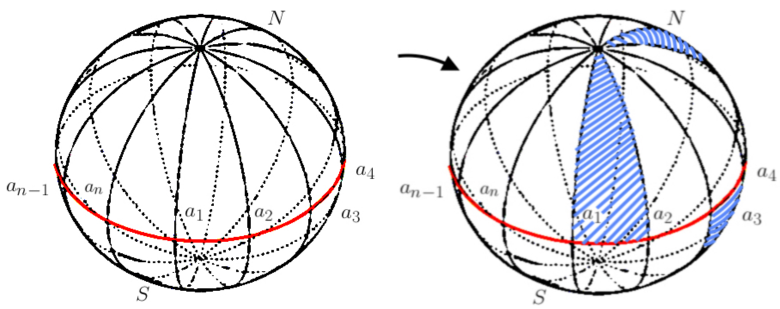


Figure 2.6: Lens spaces as quotient of the solid Ball: we identify the blue areas, following the identification of the poles N,S and the points $a_1,\dots a_n$; in the same way also the other spheric triangles are identified.

We now focus on the three dimensional lens space $L(n;-1)$ starting from another definition that explains its relation with the singular solutions space. Consider a general element $v \in S^2$, the Hopf fibration $\mathfrak{h} : S^3 \to S^2$ and an arbitrary element $w \in \mathfrak{h}^{-1}(v) = S^1_v$. We can define an action of $\mathbb{Z}_n$ on S^3 by

$$\mu_v : \mathbb{Z}_n \times S^3 \to S^3$$

$$\mu_v\big([m],u\big) := e^{\frac{2\pi m}{n}\overline{w}iw}u$$

This action doesn't depend on the choice of the element in the fiber S^1_v and it's free and properly discontinuous, hence the quotient

$$L^n_v := S^3\backslash(\mathbb{Z}_n,\mu_v)$$

is a differential manifold, in particular compact. In complex coordinates it simply becomes

$$\mu_v\big([m],(z_1,z_2)\big) := \big(e^{\frac{2\pi m}{n}\overline{w}iw}z_1, e^{-\frac{2\pi m}{n}\overline{w}iw}z_2\big)$$

Then it's clear that for $v=1$ we have the equality $L^n_1 = L(n;-1)$. Notice that in general the quotient depends on the choice of v. Consider now the double covering $\mathbf{M}$ of configuration space of the system and the cone $\mathbf{K} \cong S^2 \times S^3$. In the previous section we have seen that the map $\Phi : \mathbf{M} \to S^2 \times S^3$ defined by

$$\Phi([w_1 + lw_2]) = [\overline{w}_1 i w_1 + l(\overline{w}_1 w_2)]$$

is a diffeomorphism. From the previous discussion we know also that for ratio equal to one, we have $\mathbf{M} = \mathbf{N}_1$. The natural question is then if it is possible to generalize this map also for the other (integer) values of the ratio. It is possible to do that, but for it's inverse $\Phi^{-1} : S^2 \times S^3 \to \mathbf{M}$. Recall that it is defined as following. For $v_1 + lv_2 \in S^2 \times S^3$, if $v_1 \in S^2$ and $w_1 \in S^1_{v_1}$ then

$$\Phi^{-1}(v_1 + lv_2) := \pi(w_1 + lw_1 w_2)$$

where $\pi : S^3 \times S^3 \to \mathbf{M}$ is the projection on the quotient, that is the map

$$\pi_1 : S^3 \times S^3 \to \mathbf{N}_1$$

As we'll see, the relation between Lens spaces and the manifolds $\mathbf{N}_\rho$ is described by the generalization of this map. To define it we need first to represent quaternions in their complex representation.

Let $\mathbb{H} \cong \mathbb{C}^2$ be the complex structure on $\mathbb{H}$ such that each $w \in \mathbb{H}$ can be expressed as $w = z_1 + jz_2$ with $z_1, z_2 \in \mathbb{C}$ and consider the map

$$s_n : \mathbb{C}^2 \backslash \{0\} \to S^3$$

$$s_n(z_1, z_2) := \frac{\mathrm{pow}_n(z_1, z_2)}{\|\mathrm{pow}_n(z_1, z_2)\|}$$

where $\mathrm{pow}_n : \mathbb{C}^2 \to \mathbb{C}^2$ is defined by

$$\mathrm{pow}_n(z_1, z_2) := (z_1^n, z_2^n)$$

Proposition 43.
Consider $k \in \mathbb{Z}$, $k > 0$ and let $\pi_k : S^3 \times S^3 \to \mathbf{N}_k$ be the projection on the quotient respect to the $SO(2)_k$-action; let $s_k : \mathbb{C}^2 \backslash \{0\} \to S^3$ be as above. The map

$$p_k : S^2 \times S^3 \to \mathbf{N}_k$$

$$p_k(v_1 + lv_2) := \pi_k\big(s_k(w_1) + l(w_1 v_2)\big)$$

for an arbitrary element $w_1 \in S^1_{v_1} = \mathfrak{h}^{-1}(v_1)$ is well defined, that is it does't depend on the choice of w_1 in the fiber.

Proof.
Consider $v_1 + lv_2 \in S^2 \times S^3$ and $w_1 \in S^1_{v_1}$. If $w_1' \in S^1_{v_1}$ is another element of the fiber there exists an angle $\theta_0 \in [0, 2\pi)$ such that $w_1' = e^{\theta_0 i} w_1$ then if $w_1 = z_1 + jz_2$ and $w_1' = z_1' + jz_2'$

$$w_1' = z_1' + jz_2' = e^{i\theta_0} z_1 + je^{-\theta_0 i} z_2$$

It follows that

$$\mathrm{pow}_k(w_1') = z_1'^k + jz_2'^k = e^{k\theta_0 i} z_1^k + je^{-k\theta_0 i} z_2^k = e^{k\theta_0 i}\mathrm{pow}_k(w_1)$$

while

$$\left\|\mathrm{pow}_k(e^{\theta_0 i}w_1')\right\| = |e^{k\theta_0 i}| \cdot \|\mathrm{pow}_k(w_1)\| = \|\mathrm{pow}_k(w_1)\|$$

Hence

$$s_k(w_1') = \frac{\mathrm{pow}_k(w_1')}{\|\mathrm{pow}_k(w_1')\|} = \frac{\mathrm{pow}_k(e^{i\theta_0}w_1)}{\|\mathrm{pow}_k(e^{\theta_0 i}w_1)\|} = e^{k\theta_0 i} \cdot \frac{\mathrm{pow}_k(w_1)}{\|\mathrm{pow}_k(w_1)\|} = e^{k\theta_0 i} \cdot s_k(w_1)$$

then finally on $S^3 \times S^3$,

$$s_k(w_1') + l(w_1'v_2) = e^{k\theta_0 i}s_k(w_1) + l(e^{\theta_0 i}w_1 v_2)$$

This means that the element are related respect to the action induced by $SO(2)_k$ that is

$$\pi_k\big(s_k(w_1') + l(w_1'v_2)\big) = \pi_k\big(s_k(w_1) + l(w_1 v_2)\big)$$

and then the map is well defined.

□

We now prove some properties of this map.

Lemma 14.
Consider $k \in \mathbb{Z}$, $k > 0$. The map $p_k : S^2 \times S^3 \to \mathbf{N}_k$ is surjective. Moreover fixed a point $v_1 \in S^2$, for each choice of $v_2, v_2' \in S^3$

$$p_k(v_1 + lv_2) = p_k(v_1 + lv_2')$$

if and only if $v_2' = e^{\alpha_m \overline{w}_1 i w_1}v_2$ for one of the k angles $\alpha_m = \frac{2\pi m}{k}$ defined for $m = 1, \ldots, k$.

Proof.
i) Let's start showing that the map is surjective. Consider $[w_1, w_2] \in \mathbf{N}_k$ and let $(w_1, w_2) \in S^3 \times S^3$ be a representative; if in complex representation $w_1 = z_1 + lz_2$ with $z_1, z_2 \in \mathbb{C}$, let $\tilde{z}_1$ and $\tilde{z}_2$ be a pair of solutions for the equation $z^k = z_1$ and $z^k = z_2$. Consider

$$\tilde{w}_1 := \frac{\tilde{z}_1 + l\tilde{z}_2}{\sqrt{|\tilde{z}_1|^2 + |\tilde{z}_2|^2}} \in S^3 \qquad v_2 := \overline{\tilde{w}}_1 w_2 \in S^3$$

and $v_1 := \mathfrak{h}(\tilde{w}_1)$. Then the image of $v_1 + lv_2 \in S^2 \times S^3$ by p_k is

$$p_k(v_1 + lv_2) := \pi_k\left(\frac{\mathrm{pow}_k(\tilde{w}_1)}{\|\mathrm{pow}_k(\tilde{w}_1)\|} + l(\tilde{w}_1 v_2)\right) =$$

$$= \pi_k\big(w_1 + l\big((\tilde{w}_1(\overline{\tilde{w}}_1 w_2)\big)\big) = \pi_k\big(w_1 + l(w_2)\big) = [w_1, w_2]$$

ii) Consider now $v_1 + lv_2, v_1 + lv_2' \in S^2 \times S^3$ such that $p_k(v_1 + lv_2) = p_k(v_1 + lv_2')$ and $w_1 \in S^1_{v_1}$. We can choose w_1 in the fiber $S^1_{v_1}$ for both the points in $S^2 \times S^3$, hence $[s_k(w_1), w_1 v_2] = [s_k(w_1), w_1 v_2]$. This means that for some $\alpha \in [0, 2\pi)$

$$w_1 v_2 = e^{\alpha i}w_1 v_2'$$

if we multiply on the left by $\overline{w}_1$ we obtain

$$v_2 = \overline{w}_1 e^{\theta i}w_1 v_2' = e^{\theta \overline{w}_1 i w_1}v_2'$$

Notice that it does not depend on the choice of the representative in the equivalence class in $S^1_{v_1}$. Indeed if $w_1' = e^{\theta i} w_1$ is another choice then

$$e^{\alpha \overline{w}_1' i w_1'} v_2 = \overline{w}_1' e^{\alpha i} w_1' v_2 = \overline{w}_1 e^{-\theta i} e^{\alpha i} e^{\theta i} w_1 v_2 = e^{\alpha \overline{w}_1 i w_1} v_2$$

Consider now the points $v_1 + l(e^{\alpha \overline{w}_1 i w_1} v_2) \in S^2 \times S^3$ with $w_1 \in S^1_{v_1}$ by varying $\alpha \in [0, 2\pi)$. We have

$$p_k\big(v_1 + l\big(e^{\alpha \overline{w}_1 i w_1} v_2\big)\big) := \pi_k\big(s_k(w_1) + l(w_1 e^{\alpha \overline{w}_1 i w_1} v_2)\big) = \pi_k\big(s_k(w_1) + l(e^{\alpha i} w_1 v_2)\big) =$$

$$= [s_k(w_1), e^{\alpha i} w_1 v_2]$$

Hence the curve $\gamma : S^1 \to \mathbf{N}_k$

$$\alpha \to \gamma(\alpha) = [s_k(w_1), e^{\alpha i} w_1 v_2]$$

is a covering of the fiber $S^1_{[s_k(w_1), w_1 v_2]}$ of $\mathbf{N}_k$ as principal bundle. Let's show that for k values of $0 \le \alpha < 2\pi$ the curve returns to the same point. Let $\alpha \in [0, 2\pi)$ be such that

$$[s_k(w_1), e^{\alpha i} w_1 v_2] = [s_k(w_1), w_1 v_2]$$

then

$$(e^{k\theta_1 i} s_k(w_1), e^{(\theta_1 + \alpha) i} w_1 v_2)_m = (s_k(w_1), w_1 v_2)_m$$

hence

$$\begin{cases} k\theta_1 = 2\pi m \\ \theta_1 + \alpha = 2\pi r \end{cases} \tag{2.4}$$

where $m, r \in \mathbb{Z}$, that has solutions

$$\alpha = -\frac{2\pi m}{k} + 2\pi r = 2\pi\big(-\frac{m}{k} + r\big)$$

In particular *mod* 2π we obtain the points

$$\alpha_m = \frac{2\pi m}{k}$$

that for $m = 0, \dots k-1$ are k distinct points. The statement follows since for the first part such k points are the unique with the same image of $v_1 + l v_2$.

$\square$

The points described in the Theorem are exactly the points that are identified in the definition of lens space given at the beginning. Fixed an element of S^2 it is then possible to define a map from a Lens space to the manifold $\mathbf{N}_k$. In particular we have the following.

Theorem 44.
Consider $k \in \mathbb{Z}$, $k > 0$ and the map $\tau_v : L^k_v \to \mathbf{N}_k$ defined by

$$\tau_v([u]_k) := \pi_k\big((s_k(w), wu)\big)$$

where $w \in \mathfrak{h}^{-1}(v)$ is an element of the Hopf fiber S^1_v. The map τ_v is an embedding for each $v \in S^2$ i.e L^k_v is a smooth submanifold of $\mathbf{N}_k$.

Proof.
Consider the map $t_v : S^3 \to S^3 \times S^3$ defined by $t_v(u) = (s_k(w), wu)$. If in complex coordinates $w = z_1 + jz_2$ and $u = (z_3 + jz_4)$ the map t_v becomes a map $t_v : \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$. If we denote [9]

$$t_v(z_3, z_4) = (s_k(z_1, z_2), f(z_3, z_4) + jg(z_3, z_4))$$

in these coordinates

$$f(z_3, z_4) + jg(z_3, z_4) = wu = z_1 z_3 - \overline{z}_2 z_4 + j(z_2 z_3 + \overline{z}_1 z_4)$$

from which

$$\partial_{z_3} f = z_1 \qquad \partial_{z_4} f = -\overline{z}_2 \qquad \partial_{z_3} g = z_2 \qquad \partial_{z_4} g = \overline{z}_1$$

It's differential $J_z t_v$ is then at each point

$$J_{(z_3,z_4)} t_v = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{bmatrix}$$

The determinant of the sub matrix is $(|z_1|^2 + |z_2|^2) \neq 0$ for all $v \in S^2$ (that we have fixed), then at each point has rank max i.e. t_v is an immersion. Also $\tilde{\tau}_v := \pi_k \circ t_v : S^3 \to \mathbf{N}_k$ is an immersion and from the Lemma the other unique points with the same image of u are the points

$$e^{\alpha_m \overline{w}_1 i w_1} u \qquad m = 1, \ldots, k$$

where $\alpha_m = \frac{2\pi m}{k}$, that are the points in the equivalence class $[u]_k \in S^3 \backslash \mathbb{Z}_k$. Then the map $\tilde{\tau}_v$ extends on the quotient to a map $\tau_v : L_v^k \to \mathbf{N}_k$; moreover on the quotient the map becomes injective and then, since L_v^k is compact, it's an homeomorphism on the image that is, it's an embedding.

□

Remark that we can more generally consider an action of $\mathbb{Z}_n$ on $S^2 \times S^3$ defining it by

$$\mu : \mathbb{Z}_n \times \left(S^2 \times S^3\right) \to S^2 \times S^3$$
$$[n] \times (v, u) \to (v, e^{n\overline{w}iw} u)$$

that is still free and properly discontinuous. Moreover, the map $p_k : S^2 \times S^3 \to \mathbf{N}_k$ extends on the quotient on which the points of the form

$$v_1 + l\left(e^{\alpha_m \overline{w}_1 i w_1} v_2\right)$$

for the k angles $\alpha_m = \frac{2\pi m}{k}$ for $m = 1, \ldots, k$ on which p_k takes the same values are now identified (remark that the map doesn't become injective, since in other points it still takes same value).

[9] Remark that $s_k(z_1, z_2)$ is constant.

2.4.3 A Branched Covering on the Hopf Link

We have defined for each positive natural k a surjective map $p_k : S^2 \times S^3 \to \mathbf{N}_k$. For $k = 1$ we know from the previous section that the map is injective, while from the previous Proposition that it is not for $k > 1$. This suggests it could be a covering map for these values of k; anyway, this is not so, but it's "almost" a covering. Consider the three dimensional sphere $S^3 \subset \mathbb{H} \cong \mathbb{C}^2$ and the sets

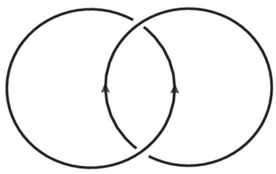


Figure 2.7: Hopf Link

$$K_1 := \big\{(z, 0) \mid z \in \mathbb{C} \ , \ \|z\| = 1\big\} \subset S^3$$
$$K_2 := \big\{(0, z) \mid z \in \mathbb{C} \ , \ \|z\| = 1\big\} \subset S^3$$

They are clearly disjoint and in particular $K_1 \cong K_2 \cong S^1$. These two sets wrap themselves in a non-trivial way in S^3 and their union $K = K_1 \cup K_2$ defines the so called *Hopf link*. Consider now the projections $\pi_k : S^3 \times S^3 \to \mathbf{N}_k$ and the images

$$\Delta_k := \pi_k\big(K \times S^3\big) \subset \mathbf{N}_k$$

Since K_1 e K_2 are closed curves, are closed sets with empty interior in the S^3 topology and then it is also their union K. It follows that also $K \times S^3 \subset S^3 \times S^3$ is closed with empty interior. We then set

$$C_1 = \pi_k(K_1) = \big\{[z, w] \mid z \in \mathbb{C}, \ w \in \mathbb{H} \ , \ |z| = |w| = 1\big\}$$
$$C_2 = \pi_k(K_2) = \big\{[jz, w] \mid z \in \mathbb{C}, \ w \in \mathbb{H} \ , \ |z| = |w| = 1\big\}$$

and $\Delta_k = C_1 \cup C_2$. Since $S^3 \times S^3$ is compact and $\mathbf{N}_k$ are Hausdorff, the projections π_k are closed maps that are also open maps, then the Δ_k are closed and have empty interior.

Definition 25.
Let X and Y be topological spaces and $p : X \to Y$ a continuous surjective function. We call p *branched covering* if there exists a closed with empty interior set $\Delta \subset Y$ such that, defined $D_\Delta = p^{-1}(Y \backslash \Delta)$ the restriction

$$p_{|_{D_\Delta}} : D_\Delta \to Y \backslash \Delta$$

is a covering map. We then call Δ and $Y \backslash \Delta$ respectively *singular set* and *regular set* of p, while $p^{-1}(\Delta)$ and $p^{-1}(Y \backslash \Delta)$ respectively *singular domain* and *regular domain* of p and finally $p_{|_D}$ *regular component* of p. If given two topological spaces X and Y there exists a such $p : X \to Y$, we'll say that X is a *branched covering* of Y on Δ.

From the definition of Δ it follows that is natural to define $\Delta(p)$ the *minimum singular set* of p, that is possible to show is a closed set. From now we will refer to Δ as this minimum set. We now want to prove that the map $p_k : S^2 \times S^3 \to \mathbf{N}_k$ is a branched covering with singular set Δ_k. To do that we first compute the pre-image of the points of $\mathbf{N}_k$ to determine the degree of the regular component of p_k and then we show that the map p_k is effective a covering map on its regular domain.

Proposition 44.
Consider $k \in \mathbb{Z}$, $k > 1$, the map $p_k : S^2 \times S^3 \to \mathbf{N}_k$ and $\Delta_k \subset \mathbf{N}_k$. Then:

i) For each $[w_1, w_2] \in \mathbf{N}_k \backslash \Delta_k$

$$\# p_k^{-1}([w_1, w_2]) = k^2$$

ii) For each $[w_1, w_2] \in \Delta_k \subset \mathbf{N}_k$

$$\# p_k^{-1}([w_1, w_2]) = k$$

Proof.
Consider $v_1 + lv_2,\ v_1' + lv_2' \in S^2 \times S^3$. Then $p_k(v_1 + lv_2) = p_k(v_1 + lv_2')$ if and only if for each choice of $w_1 \in S^1_{v_1}, w_1' \in S^1_{v_1'}$

$$[s_k(w_1), w_1 v_2] = [s_k(w_1'), w_1' v_2']$$

that is, fixed w_1 and w_1', if and only if there exists an angle $\theta_0 \in [0, 2\pi)$ such that

$$\begin{cases} e^{k\theta_0 i} s_k(w_1) = s_k(w_1') \\ e^{\theta_0 i} w_1 v_2 = w_1' v_2' \end{cases} \tag{2.5}$$

$$\begin{cases} s_k(e^{\theta_0 i} w_1) = s_k(w_1') \\ e^{\theta_0 i} w_1 v_2 = w_1' v_2' \end{cases} \tag{2.6}$$

From the first equation we get

$$\frac{\mathrm{pow}_k(w_1')}{\|\mathrm{pow}_k(w_1')\|} = \frac{\mathrm{pow}_k(e^{\theta_0 i} w_1)}{\|\mathrm{pow}_k(e^{\theta_0 i} w_1)\|} = \frac{\mathrm{pow}_k(e^{\theta_0 i} w_1)}{\|e^{k\theta_0 i}\mathrm{pow}_k(w_1)\|} = \frac{\mathrm{pow}_k(e^{\theta_0 i} w_1)}{\|\mathrm{pow}_k(w_1)\|}$$

i.e.

$$\mathrm{pow}_k\left(\frac{w_1'}{\sqrt[k]{\|\mathrm{pow}_k(w_1')\|}}\right) = \mathrm{pow}_k\left(\frac{e^{\theta_0 i} w_1}{\sqrt[k]{\|\mathrm{pow}_k(w_1)\|}}\right)$$

Then if $w_1 = z_1 + jz_2$ and $w_1' = z_1' + jz_2'$ the equation becomes

$$\left(\frac{z_1'}{\sqrt[k]{\|\mathrm{pow}_k(w_1')\|}}\right)^k + j\left(\frac{z_2'}{\sqrt[k]{\|\mathrm{pow}_k(w_1')\|}}\right)^k = \left(\frac{e^{\theta_0 i} z_1}{\sqrt[k]{\|\mathrm{pow}_k(w_1)\|}}\right) + j\left(\frac{e^{-\theta_0 i} z_2}{\sqrt[k]{\|\mathrm{pow}_k(w_1)\|}}\right)$$

hence there must exists $m, n \in \{0, 1, \ldots, k-1\}$ such that

$$\begin{cases} \dfrac{z_1'}{\sqrt[k]{\|\mathrm{pow}_k(w_1')\|}} = \dfrac{e^{(\frac{2\pi m}{k} + \theta_0)i} z_1}{\sqrt[k]{\|\mathrm{pow}_k(w_1)\|}} \\ \dfrac{z_2'}{\sqrt[k]{\|\mathrm{pow}_k(w_1')\|}} = \dfrac{e^{(\frac{2\pi n}{k} - \theta_0)i} z_2}{\sqrt[k]{\|\mathrm{pow}_k(w_1)\|}} \end{cases} \tag{2.7}$$

We now distinguish the two cases: for $[w_1, w_2] \in \Delta_k \subset \mathbf{N}_k$ and for $[w_1, w_2] \in \mathbf{N}_k \backslash \Delta_k$. Let be $s_k(w_1) = \tilde{z}_1 + j\tilde{z}_2 \in S^3$:

I°)

If $[w_1, w_2] \in \Delta_k \subset \mathbf{N}_k$: suppose $\tilde{z}_1 \neq 0$ and $\tilde{z}_2 = 0$ (the other case is analogue). If $v_1 + lv_2 \in p_k^{-1}([w_1, w_2])$ given $\tilde{w}_1 = \tilde{z}_1 + j\tilde{z}_2 \in S^1_{v_1}$ it must satisfy the relation

$$s_k(\tilde{w}_1) = \frac{\tilde{z}_1^k + j\tilde{z}_2^k}{\|\mathrm{pow}_k(\tilde{z}_1, \tilde{z}_2)\|} = z_1$$

then must be $\tilde{z}_1^k = z_1$ and $\tilde{z}_2 = 0$. This equation has exactly k distinct solutions. Notice that for each other solution $\tilde{z}_1'$ of the equation $z^k = z_1$, the point $\tilde{w}_1' = \tilde{z}_1' \in S^3$ is in the same fiber of $\tilde{w}_1$, indeed $\tilde{z}_1' = e^{\beta i}\tilde{z}_1$ for some $\beta \in [0, 2\pi)$, that is $\mathfrak{h}(\tilde{w}_1') = \mathfrak{h}(\tilde{w}_1)$. From the Lemma we then obtain for $v_1 + lv_2$ other k distinct points with the same image varying $v_2 \in S^3$.

II°)
If $[w_1, w_2] \in \mathbf{N}_k \backslash \Delta_k$: in this case $\tilde{z}_1, \tilde{z}_2 \neq 0$. Varying m and n, we obtain k^2 pairs of solutions, that are distinct points in S^3; we now show that only k of these points are in different Hopf fibers. Denote by $n_i = i$ and $m_j = j$. Fixed m_0, the pairs (m_0, n_i) for $i = 0, \dots, k-1$, are points in different fibers of $\mathfrak{h}$; let's show that given (m_1, n_j), it belongs to one of the fiber of (m_0, n_i) for some i. We have

$$\begin{cases} \dfrac{z_1'}{\sqrt[k]{\|\mathrm{pow}_k(w_1')\|}} = \dfrac{e^{(\frac{2\pi m_1}{k} + \theta_0)i} z_1}{\sqrt[k]{\|\mathrm{pow}_k(w_1)\|}} \\ \dfrac{z_2'}{\sqrt[k]{\|\mathrm{pow}_k(w_1')\|}} = \dfrac{e^{(\frac{2\pi n_j}{k} - \theta_0)i} z_2}{\sqrt[k]{\|\mathrm{pow}_k(w_1)\|}} \end{cases} \tag{2.8}$$

Consider

$$\alpha := \frac{2\pi m_0}{k} - \frac{2\pi m_1}{k}$$

and let $\tilde{w}_1 = e^{-\alpha i} w_1$ be another element of the same fiber of w_1; then if $\tilde{w}_1 = \tilde{z}_1 + j\tilde{z}_2$ we have

$$z_1 + jz_2 = w_1 = e^{\alpha i}\tilde{w}_1 = e^{\alpha i}\tilde{z}_1 + je^{-\alpha i}\tilde{z}_2$$

So substituting in the system we obtain

$$\begin{cases} \dfrac{z_1'}{\sqrt[k]{\|\mathrm{pow}_k(w_1')\|}} = \dfrac{e^{(\frac{2\pi m_1}{k} + \theta_0)i} z_1}{\sqrt[k]{\|\mathrm{pow}_k(w_1)\|}} = \dfrac{e^{(\frac{2\pi m_1}{k} + \theta_0 + \alpha)i} \tilde{z}_1}{\sqrt[k]{\|\mathrm{pow}_k(\tilde{w}_1)\|}} = \dfrac{e^{(\frac{2\pi m_0}{k} + \theta_0)i} \tilde{z}_1}{\sqrt[k]{\|\mathrm{pow}_k(\tilde{w}_1)\|}} \\ \dfrac{z_2'}{\sqrt[k]{\|\mathrm{pow}_k(w_1')\|}} = \dfrac{e^{(\frac{2\pi n_j}{k} - \theta_0)i} z_2}{\sqrt[k]{\|\mathrm{pow}_k(w_1)\|}} = \dfrac{e^{(\frac{2\pi n_j}{k} - \theta_0 - \alpha)i} \tilde{z}_2}{\sqrt[k]{\|\mathrm{pow}_k(\tilde{w}_1)\|}} = \dfrac{e^{(\frac{2\pi(n_j + m_1 - m_0)}{k} - \theta_0)i} \tilde{z}_2}{\sqrt[k]{\|\mathrm{pow}_k(\tilde{w}_1)\|}} \end{cases} \tag{2.9}$$

Thus the point is in the same fiber of the point associated to the pair $(m_0, n_j + m_1 - m_0)$. Denote by $v_1, \dots v_k$ the image of these points by $\mathfrak{h}$. By the Lemma we conclude that to each other point v_i are associated exactly k points $v_1 + l(e^{\alpha_m \overline{w}_1 i w_1} v_2)$ for $m = 1, \dots, k$ in the same preimage, that are all distincts.

□

Notice that if $v_1 + lv_2 \in S^2 \times S^3$ is such that $p_k(v_1 + lv_2) \in \Delta_h$, then $w \in S_{v_1}$ must be of the form $w = z$ or $w = jz$ for some $z \in \mathbb{C}$. Recall that the Hopf map

$\mathfrak{h} : S^3 \to S^2$ is defined by $\mathfrak{h}(w) = \overline{w} i w$ then

$$\mathfrak{h}(z) = \overline{z} i z = i|z|^2 = i$$

$$\mathfrak{h}(jz) = \overline{jz} i j z = -jzijz = -jzi\overline{z}j = -jij|z|^2 = -i$$

independently of $z \in \mathbb{C}$. We conclude that considered $i, -i \in S^2$ as the north and south poles of S^2 and denoted by $D := p_k^{-1}(\mathbf{N}_k \backslash \Delta_k)$ we simply have that $D = S^2 \backslash \{N, S\}$. Now, we want to prove p_k is a branched covering with regular domain D, i.e. that

$$p_k : S^2 \backslash \{N, S\} \times S^3 \to \mathbf{N}_k \backslash \Delta_k$$

is a covering map. In order to do that, we show that on D the differential of p_k is invertible; once proved, since by previous Proposition fibers are finite, the map is a covering.

Lemma 15.
Consider the sets

$$R_1 = \{(z_1, 0) \in \mathbb{C}^2\} \qquad R_2 = \{(0, z_2) \in \mathbb{C}^2\}$$

and the map

$$s_n : \mathbb{C}^2 \backslash \{0\} \to S^3$$

$$s_n(z_1, z_2) := \frac{\text{pow}_n(z_1, z_2)}{\|\text{pow}_n(z_1, z_2)\|}$$

Then it's differentiable on $\mathbb{C}^2 \backslash \{0\}$ and regular on $\mathbb{C} \backslash (R_1 \cup R_2)$ that is denoted by

$$(J_z s_n)_{ij} = (\partial_{z_j} (s_n)_i)$$

for each $z \in \mathbb{C}^2 \backslash (R_1 \cup R_2)$, $det(J_z s_n) \neq 0$.

Proof.
The map s_n is differentiable in each point of $\mathbb{C}^2 \backslash \{0\}$; explaining it

$$s_n(z_1, z_2) = \left(\frac{z_1^n}{\sqrt{|z_1|^{2n} + |z_2|^{2n}}}, \frac{z_2^n}{\sqrt{|z_1|^{2n} + |z_2|^{2n}}} \right) =$$

$$= \left(\frac{z_1^n}{\sqrt{z_1^{2n}\overline{z}_1^{2n} + z_2^{2n}\overline{z}_2^{2n}}}, \frac{z_2^n}{\sqrt{z_1^{2n}\overline{z}_1^{2n} + z_2^{2n}\overline{z}_2^{2n}}} \right)$$

Notice that

$$\partial_{z_1}\left(\frac{1}{\sqrt{|z_1|^{2n} + |z_2|^{2n}}} \right) = \left(\frac{2n z_1^{2n-1} \overline{z}_1^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right)$$

$$\partial_{z_2}\left(\frac{1}{\sqrt{|z_1|^{2n} + |z_2|^{2n}}} \right) = \left(\frac{2n z_2^{2n-1} \overline{z}_2^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right)$$

then the partial derivaties of s_n are

$$\partial_{z_1} s_n = \left(\frac{n z_1^{n-1}}{\sqrt{|z_1|^{2n} + |z_2|^{2n}}} + z_1^n \frac{2n z_1^{2n-1} \overline{z}_1^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3}, z_2^n \frac{2n z_1^{2n-1} \overline{z}_1^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right) =$$

$$= \left(\frac{n z_1^{n-1}(|z_1|^{2n} + |z_2|^{2n}) + 2n z_1^n z_1^{2n-1} \overline{z}_1^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3}, \frac{2n |z_1|^{2n-1} \overline{z}_1 z_2^n}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right) =$$

$$= \left(\frac{n z_1^{n-1}(|z_1|^{2n} + |z_2|^{2n}) + 2n z_1^{n-1} |z_1|^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3}, \frac{2n |z_1|^{2n-1} \overline{z}_1 z_2^n}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right)$$

$$= \left(\frac{n z_1^{n-1}(3|z_1|^{2n} + |z_2|^{2n})}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3}, \frac{2n |z_1|^{2n-1} \overline{z}_1 z_2^n}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right) \tag{2.10}$$

$$\partial_{z_2} s_n = \left(z_1^n \frac{2n z_2^{2n-1} \overline{z}_2^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3}, \frac{n z_2^{n-1}}{\sqrt{|z_1|^{2n} + |z_2|^{2n}}} + z_2^n \frac{2n z_2^{2n-1} \overline{z}_2^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right)$$

$$= \left(\frac{2n |z_2|^{2n-1} \overline{z}_2 z_1^n}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3}, \frac{n z_2^{n-1}(|z_1|^{2n} + |z_2|^{2n}) + 2n z_2^n z_2^{2n-1} \overline{z}_2^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right)$$

$$= \left(\frac{2n |z_2|^{2n-1} \overline{z}_2 z_1^n}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3}, \frac{n z_2^{n-1}(|z_1|^{2n} + |z_2|^{2n}) + 2n z_2^{n-1} |z_2|^{2n}}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right)$$

$$= \left(\frac{2n |z_2|^{2n-1} \overline{z}_2 z_1^n}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3}, \frac{n z_2^{n-1}(|z_1|^{2n} + 3|z_2|^{2n})}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \right) \tag{2.11}$$

Computing the determinant of the Jacobian

$$\det(J_z s_n) = \begin{vmatrix} \frac{n z_1^{n-1}(3|z_1|^{2n} + |z_2|^{2n})}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} & \frac{2n |z_1|^{2n-1} \overline{z}_1 z_2^n}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \\ \frac{2n |z_2|^{2n-1} \overline{z}_2 z_1^n}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} & \frac{n z_2^{n-1}(|z_1|^{2n} + 3|z_2|^{2n})}{(\sqrt{|z_1|^{2n} + |z_2|^{2n}})^3} \end{vmatrix} =$$

$$
\begin{aligned}
&= \frac{\left[n^2 z_1^{n-1} z_2^{n-1}(3|z_1|^{2n}+|z_2|^{2n})(|z_1|^{2n}+3|z_2|^{2n}) - 4n^2|z_1|^{2n-1}\overline{z}_1 z_2^n |z_2|^{2n-1}\overline{z}_2 z_1^n\right]}{(|z_1|^{2n}+|z_2|^{2n})^3} = \\
&= \frac{n^2 z_1^{n-1} z_2^{n-1}(3|z_1|^{2n}+|z_2|^{2n})(|z_1|^{2n}+3|z_2|^{2n}) - 4n^2|z_1|^{2n}|z_2|^{2n} z_2^{n-1} z_1^{n-1}}{(|z_1|^{2n}+|z_2|^{2n})^3} = \\
&= \frac{n^2 z_1^{n-1} z_2^{n-1}\left[\left(3|z_1|^{2n}+|z_2|^{2n}\right)(|z_1|^{2n}+3|z_2|^{2n}) - 4|z_1|^{2n}|z_2|^{2n}\right]}{(|z_1|^{2n}+|z_2|^{2n})^3} = \\
&= \frac{3n^2 z_1^{n-1} z_2^{n-1}\left[|z_1|^{4n}+2|z_1|^{2n}|z_2|^{2n}+|z_2|^{4n}\right]}{(|z_1|^{2n}+|z_2|^{2n})^3} = \frac{3n^2 z_1^{n-1} z_2^{n-1}\left(|z_1|^{2n}+|z_2|^{2n}\right)^2}{(|z_1|^{2n}+|z_2|^{2n})^3} \\
&= 3n^2 \frac{z_1^{n-1} z_2^{n-1}}{|z_1|^{2n}+|z_2|^{2n}}
\end{aligned}
\tag{2.12}
$$

Then the map has no critical points on the open set $\mathbb{C}^2 \backslash (R_1 \cup R_2)$.

□

Using the previous results we can finally prove the goal of this section.

Theorem 45.
For each $k \in \mathbb{Z}$, $k \geq 2$, the map $p_k : S^2 \times S^3 \to \mathbf{N}_k$ is a branched covering with singular set Δ_k which regular component has degree k^2.

Proof.
We want to prove that $p_{|_D} : D \to \mathbf{N}_k \backslash \Delta_k$ is a covering map. Since $\mathbb{O} \cong \mathbb{C}^4$ given an arbitrary element $v_1 + lv_2 \in \mathbb{O}$, if $v_1 = z_1 + jz_2$ and $v_2 = (z_3 + jz_4)$, the point has complex coordinates $\underline{z} = (z_1, z_2, z_3, z_4)$. Let's denote

$$p_k(v_1 + lv_2) = p_k(\underline{z}) = (s_k(z_1, z_2), f_2(\underline{z}))$$

In these coordinates

$$f_2(\underline{z}) = z_1 z_3 - \overline{z}_2 z_4 + j(z_2 z_3 + \overline{z}_1 z_4)$$

from which it follows

$$\partial_{z_1} f_2 = z_3 \qquad \partial_{z_2} f_2 = -z_4 + jz_3 \qquad \partial_{z_3} f_2 = z_1 + jz_2 \qquad \partial_{z_4} f_2 = -\overline{z}_2 + j\overline{z}_1$$

The determinant of the differential of the function p_k as function from $\mathbb{C}^4$ to $\mathbb{C}^4$ is then in each point

$$\det(J_{\underline{z}} p_k) = \begin{vmatrix} J_{\underline{z}} s_k & & 0 & \\ & & & \\ z_3 & -z_4 & z_1 & -\overline{z}_2 \\ 0 & z_3 & z_2 & \overline{z}_1 \end{vmatrix} = \det(J_{\underline{z}} s_k)) \cdot (|z_1|^2 + |z_2|^2) \neq 0$$

From the local invertibility theorem we conclude that p_k si a local diffeomorphism, hence a covering map.

□

Appendix A

Principal Bundles

A.1 Lie Groups Actions and Symmetries

Let M be a smooth manifold and G a group. An action of G on M

$$\mu : G \times M \to M$$

defines an equivalent relation between points of M, for which $p \sim p'$ if and only if there exists $g \in G$ such that $p' = g(p)$ and then a coset space M/G. The question is then under which conditions on μ this coset space has a structure of smooth manifold. We can distinguish two cases: the one on which the group is a generic group (that we think about as "discrete" action) and the one on which the group is a Lie group. We start looking at the first case. The meaning of the adjective "discrete" is not related to a property of the acting group but to the fact that the action generally identify "isolated points" (among them) of M . The most classical example is given by action of the group $\mathbb{Z}_2$ on S^n defined by

$$\mathbb{Z}_2 \times S^n \to S^n$$

$$(\varepsilon, x) \to \varepsilon \cdot x$$

for $\varepsilon \in \{-1, 1\}$, which coset space is the n dimensional projective space $\mathbb{P}^n$. This example suggests the following definitions.

Definition 26 (Discrete Group Actions).
Let G be a group and M a smooth manifold, $\mu : G \times M \to M$ a smooth action of G on M. We call the action
i) *Free*: if for each $g \in G$, $g \neq e_G$ and for each $p \in M$

$$\mu(g, p) \neq p$$

ii) *Properly discontinuous* : if for each $p \in M$ there exists a neighborhood U such that

$$g(U) \cap U = \emptyset$$

for each $g \in G$, $g \neq e$.

For such types of actions, the following general result can be proved.

Proposition 45.
Let M be a smooth manifold and G a group. Let $\mu : G \times M \to M$ be a smooth action of G on M. If the action is free and properly discontinuous, the coset space M/G admits a structure of differentiable manifold such that the projection

$$\pi : M \to M/G$$

is a covering map.

In the case on which the group G is a Lie group, things become more complicated. Recall that if X and Y are two Hausdorff topological spaces, a continuous map $f : X \to Y$ is by definition *proper* if for every topological space Z, the map

$$f \times \mathrm{Id} : X \times Z \to Y \times Z$$

is a closed map

Definition 27.
Let G be a Lie group and M a smooth manifold. An action $\mu : M \times G \to M$ is *proper* if it is continuous and if the map

$$G \times M \to M \times M$$

$$(g, x) \to (gp, p)$$

is proper.

An sufficient condition for a Lie group action to be proper, is given by the following Proposition.

Proposition 46.
Let G be a compact Lie group and M a smooth manifold. Then every continuous action $\mu : G \times M \to M$ is proper.

We would like to prove that under the condition for an action of being proper and free, the related coset space admits a structure of differential manifold. Before to do that, in order to better understand the meaning of this fact, we recall some properties of projections.

A.1.1 Projections, Submersions and Covering Maps

The proprieties of quotients by Lie Group actions and Fibre Bundles are related to the notion of submersion. Let M and N be two (differential) manifolds of dimension m and n respectively, with $m \geq n$. A map $f : M \to N$ is called *submersion* if in each point of M its differential $df_p : T_pM \to T_pN$ has maximum rank, i.e is surjective. More general if the differential d_pf is surjective in some point $p \in M$, then f is a submersion in a neighborhood U of p, indeed the matrix associated to d_pf in p has (maximum) rank n in p if and only if the matrix has a non-singular sub-matrix of rank $n \times n$; then, since the determinant is a continuous map, there exists a neighborhood of p on which this condition still holds, i.e. in this neighborhood the matrix has maximum rank. The most simple examples of submersions are the projection (the first fundamental ingredient of fibre bundles)

$$\pi_i : M_1 \times \cdots \times M_i \times \cdots \times M_k \to M_i$$

from the product of manifolds to one of them. On the other hand, for submersion it is possible to prove the following well known result.

Theorem 46.
Let $f : M \to N$ be a submersion. Then for each p in M there exists a neighborhood-chart (U, φ) in p and a neighborhood-chart (V, ψ) in $f(p)$ such that
i) $f(U) \subset V$
ii) If F is the representation of f in the coordinates $\underline{x}$ induced by the chart (U, φ) then

$$F(x_1, \ldots, x^n, x^{n+1}, \ldots, x^{n+m}) = (x^1, \ldots, x^n)$$

The previous Theorem tell us that the locally submersions' behavior is analogous to the projections' one. This suggests a more fundamental connection between the two concepts and the definition of *section* (the second fundamental ingredient of fibre bundles). Following this intuition, we denote a submersion with π, symbol often used to denote projections. Let $\pi : M \to N$ a smooth map. A *section* of π is a smooth map $\sigma : U \subset N \to M$ such that $\pi \circ \sigma = Id_U$:

$$\begin{tikzcd} M \arrow[d, shift right, "\pi"'] \\ U \arrow[u, shift right, bend right, "\sigma"'] \end{tikzcd}$$

The following Theorem holds.

Theorem 47 (Local Sections)**.**
Let M and N be smooth manifolds and $\pi : M \to N$ be a smooth map. Then π is a submersion if and only if every point of M is image of a (smooth) local section of π.

Let's come back to Lie groups.

Definition 28.
Let P and M be two smooth manifolds. We call P *principal bundle* on M if there exists a Lie group G and a (smooth) surjective map $\pi : P \to M$ such that
i) The group G acts freely on P.
ii) The action of G on P is transitive on the fibers: for each pairs $p, p' \in P$ such that $\pi(p) = \pi(p')$ there exists $g \in G$ such that $p' = gp$.
iii) The fiber bundle is locally trivial: for each $p \in M$ there exists a neighborhood $U \subset M$ and a diffeomorphism

$$\psi : U \times G \to \pi^{-1}(U)$$

such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[d, "\pi"'] & U \times G \arrow[l, "\psi"'] \arrow[dl, "\mathrm{pj}_1"] \\ U & \end{tikzcd}$$

where $\mathrm{pj}_1 : U \times G \to U$ is the factor projection, commutes and

$$\psi(p, g_1 g_2) = g_1 \psi(p, g_2)$$

The Lie group G associated to a principal bundle is usually called *structure group* of P and then P is called G-principal bundle. We can now state the fundamental Theorem establishing the differential structure of the coset space respect to a Lie group action, under the sufficient conditions explained above.

Theorem 48 (Lie group Action)**.**
Let M be a smooth manifold, G be a Lie group and let $\mu : G \times M \to M$ be a proper and free smooth action. Then M/G is a principal G-bundle of dimension $dim(M) - dim(G)$ and the canonical projection

$$\pi : M \to M/G$$

is a submersion. Moreover, there exists a unique smooth structure on M/G with this property.

We finally report a result that tell us when two different actions give the same coset space.

Definition 29.
Let M and N be two smooth manifolds with two actions of two Lie groups G and H respectively on M and N; we call the action *diffeomorphic* if there exists a pair of maps (φ, σ) where $\varphi : G \to H$ homomorphism and $\sigma : M \to N$ smooth such that for each $g \in G$ and $h \in H$

$$\sigma(g \cdot x) = \varphi(g) \cdot \sigma(x)$$

$$\begin{tikzcd} M \arrow[r, "\sigma"] \arrow[d, "g"] & N \arrow[d, "h"] \\ M \arrow[r, "\sigma"'] & N \end{tikzcd}$$

Proposition 47.
If two actions are diffeomorphic then the coset spaces are diffeomorphic.

A.1.2 Connections on Principal Bundles

We now want to give a more general definition of connection on a smooth manifold M, compatible with a Lie group action. To do that, we need to define it on a principal bundle $\pi : P \to M$. We can think about a connection on M as the object that encodes how the tangent spaces T_pM change varying p. The idea is then to define a subspace of P varying following the action on G and then move it to the manifold M via the projection.

Definition 30.
Let $\pi : P \to M$ be a G-principal bundle and $u \in P$. The *vertical subspace V_uP* at $u \in P$ is the subspace of T_uP defined by

$$V_uP = \ker(d\pi_u)$$

From the definition it follows that the vertical subspace V_uP in a point $u \in P$ is tangent to the fiber G_u. Since it depends on the definition of π, this subspace is fixed. We then define the connection imposing how the remained subspace have to change.

Definition 31.
Let $\pi : P \to M$ be a G-principal bundle. An *Ehreshamann connection* on P is given by an vector distribution $H \subset TP$ on P, called *horizontal distribution*

such that:

i) It is invariant with respect to the G action on P: denoted by $L_g : P \to P$ the left translation associated to $g \in G$, for all $g \in G$

$$\left(L_g\right)_* H = H$$

ii) In each point $u \in P$ the projection $\pi : P \to M$ induces via its differential an isomorphism

$$d\pi_u : H_u \to T_{\pi(u)}M$$

that is, H_u and $T_{\pi(u)}M$ can be in each point identified.

In particular a connection is said *flat* at $u \in P$ if there exists a local trivialization of the bundle in a neighborhood of u under which the given horizontal subspaces is mapped to the horizontal subspaces of the trivial connection on the corresponding trivial bundle. It follows that at each point $u \in P$ we have a decomposition

$$T_uP = V_u \oplus H_u$$

of the tangent space T_uP and then a decomposition of each vector in $v \in T_pP$

$$v = hor(v) + vert(v)$$

This geometric definition of connection is however not comfortable for computation. An equivalent way to specify a principal bundle connection is to give a $\mathfrak{g}$-valued one-form ω on P, called *connection form*, that satisfies two conditions relating to the G-action on P. Let M be a smooth manifold and V a real vector space. A k-form ω on M with values on V is function that associates to each point p of M a multilinear antisymmetric map

$$\omega_p : T_pM \times \cdots \times T_pM \to V$$

smooth by varying p; similarly to the $\mathbb{R}$ valued k-forms, the set $\Omega_k(M, V)$ of such maps is a vector bundle over M. Remark that, if $v_1, \ldots, v_n$ is a basis of V, there exist n $\mathbb{R}$-valued k-forms $\omega^1, \ldots, \omega^n$ such that given $X_p^1, \ldots, X_p^k \in T_pM$ each V-valued form can be expressed as the sum

$$\omega_p(X_p^1, \ldots, X_p^k) = \sum_{i=1}^{n} \omega_p^i(X_p^1, \ldots, X_p^k)v_i$$

This allows to define the *differential* of a V-valued k form as

$$d\omega_p(X_p^1, \ldots, X_p^k, X_p^{k+1}) := \sum_{i=1}^{n} d\omega_p^i(X_p^1, \ldots, X^k, X_p^{k+1})v_i$$

that is independent to the choice of the basis $v_1 \ldots, v_n$ of V. We would like to extend also the notion of wedge product. Anyway, to do that it is necessary to define a "product " between elements of V, that is to fix a bilinear form on V. More formally let

$$s : V \times V \to V$$

be a V-valued bilinear form on V and $v_1, \dots, v_n$ a basis of V. Let

$$\omega = \sum_{i=1}^{n} \omega^i v_i \qquad \eta = \sum_{k=1}^{n} \eta^k v_k$$

be two V-valued k forms on M. We define their s-wedge product the V-valued $(k+l)$-form defined by

$$\omega \wedge \eta = \sum_{i,k=1}^{n} \omega^i \wedge \eta^k s(v_i, v_k)$$

Remark that if G is a Lie group it is possible to define a natural $\mathfrak{g}$-valued 1-form in the following way. Let $L : G \times G \to G$ be the left translation; for each $g \in G$ we have a map $\left(L_{g^{-1}}\right)_* : T_gG \to T_eG$. The so called *Maurer-Cartan form* on G is the $\mathfrak{g}$-valued one-form defined at each point $g \in G$ by

$$\omega_g(v_p) := \left(L_{g^{-1}}\right)_* v_p$$

Since on $\mathfrak{g}$ is defined the bilinear form given by the commutator

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

if $E_1, \dots, E_n$ is a set of generators of $\mathfrak{g}$ with constants structure c_{ij}^k the wedge product can be defined by

$$\omega \wedge \eta = \sum_{i,j=1}^{n} \omega^i \wedge \eta^j [E_i, E_j]$$

that means

$$\omega \wedge \eta = \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} c_{ij} \omega^i \wedge \eta^j \right) E_k$$

Let's come back to principal bundles. Consider a smooth action $G \times M \to M$ on a smooth manifold M. The action induces a one parameter group $\mathbb{R} \times M \to M$ defined by

$$(t, p) \to \exp(tA)p$$

and then a vector field on M, called *fundamental vector field* associated to A defined at each point $p \in M$ by

$$X_p^A = \frac{d}{dt}\big(\exp(tA)p\big)_{|t=0}$$

Theorem 49.
Let G be a Lie group acting on a smooth manifold M. The map

$$i : \mathfrak{g} \to Vect(M)$$

$$A \to X^A$$

is a Lie algebra homomorphism. Moreover, if the action is free, for each not null vector field $A \in \mathfrak{g}$ the associated fundamental vector field X^A is a never null vector field on M.

Let now $\pi : P \to M$ be a G-principal bundle. From the previous proposition, the action of G on P defines a map $i : \mathfrak{g} \to \mathrm{Vect}(P)$

$$A \to X^A$$

and then at each $u \in P$ a map $i_u : \mathfrak{g} \to T_uP$

$$A \to X_u^A$$

Since the action of G sends each fiber in itself, for each $u \in P$ the vector X_u^A is tangent to the fiber G_u and, since the action of G is free on P, at each $u \in P$ and for each $A \in \mathfrak{g}$ we have $X_u^A \neq 0$. Then, since the fiber is diffeomorphic to G, the map i_p defines a linear isomorphism on the tangent space to the fiber $T_u(G_u) \subset T_uP$.

Definition 32.
Let $\pi : P \to M$ be a principal G-bundle and H an *Ehreshamann connection* on P. The *connection form* ω on P is the $\mathfrak{g}$-valued one-form defined by

$$\omega_p(X_p) := i_p^{-1}\big(\mathrm{vert}(X)\big)$$

From the definition it follows that the horizontal subspaces H can be obtained from ω, since

$$H_u := \ker(\omega_u)$$

for each $u \in P$.

Theorem 50.
Let $\pi : P \to M$ be a principal G-bundle, H an Ehreshamann connection on P and ω the associated connection one-form. Then ω is smooth and the following properties hold
i) $\omega_p(X_p^A) = A$ for each fundamental vector field $X^A \in Vect(P)$
ii) $\big((L_g)^*\omega\big)_p(X_p) = (Ad_{g^{-1}})_*(\omega_p(X_p))$

Moreover it is possible to prove that a $\mathfrak{g}$-valued 1-form on P with the two properties of the last Theorem, defines an Ehreshamann connection on P and then the two definitions are equivalent.

Appendix B

Topology of Fiber Bundles

Consider a generic fiber bundle (as example a principal bundle) $E \to B$ with fiber F. We can explicitly describe the structure of E only locally while we (usually) know the one of the base space B and the fiber F. It is then natural to wonder if it is possible to compute the topological invariants of E from the ones of B and F. To try to answer this question, we first summarize two fundamental topological invariants of differential manifolds that are Homotopy and (singular) Homology groups.

B.1 Homotopy and Homology

Homotopy groups are the natural extension of the fundamental group. Let X be a topological space, $S^n \subset \mathbb{R}^{n+1}$ be the n-dimensional sphere and $s_0 \in S^n$. For each $x_0 \in X$, we define the set of n-dimensional continuous loops based at x_0 by

$$L_n(X, x_0) := \{f : S^n \to X \mid f(s_0) = x_0\}$$

and then an equivalence relation on $L_n(X, X_0)$ imposing that $f \sim g$ if and only if there exists a continuous homotpy $H : [0,1] \times S^n \to X$ such that

$$H(0, x_0) = f(x_0) \qquad H(1, x_0) = g(x_0)$$

We then define the *n-dimensional homotopy group based in x_0* as the quotient

$$\pi_n(X, x_0) := L_n(X, x_0)/ \sim$$

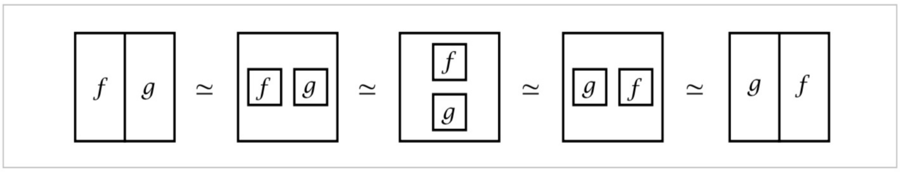
For $n \geq 2$ we can provide a group structure on $L_n(X, x_0)$ setting the sum of two loops $f, g \in L_n(X, x_0)$ as

$$(f+g)(t_1, \dots, t_n) := \begin{cases} f(2t_1, t_2, \dots, t_n) & t_1 \in \left[0, \frac{1}{2}\right] \\ \\ g(2t_1 - 1, t_2, \dots t_n) & t_1 \in \left[\frac{1}{2}, 1\right] \end{cases}$$

It is easy to check that the inverse element of $f \in L_n(X, x_0)$ is given by

$$(-f)(t_1, t_2, \dots, t_n) := f(1 - t_1, t_2, \dots, t_n)$$

therefore, the sum and the inverse are well define on the quotient and then $\pi_n(X, x_0)$ is a group. Moreover, for $n \geq 2$ the operation is commutative (this is not true in general for the fundamental group). The reason is that in dimension bigger than one loops can move in an "additional dimension"; this suggestion is showed in the following figure.



This means that $\pi_n(X, x_0)$ is abelian for $n \geq 2$; for $n = 1$ it instead coincides with the fundamental group while for $n = 0$, since S^0 is a point, it is in fact a single point. If X is path connected the group $\pi_n(X, x_0)$ clearly doesn't depend on the base point then under this assumption we can simply set

$$\pi_n(X) := \pi_n(X, x)$$

for an arbitrary $x \in X$. The relation between fibre bundles and these groups is given by the following Theorem which answer positively to our initial question.

Theorem 51 (Homotopy of Fiber Bundle)**.**
Let $E \to B$ be a fiber bundle with fiber F. Then there exists a long exact sequence:

$$\cdots \to \pi_2(F) \to \pi_2(E) \to \pi_2(B) \to \pi_1(F) \to$$

$$\to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B)$$

In the next section we'll see how, if we are lucky, from homotopy groups we can compute also the homology groups of a bundle.

B.2 Smooth Homology and dR Cohomology

Let M be a smooth manifold and consider the standard k-simplex in $\mathbb{R}^k$ defined by

$$\Delta^k := \left\{ x \in \mathbb{R}^k \mid \sum_{i=1}^{k} x_i \leq 1,\ x_i \geq 0\ \forall i \right\}$$

A *simplex* of class r in M is a map $\sigma : \Delta^k \to M$ obtained by the restriction of a class r map $U \subset \mathbb{R}^k \to M$ where U is an open subset such that $\Delta \subset U$. Denote by

$$E_k^r(M) := \left\{ \sigma : \Delta^k \to M \ \mid \sigma \in C^r(U) \right\}$$

the set of k-simplices of regularity r, for each $i = 0 \ldots k$ we define the *i-face map* of Δ^k by the function

$$\delta_i^{k-1} : \Delta^{k-1} \to \Delta^k$$

$$\delta_i^{k-1}(x_1,\dots,x_{k-1}) := \begin{cases} (1-\sum_{j=1}^{k-1}, x_j, x_1, \dots x_{k-1}) i = 0 \\ (x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_{k-1}) \end{cases}$$

thus given $\sigma \in E_k^r$ we call *face* of σ the $k-1$-simplex

$$\sigma_i := \sigma \circ \delta_i^{k-1}$$

We can now give the general definition of homology groups. Let A be an abelian group; we define the set of *k-chains of class r counted by A* as the free abelian group with coefficients in A generated by the k-chains of class r, that is the set of formal linear combinations

$$S_k^r(M,A) = \{a_1\sigma_1 + \cdots + a_n\sigma_n\} | \ \sigma_i \in E_k^r(M), \ a_i \in A\}$$

and the *boundary map*

$$\partial_k : S_k^r(M,A) \to S_{k-1}^r(M,A)$$

defined on the simplices by

$$\partial_k\sigma := \sum_{i=0}^{k}(-1)^i\sigma_i$$

where σ^i is defined as above. Remark that the image of this map are the geometric boundary of the simplices. Moreover from the definition easy follows that for each abelian group A the composition $\partial_{k-1} \circ \partial_k = 0$ that is

$$\mathrm{Im}(\partial_{k+1}) \subset \ker(\partial_k)$$

This allows to define the k-th *homology group* of M *of class r with coefficients in A* as the quotient group

$$H_k^s(M,A) := \frac{\ker(\partial_k)}{\mathrm{Im}(\partial_{k+1}))}$$

We can think about this quotient group as the *"group of chains that are not boundary of something smaller"*. To understand the geometrical intuition behind this definition, that is why this group encodes the topological structure of a topological space, think about the following example. Consider a square $Q = [-1,1] \times [-1,1]/\{(0,0)\} \subset \mathbb{R}^2$ and a circle around the point $(0,0)$, that is a 1-simplex in Q; can it be the boundary of an injective enough regular 2-simplex? The answer is negative. Indeed, since the image of Δ^k in M is a (simply connected) smooth submanifold of M, it can not be the internal neither the external region respect to the circle, then the first homology group can not be zero. Moreover, the topological properties don't depend on the regularity of the simplices, indeed the following Theorem holds.

Theorem 52.
Let M be a smooth manifold and A an abelian group. Then for each $k \geq 0$

$$H_k^0(M,A) \cong H_k^\infty(M,A)$$

This Theorem tell us that the topological properties of a smooth manifold M encoded by these invariants, don't depend on its differentiable structure. As consequence of the Theorem we can simply set

$$H_k(M, A) := H_k^0(M, A) \cong H_k^\infty(M, A)$$

Usually the standard choice of the abelian group A is the abelian group $\mathbb{Z}$, we then denote the *standard homology group* by $H_k(M) := H_k(M, \mathbb{Z})$. The reason behind this choice is that standard homology group contains more information respect to the others. Indeed, is possible to prove the following Theorem.

Theorem 53 (Universal Coefficients)**.**
Let X be a topological space and A an abelian group. There exists a (natural) short exact sequence

$$\{0\} \to H_k(X) \otimes A \to H_k(X, A) \to \mathit{Tor}\big[H_{k-1}(X), A\big] \to \{0\}$$

Remark that, if $H_{k-1}(X)$ are free groups, then $\mathrm{Tor}\big[H_{k-1}(X), A\big] = \{0\}$ hence we have the identification

$$H_k(X) \otimes A \cong H_k(X, A)$$

We now see how the Homology and Homotopy groups are related. First recall the following definition. Let X be a path-connected topological space. The space X is said n-connected (for positive n) when its first n homotopy groups vanish identically, that is

$$\pi_i(X) \cong \{0\} \ , \quad 1 \leq i \leq n$$

Now, the elements of $\pi_n(X)$ are equivalent classes of maps

$$f : S^n \to X$$

then if $\sigma : \Delta^k \to S^n$ is a n-simplex in S^n we can take the composition $f_*\sigma$ obtaining a map

$$f_*\sigma : \Delta^k \to X$$

For any path-connected space X it is possible to define a group homomorphism

$$h_* \colon \pi_n(X) \to H_n(X)$$

called the *Hurewicz homomorphism* that send a canonical generator $\sigma_n \in H_n(S^n)$ to $f_*(u_n) \in H_n(X)$.

Theorem 54. *Hurewicz*
Let X be a path connected topological space and $h_* \colon \pi_n(X) \to H_n(X)$ the Hurewicz homomorphism. Then
i) For $n = 1$ this homomorphism induces an isomorphism

$$\tilde{h}_* \colon \pi_1(X)/[\pi_1(X), \pi_1(X)] \to H_1(X)$$

between the abelianization of the first homotopy group (the fundamental group) and the first homology group.
ii) If $n \geq 2$ and X is $(n-1)$-connected, the Hurewicz map

$$h_* \colon \pi_n(X) \to H_n(X)$$

is an isomorphism. In addition, the Hurewicz map

$$h_*\colon \pi_{n+1}(X) \to H_{n+1}(X)$$

is an epimorphism.

It follows that given a fiber bundle (as example a principal bundle) $E \to B$ with fiber F, if we are lucky, we can compute the homotopy groups of E from the ones of F and B and then, if we are twice lucky, from them also the Homology groups, using Hurewicz Theorem.

Bibliography

[1] A. Agrachev, *Rolling Balls and Octonions*, SISSA, Trieste, 2006 .

[2] R. Montgomery, *A tour of sub-Riemannian Geometry*, AMS, 2001.

[3] A. Agrachev, Yu. L. Sachkov, *Control Theory from the Geometric Viewpoint*, Moscow, 2004.

[4] A. Agrachev, D. Barilari, U. Boscain, *A Comprehensive Introduction to sub-Riemannian Geometry.*

[5] G. Bor, R. Montgomery, *G_2 and the "Rolling Distribution"*, 2006.

[6] John C. Baez, J. Huerta, *G_2 and the Rolling Ball*, 2012.

[7] D. An, P. Nurowski, *Twistor Space for Rolling Bodies*, Communications in Mathematical Physics, 2013.

[8] U. Bruzzo, M. Pedroni, *Appunti del Corso di Metodi Geometrici in Fisica Matematica*, Università degli Studi di Genova, 2002/2003.

[9] F. Benini, *Basics of Differential Geometry and Group Theory*, SISSA-PhD course Fall 2019.

[10] V. Susic, *Classification of Semisimple Lie Algebras* , Seminar for Symmetries in Physics, 2011.

[11] N. J. Wildberger, *An easy construction of G_2*, School of Mathematics UNSW, Sydney.

[12] C. Draper Fontanals, *Notes on G_2 :The Lie Algebra and the Lie Group*, Universidad de Málaga, Málaga, 2017.

[13] Tonny A. Springer, F. D. Veldkamp, *Octonions, Jordan Algebras and Exceptional Groups*, ed. Springer.

[14] T. Dray, C. AManogue, *The Geometry of Octonions*, ed. Word Scientific.

[15] M. Taylor, *The Octonions* .

[16] A. Lerario, *Lecture Notes on Basic Differential Topology*, SISSA, Trieste, 2020.

[17] J. M. Lee, *Introduction to Smooth Manifolds* , University of Washington.

[18] F. Perroni, *Introduction to Differentiable Manifolds and de Rham Cohomology* , Università degli Studi di Trieste, 2017-2018.

[19] A. Hatcher, *Algebric Topology*, Cambridge University Press, 2002.

[20] S. Schmitt, *The Quotient Manifold Theorem* , University of Heidelberg, 2015/2016.

[21] M. M. Maglio, *Gli aspetti geometrici delle teorie di gauge: i monopoli di Dirac e gli istantoni di Yang Mills* Università del Salento, 2015/2016.

[22] T. E. Goldberg, *What is a connection, and what is it good for?*, 2008.

[23] F. Bracci, *Teoria dei Fibrati* , Università di Roma Tor Vergata.

[24] C. Bartocci, *Fibrati Vettoriali, Connessioni e Forme Differenziali* , 1998.

[25] Kobayashi, Nomizu *Fundations of Differential Geometry-Volume 1* , 1963.

[26] Robert L. Bryant, *Elie Cartan and Geometric Duality* , A lecture given at the Institut d'Elie Cartan, 1998.