assignment4

February 5, 2024

1 Stanford CME 241 (Winter 2024) - Assignment 4

Due: Feb 5 @ 11:59pm Pacific Time on Gradescope.

Assignment instructions: - **Please solve questions 1 and 2, and choose one of questions 3 or 4.** - Empty code blocks are for your use. Feel free to create more under each section as needed.

Submission instructions: - When complete, fill out your publicly available GitHub repo file URL and group members below, then export or print this .ipynb file to PDF and upload the PDF to Gradescope.

Link to this ipynb file in your public GitHub repo (replace below URL with yours):

https://github.com/my-username/my-repo/assignment-file-name.ipynb

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1.1 Imports

[3]:
```
from matplotlib import pyplot as plt
from rl.approximate_dynamic_programming import (
    value_iteration,
    approximate_policy_iteration
)

from rl.function_approx import Tabular
from rl.markov_decision_process import FiniteMarkovDecisionProcess,␣
 ↪StateActionMapping
from numpy import allclose
from rl.distribution import Choose
import numpy as np
from typing import Mapping, Iterable, Tuple, Dict
from rl.distribution import Categorical, Choose
from rl.markov_decision_process import FiniteMarkovDecisionProcess,␣
 ↪StateActionMapping
```

```python
from dataclasses import dataclass, field, replace
from rl.dynamic_programming import policy_iteration, value_iteration_result
from rl.function_approx import FunctionApprox, Tabular
from dataclasses import dataclass
from itertools import islice
```

1.2 Question 1

Implement *Approximate Policy Iteration*, generalization of the tabular Policy Iteration we covered in the previous class. In order to implement Approximate Policy Iteration, first review the interface and implementation of *Approximate Policy Evaluation* and *Approximate Value Iteration* (in file rl/approximate_dynamic_programming.py), then design the interface of *Approximate Policy Iteration* to be the same as that of *Approximate Value Iteration*. Note that your implementation of *Approximate Policy Iteration* would need to invoke *Approximate Policy Evaluation* since Policy Evaluation is a component of Policy Iteration. Test that your implementation is correct in two ways:

- Ensure that *Approximate Policy Iteration* gives the same Optimal Value Function/Optimal Policy as that obtained by *Approximate Value Iteration*.
- Ensure that *Approximate Policy Iteration* produces the same result as our prior implementation of Policy Iteration (in file rl/dynamic_programming.py). For this you need to pass to your implementation of *Approximate Policy Iteration* a `FiniteMarkovDecisionProcess` input and a `Tabular` instance for the `FunctionApprox` input.

[ ]:
```python
def approximate_policy_iteration(
    mdp: MarkovDecisionProcess[S, A],
    gamma: float,
    approx_0: ValueFunctionApprox[S],
    non_terminal_states_distribution: NTStateDistribution[S],
    num_state_samples: int,
    num_iterations: int
) -> Iterator[Tuple[ValueFunctionApprox[S], DeterministicPolicy[S, A]]]:
    """
    Approximate Policy Iteration function.

    Parameters:
    mdp: MarkovDecisionProcess[S, A] - The given Markov Decision Process.
    gamma: float - Discount factor.
    approx_0: ValueFunctionApprox[S] - Initial value function approximation.
    non_terminal_states_distribution: NTStateDistribution[S] - Distribution of␣
 ↪non-terminal states.
    num_state_samples: int - Number of state samples for approximation␣
 ↪calculation.
    num_iterations: int - Number of iterations.

    Returns:
```

```
    Iterator[Tuple[ValueFunctionApprox[S], DeterministicPolicy[S, A]]] -␣
↪Iterator of value function approximations and policies.
    """
    value_function = approx_0
    policy = DeterministicPolicy(lambda s: np.random.choice(mdp.
↪actions(NonTerminal(s))))

    for _ in range(num_iterations):
        # Approximate Policy Evaluation
        value_function = evaluate_mrp(
            mdp.apply_policy(policy),
            gamma,
            value_function,
            non_terminal_states_distribution,
            num_state_samples
        ).__next__()

        # Policy Improvement
        def action_for_state(s: NonTerminal[S]) -> A:
            return max(
                ((mdp.step(s, a).expectation(lambda s_r: s_r[1] + gamma *␣
↪extended_vf(value_function, s_r[0])), a)
                 for a in mdp.actions(s)),
                key=itemgetter(0)
            )[1]

        policy = DeterministicPolicy(action_for_state)

        yield (value_function, policy)
```

1.2.1 Test1: Verify that Approximate Policy Iteration gives results as that obtained by Approximate Value Iteration.

[ ]:
```
# design a simple mdp for testing
class SimpleGamblingMDP(FiniteMarkovDecisionProcess[int, int]):
    def __init__(self, max_capital: int, win_prob: float):
        """
        Initialize the Simple Gambling MDP.

        Parameters:
        max_capital (int): The maximum capital.
        win_prob (float): Probability of winning a bet.
        """
        self.max_capital = max_capital
        self.win_prob = win_prob
        super().__init__(self.get_action_transition_reward_map())
```

```python
    def get_action_transition_reward_map(self) -> StateActionMapping[int, int]:
        """
        Get the action transition reward map for the MDP.

        Returns:
        StateActionMapping[int, int]: The action transition reward mapping.
        """
        d: Dict[int, Dict[int, Categorical[Tuple[int, float]]]] = {}
        for capital in range(1, self.max_capital):
            action_dict: Dict[int, Categorical[Tuple[int, float]]] = {}
            for bet in range(1, min(capital, self.max_capital - capital) + 1):
                outcomes = {
                    (capital + bet, bet): self.win_prob,  # Winning outcome
                    (capital - bet, -bet): 1 - self.win_prob  # Losing outcome
                }
                action_dict[bet] = Categorical(outcomes)
            d[capital] = action_dict
        return d


max_capital = 100
win_prob = 0.5
gamma = 0.9

# Creating an instance of MDP
gambling_mdp = SimpleGamblingMDP(max_capital=max_capital, win_prob=win_prob)

initial_values = {s: 0.0 for s in range(1, max_capital)}
approx_0 = Tabular(values_map=initial_values, count_to_weight_func=lambda n: 1.
 ↪0 / n)

# Approximate value iteration
value_function_approx = value_iteration(
    mdp=gambling_mdp,
     =gamma,
    approx_0=approx_0,
    non_terminal_states_distribution=Choose(range(1, max_capital)),
    num_state_samples=1000
).__next__()

# Approximate policy iteration
policy_iteration_result = approximate_policy_iteration(
    mdp=gambling_mdp,
    gamma=gamma,
    approx_0=approx_0,
    non_terminal_states_distribution=Choose(range(1, max_capital)),
    num_state_samples=1000,
```

```
    num_iterations=100
).__next__()

# Check if value functions from both methods are close enough
tolerance = 1e-5
similar = all(
    abs(value_function_approx.values_map[s] - policy_iteration_result[0].
↪values_map.get(s, 0)) <= tolerance
    for s in value_function_approx.values_map
)
print(f"Similar Value Functions: {similar}")
```

```
Similar Value Functions: True
```

1.2.2 Test2: Ensure that Approximate Policy Iteration produces the same results as our prior implementation of Policy Iteration

[ ]:
```
# Test code
max_capital = 100
win_prob = 0.5
gamma = 1.0
tolerance = 0.001

gambling_mdp = SimpleGamblingMDP(max_capital=max_capital, win_prob=win_prob)

# Use policy_iteration function
policy_iter_result = policy_iteration(gambling_mdp, gamma)

# Retrieve the final value function and policy
optimal_value_function, optimal_policy = next(islice(policy_iter_result, 100,␣
↪None))

# Retrieve the final results from approximate policy iteration
approximate_value_function = policy_iteration_result[0].values_map

#function to test whether two output are close
def within_tolerance(v1: Dict[int, float], v2: Dict[int, float], tolerance:␣
↪float) -> bool:
    return all(abs(v1[s] - v2.get(s, 0.0)) <= tolerance for s in v1)
# Check if the optimal and approximate value functions are close enough
are_close = within_tolerance(optimal_value_function,␣
↪approximate_value_function, tolerance)
print(f"Are the value functions close enough (tolerance {tolerance})? {'Yes' if␣
↪are_close else 'No'}")
```

```
Are the value functions close enough (tolerance 0.001)? Yes
```

1.3 Question 2

Assume the Utility function is $U(x) = x - \frac{\alpha x^2}{2}$. Assuming $x \sim \mathcal{N}(\mu, \sigma^2)$, calculate:

- Expected Utility $\mathbb{E}[U(x)]$
- Certainty-Equivalent Value x_{CE}
- Absolute Risk-Premium π_A

Assume you have a million dollars to invest for a year and you are allowed to invest z dollars in a risky asset whose annual return on investment is $\mathcal{N}(\mu, \sigma^2)$ and the remaining (a million minus z dollars) would need to be invested in a riskless asset with fixed annual return on investment of r. You are not allowed to adjust the quantities invested in the risky and riskless assets after your initial investment decision at time $t = 0$ (static asset allocation problem). If your risk-aversion is based on this Utility function, how much would you invest in the risky asset? In other words, what is the optimal value for z, given your level of risk-aversion (determined by a fixed value of α)?

Plot how the optimal value of z varies with α.

1). Expected Utility:

$$\begin{aligned}
\mathbb{E}[U(x)] &= \mathbb{E}[x] - \frac{\alpha}{2}\mathbb{E}[x^2] && (1)\\
&= \mathbb{E}[x] - \frac{\alpha}{2}(\mathrm{Var}[x] + \mathbb{E}[x]^2) && (2)\\
&= \mu - \frac{\alpha}{2}(\sigma^2 + \mu^2) && (3)
\end{aligned}$$

2). Certainty-Equivalent Value:

$$\begin{aligned}
x_{CE} &= U^{-1}(\mathbb{E}[U(x)]) && (4)\\
&= \frac{1 \pm \sqrt{1 - 2\alpha\mathbb{E}[U(x)]}}{\alpha} && (5)\\
&= \frac{1 \pm \sqrt{\alpha^2\sigma^2 + (\alpha\mu - 1)^2}}{\alpha} && (6)
\end{aligned}$$

3). Absolute Risk-Premium:

$$\begin{aligned}
\pi_A &= \mathbb{E}[x] - x_{CE} && (7)\\
&= \mu - \frac{1 \pm \sqrt{\alpha^2\sigma^2 + (\alpha\mu - 1)^2}}{\alpha} && (8)\\
&= \frac{\alpha\mu - 1 \pm \sqrt{\alpha^2\sigma^2 + (\alpha\mu - 1)^2}}{\alpha} && (9)
\end{aligned}$$

For the situation of investing z dollars in risky asset, $1M - z$ dollars in rates, the synthetic portfolio has a return of

$$X_z \sim \mathcal{N}(z\mu + (1M - z)r, z^2\sigma^2).$$

From the expression of expected utility of a Gaussian R.V., we get

$$
\begin{aligned}
\mathbb{E}[U(X_z)] &= z\mu + (1M - z)r - \frac{\alpha}{2}(z^2\sigma^2 + (z\mu + (1M - z)r)^2) && (10)\\
&= (\mu - r)z + 1M \cdot r - \frac{\alpha}{2}[\sigma^2 + (\mu - r)^2]z^2 - \frac{\alpha}{2}2(1M \cdot r) \cdot (\mu - r)z + \frac{\alpha}{2}(1M \cdot r)^2 && (11)\\
&= -\frac{\alpha[\sigma^2 + (\mu - r)^2]}{2}z^2 + [1 - \alpha(1M \cdot r)](\mu - r)z + 1M \cdot r + \frac{\alpha}{2}(1M \cdot r)^2 && (12)\\
& && (13)
\end{aligned}
$$

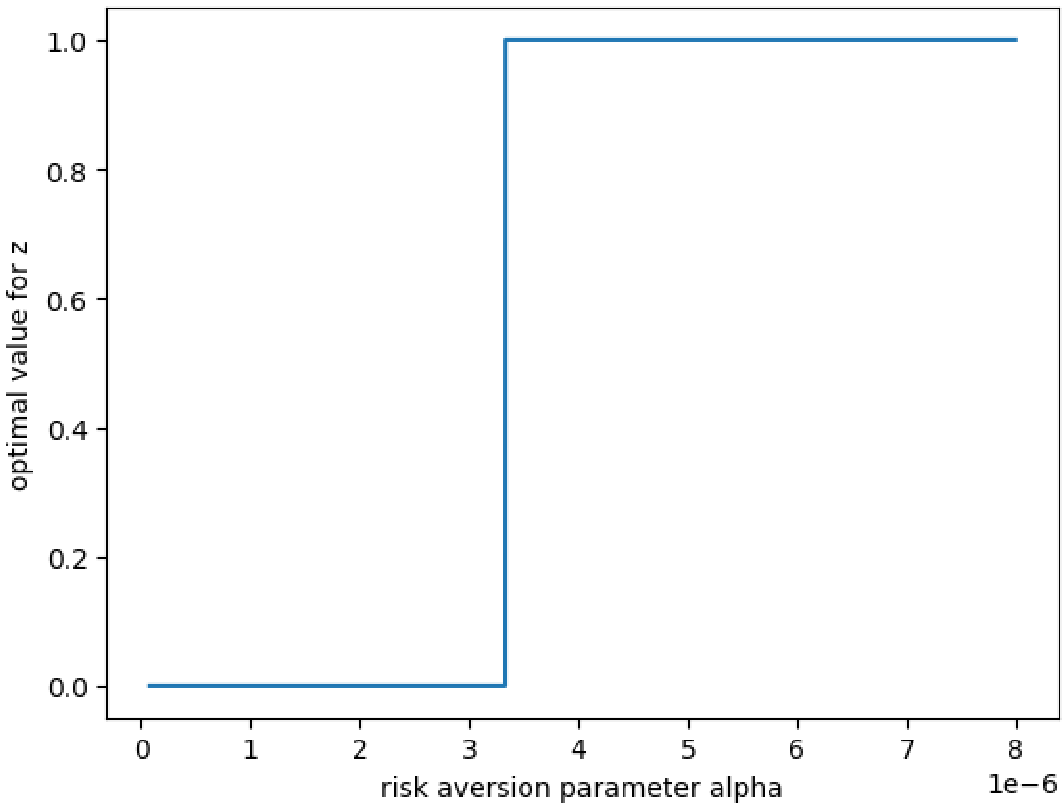
Thus, if the risk-aversion is perfectly measured by certainty equivalent value x_{CE} with parameter α, the optimal value is given by

$$
\begin{aligned}
z^* &= \text{argmax}_z\ U^{-1}(\mathbb{E}[U(X_z)]) && (14)\\
&= \text{argmax}_z\ \frac{1 \pm \sqrt{\alpha^2\sigma^2z^2 + (\alpha \cdot 1M \cdot r + \alpha(\mu - r)z - 1)^2}}{\alpha} && (15)\\
&= \text{argmax}_z\ \alpha^2[\sigma^2 + (\mu - r)^2]z^2 - 2\alpha[1 - \alpha(1M \cdot r)](\mu - r)z + [1 - \alpha(1M \cdot r)]^2 && (16)\\
&= \begin{cases} 0, & \dfrac{[1 - \alpha(1M \cdot r)](\mu - r)}{\alpha[\sigma^2 + (\mu - r)^2]} \geq 0.5M \\ 1M, & o.w. \end{cases} && (17)\\
&= \begin{cases} 0, & \alpha \geq \dfrac{2(\mu - r) * 10^{-6}}{\sigma^2 + \mu^2 - r^2} \\ 1M, & o.w. \end{cases} && (18)\\
& && (19)
\end{aligned}
$$

[4]:
```
import numpy as np
from matplotlib import pyplot as plt
```

[18]:
```
mu = 0.1
r = 0.05
sigma = 0.15
alpha0 = 0.4/(1e6*r)
alpha = np.linspace(alpha0/100, alpha0, 10000)
z = ((1 - alpha*1e6*r)*(mu - r)/(alpha * (sigma**2 + (mu-r)**2)) < 5e5)
plt.plot(alpha,z)
plt.xlabel('risk aversion parameter alpha')
plt.ylabel('optimal value for z')
```

[18]:
```
Text(0, 0.5, 'optimal value for z')
```



1.4 Question 3

Assume you are playing a casino game where at every turn, if you bet a quantity x, you will be returned $x \cdot (1 + \alpha)$ with probability p and returned $x \cdot (1 - \beta)$ with probability $q = 1 - p$ for $\alpha, \beta \in \mathbb{R}^+$ (i.e., the return on bet is α with probability p and $-\beta$ with probability $q = 1 - p$) . The problem is to identify a betting strategy that will maximize one's expected wealth over the long run. The optimal solution to this problem is known as the Kelly criterion, which involves betting a constant fraction of one's wealth at each turn (let us denote this optimal fraction as f^*).

It is known that the Kelly criterion (formula for f^*) is equivalent to maximizing the Expected Utility of Wealth after a single bet, with the Utility function defined as: $U(W) = \log(W)$. Denote your wealth before placing the single bet as W_0. Let f be the fraction (to be solved for) of W_0 that you will bet. Therefore, your bet is $f \cdot W_0$.

- Write down the two outcomes for wealth W at the end of your single bet of $f \cdot W_0$.
- Write down the two outcomes for log (Utility) of W.
- Write down $\mathbb{E}[\log(W)]$.
- Take the derivative of $\mathbb{E}[\log(W)]$ with respect to f.
- Set this derivative to 0 to solve for f^*. Verify that this is indeed a maxima by evaluating the second derivative at f^*. This formula for f^* is known as the Kelly Criterion.

- Convince yourself that this formula for f^* makes intuitive sense (in terms of it's dependency on α, β and p).

[ ]:

1.5 Question 4

Derive the solution to Merton's Portfolio problem for the case of the $\log(\cdot)$ Utility function. Note that the derivation in the textbook is for CRRA Utility function with $\gamma \neq 1$ and the case of the $\log(\cdot)$ Utility function was left as an exercise to the reader.

We denote the fraction of wealth allocated to the risky asset at time t as \$ (t, W_t)\$ and the fraction of wealth allocated to the riskless asset at time t is $1 - \pi(t, W_t)$. The infinitesimal change in wealth dW_t from time t to time $t + dt$ is given by:

$$dW_t = ((r + \pi_t \cdot (\mu - r)) \cdot W_t - c_t) \cdot dt + \pi_t \cdot \sigma \cdot W_t \cdot dz_t$$

The goal is to determine optimal $\pi(t, W_t), c(t, W_t)$ at any time t to maximize:

$$\mathbb{E}[\int_t^T e^{-\rho(s-t)} \cdot \log(c_s) \cdot ds + e^{-\rho(T-t)} \cdot B(T) \cdot \log(W_T) | W_t]$$

where $B(T)$ is the bequest and take $B(T) = 1/\rho$ for simplicity. The Hamilton-Jacobi-Bellman equation is then

$$\max_{\pi_t, c_t} \{\mathbb{E}_t[dV^*(t, W_t)] + \log(c_t) \cdot dt\} = \rho \cdot V^*(t, W_t) \cdot dt$$

Then, use Ito's Lemma on dV^*, remove the dz_t term, and divide throughout by dt to get

$$\max_{\pi_t, c_t} \{\frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t} \cdot ((r + \pi_t \cdot (\mu - r)) \cdot W_t - c_t) + \frac{\partial^2 V^*}{\partial W_t^2} \cdot \frac{\pi_t^2 \cdot \sigma^2 \cdot W_t^2}{2} + \log(c_t)\} = \rho \cdot V^*(t, W_t)$$

with the terminal condition

$$V^*(T, W_T) = \frac{1}{\rho} \log(W_T)$$

Take partial derivative with respect to π_t^*, c_t^* and equate to 0 to get

$$(\mu - r) \cdot \frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2} \cdot \pi_t \cdot \sigma^2 \cdot W_t = 0 \implies \pi_t^* = \frac{-\frac{\partial V^*}{\partial W_t} \cdot (\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \cdot \sigma^2 \cdot W_t}$$

$$-\frac{\partial V^*}{\partial W_t} + \frac{1}{c_t^*} = 0 \implies c_t^* = \frac{\partial W_t}{\partial V^*}$$

Substitute π_t^* and c_t^* back to get the Optimal Value Function Partial Differential Equation:

$$\frac{\partial V^*}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{(\frac{\partial V^*}{\partial W_t})^2}{\frac{\partial^2 V^*}{\partial W_t^2}} + \frac{\partial V^*}{\partial W_t} \cdot r \cdot W_t + \log(\frac{\partial W_t}{\partial V^*}) - 1 = \rho \cdot V^*(t, W_t)$$

We surmise with a guess solution in terms of a deterministic function (f) of time:

$$V^*(t, W_t) = f(t) + \frac{1}{\rho} log(W_t)$$

then

$$\frac{\partial V^*}{\partial t} = f'(t)$$

$$\frac{\partial V^*}{\partial W_t} = \frac{1}{\rho W_t}$$

$$\frac{\partial^2 V^*}{\partial W_t^2} = -\frac{1}{\rho W_t^2}$$

In this case, $\pi_t^*(t, W_t) = \dfrac{\mu - r}{\sigma^2}$ and $c_t^*(t, W_t) = \rho W_t$. Substituting the guess solution in the PDE, we get the simple ODE: $f'(t) - \rho f(t) + \nu = 0$ where $\nu = \dfrac{(\mu - r)^2}{2\rho\sigma^2} + \dfrac{r - \rho}{\rho} + log(\rho)$ with boundary condition $f(T) = 0$. The solution of the ODE is then

$$f(t) = \frac{\nu}{\rho}(1 - e^{-\rho(T-t)})$$

Substituting $f(t)$ into $V^*(t, W_t)$ to get

$$V^*(t, W_t) = \frac{\nu}{\rho}(1 - e^{-\rho(T-t)}) + \frac{1}{\rho} log(W_t)$$