Κυματική και Κβαντική Φυσική 2021-2022

4η Σειρά Ασκήσεων

ΟΝΟΜΑΤΕΠΩΝΥΜΟ: Αλεξοπούλου Γεωργία

ΑΜ: 03120164

Άσκηση IV 15

$$V(x) = \frac{\hbar^2}{8mx^2} + \frac{\hbar^2}{2mL^3}x$$

Ισχύει πως:

$$H(p,x) = \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} + \frac{\hbar^2}{8mx^2} + \frac{\hbar^2}{2mL^3}x$$

Από την αρχή της αβεβαιότητας έχουμε ότι:

$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$, όμως $\Delta x$: σταθερό, άρα

$$\Delta p \geq \frac{\hbar}{2 \cdot \Delta x} \quad (1)$$

Ισχύει επίσης πως $x \simeq \Delta x$ και $p \simeq \Delta p$:

$$E \simeq \frac{\Delta p^2}{2m} + \frac{\hbar^2}{8m \cdot \Delta x^2} + \frac{\hbar^2}{2mL^3}\Delta x \quad (2)$$

Από τις (1) και (2) έχουμε:

$$E \geq \frac{1}{2m} \cdot \frac{\hbar^2}{4\Delta x^2} + \frac{\hbar^2}{8m\Delta x^2} + \frac{\hbar^2}{2mL^3}\Delta x$$

$$E \geq \frac{\hbar^2}{4m\Delta x^2} + \frac{\hbar^2}{2mL^3}\Delta x$$

Έχουμε $E_{min}$ (ελάχιστη ενέργεια E) για κάποιο $\Delta x_0$. Για $\Delta x_0$ ισχύει:

$$\left.\frac{dE}{d(\Delta x)}\right|_{\Delta x_0} = 0 \Leftrightarrow -\frac{\hbar^2}{2m(\Delta x_0)^3} + \frac{\hbar^2}{2mL^3} = 0 \Leftrightarrow \Delta x_0 = L$$

Συνεπώς: $\Delta p_0 = \frac{\hbar}{2L}$, ~~όπου~~ άρα

$$E_{min} = \frac{\Delta p_0^2}{2m} + \frac{\hbar^2}{8m\Delta x_0^2} + \frac{\hbar^2}{2mL^3}\Delta x_0 = \frac{\hbar^2}{4L^2} \cdot \frac{1}{2m} + \frac{\hbar^2}{8m} \cdot \frac{4}{L^2} + \frac{\hbar^2}{2mL^3} \cdot L$$

$$E_{min} = \frac{\hbar^2}{8mL^2} + \frac{\hbar^2}{8mL^2} + \frac{\hbar^2}{2mL^2} = \frac{\hbar^2}{2mL^2}\left[\frac{1}{4} + \frac{1}{4} + 1\right] = \frac{\hbar^2}{2mL^2} \cdot \frac{1+1+4}{4}$$

$$E_{min} = \frac{3\hbar^2}{4mL^2}$$

Άσκηση IV 16

Η V(x) είναι τμηματικά σταθερή.

Ισχύει: $-\frac{\hbar^2}{2m}y'' + Vy = Ey \Leftrightarrow -\frac{\hbar^2}{2m}y'' = (E-V)y \Leftrightarrow +\frac{\hbar^2}{2m}y'' = (V-E)y$

$$y'' = \frac{2m(V-E)}{\hbar^2}y \qquad y'' = \left(\frac{2m(V-E)}{\hbar^2}\right)y$$

- Για $V-E>0 \Leftrightarrow V>E$: Θέτουμε $k_1^2 = \frac{2m}{\hbar^2}(V-E) \Leftrightarrow k_1 = \frac{\sqrt{2m}}{\hbar}\cdot\sqrt{V-E}$

  Άρα $y = Ae^{k_1x} + Be^{-k_1x}$
- Για $V-E<0 \Leftrightarrow V<E$: Θέτουμε $-k_2^2 = \frac{2m}{\hbar^2}(V-E) \Leftrightarrow k_2 = \frac{\sqrt{2m(E-V)}}{\hbar}$

  Άρα $y = Ae^{ik_2x} + Be^{-ik_2x}$
- Για $V-E=0 \Leftrightarrow V=E$: $y'=0 \Rightarrow y = ax+b$

α) Για $E \in (0eV, 1eV)$:

Για $x \in (-\infty, 0Å] \cup [0Å, 2Å] \cup [3Å, +\infty)$, ισχύει πως $V>E$, συνεπώς $y = Ae^{k_1x} + Be^{-k_1x}$, όπου $k_1 = \frac{\sqrt{2m(V-E)}}{\hbar}$

Για $x \in (-\infty, 0Å]$: $V=3 \Rightarrow k_1 = \frac{\sqrt{2m(3-E)}}{\hbar^2}$

Για $x \in [0Å, 2Å]$: $V=1 \Rightarrow k_1 = \frac{\sqrt{2m(1-E)}}{\hbar}$

Για $x \in [3Å, +\infty)$: $V=2 \Rightarrow k_1 = \frac{\sqrt{2m(2-E)}}{\hbar}$

Για $x \in [2Å, 3Å]$, ισχύει πως $V<E$, άρα $y = Ae^{ik_2x} + Be^{-ik_2x}$,
όπου $k_2 = \frac{\sqrt{2m(E-V)}}{\hbar}$, όμως $V=0$, άρα $k_2 = \frac{\sqrt{2mE}}{\hbar}$

β) Για $E \in (1eV, 2eV)$:

Για $x \in (-\infty, 0Å] \cup [3Å, +\infty)$ ισχύει $V > E$, συνεπώς

$y = Ae^{k_1 x} + Be^{-k_1 x}$

Για $x \in [0Å, 2Å] \cup [2Å, 3Å]$ ισχύει $V < E$, συνεπώς

$y = Ae^{ik_2 x} + Be^{-ik_2 x}$

γ) Για $E \in (2eV, 3eV)$:

Για $x \in (-\infty, 0Å]$ ισχύει $V > E$, άρα: $y = Ae^{k_1 x} + Be^{-k_1 x}$

Για $x \in [0Å, +\infty)$ ισχύει $V < E$, άρα: $y = Ae^{ik_2 x} + Be^{-ik_2 x}$

δ) Για $E > 3eV$:

Ισχύει $V < E$, άρα $y = Ae^{ik_2 x} + Be^{-ik_2 x}$

Συνθήκες συνέχειας:

- $y_I(-\infty) = 0$ και $y_{IV}(+\infty) = 0$
- $y_I(0) = y_{II}(0)$
- $y_{II}(2) = y_{III}(2)$
- $y_{III}(3) = y_{IV}(3)$

Άσκηση IV.19

$y(x) = N \exp\left\{ -\frac{\mu^2 x^2}{2} + iax \right\}, \ \mu > 0$

α) $\int_{\mathbb{R}} y(x) \cdot y^*(x) dx = \int_{\mathbb{R}} N \cdot e^{\left(-\frac{\mu^2 x^2}{2} + iax\right)} \cdot N \cdot e^{\left(-\frac{\mu^2 x^2}{2} - iax\right)} \cdot e^{-ibt} \cdot e^{ibt} dx =$

$= \int_{\mathbb{R}} N^2 \cdot e^{-\mu^2 x^2} dx = N^2 \int_{-\infty}^{+\infty} e^{-\mu^2 x^2} dx = N^2 \cdot \frac{\sqrt{\pi}}{\mu} = 1$

$N > 0$, άρα: $N = \sqrt{\frac{\mu\sqrt{\pi}}{\pi}}$

β) • $\int_{\mathbb{R}} x y^*(x) y(x) dx = \frac{\mu\sqrt{\pi}}{\pi} \cdot \int_{-\infty}^{+\infty} x e^{-\mu^2 x^2} dx = \frac{\mu\sqrt{\pi}}{\pi} \cdot 0 = 0$

$\Rightarrow \langle x \rangle = 0$

• $\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 y^*(x) y(x) dx = \frac{\mu\sqrt{\pi}}{\pi} \cdot \int_{-\infty}^{+\infty} x^2 e^{-\mu^2 x^2} dx = \frac{\mu\sqrt{\pi}}{\pi} \cdot \frac{\sqrt{\pi}}{2\mu^3} = \frac{1}{2\mu^2} \Leftrightarrow \langle x^2 \rangle = \frac{1}{2\mu^2}$

• $\langle p \rangle = \int_{-\infty}^{+\infty} y^*(x) p y(x) dx = \int_{-\infty}^{+\infty} y^* (-i\hbar) \frac{dy}{dx} dx =$

$$= -i\hbar N^2 \int_{-\infty}^{+\infty} e^{\left(-\frac{\mu^2 x^2}{2} - iax\right)} (-\mu^2 x + ia) \cdot e^{\left(-\frac{\mu^2 x^2}{2} + iax\right)} dx =$$

$$= -i\hbar N^2 \cdot \int_{-\infty}^{+\infty} e^{-\mu^2 x^2} (-\mu^2 x + ia) dx =$$

$$= -i\hbar N^2 \cdot \int_{-\infty}^{+\infty} e^{-\mu^2 x^2} (-\mu^2 x) dx - i\hbar N^2 \int_{-\infty}^{+\infty} e^{-\mu^2 x^2} \cdot ia\, dx =$$

$$= -i\hbar N^2 \left(0 + \int_{-\infty}^{+\infty} e^{-\mu^2 x^2} \cdot ia\, dx\right) \Rightarrow \underline{\langle p \rangle = a\hbar}$$

- $$\langle p^2 \rangle = -\hbar^2 N^2 \int_{\mathbb{R}} e^{\left(-\frac{\mu^2 x^2}{2} + iax\right)} \cdot (\mu^4 x^2 - i2a\mu^2 x - \mu^2 - a^2) e^{\left(-\frac{\mu^2 x^2}{2} + iax\right)} dx =$$

$$= -\hbar^2 N^2 \int_{\mathbb{R}} e^{(-\mu^2 x^2 + i2ax)} (\mu^4 x^2 - i2a\mu^2 x - \mu^2 - a^2) dx =$$

$$= -\hbar^2 N^2 \cdot \mu^4 \int_{\mathbb{R}} x^2 \cdot e^{(-\mu^2 x^2 + i2ax)} dx + \hbar^2 N^2 \cdot i2a\mu^2 \cdot \int_{\mathbb{R}} x \cdot e^{(-\mu^2 x^2 + i2ax)} dx +$$

$$+ \hbar^2 N^2 (\mu^2 + a^2) \cdot \int_{\mathbb{R}} e^{(-\mu^2 x^2 + i2ax)} dx =$$

$$= -\hbar^2 N^2 \mu^{\cancel{4}} \cdot \frac{\sqrt{\pi}}{2\cancel{\mu^3}} + i\hbar^2 N^2 \cdot 2a\mu^2 \cdot 0 + \hbar^2 N^2 (\mu^2 + a^2) \cdot \frac{\sqrt{\pi}}{\mu} \Rightarrow$$

$$\Rightarrow \underline{\langle p^2 \rangle = \frac{\hbar^2 \mu^2}{2} + \hbar^2 a^2}$$

γ) $\Delta x^2 = \langle x^2 \rangle - \cancel{(\langle x \rangle)^2}^{\,0} \Rightarrow \Delta x^2 = \frac{1}{\mu\sqrt{2}}$

$$\Delta p^2 = \langle p^2 \rangle - (\langle p \rangle)^2 = \frac{\hbar^2 \mu^2}{2} + \hbar^2 a^2 - a^2 \hbar^2 \Rightarrow \Delta p^2 = \frac{\hbar^2 \mu^2}{2} \Rightarrow \Delta p = \frac{\hbar \mu}{\sqrt{2}}$$

$\Delta x \cdot \Delta p = \frac{1}{\mu\sqrt{2}} \cdot \frac{\hbar\mu}{\sqrt{2}} \Rightarrow \underline{\Delta x \cdot \Delta p = \frac{\hbar}{2}}$, άρα προκύπτει ισχύει η αρχή της αβεβαιότητας

Άσκηση IV 23

$y = Nx(L-x)$

$$\int_0^L y^*(x) y(x) dx = \int_0^L y^2(x) dx = \int_0^L N^2 x^2 (L-x)^2 dx = N^2 \cdot \int_0^L (x^4 - 2Lx^3 + L^2 x^2) dx$$

$$= N^2 \left[\frac{x^5}{5} + \frac{(-2)x^5}{4} + \frac{x^5}{3}\right]_0^L = N^2 \left[\frac{L^5}{5} - \frac{L^5}{2} + \frac{L^5}{3}\right] = N^2 \frac{L^5}{30}$$

Πρέπει $\int \psi^*(x)\psi(x)dx = 1$, άρα

$N^2 \cdot \frac{L^5}{30} = 1 \Rightarrow N = \sqrt{\frac{30}{L^5}}$ (εφόσον $N>0$).

Οι ιδιοσυναρτήσεις θα έχουν μορφή:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \cdot \sin\left(\frac{n\pi x}{L}\right)$$

Η πιθανότητα εμφάνισης της $E_1$ είναι $|c_1|^2$, όπου:

$$c_n = \int_0^L \psi_n(x)\cdot\psi(x,0)dx = \int_0^L \sqrt{\frac{2}{L}} \cdot N \cdot \sin\left(\frac{n\pi x}{L}\right) x(L-x)dx =$$

$$= \sqrt{\frac{2}{L}} \cdot \sqrt{\frac{30}{L^5}} \cdot \int_0^L x \sin\left(\frac{n\pi x}{L}\right)(L-x)dx$$

Για $n=1$:

$$c_1 = \sqrt{\frac{60}{L^6}} \cdot \int_0^L x \cdot \sin\left(\frac{\pi x}{L}\right)(L-x)dx$$

$$= \frac{2\sqrt{15}}{L^3} \cdot \left[\cancel{\frac{L^3}{\pi}} - \cancel{\frac{L^3}{\pi}} + \frac{4L^3}{\pi^3}\right] = \frac{8\sqrt{15}}{\pi^3}$$

$|c_1|^2 = \frac{960}{\pi^6} \simeq 0{,}998$

Συνεπώς: $\psi_1(x) = \sqrt{\frac{2}{L}} \cdot \sin\left(\frac{\pi x}{L}\right)$, άρα

$$\psi_1(x) = \sqrt{\frac{2}{L}} \cdot \sin\left(\frac{\pi x}{L}\right) \cdot e^{-i\frac{E_1 t}{\hbar}}$$

## Άσκηση IV 25

α) Ισχύει ότι: $\psi_n(x) = \sqrt{\frac{2}{L}} \cdot \sin\left(\frac{n\pi x}{L}\right)$

$$\langle x \rangle = \int_0^L x \cdot \psi_n^*(x)\psi_n(x)dx = \int_0^L x \cdot \psi_n^2(x)dx = \frac{2}{L} \cdot \int_0^L x \cdot \sin^2\left(\frac{n\pi x}{L}\right)dx =$$

$$= -\frac{2}{L} \cdot \frac{L^2[2\pi n \sin(2\pi n) + \cos(2\pi n) - 2\pi^2 n^2 - 1]}{8\pi^2 n^2} \Rightarrow \langle x \rangle = \frac{L}{2}$$

$$\langle x^2 \rangle = \int_0^L x^2 \psi_n(x) \cdot \psi_n^*(x)dx = \frac{2}{L} \int_0^L x^2 \sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_0^L x^2\left(1 - \cos\frac{2n\pi x}{L}\right)dx =$$

$$= \frac{1}{L}\int_0^L x^2 dx - \frac{1}{L}\int_0^L x^2 \cdot \cos\left(\frac{2n\pi x}{L}\right) dx = \frac{1}{L}\cdot\left[\frac{L^3}{3}\right]_0^L - \frac{1}{L}\overset{0}{\cancel{\left[\frac{L}{2n\pi}\cdot x^2 \sin\left(\frac{2n\pi x}{L}\right)\right]_0^L}} +$$

$$+ \frac{1}{L}\int_0^L \frac{L}{2n\pi}\cdot 2x \cdot \sin\left(\frac{2n\pi x}{L}\right) dx =$$

$$= \frac{L^2}{3} + \frac{1}{n\pi}\left[-\frac{L}{2n\pi}\cdot x \cdot \cos\left(\frac{2n\pi x}{L}\right)\right]_0^L + \frac{1}{n\pi}\int_0^L \frac{L}{2n\pi}\cdot \cos\left(\frac{2n\pi x}{L}\right) dx =$$

$$= \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2} + \frac{L}{2n^2\pi^2}\cdot\int_0^L \cos\left(\frac{2n\pi x}{L}\right) dx =$$

$$= \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2} + \frac{L}{2n^2\pi^2}\cdot \overset{0}{\cancel{\left[\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_0^L}} =$$

$$\langle x^2 \rangle = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$$

β) $\Delta x^2 = \langle x^2 \rangle - (\langle x \rangle)^2 = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2} - \frac{L^2}{4}$

$$\Delta x^2 = \frac{L^2}{12} - \frac{L^2}{2n^2\pi^2} \Rightarrow \Delta x = L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

$$\Delta x = \frac{L}{2n\pi}\cdot\sqrt{\frac{n^2\pi^2 - 6}{3}}$$

Ισχύει πως: $\frac{\langle p \rangle}{m} = \frac{\partial \langle x \rangle}{\partial t} = \frac{\partial}{\partial t}\left(\frac{L}{2}\right) = 0 \Rightarrow \langle p \rangle = 0.$

$$\langle p^2 \rangle = \int_0^L \psi_n^*(x)\cdot \psi_n(x) p^2 \, dx = \int_0^L (-i\hbar)^2 \cdot \sqrt{\frac{2}{L}}\cdot \sin\left(\frac{n\pi x}{L}\right)\cdot \frac{\partial^2}{\partial x^2}\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right) dx =$$

$$= -\sqrt{\frac{2}{L}}\cdot \hbar^2 \cdot \int_0^L -\sin\left(\frac{n\pi x}{L}\right)\cdot\sqrt{\frac{2}{L}}\cdot\frac{n^2\pi^2}{L^2}\sin\left(\frac{n\pi x}{L}\right) dx =$$

$$= \frac{2}{L}\cdot\frac{n^2\pi^2}{L^2}\cdot\hbar^2\cdot\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{2n^2\pi^2\hbar^2}{L^3}\cdot\int_0^L \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{L}\right) dx =$$

$$= \frac{\cancel{2}n^2\pi^2\hbar^2}{L^3}\cdot\frac{L}{\cancel{2}} - \frac{2n^2\pi^2\hbar^2}{L^3}\overset{0}{\cancel{\left[\frac{L}{2n\pi}\cdot \sin\left(\frac{2n\pi x}{L}\right)\right]_0^L}} = \frac{n^2\pi^2\hbar^2}{L^2}$$

$$\Delta p^2 = \langle p^2 \rangle - \overset{0}{\cancel{(\langle p \rangle)^2}} = \left(\frac{n\pi\hbar}{L}\right)^2 \Rightarrow \Delta p = \frac{n\pi\hbar}{L}$$