Φυλλάδιο VI

① $f(x) = ce^{-4|x|}, \; x \in \mathbb{R}, \; c > 0$

α) Από ιδιότητες σπη: $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = \int_{-\infty}^{0} ce^{4x}dx + \int_{0}^{\infty} ce^{-4x}dx = c\left\{\left[\frac{e^{4x}}{4}\right]_{-\infty}^{0} + \left[\frac{e^{-4x}}{-4}\right]_{0}^{+\infty}\right.$$

$$= c\left[\left(\frac{1}{4} - 0\right) + \left(0 + \frac{1}{4}\right)\right] = \frac{c}{2} = 1 \Rightarrow \underline{c = 2}$$

β) $$E(x) = \int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{0} 2xe^{4x}dx + \int_{0}^{+\infty} 2xe^{-4x}dx \underset{dy = -dx}{\overset{y=-x}{=}} + \int_{-\infty}^{0} 2ye^{-4y}dy + \int_{0}^{+\infty} 2xe^{-4x}dx = 0$$

$\underline{E(x) = 0}$

γ) $Var(x) = E[(x - E(x))^2] = E[(x - 0)^2] = E(x^2)$

$$= \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_{-\infty}^{0} 2x^2e^{4x}dx + \int_{0}^{+\infty} 2x^2e^{-4x}dx \overset{y=-x}{=}$$

$$= -2\int_{-\infty}^{0} y^2e^{-4y}dy + 2\int_{0}^{+\infty} x^2e^{-4x}dx = 2\int_{0}^{+\infty} y^2e^{-4y}dy + 2\int_{0}^{+\infty} x^2e^{-4x}dx =$$

$$= 4\int_{0}^{+\infty} x^2e^{-4x}dx = -\left[x^2e^{-4x}\right]_{0}^{+\infty} + \int_{0}^{+\infty} 2xe^{-4x}dx$$

$$4\int_{0}^{+\infty} x^2e^{-4x}dx = -(0-0) - \frac{1}{2}\int_{0}^{+\infty} x(e^{-4x})'dx = -\frac{1}{2}\left[\left[xe^{-4x}\right]_{0}^{+\infty} + \int_{0}^{+\infty} e^{-4x}dx\right] =$$

$$= -\frac{1}{2}\left[0 + \left[\frac{e^{-4x}}{-4}\right]_{0}^{+\infty}\right] = -\frac{1}{2}\left[-\left(0 + \frac{1}{4}\right)\right] = -\frac{1}{2}\left(-\frac{1}{4}\right)$$

$\underline{Var(x) = \frac{1}{8}}$

δ) $$P[|X| \geq \tfrac{1}{2}] = P[X \geq \tfrac{1}{2}] + P[X \leq -\tfrac{1}{2}] = \int_{1/2}^{+\infty} 2e^{-4x}dx + \int_{-\infty}^{-1/2} 2e^{4x}dx =$$

$$= 4\int_{\frac{1}{2}}^{+\infty} e^{-4x}dx = 4\left[\frac{e^{-4x}}{-4}\right]_{1/2}^{+\infty} = -e^{-2} \Rightarrow \underline{P[|X| \geq \tfrac{1}{2}] = -e^{-2}}$$

② α) $Y_i = \begin{cases} 1, \text{με } p=0{,}14 \\ 0, \text{με } p=0{,}86 \end{cases}$

Μιλάμε για ανεξάρτητες ~~ίσες~~ δοκιμές με $p=0{,}14$, επειδή για i δοκιμές Bernoulli $p=0{,}14$. Άρα η Y_i έχει κατανομή Bernoulli

$$Y_i = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{όπου } n=i \text{ και } p=0{,}14 \text{ δηλ.}$$

$$Y_i = \binom{i}{k}(0{,}14)^k \cdot (0{,}86)^{i-k}$$

β) $X = \sum_{i=1}^{30} Y_i$, επειδή ακολουθεί τη διωνυμική κατανομή με $N=30$, $k=k$ και $p=0{,}14$, συνεπώς

$$X = \binom{30}{k}(0{,}14)^k \cdot (0{,}86)^{30-k}$$

γ) $P[E] = P[10 \le X \le 12] = P[10] + P[11] + P[12] =$

$$P[E] = \binom{30}{10}(0{,}14)^{10}(0{,}86)^{20} + \binom{30}{11}(0{,}14)^{11}(0{,}86)^{19} + \binom{30}{12}(0{,}14)^{12}(0{,}86)^{18}$$

δ) $Z = 100 + \sum_{i=1}^{60} Y_i$

$$E(Z) = E(100 + \sum_{i=1}^{60} Y_i) = 100 + E(\sum_{i=1}^{60} Y_i) = 100 + 60 \cdot 0{,}14 = 100 + 8{,}4$$

$E(Z) = 108{,}4$

$$Var(Z) = Var(100 + \sum_{i=1}^{60} Y_i) = Var(\sum_{i=1}^{60} Y_i) = 60 \cdot 0{,}14 \cdot (1-0{,}14) = 8{,}4 \cdot 0{,}86$$

$Var(Z) = 7{,}224$

ε) $\Theta = 100 + \sum_{i=1}^{60} 2Y_i = 100 + 2\sum_{i=1}^{60} Y_i$

$E(\Theta) = 100 + 2E(\sum_{i=1}^{60} Y_i) = 100 + 2 \cdot 8{,}4 = 100 + 16{,}8 \Rightarrow E(Z) = 116{,}8$

$Var(\Theta) = Var(100 + 2\sum_{i=1}^{60} Y_i) = 2^2 Var(\sum_{i=1}^{60} Y_i) = 2 Var(Z) = 2^2 \cdot 7{,}224$

$Var(\Theta) = 28{,}896$

③ X συνεχής, $S = [2, +\infty)$, $f(x) = \frac{c}{x^2}$, $x \in S$

α) Από ιδιότητες σππ: $\int_{-\infty}^{+\infty} f(x) = 1$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_2^{+\infty} \frac{c}{x^2}dx = c\left[\frac{-1}{x}\right]_2^{+\infty} = c\left(0 + \frac{1}{2}\right) = \frac{c}{2} = 1 \Rightarrow c = 2$$

β) $E(X) = \int_{-\infty}^{+\infty} x \frac{2}{x^2} dx = 2\int_{2}^{+\infty} \frac{1}{x} dx = 2[\ln x]_2^{+\infty} = 2(+\infty - \ln 2) \Rightarrow E(X) = +\infty$

⑥ α) (i) $Y = X^2$

$\cancel{\quad}$

$F_Y(t) = P[Y \le t] = P[X^2 \le t] = P[-t \le X \le t] = P[X \ge -t] + P[X \le t] =$

$= 1 - P[X < t] + F_X(t) = 1 - \lim_{x \to -t^-} F_X(x) + F_X(t)$

άρα $f_Y(t) = (-\lim_{x \to -t^-} f_X(x))' + f_X(t)$

όμως $X \sim N(0,1)$ άρα $f_X(t) = \frac{1}{\sqrt{2t}} e^{-t^2/2}$ και η X είναι συνεχής, άρα

$\Rightarrow \lim_{x \to -t^-} F_X(x) = F_X(-t)$ και άρα $f_X(-t) = (\lim_{x \to -t^-} F_X(x))$ όπου

$f_X(t) = \lim_{x \to -t} \frac{F_X(x) - F_X(-t)}{x + t}$ όμως η f_X ορίζεται για $t > 0$ άρα $f_Y(t) = f_X(t)$

Συνεπώς $f_Y(t) = \frac{1}{\sqrt{2t}} e^{-t^2/2}$.

$E[Y] = \int_{-\infty}^{+\infty} y f_Y(y) dy = \int_0^{+\infty} \sqrt{\frac{y}{2}} e^{-y^2/2} dy \Rightarrow E[y] = \frac{\Gamma(3/4)}{2^{3/4}} \approx 0,73$

(ii) $E[y] = \int_{-\infty}^{+\infty} y f_Y(y) dy$

β) $F_Y(t) = P[Y \le t] = P[e^X \le t] = P[X \le \ln t] = F_X(\ln t)$

άρα $f_Y(t) = \frac{1}{t} f_X(t) = \frac{1}{t} \cdot \frac{1}{\sqrt{2t}} e^{-t^2/2}$

⑦ α) Από ιδιότητες πυκνότητας : $\sum_{k=1}^{\infty} P[X \ge k] = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} P[X = j]$. ①

Εφόσον $k \in \mathbb{N}$, μπορούμε να πούμε ότι: $\sum^{+\infty} P[X \ge k] = \sum^{\infty} P[X > k]$. ②

Από τις ① και ② $\Rightarrow \sum_{k=0}^{\infty} P[X > k] = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} P[X = j] = \sum_{j=1}^{\infty} j\, P[X = j] \Rightarrow$

$\Rightarrow \sum_{k=0}^{\infty} P[X > k] = E(X)$

Για $\forall k \in \mathbb{N}$, $P[X > k] = 1 - P[X \le k] = \int_k^{+\infty} f_X(t) dt$

$\sum_{k=0}^{\infty} P[X > k] = \sum_{k=0}^{\infty} (1 - P[X \le k]) = \sum_{k=0}^{\infty} \int_k^{+\infty} f_X(t) dt$

$\sum_{k=0}^{\infty} P[X > k] = \int_0^{+\infty} P[X > t] dt = E(X)$

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$$\beta)\ E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_{-\infty}^{+\infty} P[|X| \geq t]\,dt = -\int_{-\infty}^{0} P[X < -t]\,dt + \int_{0}^{+\infty} P[X < t]\,dt$$

④ a) $f(x) = \begin{cases} a x^2 e^{-\beta x^2}, & x > 0 \\ 0, & x \notin (0,+\infty) \end{cases}$, $\beta = \dfrac{m}{2kT}$

Από ιδιότητες σππ: $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$1 = \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{+\infty} a x^2 e^{-\beta x^2}\,dx = a\left[\int_{0}^{+\infty} x^2 \cdot \left(\frac{e^{-\beta x^2}}{-2\beta}\right)' dx\right]\cdot(-2\beta) =$$

$$= \frac{-a}{2\beta}\left[x^2 \cdot e^{-\beta x^2}\right]_0^{+\infty} + \frac{a}{2\beta}\int_0^{+\infty} 2x e^{-\beta x^2}\,dx = \frac{-a}{2\beta^2}\left[x e^{-\beta x^2}\right]_0^{+\infty} + \frac{a}{2\beta^2}\int_0^{+\infty} e^{-\beta x^2}\,dx$$

$$= \frac{a}{2\beta^2}\cdot\frac{-1}{2\beta}\left[e^{-\beta x^2}\right]_0^{+\infty} = +\frac{a}{4\beta^3}\cdot\frac{\sqrt{\pi}}{\beta^{-3/2}} = \frac{a\sqrt{\pi}}{4\beta^{3/2}}$$

$$a = \frac{4\beta^{3/2}}{\sqrt{\pi}} \Rightarrow a = \frac{4\cdot\sqrt{\left(\frac{m}{2kT}\right)^3}}{\sqrt{\pi}} \Rightarrow a = 4\sqrt{\frac{m^3}{\pi(2kT)^3}}$$

$$\beta)\ \mu_1 = E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_{-\infty}^{+\infty} x f(x)\,dx = a\int_0^{+\infty} x^3 e^{-\beta x^2}\,dx = \frac{a}{2\beta^2}$$

$$\mu_1 = \frac{4\beta}{2\beta^2}\sqrt{\frac{\beta}{\pi}} = \frac{2}{\beta}\sqrt{\frac{\beta}{\pi}} \Rightarrow \mu_1 = \frac{2}{\sqrt{\beta\pi}}$$

$\gamma)$ $\mu_2 = E(y) = \displaystyle\int_{-\infty}^{+\infty} y f(y)\,dy$, όπου $f(y) = E = \frac{1}{2}mx^2$

$$\mu_2 = \int_{-\infty}^{+\infty} yE\,dy$$

$$Y = E = \frac{1}{2}mX^2 \Rightarrow F_Y(t) = P[Y \leq t] = P\left[X^2 \leq \frac{2t}{m}\right] = P\left[0 \leq X \leq \frac{2t}{m}\right] =$$

$$= P[X > 0] + P\left[X \leq \frac{2t}{m}\right] = 1 - \cancel{P[X \leq 0]} + P\left[X \leq \frac{2t}{m}\right]$$