Φυλλάδιο IX

② $X \sim Bern(\frac{1}{4})$, $Y \sim Bern(\frac{1}{2})$, $Z = X \text{ XOR } Y$

α) Οι X και Y είναι ανεξάρτητες μεταβλητές

$p = P[Z=1] = P[\{X=1\} \cup \{Y=0\}] + P[\{X=0\} \cup \{Y=1\}] = \frac{1}{4}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{1}{2}$

$p = 1/2$

β) Αν οι X και Z είναι ανεξάρτητες, θα ισχύει $P[\{X=x\}\cup\{Z=z\}] = P[X=x]P[Z=z]$

- $P[\{X=1\}\cup\{Z=1\}] = P[\{X=1\}\cup\{Y=0\}] = P[X=1]P[Y=0] = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$
  και $P[X=1]P[Z=1] = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$
- $P[\{X=0\}\cup\{Z=1\}] = P[X=0]P[Y=1] = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8}$
  και $P[X=0]P[Z=1] = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8}$
- $P[\{X=1\}\cup\{Z=0\}] = P[X=1]P[Y=1] = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$
  και $P[X=1]P[Z=0] = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$
- $P[\{X=0\}\cup\{Z=0\}] = P[X=0]P[Y=0] = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8}$
  και $P[X=0]P[Z=0] = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8}$

άρα οι X και Z είναι ανεξάρτητες.

③ α) Τα πιθανά ενδεχόμενα της ζαριάς είναι $x_1, x_2 = \{1,2,3,4,5,6\}$

$P[\{X_1=x_1\}\cup\{X_2=x_2\}] = P[X_1=x_1]P[X_2=x_2] = \frac{1}{6}\cdot\frac{1}{6} \Leftrightarrow P[\{X_1=x_1\}\cup\{X_2=x_2\}] = \frac{1}{36}$

β) $|\Omega| = 6^2 = 36$

X \ Y	1	2	3	4	5	6	$P_X(x)$
1	1/36	2/36	2/36	2/36	2/36	2/36	11/36
2	0	1/36	2/36	2/36	2/36	2/36	1/4
3	0	0	1/36	2/36	2/36	2/36	7/36
4	0	0	0	1/36	2/36	2/36	5/36
5	0	0	0	0	1/36	2/36	1/12
6	0	0	0	0	0	1/36	1/36
$P_Y(y)$	1/36	1/12	5/36	7/36	1/4	11/36	

γ) Οι πυκνότητες των X και Y έχουν υπολογιστεί παραπάνω.

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⑥ $f_{XYZ}(x,y,z) = \begin{cases} \frac{1}{k}(2x+3y^2+4z^3), & 0\le x\le 3,\ 0\le y\le 2,\ 0\le z\le 1 \\ 0, & \text{αλλού} \end{cases}$

α) Για το σύνολο τιμών των (X,Y,Z) ισχύει

$S = [0,3]\cdot[0,2]\cdot[0,1]$

Από ιδιότητες οτι $\iiint_{\mathbb{R}^3} f_{XYZ}(x,y,z)\,dx\,dy\,dz = 1$

$1 = \int_0^1\int_0^2\int_0^3 \frac{1}{k}(2x+3y^2+4z^3)\,dx\,dy\,dz \Leftrightarrow k = \int_0^1\int_0^2\int_0^3 (2x+3y^2+4z^3)\,dx\,dy\,dz$

$k = \int_0^1\int_0^2\int_0^3 2x\,dx\,dy\,dz + \int_0^1\int_0^2\int_0^3 3y^2\,dx\,dy\,dz + \int_0^1\int_0^2\int_0^3 4z^3\,dx\,dy\,dz$

$k = 2\int_0^3 2x\,dx + 3\int_0^2 3y^2\,dy + 4\int_0^1 4z^3\,dz = 2[x^2]_0^3 + 3[y^3]_0^2 + 4[z^4]_0^1$

$k = 2\cdot 9 + 3\cdot 8 + 4\cdot 1 = 18 + 24 + 4 = 48 \Leftrightarrow \underline{k=48}$

β) $E(XYZ) = \iiint_{\mathbb{R}^3} xyz\, f_{XYZ}(x,y,z)\,dx\,dy\,dz$

$= \frac{1}{48}\int_0^1\int_0^2\int_0^3 xyz(2x+3y^2+4z^3)\,dx\,dy\,dz$

$= \frac{1}{48}\int_0^1\int_0^2\int_0^3 2x^2yz\,dx\,dy\,dz + \frac{1}{48}\int_0^1\int_0^2\int_0^3 x3y^3z\,dx\,dy\,dz + \frac{1}{48}\int_0^1\int_0^2\int_0^3 4xyz^4\,dx\,dy\,dz$

$= \frac{1}{48}\left[2\left(\int_0^3 x^2dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^1 z\,dz\right)\right] + \frac{1}{48}\left[\left(\int_0^3 x\,dx\right)\cdot 3\left(\int_0^2 y^3dy\right)\left(\int_0^1 z\,dz\right)\right] +$

$+ \frac{1}{48}\left[\left(\int_0^3 x\,dx\right)\left(\int_0^2 y\,dy\right)4\left(\int_0^1 z^4\,dz\right)\right]$

$= \frac{1}{24}\left\{\left[\frac{x^3}{3}\right]_0^3\left[\frac{y^2}{2}\right]_0^2\left[\frac{z^2}{2}\right]_0^1\right\} + \frac{1}{16}\left\{\left[\frac{x^2}{2}\right]_0^3\left[\frac{y^4}{4}\right]_0^2\left[\frac{z^2}{2}\right]_0^1\right\} + \frac{1}{12}\left\{\left[\frac{x^2}{2}\right]_0^3\left[\frac{y^2}{2}\right]_0^2\left[\frac{z^5}{5}\right]_0^1\right\}$

$= \frac{1}{24}\left[9\cdot\cancel{2}\cdot\frac{1}{\cancel{2}}\right] + \frac{1}{16}\left[\frac{9}{2}\cdot 4\cdot\frac{1}{2}\right] + \frac{1}{12}\left[\frac{9}{\cancel{2}}\cdot\cancel{2}\cdot\frac{1}{5}\right] = \frac{9}{24} + \frac{9}{16} + \frac{9}{60}$

$\underline{E(XYZ) = 37/180}$

γ) Θεωρούμε το σύνολο $B = \{(x,y,z) : x+y+z\le 1\}\subset\mathbb{R}^3$

$P[X+Y+Z\le 1] = \iiint_B f_{XYZ}(x,y,z)\,dx\,dy\,dz = \frac{1}{48}\int_0^1\left(\int_0^{1-z}\left(\int_0^{1-y-z}(2x+3y^2+4z^3)\,dx\right)dy\right)dz$

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$$\int_0^{1-y-z} (2x+3y^2+4z^3)\,dx = \left[x^2 + (3y^2+4z^3)x\right]_0^{1-y-z} = (1-y-z)^2 + (3y^2+4z^3)(1-y-z)$$

$$= (1+z^2-2z+4z^3-4z^4) + y(-2+z-4z^3) + y^2(4-3z) - 3y^3$$

$$\int_0^{1-z} \left[(1+z^2-2z+4z^3-4z^4) + y(-2+z-4z^3) + y^2(4-3z) - 3y^3\right] dy =$$

$$= \left[(1+z^2-2z+4z^3-4z^4)y + \frac{y^2}{2}(z-2-4z^3) + \frac{y^3}{3}(4-3z) - \frac{3y^4}{4}\right]_0^{1-z} =$$

$$= 2z^5 - \frac{15}{4}z^4 + \frac{1}{6}z^3 + \frac{7}{2}z^2 - \frac{5}{2}z + \frac{7}{12}$$

$$\int_0^1 \left(2z^5 - \frac{15}{4}z^4 + \frac{1}{6}z^3 + \frac{7}{2}z^2 - \frac{5}{2}z + \frac{7}{12}\right) dz = \left[\frac{1}{3}z^6 - \frac{3}{4}z^5 + \frac{1}{24}z^4 + \frac{7}{6}z^3 - \frac{5}{4}z^2 + \frac{7}{12}z\right]_0^1$$

$$= \frac{1}{3} - \frac{3}{4} + \frac{1}{24} + \frac{7}{6} - \frac{5}{4} + \frac{7}{12} = \frac{8-18+1+28-30+14}{24} = \frac{+3}{24} = +\frac{1}{8}$$

άρα $\mathbb{P}[X+Y+Z \leq 1] = \frac{1}{48} \cdot \frac{1}{8} \iff \underline{\mathbb{P}[X+Y+Z \leq 1] = \frac{1}{384}}$

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