Concepto de Probabilidad: teorema de Bayes

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Teorema de la partición (o de la probabilidad total)

Dados $A_i \in \mathcal{A}, \quad i = 1,\ldots,n$ tales que:

$$\bigcup_{i=1}^{n} A_i = \Omega$$

$$A_i \cap A_j = \varnothing \quad \forall i, j = 1,\ldots,n \,/\, i \neq j$$

$$P(A_i) > 0 \,, i = 1,\ldots,n$$

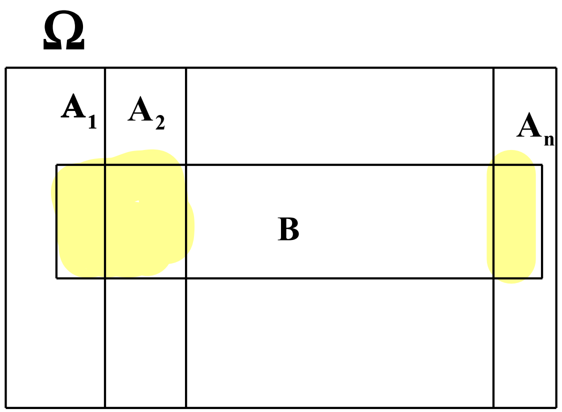
Sea $B \in \mathcal{A}$ otro suceso cualesquiera, entonces:

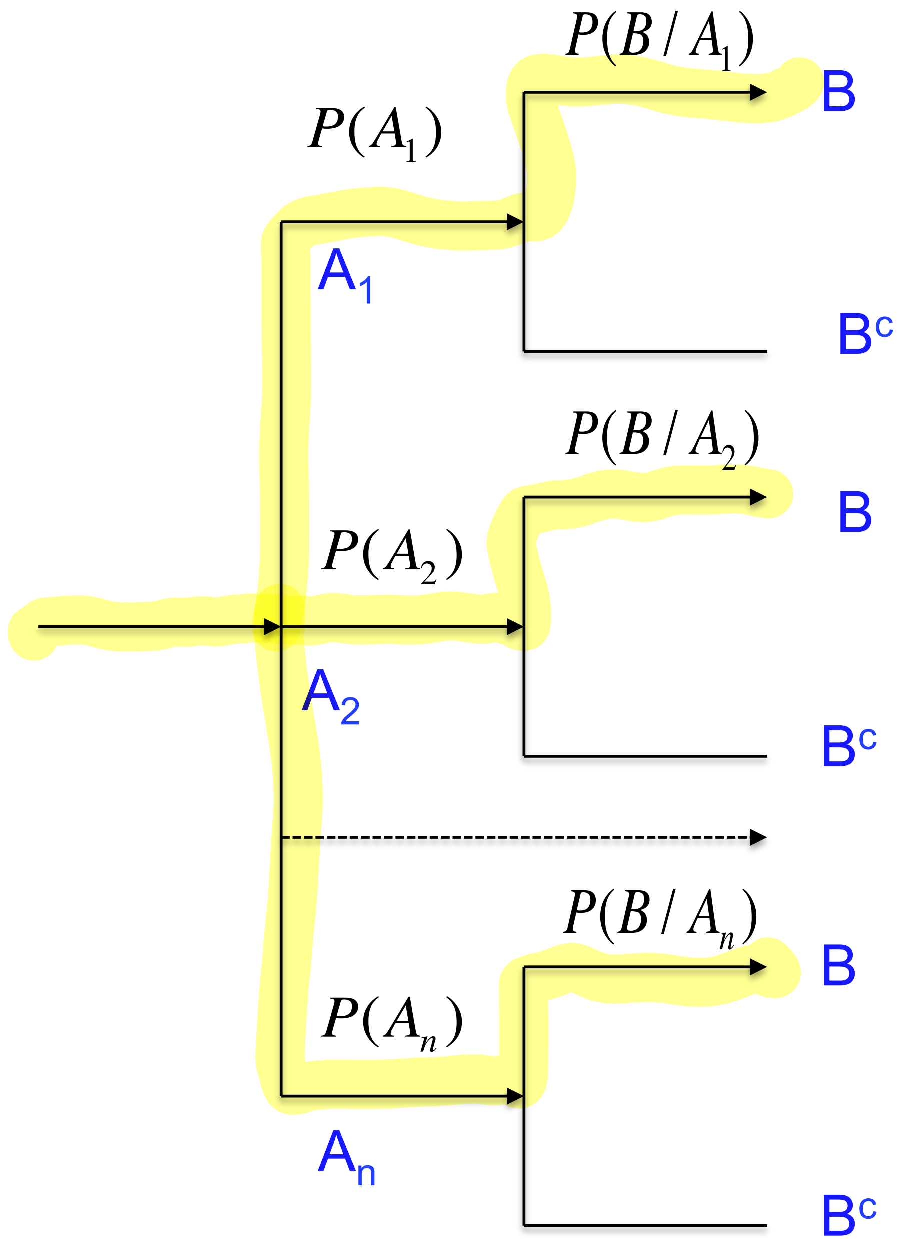
$$P(B) = \sum_{i=1}^{n} P(A_i)\, P(B / A_i)$$

Demostración: Basta considerar

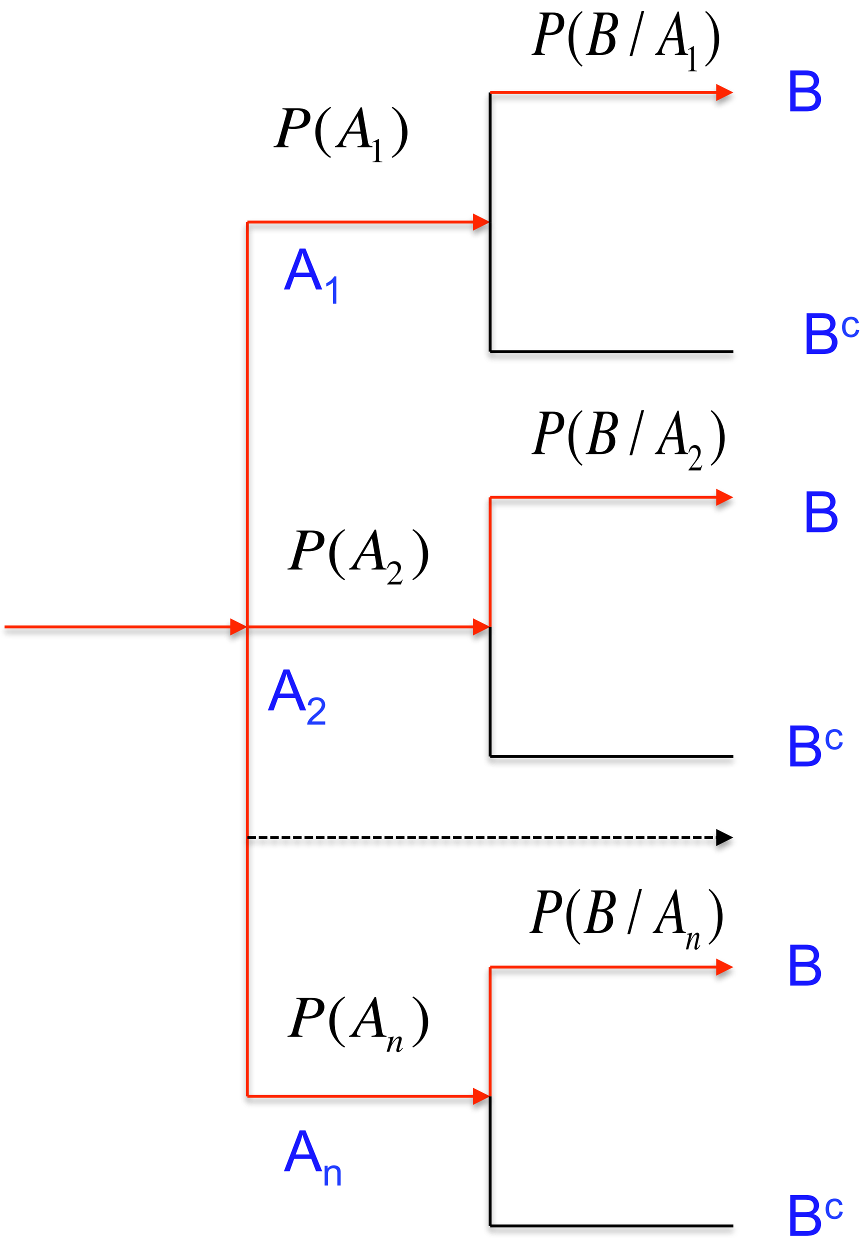
$$B = \bigcup_{i=1}^{n} (A_i \cap B)$$

$$P(A_i \cap B) = P(A_i)\, P(B / A_i)$$

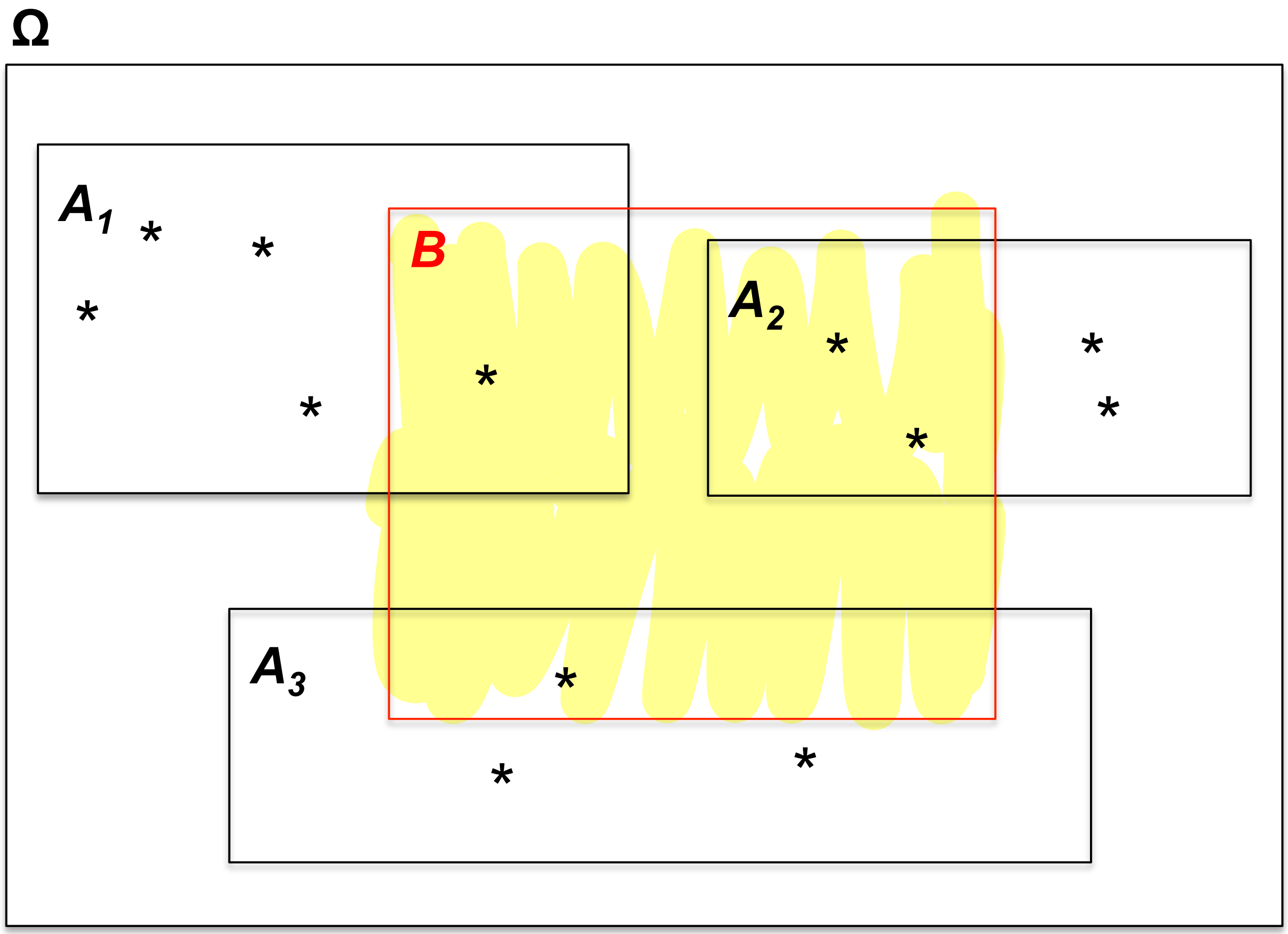




$P(A_1)$
A_1
$P(B/A_1)$
B
B^c
$P(A_2)$
A_2
$P(B/A_2)$
B
B^c
$P(A_n)$
A_n
$P(B/A_n)$
B
B^c

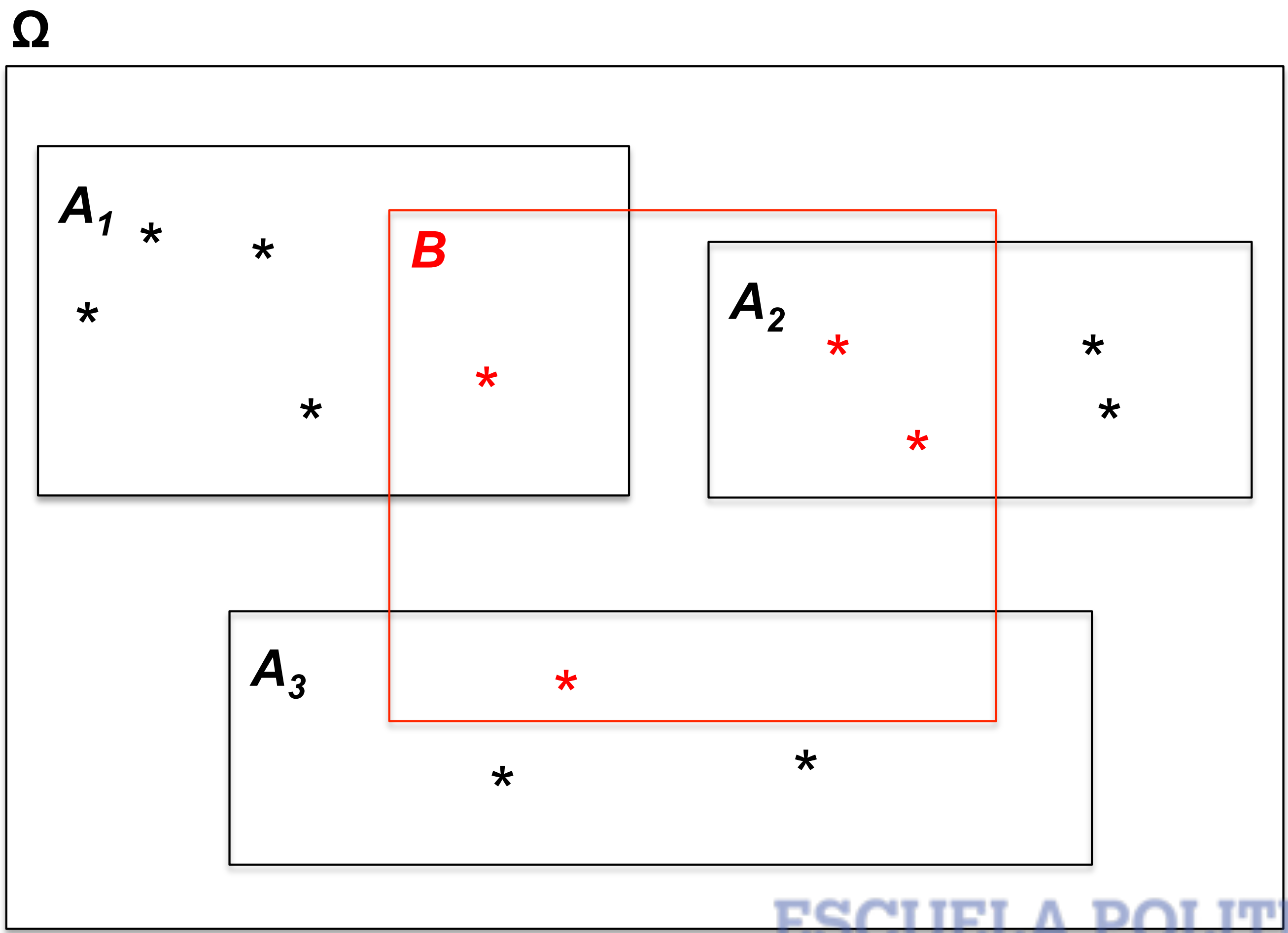


$$P(B)=\sum_{i=1}^{n} P(A_i)P(B/A_i)$$



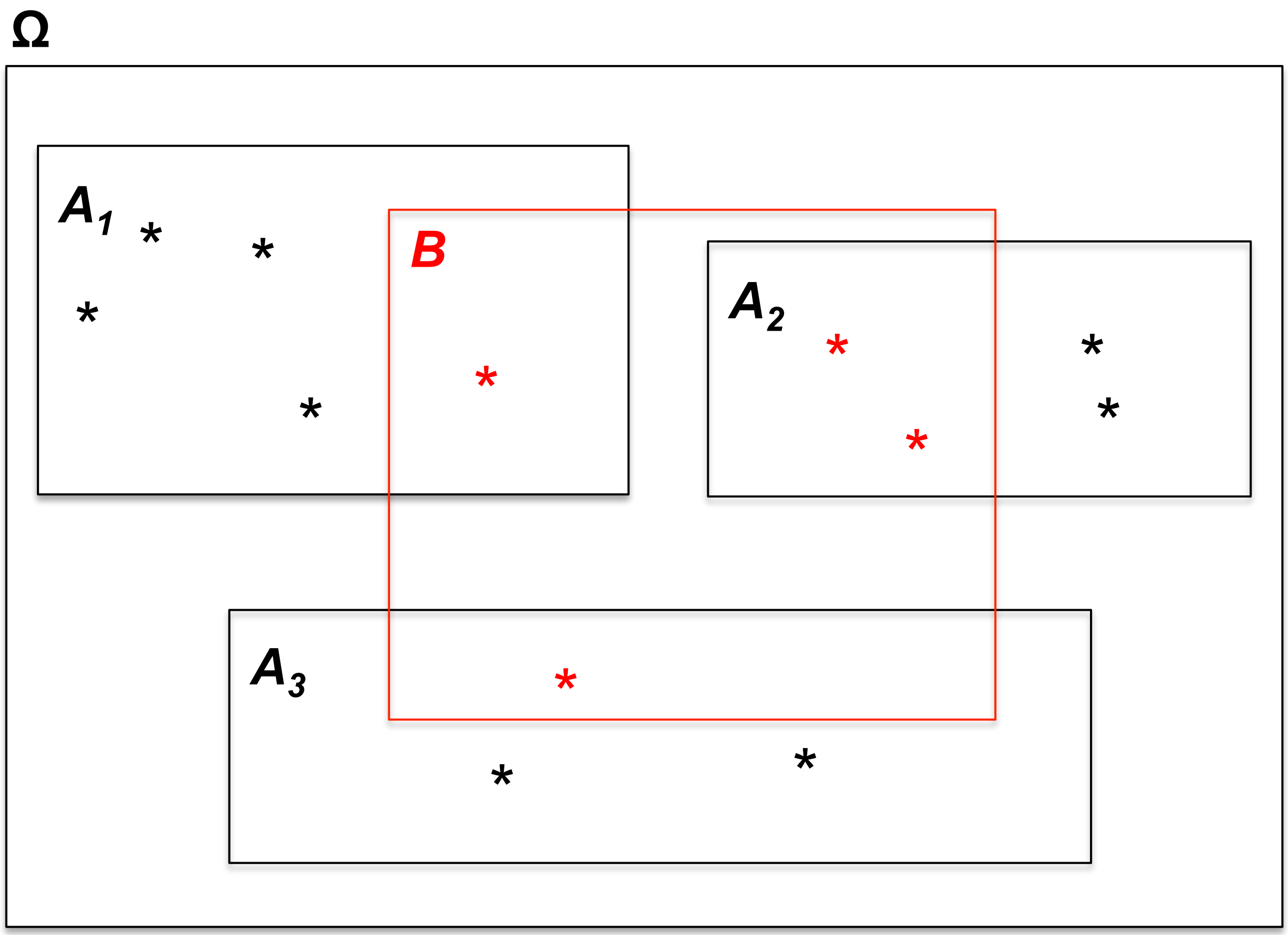
$P(B)$?

$$A_1 \cup A_2 \cup A_3 = \Omega \qquad A_1 \cap A_2 \cap A_3 = \varnothing \qquad A_1 \cap A_2 = \varnothing \quad A_1 \cap A_3 = \varnothing \quad A_2 \cap A_3 = \varnothing$$



$$\boxed{P(B) = \frac{4}{12} = \frac{1}{3}}$$

$$A_1 \cup A_2 \cup A_3 = \Omega \qquad A_1 \cap A_2 \cap A_3 = \varnothing \qquad A_1 \cap A_2 = \varnothing \qquad A_1 \cap A_3 = \varnothing \qquad A_2 \cap A_3 = \varnothing$$



$$\boxed{P(B) = \frac{4}{12} = \frac{1}{3}}$$

$$P(A_1) = \frac{5}{12}$$

$$P(A_2) = \frac{4}{12} = \frac{1}{3}$$

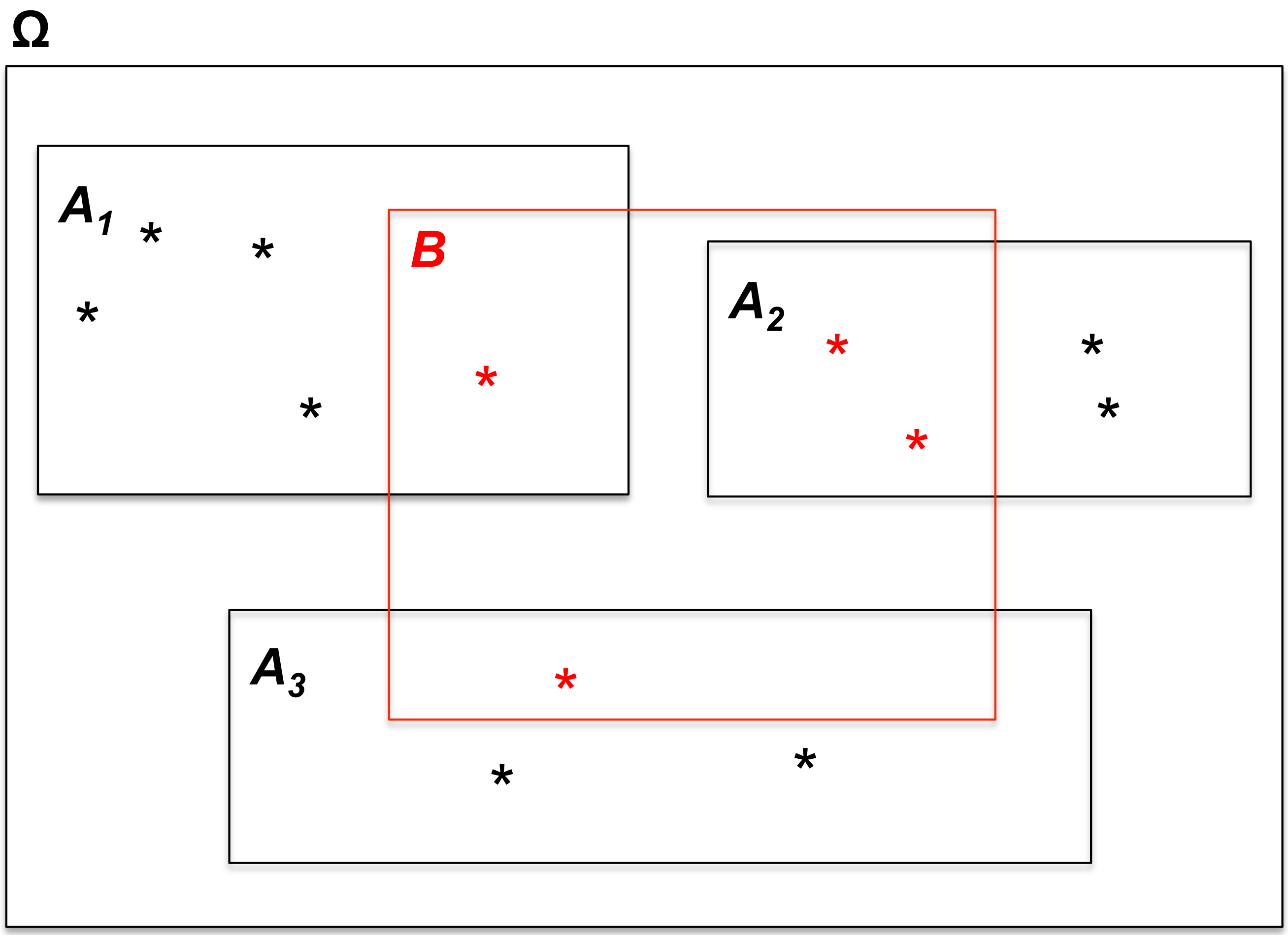
$$P(A_3) = \frac{3}{12} = \frac{1}{4}$$

$$P(B/A_1) = \frac{1}{5}$$

$$P(B/A_2) = \frac{2}{4} = \frac{1}{2}$$

$$P(B/A_3) = \frac{1}{3}$$

$$A_1 \cup A_2 \cup A_3 = \Omega \qquad A_1 \cap A_2 \cap A_3 = \varnothing \qquad A_1 \cap A_2 = \varnothing \qquad A_1 \cap A_3 = \varnothing \qquad A_2 \cap A_3 = \varnothing$$



$$\boxed{P(B) = \frac{4}{12} = \frac{1}{3}}$$

$$P(A_1) = \frac{5}{12}$$

$$P(A_2) = \frac{4}{12} = \frac{1}{3}$$

$$P(A_3) = \frac{3}{12} = \frac{1}{4}$$

$$P(B/A_1) = \frac{1}{5}$$

$$P(B/A_2) = \frac{2}{4} = \frac{1}{2}$$

$$P(B/A_3) = \frac{1}{3}$$

$$P(B) = P(A_1)P(B/A_1) + P(A_2)P(B/A_2) + P(A_3)P(B/A_3) = \frac{5}{12}\cdot\frac{1}{5} + \frac{4}{12}\cdot\frac{2}{4} + \frac{3}{12}\cdot\frac{1}{3} = \frac{4}{12} = \frac{1}{3}$$

Teorema de Bayes

Dadas las condiciones del teorema de la partición:

$$P(A_j / B) = \frac{P(A_j)P(B / A_j)}{\sum_{i=1}^{n} P(A_i)P(B / A_i)}$$

Donde:

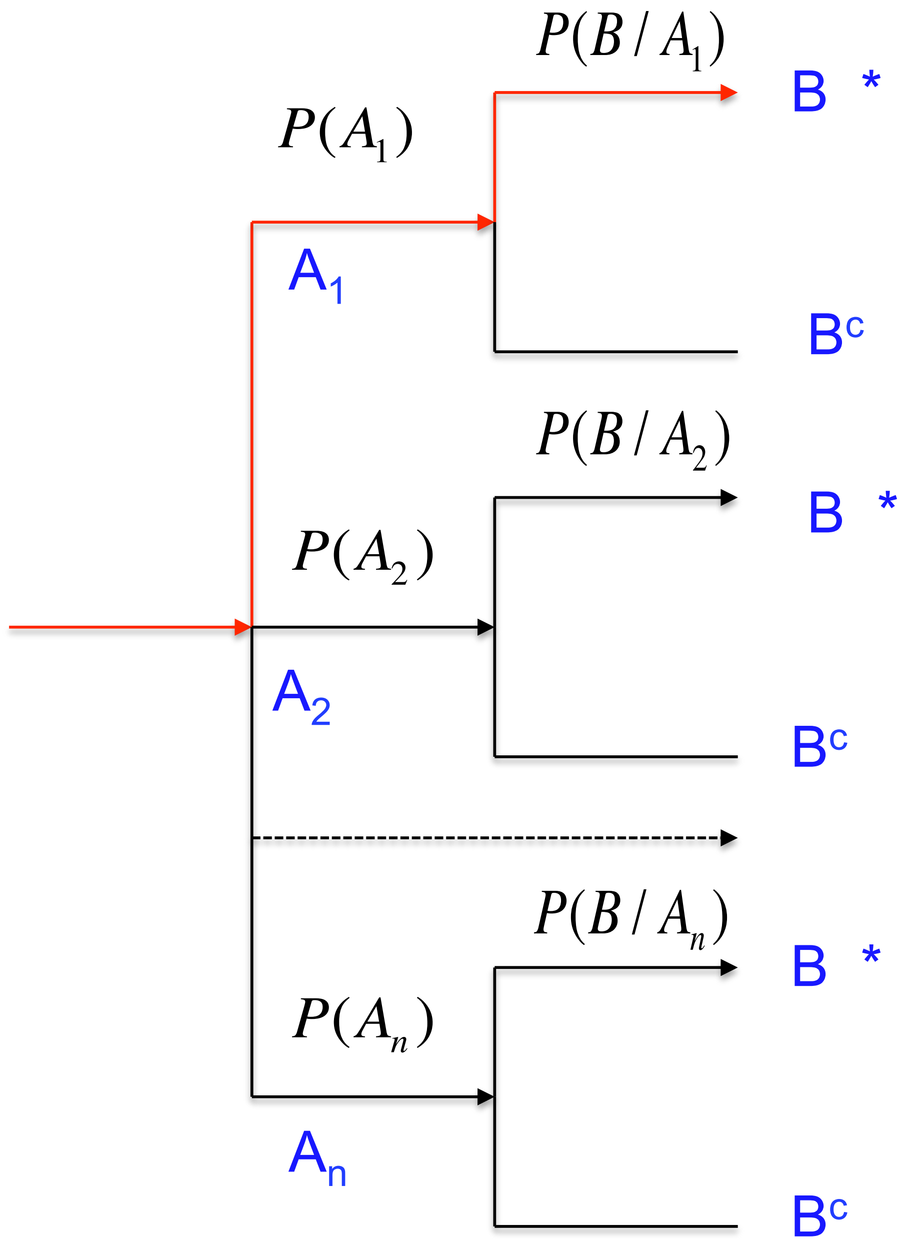
$P(A_j / B) \rightarrow$ Probabilidad a posteriori.

$P(A_i) \rightarrow$ Probabilidad a priori.

$P(B / A_j) \rightarrow$ Verosimilitudes.

Demostración:

$$P(A_j / B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(A_j)P(B / A_j)}{P(B)} \overset{T.P.}{=} \frac{P(A_j)P(B / A_j)}{\sum_{i=1}^{n} P(A_i)P(B / A_i)}$$

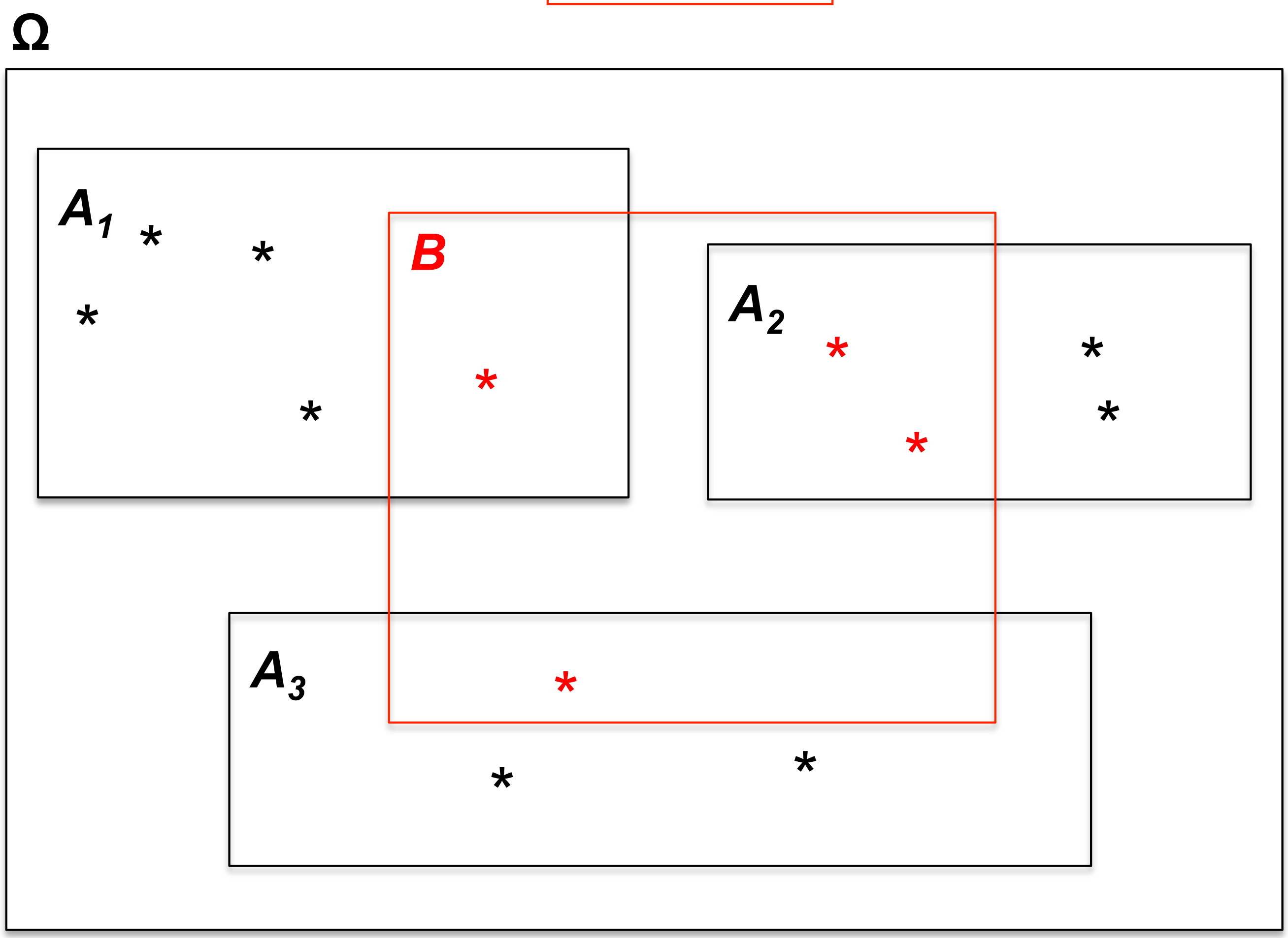


Ha ocurrido B

$$P(A_1/B) = \frac{P(A_1)P(B/A_1)}{P(B)}$$

$$P(B)=\frac{4}{12}=\frac{1}{3}$$

Ha ocurrido B



$$P(A_1)=\frac{5}{12}$$

$$P(A_2)=\frac{4}{12}=\frac{1}{3}$$

$$P(A_3)=\frac{3}{12}=\frac{1}{4}$$

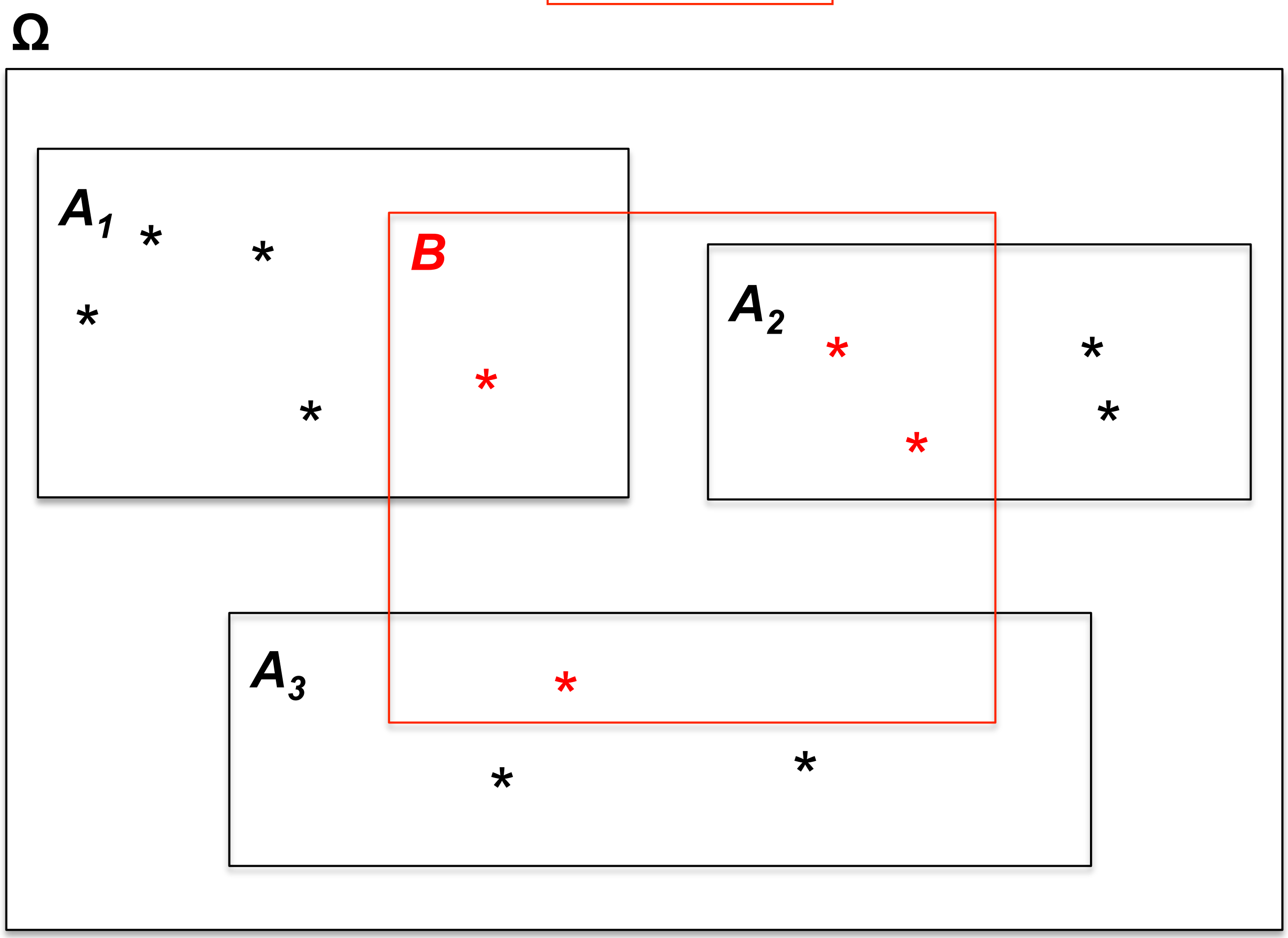
$$P(B/A_1)=\frac{1}{5}$$

$$P(B/A_2)=\frac{2}{4}=\frac{1}{2}$$

$$P(B/A_3)=\frac{1}{3}$$

$$P(B)=\frac{4}{12}=\frac{1}{3}$$

Ha ocurrido B



$$P(A_1)=\frac{5}{12}$$

$$P(A_2)=\frac{4}{12}=\frac{1}{3}$$

$$P(A_3)=\frac{3}{12}=\frac{1}{4}$$

$$P(B/A_1)=\frac{1}{5}$$

$$P(B/A_2)=\frac{2}{4}=\frac{1}{2}$$

$$P(B/A_3)=\frac{1}{3}$$

$$P(A_1/B)=\frac{P(A_1)P(B/A_1)}{P(B)}=\frac{\frac{5}{12}\cdot\frac{1}{5}}{\frac{1}{3}}=\frac{3}{12}=\frac{1}{4}$$

$$P(A_2/B)=\frac{P(A_2)P(B/A_2)}{P(B)}=\frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}}=\frac{1}{2}$$