First to Default Pricing

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1 Problem Statement

On the 15^{th} February 2008 at 10:45 C.E.T., consider two obligors: ISP and UCG. Assume that the obligor ISP has a recovery π equal to 40% and CDS spreads (annual bond): 1y 29 bps, 2y 32 bps, 3y 35 bps, 4y 39 bps, 5y 40 bps, 7y 41 bps, while obligor UCG has a recovery π equal to 45% and CDS spreads (annual bond): 1y 34 bps, 2y 39 bps, 3y 45 bps, 4y 46 bps, 5y 47 bps, 7y 47 bps. Price a first-to-default with maturity on the 20^{th} February 2012 on the obligors ISP and UCG with the correlation of default between the two obligors equal to $\rho = 20\%$.

2 Bootstrap

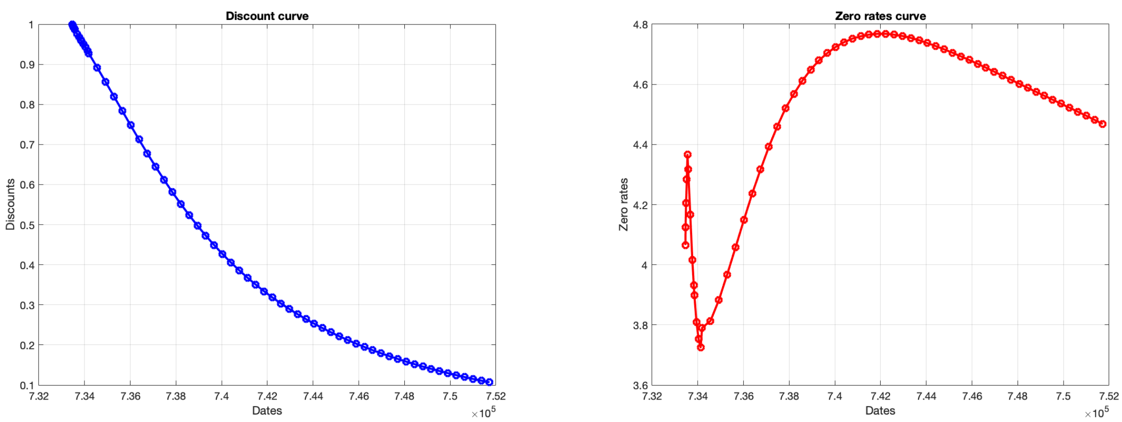
First, it is necessary to retrieve the Discount Factors curve. To do this, I use the Bootstrap technique, a non-parametric approach that exactly reproduces quoted prices of liquid instruments in the IR market.

In order to do this, I consider the interbank (IB) market on the 15^{th} February 2008 at 10:45 C.E.T. **MktData_CurveBootstrap** contains quoted rates relative to different instruments - Interbank deposits, STIR Futures (short-term interest rate futures), and IR Swaps (interest rate swaps). Among these, I use the most liquid instruments to build the discount factor curve.

In particular, I construct the curve via IB deposits up to the first STIR Future. Secondly, I refer to Future contracts. These are issued every 3 months, and the first seven contracts are known to be very liquid. Finally, I move to Swaps, whose liquidity is highlighted by a relatively small bid-ask spread.

I interpolate using linear interpolation on zero rates instead of discount factors and use the Act/365 convention whenever exponential argument time-to-maturity.

Figure 1a shows that the obtained discount curve follows a log-linear evolution in time. On the other hand, the zero rates curve could hypothetically be divided into three different sections, each one reflecting the corresponding zero rates curve (zero rates corresponding to IB Deposits, STIR Futures rates, and zero rates of Swaps) after stages of adjustment (figure 1b).



(a) Discount curve

(b) Zero rate curve

Figure 1: Discount and Zero Rate Curves

3 Survival Probabilities Computation

The approach to pricing the First to Default is based on the Li Model, where I opt for a Gaussian copula. To achieve this, it is necessary to retrieve the survival probability curve for both obligors using an intensity-based model. In this model, the probability of default in the interval $(t, t+\delta)$ is modeled as a Poisson process with intensity $\lambda(t)$. The simplest approach is to assume that $\lambda(t)$ is piecewise constant throughout each year, with jumps reflecting market information.

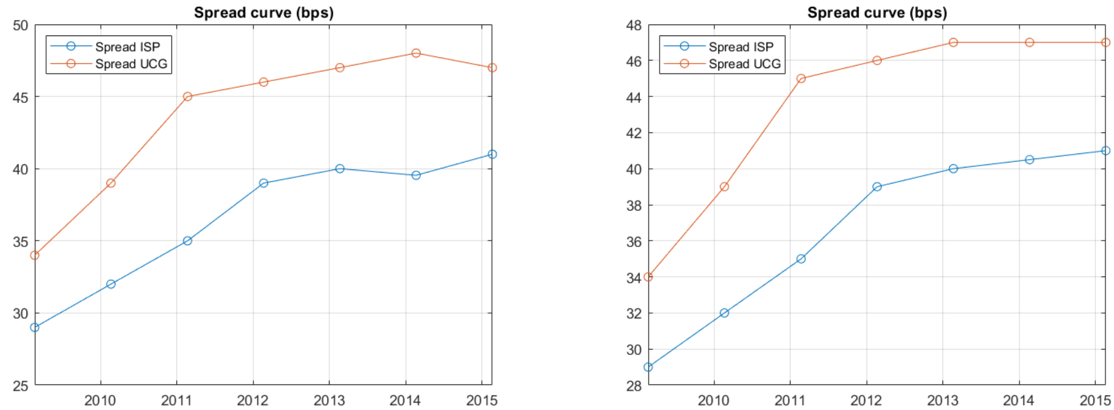
In this case, the survival probability is given by:

$$P(t_0, t_i) = \exp\left(-\sum_{j=1}^{i} \lambda(t_j) \cdot \delta(t_{j-1}, t_j)\right)$$

Thus, it is necessary to calibrate the intensities $\lambda(t)$ for each obligor to the market. To achieve this, I use the information from the Credit Default Swaps (CDS) available for each obligor, which will be further explained in the following sections.

Even though it is not strictly necessary for the problem, I have decided to construct $\lambda(t)$ for each obligor and compute the survival probability $P(t_0, t_i)$ for $i = 1\,\text{year}, \ldots, 7\,\text{years}$. The maturity of the contract is the 20^{th} February 2012, so it is not required to calibrate $\lambda(t_i)$ up to $i = 7$. However, I have chosen to do so for discussion purposes using different approaches. Then, I will select one approach and consider only the required time interval.

To proceed, it is important to note that the 6-year CDS spread is not provided for either obligor. Therefore, I perform an interpolation using a spline and a linear interpolation to obtain a complete set of CDS spreads. The results are presented in Figures 2a and 2b below.



(a) CDS spreads curve of ISP and UCG using spline interpolation

(b) CDS spreads curve of ISP and UCG using linear interpolation

Figure 2: Comparison of CDS spreads curve using different interpolations

An unusual behavior in the trend of the spreads when using spline interpolation can be noticed. Specifically, the 6-year spread is smaller than the 5-year spread for ISP, and the 7-year spread is smaller than the 6-year spread for UCG. This anomaly is not present when using linear interpolation. Therefore, I have decided to proceed with linear interpolation for the remainder of the analysis.

With this interpolation method in place, I can now retrieve the intensities and calculate the survival probabilities. I will explore three different approaches:

- $\lambda(t)$ piecewise constant, neglecting the "accrual" part.
- $\lambda(t)$ piecewise constant, considering the "accrual" part.
- Jarrow-Turnbull approximation.

3.1 $\lambda(t)$ Piecewise Constant, Neglecting the "Accrual" Part

To deduce the intensities of the CDS, I first compute the probabilities of default $P(t_0, t_i)$ for $i = 1, \ldots, 7$, derived from the structure of the CDS. I set the Net Present Value (NPV) of the fee leg, neglecting the accrual part, equal to the NPV of the contingent leg, as shown below:

$$S_N \sum_{i=1}^{N} \delta(t_{i-1}, t_i) B(t_0, t_i) P(t_0, t_i) = (1 - \pi) \sum_{i=1}^{N} B(t_0, t_i) \left[P(t_0, t_{i-1}) - P(t_0, t_i)\right]$$

It is possible to solve these equations iteratively: after computing $P(t_0, t_1)$, the subsequent $P(t_0, t_i)$ values can be computed using the previously determined $P(t_0, t_j)$ values for $j = 1, \ldots, i-1$. The factors $\delta(t_{i-1}, t_i)$ are computed using the 30/360 convention. Table 1 shows the obtained results for both obligors.

$P(t_0, t_i)$	1y	2y	3y	4y	5y	6y	7y
ISP	0.9952	0.9894	0.9826	0.9741	0.9670	0.9600	0.9529
UCG	0.9939	0.9859	0.9756	0.9670	0.9580	0.9499	0.9418

Table 1: Survival probabilities neglecting accrual for both ISP and UCG

As expected, the survival probabilities of UCG are smaller than those of ISP, as UCG's CDS spreads are higher. To obtain the intensities, I use the relationship:

$$P(t_0, t_i) = \exp\left(-\sum_{j=1}^{i} \lambda(t_j) \cdot \delta(t_{j-1}, t_j)\right)$$

By rearranging this equation, the intensities $\lambda(t_j)$ can be derived (see Table 2). The convention used to compute the time intervals is ACT/365.

λ_i	1y	2y	3y	4y	5y	6y	7y
ISP	0.0048	0.0058	0.0069	0.0087	0.0074	0.0072	0.0074
UCG	0.0061	0.0080	0.0105	0.0089	0.0093	0.0085	0.0085

Table 2: Intensities neglecting accrual for both ISP and UCG

3.2 $\lambda(t)$ Piecewise Constant, Considering the "Accrual" Part

In the previous subsection, the accrual part in the contingent premium leg was neglected. Here, I proceed in the same way as before, but now I take the accrual part into account. By equating the NPV of the two legs, I obtain:

$$S_N \sum_{i=1}^{N} \delta(t_{i-1}, t_i) B(t_0, t_i) P(t_0, t_i) + S_N \sum_{i=1}^{N} \frac{\delta(t_{i-1}, t_i)}{2} B(t_0, t_i) \left[P(t_0, t_{i-1}) - P(t_0, t_i)\right] =$$

$$(1 - \pi) \sum_{i=1}^{N} B(t_0, t_i) \left[P(t_0, t_{i-1}) - P(t_0, t_i)\right]$$

From this equation, I obtain the following survival probabilities as above (see table 3).

$P(t_0, t_i)$	1y	2y	3y	4y	5y	6y	7y
ISP	0.9952	0.9894	0.9826	0.9742	0.9670	0.9601	0.9531
UCG	0.9938	0.9859	0.9756	0.9670	0.9581	0.9501	0.9420

Table 3: Survival probabilities considering accrual for both ISP and UCG

It can be observed that these results are very similar to those obtained with the previous method. By calculating the intensities in the same manner as above, I obtain the following:

λ_i	1y	2y	3y	4y	5y	6y	7y
ISP	0.0048	0.0058	0.0069	0.0086	0.0073	0.0072	0.0074
UCG	0.0062	0.0080	0.0104	0.0089	0.0093	0.0085	0.0085

Table 4: Intensities considering accrual for both ISP and UCG

The intensities obtained using both the approximation and the exact method are remarkably close, with an error less than the order of 10^{-4}. This demonstrates that the accrual part can indeed be neglected in the computation of survival probabilities.

3.3 Jarrow-Turnbull Approximation

If we assume that the spread is paid continuously and the intensities are constant over time, it is also possible to use the Jarrow-Turnbull (JT) approximation to compute the intensities curve. In this rule of thumb, the intensities are obtained as follows:

$$\lambda_i = \frac{S_i}{1-\pi}$$

which yields (Table 5):

λ_i	1y	2y	3y	4y	5y	6y	7y
ISP	0.0048	0.0053	0.0058	0.0065	0.0067	0.0067	0.0068
UCG	0.0062	0.0071	0.0082	0.0084	0.0085	0.0085	0.0085

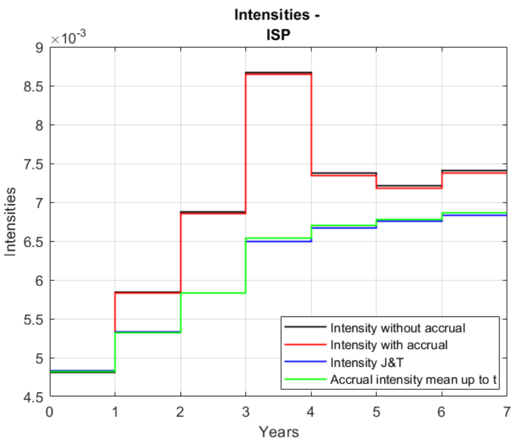
Table 5: Intensities using JT for both ISP and UCG

Comparing this rule of thumb with the mean values of the intensities obtained using the exact formula, the difference is on the order of 10^{-3}, indicating that the rule of thumb provides a good approximation of the result. From the intensities, it is also possible to directly compute the survival probabilities as shown above, reported in Table 6.

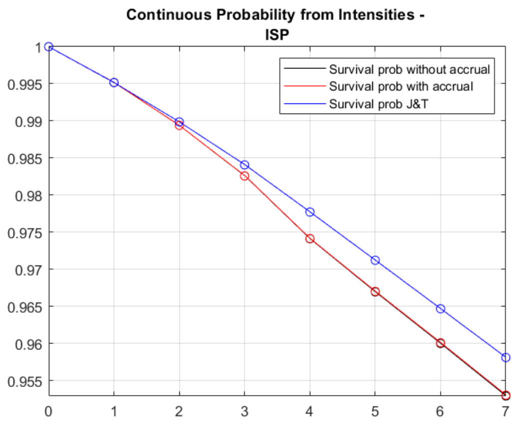
$P(t_0, t_i)$	1y	2y	3y	4y	5y	6y	7y
ISP	0.9952	0.9899	0.9841	0.9777	0.9712	0.9647	0.9581
UCG	0.9938	0.9868	0.9787	0.9706	0.9623	0.9541	0.9460

Table 6: Survival probabilities using JT for both ISP and UCG

The following figures show the plots of the intensities and survival probabilities for each obligor, comparing the three approaches discussed.

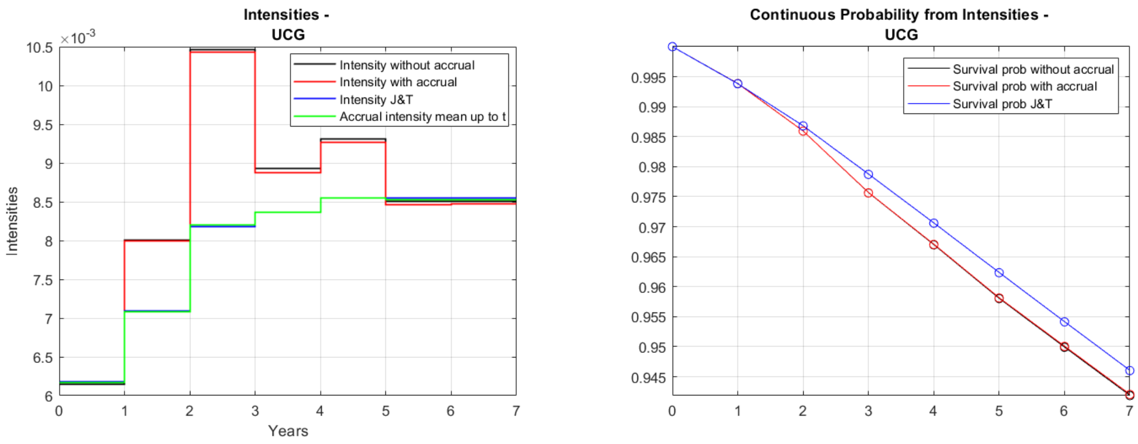


(a) Intensities curve ISP



(b) Survival probabilities curve ISP

Figure 3: ISP curves



(a) Intensities curve UCG

(b) Survival probabilities curve UCG

Figure 4: UCG curves

As noted earlier, the error between considering and neglecting the accrual factor is negligible. Therefore, I will adopt the approach of neglecting the accrual from this point onward. Regarding the Jarrow-Turnbull intensities, they closely follow the mean path of the exact intensities. Another point worth noting is that, for the first year, all results essentially coincide, which is consistent with the underlying theory.

4 First to Default Pricing

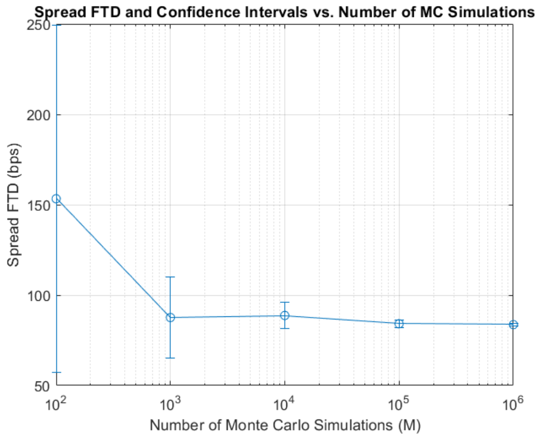
Having computed the survival probabilities for both obligors, it is now possible to proceed with the Li model using a Gaussian Copula. The process is as follows:

1. For each Monte Carlo simulation, generate 2 normal random variables.
2. Use Cholesky decomposition to correlate them and apply the Gaussian cumulative density function to get uniforms.
3. For each uniform drawn, invert the survival probability function (using bootstrapped intensities) and get the random default times τ.
4. For each sample, select the smaller τ between the two issuers.
5. For each simulation, compute the two legs:
   - Premium leg:
   $$\sum_{i=1}^{j} \delta(t_{i-1}, t_i) \cdot B(t_0, t_i) + (\tau - t_j) \cdot B(t_0, \tau) \quad \text{if} \quad \tau \leq T \quad \text{otherwise full BPV}$$
   - Recovery leg:
   $$(1 - \pi) \cdot B(t_0, \tau) \quad \text{if} \quad \tau \leq T \quad \text{otherwise } 0$$
6. Average the results above to calculate the pointwise FTD spread as $\frac{\text{Recovery Leg}}{\text{Premium Leg}}$ and compute the confidence interval.

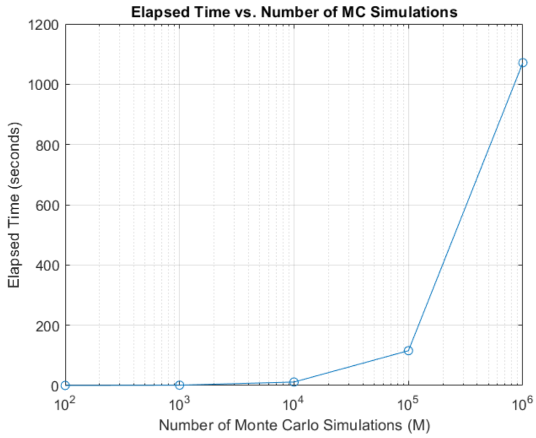
Table 7 and Figure 5 show the means and confidence intervals for the FTD spread and the elapsed times, setting $\alpha = 5\%$, while varying the number of Monte Carlo simulations, M.

M	Mean Spread (bps)	CI Lower (bps)	CI Upper (bps)	Elapsed Time (sec)
100	153.31	57.329	249.29	0.14884
1,000	87.584	65.24	109.93	1.0791
10,000	88.731	81.626	95.836	11.468
10^5	84.139	81.949	86.328	115.56
10^6	83.671	82.981	84.362	1071.3

Table 7: Mean Spreads, Confidence Intervals (in bps), and Elapsed Time (in seconds) for Different Values of M



(a) SFTD spread 95% confidence intervals (in bps) as a function of M



(b) Elapsed times (in seconds) as a function of M

Figure 5: Monte Carlo results

It can be observed that, as expected, as the number of Monte Carlo simulations increases, the FTD spread stabilizes, and the length of the confidence intervals decreases. However, the computational time increases significantly.

5 Final Remarks

It may be interesting to plot the First-to-Default spread with respect to different values of the correlation ρ. Figure 6 illustrates this behavior for the case where the number of Monte Carlo simulations is 10^5, varying ρ in the range $[-1, 1]$. The total elapsed time was 998.23 seconds.

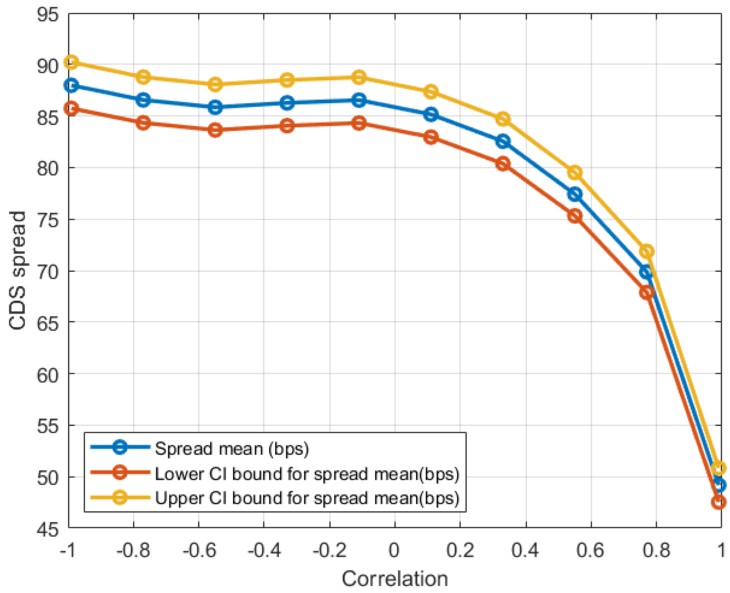


Figure 6: FTD spread and 95% confidence interval as a function of rho

It can be noticed that the spread is minimum when the correlation is maximum (i.e., $\rho = 1$) and is maximum when the correlation is minimum (i.e., $\rho = -1$); overall, it decreases as ρ increases. This is intuitive: if the correlation is high and positive, the defaults of the two names are more likely to occur close to each other. However, as the correlation decreases to zero, the two names default independently. If the correlation is negative, the probability that one obligor defaults while the other does not increases, so one default is more likely.