Structured bond: pricing and hedging

Alessandro John Howe, Cyril Andrea Hauser, Luigi Lacarpia

Contents

1 Problem Statement

For the problem statement, please see the **readme** in the GitHub repository.

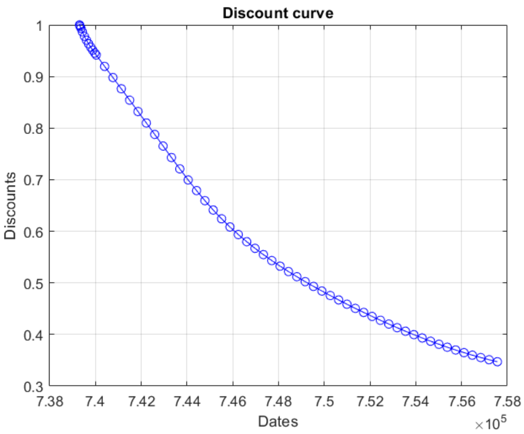
2 Bootstrap

First, it is necessary to retrieve the Discount Factors curve. To do this, we use the Bootstrap technique, a non-parametric approach that exactly reproduces quoted prices of liquid instruments in the IR market.

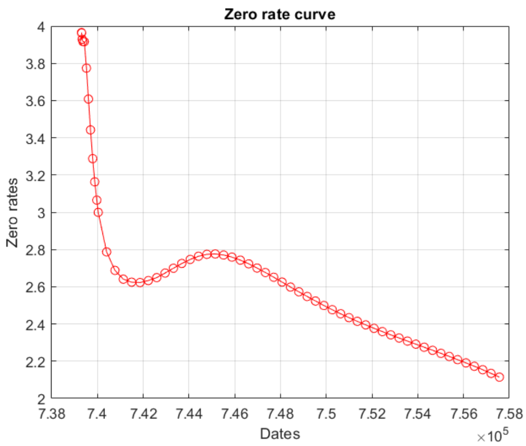
We consider the interbank (IB) market on the 16th of February 2024 at 10:45 C.E.T. The **MktData_CurveBootstrap_20-2-24** file contains quoted rates for various instruments—Interbank deposits, STIR Futures (short-term interest rate futures), and IR Swaps (interest rate swaps). Among these, we use the most liquid instruments to build the discount factor curve.

In particular, we construct the curve using IB deposits up to the first STIR Future. Next, we refer to Future contracts, which are issued every 3 months. The first seven contracts are known to be very liquid. Finally, we move to Swaps, whose liquidity is evidenced by relatively small bid-ask spreads. We build a complete set of swap rates from 1 year up to 50 years by computing the mean between the bid and ask prices of the existing values and interpolating them using a spline interpolation. For this purpose, we consider only business days, adjusting the dates to the European calendar.

Having identified the correct set of instruments, we proceed with the bootstrap. Figures 1a and 1b show the obtained curves.



(a) Discount curve



(b) Zero rate curve

Figure 1: Discount and Zero Rate Curves

We observe that the zero rates are decreasing over time, indicating that the market expects rates to fall in the future.

3 Computation of the Spot Volatilities

Before determining the upfront payment of the contract reported in the annex, we need to bootstrap the set of spot Cap volatilities given the flat ones. **Caps_vol_20-2-24** shows the flat volatilities at which the Cap at the corresponding strike (in percentage) and maturity (up to 30 years) is traded on the market (i.e., all the Caplets of the Cap at such strike and maturity are priced using the same flat volatility). To bootstrap the corresponding spot volatilities, we need to compute the price of each caplet for every quarter. We do this relying on Bachelier's equation:

$$\text{Caplet}_i(T_0) = B(T_0, T_{i+1})\delta_i \left(L_i(T_0) - K\right) N(d^n) + \sigma_i \sqrt{T_i - T_0}\phi(d^n)$$

where

$$d^n = \frac{L_i(T_0) - K}{\sigma_i \sqrt{T_i - T_0}}$$

and the following notation is used:

$$\delta_i = \delta(T_i, T_{i+1})$$

$$L_i(T_0) = L(T_0, T_i, T_{i+1})$$

To price a cap with maturity in i years, we sum all the prices of the caplets from the first up to the caplet with maturity in i years. We do this using the flat volatility, which is the same for all the caplets contained in the cap with a given maturity. We use the ACT/360 convention for the year fraction in the caplet.

Having done this, we follow these steps:

- Set all the spot volatilities equal to the flat volatilities for maturities lower than one year.
- Compute the difference between two consecutive Caps $\Delta C_{T_\alpha, T_\beta}$, where $T(\alpha)$ and $T(\beta)$ are Cap maturities from the previous table, using Bachelier's pricing formula above and flat volatilities:

$$\Delta C = C(T_0, T_\beta, \sigma_\beta^{\text{flat}}) - C(T_0, T_\alpha, \sigma_\alpha^{\text{flat}}) = \sum_{i:T_i \le T_\beta} \text{Caplet}_i(\sigma_i^{\text{flat}}) - \sum_{i:T_i \le T_\alpha} \text{Caplet}_i(\sigma_i^{\text{flat}})$$

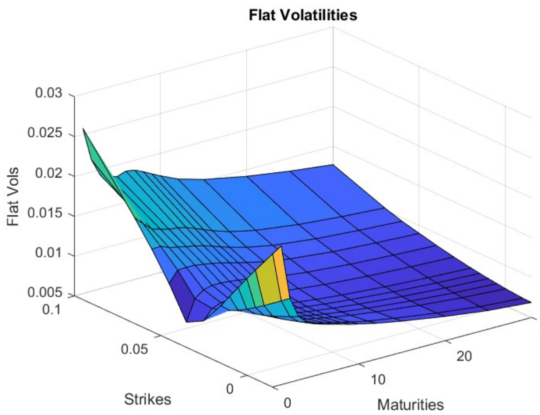
- Set this difference equal to the sum of the caplets in the previous time window priced via Bachelier's formula but with spot volatilities:

$$\Delta C = \sum_{i:T_\alpha \le T_i \le T_\beta} \text{Caplet}_i(\sigma_i^{\text{spot}})$$

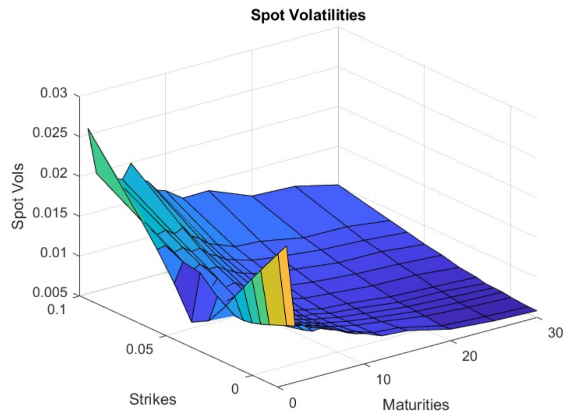
- By imposing a linear relation between the spot volatilities (interpolated directly on the dates from the table), it is possible to retrieve the spot volatilities:

$$\sigma_\alpha^{\text{spot}} = \sigma_{\alpha-1}^{\text{spot}} + \frac{T_\alpha - T_{\alpha-1}}{T_{\alpha+1} - T_{\alpha-1}} \left(\sigma_{\alpha+1}^{\text{spot}} - \sigma_\alpha^{\text{spot}}\right)$$

The following figures show the flat and spot volatilities as functions of maturity and strike.



(a) Flat volatilities



(b) Spot volatilities

We notice that the two surfaces are very similar. However, the spot volatility surface seems to be more irregular in terms of smoothness, likely due to the short time-to-maturity nature of caplets.

A smile with positive convexity and a minimum around the ATM option is observed, flattening with the increase in time to maturity, as expected.

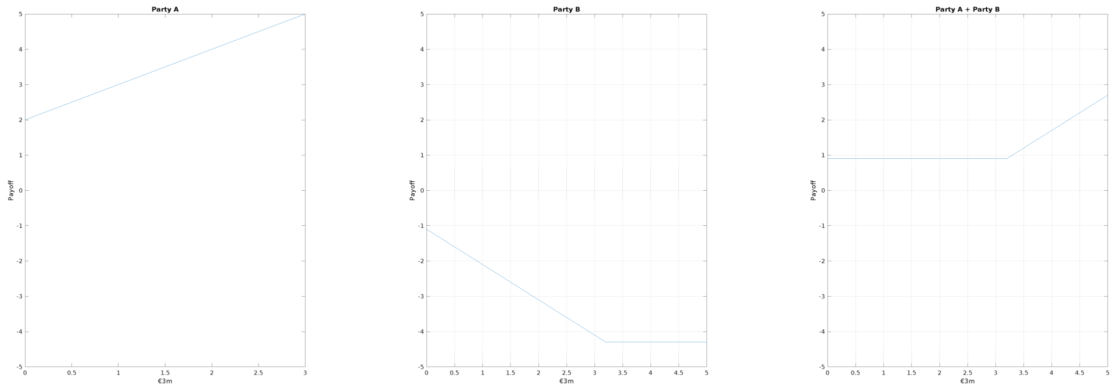
4 Pricing

It is now possible to determine the Upfront X% paid by Party B to Party A at the start date. We consider the cash flows and impose a null total Net Present Value (NPV). We split the cash flows into different parts and then compute the sum of the net cash flows.

$$N\left(-X + [(L(T_0,T_1) + \text{spol}\cdot\delta(T_0,T_1))B(T_0,T_1) - c_{3m}\delta(T_0,T_1)B(T_0,T_1)]\right.$$

$$\left.+ \sum_{i:3m\leq T_i\leq 14y,9m} \text{Caplet}(\sigma_i^{\text{spot}}, \text{Cap rate} - \text{spol}_B) + \sum_{i:3m\leq T_i\leq 14y,9m} B(T_0,T_{i+1})\delta(T_i,T_{i+1})(\text{spol}_B - \text{spol}_A)\right) = 0$$

We observe that the payoff paid every 3 months by Parties A and B is equivalent to the payoff of a translated Caplet. Consider, for instance, the case up to the 5th year (payoffs from B's perspective, where the figure on the right is the sum of the payoffs on the left):



To calculate this, we apply the previous Bachelier's formula with 3 different strikes equal to the differences between the cap rates and the spol of Party B. Regarding the translation, the NPV of such a term is obtained by considering a coupon every 3 months given by the difference between the spol of the two parties.

By imposing that the NPV of the contract flows equals 0, we find that Party B needs to pay Party A at the start date the following:

Payment (%)	**Payment (Notional in Mio€)**
18.94	9.4694

If, instead of using spot volatilities, flat volatilities are used, a higher price is yielded:

Payment (%)	**Payment (Notional in Mio€)**
19.73	9.8650

Indeed, flat volatilities consistently exceeding spot volatilities lead to an overestimation of the present value of caps, and in turn, the upfront payment needed to zero out the contract's NPV.

5 Delta-Bucket Sensitivities

At this point in our analysis, we want to manage the risk of holding the contract. In this section, we focus on evaluating the sensitivity of our contract to variations in market rates, i.e., the delta sensitivities. We start by computing the Delta-Bucket sensitivities. To do so, we adjust the rates of the most liquid financial instruments involved in the bootstrap process by adding 1 basis point at a time to each of them (so we consider the first 4 deposits, the first 7 futures, and 17 swaps, neglecting the first since it is useless for the bootstrap). Then, for each shift, we evaluate the difference between the NPV obtained with the discount factors coming from the bootstrap after shifting the rates and the previous one, which was set to 0 in the previous item.

Figure 3 shows the result.

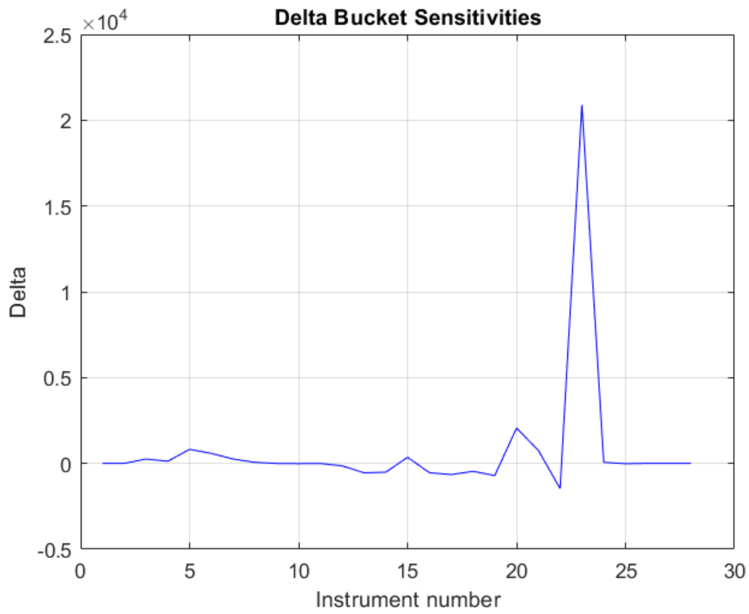


Figure 3: Spot volatilities Delta-Bucket

We notice that the contract value is not very sensitive to changes in the short-term future rates nor in the long-term future rates after the maturity of the contract. However, the latter is not exactly equal to zero (even if very small). This is due to the spline interpolation on the swap rates. Using spline interpolation can affect points far from the point of interest, which also causes complications in hedging strategies, as explained later. If we adopt linear interpolation on the swap rates, the above issue does not occur.

We also notice that the DV01 of the first two buckets, corresponding to the first two deposits, is null. The following two DV01s, corresponding to the next two deposits, are different from zero, even though they correspond to a date before any future flow of our contract. This is because the discount corresponding to the first future—the next one in the bootstrap—depends on the linear interpolation of these deposits.

6 Total Vega

We also evaluate the sensitivity of our contract to movements in market flat volatilities, i.e., the vega of our contract. First, we focus on the total vega.

To compute it, we shift all the flat volatilities by 1bp. We then recompute the spot volatilities using the procedure described above and calculate the new NPV. By evaluating the difference in NPV before and after the shift (which is 0 when choosing $X = 18.94\%$), we obtain:

ν	56277.693

7 Vega-Bucket Sensitivities

Regarding the Vega-Bucket sensitivities, we shift the available flat volatilities for each maturity (i.e., a row at a time) in the excel table by 1bp, one group at a time.

Figure 4 and Table 1 show the results.

1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	12y	15y	20y	25y	30y
89.31	-20.57	16.61	-10.91	952.58	-14.98	19.89	-3.34	-8.01	3418.84	0.00	51877.17	0.00	0.00	0.00

Table 1: Vega Bucket Sensitivities

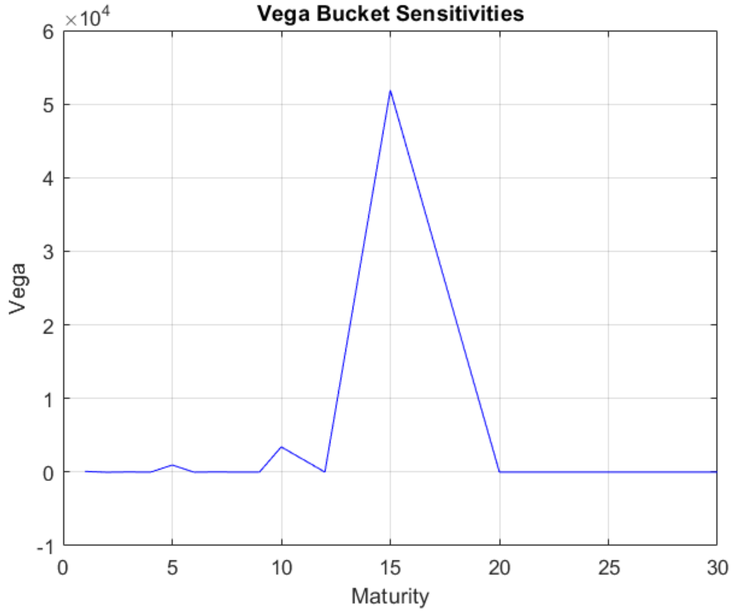


Figure 4: Vega Bucket Sensitivity

In this case, we observe that the vega bucket sensitivities are 0 after the contract's expiry. As explained above, this is because we used linear interpolation among maturities of spot volatilities, having fixed the strike (we used spline interpolation only between different strikes). This is useful in hedging strategies. We also note that changes in volatility near the contract's expiry have a more significant impact on the NPV.

To validate this result, we calculate the sum of the vega bucket sensitivities and compare it with the total vega computed earlier.

We have:

Vega bucket's sum	56322.304

The result is consistent with the findings in the previous section.

8 Hedging Delta - Coarse-Grained Buckets

The aim now is to hedge the delta risk using swaps, considering 4 coarse-grained buckets (0-2 years, 2-5 years, 5-10 years, 10-15 years). In risk management, it is often more convenient to aggregate information: sensitivities are often relevant only within a set of macro-buckets. We collect the previous Delta-Bucket DV01 by applying the scalar product with each of the following weights (Figure 5).

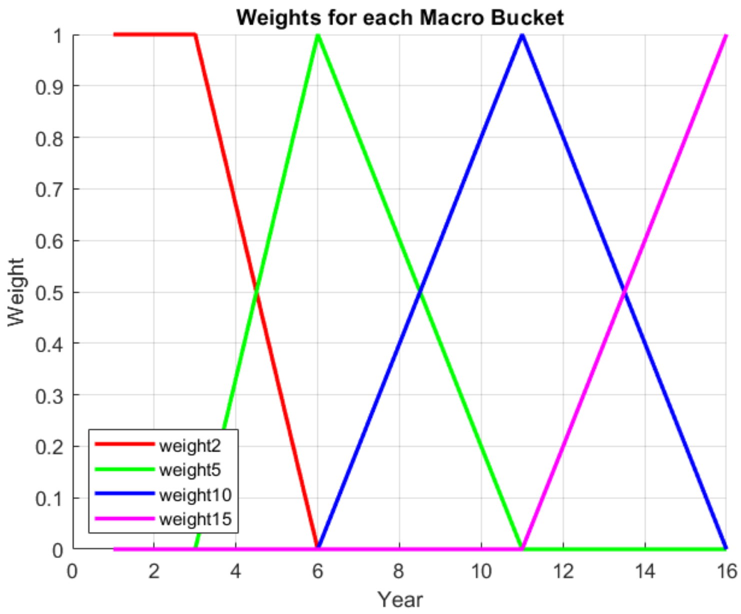


Figure 5: Weights

Obtaining the following results:

Bucket	Delta Coarse-Grained Sensitivity
0-2y	1410.295
2y-5y	-1315.087
5-10y	578.898
10-15y	20456.957

Table 2: Delta Coarse-Grained Sensitivities

The results are consistent with the yearly buckets, as both delta sensitivities sum up to 2.1181e4.

To hedge the delta risk, we compute the swap notionals N_{2y}, N_{5y}, N_{10y}, N_{15y} (corresponding to the coarse-grained buckets mentioned before). We observe that the suggestion to "start with the longest swap" would be very useful if we had linear interpolation on swap rates. In such a case, as observed before, a shift in a rate after the swap's maturity would not impact the coarse-grained DV01 buckets after expiry, so by starting with the longest swap, the linear system would be easier:

$$\begin{cases} DV01_{swap15y}(10y,15y)\cdot N_{15y}+DV01_{10y,15y}\cdot N=0 \\ DV01_{swap15y}(5y,10y)\cdot N_{15y}+DV01_{swap10y}(5y,10y)\cdot N_{10y}+DV01_{5y,10y}\cdot N=0 \\ DV01_{swap15y}(2y,5y)\cdot N_{15y}+DV01_{swap10y}(2y,5y)\cdot N_{10y}+DV01_{swap5y}(2y,5y)\cdot N_{5y}+DV01_{2y,5y}\cdot N=0 \\ DV01_{swap15y}(0,2y)\cdot N_{15y}+DV01_{swap10y}(0,2y)\cdot N_{10y}+DV01_{swap5y}(0,2y)\cdot N_{5y}+DV01_{swap2y}(0,2y)\cdot N_{2y}+DV01_{0,2y}\cdot N=0 \end{cases}$$

However, as mentioned before, with spline interpolation, this is no longer true. The problem persists even after the contract's maturity, but with such a choice of weights, this issue disappears. In this case, we need to solve a 4×4 linear system, and not a triangular system as before:

$$\begin{cases} DV01_{swap15y}(10y,15y)\cdot N_{15y}+DV01_{swap10y}(10y,15y)\cdot N_{10y}+DV01_{swap5y}(10y,15y)\cdot N_{5y}+DV01_{swap2y}(10y,15y)\cdot N_{2y}+DV01_{10y,15y}\cdot N=0 \\ DV01_{swap15y}(5y,10y)\cdot N_{15y}+DV01_{swap10y}(5y,10y)\cdot N_{10y}+DV01_{swap5y}(5y,10y)\cdot N_{5y}+DV01_{swap2y}(5y,10y)\cdot N_{2y}+DV01_{5y,10y}\cdot N=0 \\ DV01_{swap15y}(2y,5y)\cdot N_{15y}+DV01_{swap10y}(2y,5y)\cdot N_{10y}+DV01_{swap5y}(2y,5y)\cdot N_{5y}+DV01_{swap2y}(2y,5y)\cdot N_{2y}+DV01_{2y,5y}\cdot N=0 \\ DV01_{swap15y}(0,2y)\cdot N_{15y}+DV01_{swap10y}(0,2y)\cdot N_{10y}+DV01_{swap5y}(0,2y)\cdot N_{5y}+DV01_{swap2y}(0,2y)\cdot N_{2y}+DV01_{0,2y}\cdot N=0 \end{cases}$$

We solve both systems. Table 3 shows the results of solving the first system, while Table 4 shows the results for the second system.

N_{2y}	7.3779 Mio €	Receiver
N_{5y}	2.8811 Mio €	Payer
N_{10y}	0.6396 Mio €	Receiver
N_{15y}	16.8161 Mio €	Receiver

Table 3: Swap Notionals to Hedge the Position (First System)

N_{2y}	7.3901 Mio €	Receiver
N_{5y}	2.8565 Mio €	Payer
N_{10y}	0.6686 Mio €	Receiver
N_{15y}	16.8161 Mio €	Receiver

Table 4: Swap Notionals to Hedge the Position (Second System)

We observe that the results are similar, though not identical. This difference arises because the sensitivity of the swap to changes in rates after expiry is not significant, even if it is non-zero. A solution to mitigate this discrepancy would be to perform the shift after interpolating the rates. In such a case, solving only the first system would suffice.

9 Hedging total Vega and Delta - Coarse-Grained Buckets

To hedge the total Vega, we first compute the Vega of our ATM 5y cap. To compute the value of the strike, we use the cap-floor parity:

$$\text{Cap}(K) - \text{Floor}(K) = \text{Swap}(K)$$

We have that the contracts are ATM if the cap's price equals the floor's price, given the same characteristics:

$$\sum_i B(T_0, T_{i+1})\delta_i[L_i(T_0) - K]N(d^N) + \sigma_i\sqrt{T_i - T_0}\Phi(d^N) = \sum_i B(T_0, T_{i+1})\delta_i[K - L_i(T_0)]N(-d^N) + \sigma_i\sqrt{T_i - T_0}\Phi(-d^N)$$

This implies that the cap-floor parity ensures the swap's value is zero. When the swap is placed on the market, the swap rate is the value that makes the swap's NPV equal to zero. Hence, the swap rate is exactly the cap rate for which we have an ATM cap.

Next, we compute the difference between the NPV of the cap obtained using Bachelier's formula and after shifting the flat volatilities as before. Referring to the previously calculated total Vega, we determine N_{cap} by solving:

$$N_{\text{cap}}\nu_{\text{cap}} + N\nu = 0$$

From which we obtain:

N_{cap}	208.0707 Mio €	Sell

Table 5: Cap Notional to Hedge the Position

We now proceed to hedge the coarse-grained bucketed delta of the entire portfolio. Since swaps are not sensitive to volatility movements, adding swaps to the portfolio would not alter our Vega-hedged position. We proceed as before, but we must also consider the cap's delta sensitivity. We choose the cap notionals that match the coarse-grained bucketed deltas of the portfolio:

$$\begin{cases} N_{15y}DV01_{\text{s15y}}(10y,15y) + N_{10y}DV01_{\text{s10y}}(10y,15y) + N_{5y}DV01_{\text{s5y}}(10y,15y) + N_{2y}DV01_{\text{s2y}}(10y,15y) + NDV01_{10,15y} + N_{\text{cap}}DV01_{\text{Cap5y}}(10y,15y) = 0 \\ N_{15y}DV01_{\text{s15y}}(5y,10y) + N_{10y}DV01_{\text{s10y}}(5y,10y) + N_{5y}DV01_{\text{s5y}}(5y,10y) + N_{2y}DV01_{\text{s2y}}(5y,10y) + NDV01_{5,10y} + N_{\text{cap}}DV01_{\text{Cap5y}}(5y,10y) = 0 \\ N_{15y}DV01_{\text{s15y}}(2y,5y) + N_{10y}DV01_{\text{s10y}}(2y,5y) + N_{5y}DV01_{\text{s5y}}(2y,5y) + N_{2y}DV01_{\text{s2y}}(2y,5y) + NDV01_{2y,5y} + N_{\text{cap}}DV01_{\text{Cap5y}}(2y,5y) = 0 \\ N_{15y}DV01_{\text{s15y}}(0y,2y) + N_{10y}DV01_{\text{s10y}}(0y,2y) + N_{5y}DV01_{\text{s5y}}(0y,2y) + N_{2y}DV01_{\text{s2y}}(0y,2y) + NDV01_{0,2y} + N_{\text{cap}}DV01_{\text{Cap5y}}(0y,2y) = 0 \end{cases}$$

In this scenario, we also consider the sensitivities after the swaps' maturities, as discussed in previous sections (although the results do not change much, as seen before).

We obtain the following results:

N_{2y}	2.6159 Mio €	Payer
N_{5y}	102.3511 Mio €	Payer
N_{10y}	2.0106 Mio €	Receiver
N_{15y}	17.3942 Mio €	Receiver

Table 6: Swap Notionals to Hedge the Position

10 Hedging Vega - Coarse-Grained Buckets

In this section, we consider the coarse-grained buckets 0-5y and 5-15y. The objective is to hedge the structured bond with respect to the Vega-bucket sensitivity. After evaluating the coarse-grained buckets for Vega, stopping the computation at the contract's maturity (thanks to linear interpolation on the volatilities), and using a similar procedure to set the weights, we compute the cap notionals N_{5y} and N_{15y} that satisfy the following:

$$\begin{cases} \nu_{\text{Cap15y}}(5y, 15y) \cdot N_{15y} + \nu_{5y,15y} \cdot N = 0 \\ \nu_{\text{Cap15y}}(0y, 5y) \cdot N_{15y} + \nu_{\text{Cap10y}}(0y, 5y) \cdot N_{10y} + \nu_{0y,5y} \cdot N = 0 \end{cases}$$

The results are as follows:

Cap 5y	11.7555 Mio €	Sell
Cap 15y	44.3481 Mio €	Sell

Table 7: Cap Notionals to Hedge the Vega Bucket Sensitivity