Defects across dimensions

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1 Superconformal algebra

We use the conventions from Bobev and friends `1503.02081`, which we now summarize. We work in euclidean signature, so upper and lower indices do not matter. The conformal algebra is:

$$
\begin{aligned}
[D, P_\mu] &= P_\mu, && (1)\\
[D, K_\mu] &= -K_\mu, && (2)\\
[K_\mu, P_\nu] &= 2\left(\delta_{\mu\nu} D - M_{\mu\nu}\right), && (3)\\
[M_{\mu\nu}, M_{\rho\sigma}] &= -\delta_{\mu\rho} M_{\nu\sigma} \pm \ldots, && (4)\\
[M_{\mu\nu}, P_\rho] &= -\delta_{\mu\rho} P_\nu + \delta_{\nu\rho} P_\mu, && (5)\\
[M_{\mu\nu}, K_\rho] &= -\delta_{\mu\rho} K_\nu + \delta_{\nu\rho} K_\mu. && (6)
\end{aligned}
$$

The supercharges transform as:

$$
\begin{aligned}
[D, Q^+_\alpha] &= \tfrac{1}{2} Q^+_\alpha, && (7)\\
[D, Q^-_{\dot\alpha}] &= \tfrac{1}{2} Q^-_{\dot\alpha}, && (8)\\
[R, Q^+_\alpha] &= Q^+_\alpha, && (9)\\
[R, Q^-_{\dot\alpha}] &= -Q^-_{\dot\alpha}, && (10)\\
[K^\mu, Q^+_\alpha] &= \Sigma^\mu_{\alpha\dot\alpha} S^{\dot\alpha +}, && (11)\\
[K^\mu, Q^-_{\dot\alpha}] &= \Sigma^\mu_{\alpha\dot\alpha} S^{\alpha -}, && (12)\\
[M_{\mu\nu}, Q^+_\alpha] &= (m_{\mu\nu})_\alpha{}^\beta Q^+_\beta, && (13)\\
[M_{\mu\nu}, Q^-_{\dot\alpha}] &= (\tilde m_{\mu\nu})^{\dot\beta}{}_{\dot\alpha} Q^-_{\dot\beta}, && (14)
\end{aligned}
$$

and

$$
\begin{aligned}
[D, S^{\dot{\alpha}+}] &= -\tfrac{1}{2} S^{\dot{\alpha}+}, && (15)\\
[D, S^{\alpha-}] &= -\tfrac{1}{2} S^{\alpha-}, && (16)\\
[R, S^{\dot{\alpha}+}] &= S^{\dot{\alpha}+}, && (17)\\
[R, S^{\alpha-}] &= -S^{\alpha-}, && (18)\\
[P_\mu, S^{\dot{\alpha}+}] &= -\bar{\Sigma}_\mu^{\dot{\alpha}\alpha} Q_\alpha^+, && (19)\\
[P_\mu, S^{\alpha-}] &= -\bar{\Sigma}_\mu^{\dot{\alpha}\alpha} Q_{\dot{\alpha}}^-, && (20)\\
[M_{\mu\nu}, S^{\dot{\alpha}+}] &= -(\tilde{m}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} S^{\dot{\beta}+}, && (21)\\
[M_{\mu\nu}, S^{\alpha-}] &= -(m_{\mu\nu})_\beta{}^\alpha S^{\beta-}. && (22)
\end{aligned}
$$

The anticommutators are:

$$
\begin{aligned}
\{Q_\alpha^+, Q_{\dot{\alpha}}^-\} &= \Sigma^\mu_{\alpha\dot{\alpha}} P_\mu, && (23)\\
\{S^{\dot{\alpha}+}, S^{\alpha-}\} &= \bar{\Sigma}_\mu^{\dot{\alpha}\alpha} K_\mu, && (24)\\
\{S^{\dot{\alpha}+}, Q_{\dot{\beta}}^-\} &= \delta^{\dot{\alpha}}{}_{\dot{\beta}} \left(D + \frac{d-1}{2} R \right) - (\tilde{m}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} M_{\mu\nu} + \ldots, && (25)\\
\{S^{\alpha-}, Q_\beta^+\} &= \delta^\alpha{}_\beta \left(D - \frac{d-1}{2} R \right) - (m_{\mu\nu})_\beta{}^\alpha M_{\mu\nu} + \ldots. && (26)
\end{aligned}
$$

The gamma matrices are defined from

$$
\begin{aligned}
\Sigma_i \bar{\Sigma}_j + \Sigma_j \bar{\Sigma}_i &= 2\delta_{ij}, && (27)\\
\bar{\Sigma}_i \Sigma_j + \bar{\Sigma}_j \Sigma_i &= 2\delta_{ij}, && (28)\\
\bar{\Sigma}_\mu^{\dot{\alpha}\alpha} &= \left(\Sigma^\mu_{\alpha\dot{\alpha}}\right)^*, && (29)
\end{aligned}
$$

and then

$$
\begin{aligned}
m_{\mu\nu} &= \frac{1}{4}\left(\Sigma_\nu \bar{\Sigma}_\mu - \Sigma_\mu \bar{\Sigma}_\nu\right), && (30)\\
\tilde{m}_{\mu\nu} &= \frac{1}{4}\left(\bar{\Sigma}_\mu \Sigma_\nu - \bar{\Sigma}_\nu \Sigma_\mu\right). && (31)
\end{aligned}
$$

The Casimir is

$$
C_2 = D^2 - \frac{1}{2}\{P_\mu, K^\mu\} - \frac{1}{2} M_{\mu\nu} M^{\mu\nu} - \frac{1}{2} R^2 + \frac{1}{2}[S^{\dot{\alpha}+}, Q_{\dot{\alpha}}^-] + \frac{1}{2}[S^{\alpha-}, Q_\alpha^+]. \tag{32}
$$

Acting with it on an operator with quantum numbers Δ, ℓ, r gives the eigenvalue

$$
\lambda_C = \Delta(\Delta - d + 2) + \ell(\ell + d - 2) - \frac{r^2}{2}. \tag{33}
$$

When we look at the particular case $d = 3$ we take indices $\mu, \nu = 1, 2, 3$ and build the gamma matrices from the Pauli matrices

$$
\Sigma^1 = \sigma^1, \quad \Sigma^2 = \sigma^2, \quad \Sigma^3 = \sigma^3, \tag{34}
$$

When we look at the particular case $d = 4$ we take indices $\mu, \nu = 1, 2, 3, 4$ and build the remaining matrix as

$$\Sigma^4 = i\mathbb{1}_2. \tag{35}$$

1.1 Action of generators on fields

We will consider two-point functions $\langle \phi_1(x_1)\phi_2(x_2)\rangle$, where ϕ_1 is chiral and ϕ_2 is antichiral

$$[Q^+_\alpha, \phi_1(0)] = 0, \tag{36}$$
$$[Q^-_{\dot\alpha}, \phi_2(0)] = 0. \tag{37}$$

Consistency with the superconformal algebra fixes the R-charge in terms of the dimension:

$$[D, \phi_1(0)] = \Delta_\phi, \tag{38}$$
$$[D, \phi_2(0)] = \Delta_\phi, \tag{39}$$
$$[R, \phi_1(0)] = \frac{2}{d-1}\Delta_\phi, \tag{40}$$
$$[R, \phi_2(0)] = \frac{2}{d-1}\Delta_\phi. \tag{41}$$

The operators are translated as

$$\phi(x) = e^{x\cdot P}\phi(0)e^{-x\cdot P}, \tag{42}$$

which implies the usual

$$[P_\mu, \phi(x)] = \partial_\mu\phi(x), \tag{43}$$
$$[D, \phi(x)] = (x^\mu\partial_\mu + \Delta)\phi(x), \tag{44}$$
$$[M_{\mu\nu}, \phi(x)] = (x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x), \tag{45}$$
$$\dots \tag{46}$$

The shortening condition implies

$$[Q^+_\alpha, \phi_1(x_1)] = 0, \tag{47}$$
$$[S^{\dot\alpha+}, \phi_1(x_1)] = 0, \tag{48}$$
$$[Q^-_{\dot\alpha}, \phi_2(x_2)] = 0, \tag{49}$$
$$[S^{\alpha-}, \phi_2(x_2)] = 0. \tag{50}$$

For the remaining S generators we get

$$[S^{\alpha-}, \phi_1(x_1)] = x_1^\mu\bar\Sigma_\mu^{\dot\alpha\alpha}[Q^-_{\dot\alpha}, \phi_1(x_1)], \tag{51}$$
$$[S^{\dot\alpha+}, \phi_2(x_2)] = x_2^\mu\bar\Sigma_\mu^{\dot\alpha\alpha}[Q^+_\alpha, \phi_2(x_2)]. \tag{52}$$

2 Half-BPS boundaries

2.1 Non-supersymmetric

We can obtain a subalgebra restricting to

$$D, \quad R, \quad M_{ab}, \quad P_a, \quad K_a, \tag{53}$$

where $a, b = 1, \ldots, d-1$ are parallel to the boundary, which sits at $x^d = 0$. The defect Casimir is obtained from the full one restricting to the defect operators

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a, K^a\} - \frac{1}{2} M_{ab} M^{ab} \tag{54}$$

The Ward identities fix the two-point function

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{f(\xi)}{(2x_1^d)^{\Delta_1}(2x_2^d)^{\Delta_2}}, \qquad \xi = \frac{x_{12}^2}{4x_1^d x_2^d}. \tag{55}$$

2.1.1 Defect channel

Acting with the bosonic part of the defect Casimir and using the non-supersymmetric Casimir $\lambda = \Delta(\Delta - d + 1)$ we get in general dimension

$$\frac{(C_{\text{def,bos}} - \lambda)\langle \phi_1(x_1)\phi_2(x_2) \rangle}{\langle \phi_1(x_1)\phi_2(x_2) \rangle} = \xi(\xi+1)f''(\xi) + \frac{d}{2}(2\xi+1)f'(\xi) - \Delta(\Delta - d + 1)f(\xi) = 0, \tag{56}$$

so the defect block that solves this is

$$f_\Delta(\xi) = \xi^{-\Delta}\, {}_2F_1\left(\Delta, \Delta - \frac{d}{2} + 1; 2\Delta - d + 2; -\frac{1}{\xi}\right). \tag{57}$$

2.1.2 Bulk channel

We should act with the full Casimir. The bosonic differential equation is

$$\begin{aligned} \frac{(C_{\text{bulk,bos}} - \lambda)\langle \phi_1(x_1)\phi_2(x_2) \rangle}{\langle \phi_1(x_1)\phi_2(x_2) \rangle} &= 4\xi^2(\xi+1)f''(\xi) + \big(4(2\Delta_\phi + 1)\xi(\xi+1) - 2d\xi\big)f'(\xi) \\ &\quad + \big(-\Delta(\Delta - d) - 2d\Delta_\phi + 4\Delta_\phi^2(\xi+1)\big)f(\xi) = 0, \end{aligned} \tag{58}$$

where the bosonic eigenvalue is $\lambda = \Delta(\Delta - d)$. The block is usually written in terms of $G(\xi) = \xi^{\Delta_\phi} f(\xi)$ because the dependance on the external dimensions drops:

$$G_\Delta(\xi) = \xi^{\Delta/2}\, {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta - \frac{d}{2} + 1; -\xi\right). \tag{59}$$

2.2 $\mathcal{N} = (2,0)$ subalgebra for $d = 3$

We can obtain a subalgebra restricting to

$$D, \quad R, \quad M_{12}, \quad P_1, P_2, \quad K_1, K_2, \quad Q_1^+, Q_2^-, \quad S^{1-}, S^{2+}. \tag{60}$$

This can be mapped to the $(2,0)$ algebra in $2d$. The defect Casimir is obtained from the full one restricting to the defect operators

$$C_{\rm def} = D^2 - \frac{1}{2}\{P_a, K^a\} - M_{12}^2 - \frac{1}{2}R^2 + \frac{1}{2}[S^{2+}, Q_2^-] + \frac{1}{2}[S^{1-}, Q_1^+]. \tag{61}$$

Acting with it on an operator with quantum numbers Δ, ℓ, r gives the eigenvalue:

$$\lambda_C = \Delta(\Delta-1) + \ell(\ell-1) - \frac{r^2}{2}. \tag{62}$$

2.2.1 Defect channel

We should act with the defect Casimir at position x_1. The new contribution from susy is

$$C_{\rm ferm} = -\frac{1}{2}R^2 + \frac{1}{2}[S^{2+}, Q_2^-] + \frac{1}{2}[S^{1-}, Q_1^+] = -Q_2^- S^{2+} + S^{1-}Q_1^+ - \frac{1}{2}R^2 + R. \tag{63}$$

Acting on one point and using the chirality of ϕ_1 we get

$$[C_{\rm ferm}, \phi_1(x_1)]|0\rangle = \left(-\frac{1}{2}\Delta_\phi^2 + \Delta_\phi\right)\phi_1(x_1)|0\rangle. \tag{64}$$

This suggests that to obtain the supersymmetric block we should just add a term $(-\frac{1}{2}\Delta_\phi^2 + \Delta_\phi)f(\xi)$ in the Casimir equation. Consider a superprimary $\hat{\mathcal{O}}_p$ of dimension Δ, spin $\ell = 1/2$ and charge $r = r_\phi - 1$, which has Casimir

$$\lambda = \Delta(\Delta-1) + \ell(\ell-1) - \frac{r^2}{2} = \Delta(\Delta-1) - \frac{1}{2}\Delta_\phi^2 + \Delta_\phi - \frac{3}{4}. \tag{65}$$

Then only the descendant $Q_1^+\hat{\mathcal{O}}_p$ has the right quantum numbers to appear in the defect OPE $\phi \sim Q_1^+\hat{\mathcal{O}}_p$, so the superconformal block should reduce to a single non-supersymmetric block. Indeed, in the differential equation all Δ_ϕ terms drop and we get

$$\xi(\xi+1)f''(\xi) + \frac{3}{2}(2\xi+1)f'(\xi) - \left(\Delta + \frac{1}{2}\right)\left(\Delta + \frac{1}{2} - 2\right)f(\xi) = 0, \tag{66}$$

which is solved by a shifted non-supersymmetric block $f_{\Delta+\frac{1}{2}}(\xi)$ as expected.

2.2.2 Bulk channel

We should act with the full Casimir. The only new contributions from susy are

$$C_{\rm ferm} = \frac{1}{2}[S^{\dot\alpha+}, Q_{\dot\alpha}^-] + \frac{1}{2}[S^{\alpha-}, Q_\alpha^+] = -Q_{\dot\alpha}^- S^{\dot\alpha+} + S^{\dot\alpha-}Q_\alpha^+ + 2R. \tag{67}$$

There is a contribution $-\frac{1}{2}R^2$ that cancels between the two sides of the Casimir equation. Acting on the two points we get

$$\begin{aligned}
&[C_{\rm ferm}, \phi_1(x_1)\phi_2(x_2)]|0\rangle \\
&\quad = \left([S^{\alpha-}, \phi_1(x_1)][Q_\alpha^+, \phi_2(x_2)] - [Q_{\dot\alpha}^-, \phi_1(x_1)][S^{\dot\alpha+}, \phi_2(x_2)] + 4\Delta_\phi\, \phi_1(x_1)\phi_2(x_2)\right)|0\rangle \\
&\quad = \left(x_{12}^\mu \bar\Sigma_\mu^{\dot\alpha\alpha}[Q_{\dot\alpha}^-, \phi_1(x_1)][Q_\alpha^+, \phi_2(x_2)] + 4\Delta_\phi\, \phi_1(x_1)\phi_2(x_2)\right)|0\rangle
\end{aligned} \tag{68}$$

We use Ward identities to figure them out. We are only allowed to use Q_1^+, Q_2^-, S^{1-}, S^{2+} to get Ward identities, because they are the only supercharges preserved by our defect. The first Ward identity

$$\langle\{Q_1^+,[Q_{\dot{\alpha}}^-,\phi_1(x_1)]\phi_2(x_2)\}\rangle = 0 \tag{69}$$

implies

$$\begin{aligned}\langle[Q_{\dot{\alpha}}^-,\phi_1(x_1)][Q_1^+,\phi_2(x_2)]\rangle &= \langle[\{Q_1^+,Q_{\dot{\alpha}}^-\},\phi_1(x_1)]\phi_2(x_2)\rangle \qquad (70)\\ &= \Sigma^\mu_{1\dot{\alpha}}\partial^{x_1}_\mu\langle\phi_1(x_1)\phi_2(x_2)\rangle. \qquad (71)\end{aligned}$$

We can play the same game with the other $Q^\pm$ and $S^\pm$ preserved by the defect. All in all, when we act with the fermionic part of the Casimir on the two-point function we get

$$\frac{C_{\text{bulk,ferm}}\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = 4\xi(\xi+1)f'(\xi) + 4\Delta_\phi(\xi+1)f(\xi). \tag{72}$$

It is fairly non-trivial to get a differential operator that acting on the two-point function can be rewritten in terms of ξ. This is a sanity check for our calculation. The supersymmetric Casimir eigenvalue is $\lambda = \Delta(\Delta-1) = \lambda_{bos}+2\Delta$, so we should also add a term $-2\Delta f(\xi)$ to the LHS. Adding all the contributions we get

$$\begin{aligned}4\xi^2(\xi+1)f''(\xi) + 2\xi(4\Delta_\phi(\xi+1)+4\xi+1)f'(\xi)&\\ + \left(2\Delta_\phi(2\Delta_\phi(\xi+1)+2\xi-1)-\Delta(\Delta-1)\right)f(\xi) = 0.&\end{aligned} \tag{73}$$

If we define $G(\xi) = \xi^{\Delta_\phi}f(\xi)$ the dependence on Δ_ϕ drops

$$4\xi^2(\xi+1)G''(\xi) + 2\xi(4\xi+1)G'(\xi) - \Delta(\Delta-1)G(\xi) = 0, \tag{74}$$

and we find the solution

$$G(\xi) = \xi^{\Delta/2}\,{}_2F_1\left(\frac{\Delta}{2}+1,\frac{\Delta}{2};\Delta+\frac{1}{2};-\xi\right), \tag{75}$$

which is exactly what Philine already found!

2.3 $\mathcal{N}=(1,1)$ subalgebra for $d=3$

We can obtain a subalgebra restricting to

$$D, \quad M_{12}, \quad P_1,P_2, \quad K_1,K_2, \quad Q_1^+ + Q_2^-, Q_2^+ + Q_1^-, \quad S^{1+}+S^{2-}, S^{2+}+S^{1-}. \tag{76}$$

This can be mapped to the $(1,1)$ algebra in $2d$. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a,K^a\} - M_{12}^2 + \frac{1}{4}[S^{2+}+S^{1-},Q_1^+ + Q_2^-] + \frac{1}{4}[S^{1+}+S^{2-},Q_2^+ + Q_1^-]. \tag{77}$$

Acting with it on an operator with quantum numbers Δ,ℓ gives the eigenvalue:

$$\lambda_C = \Delta(\Delta-1)+\ell^2. \tag{78}$$

2.3.1 Defect channel

The fermionic contribution to the Casimir equation is

$$\frac{1}{4}[S^{2+}+S^{1-},Q_1^+ + Q_2^-] + \frac{1}{4}[S^{1+}+S^{2-},Q_2^+ + Q_1^-] \to R - \frac{1}{2}\left(Q_2^- S^{1-} + Q_1^- S^{2-}\right) + \dots, \tag{79}$$

where we dropped all terms that vanish when acting on $\phi_1(x_1)$:

$$[C_{\text{def,ferm}}, \phi_1(x_1)]|0\rangle = \left(\Delta_\phi \phi_1(x_1) + x_1^3 \{Q_1^-, [Q_2^-, \phi_1(x_1)]\}\right)|0\rangle. \tag{80}$$

Using Ward identities we express $Q_1^- Q_2^-$ as a differential operator acting at x_1, and we get

$$\frac{C_{\text{bulk,ferm}}\langle \phi_1(x_1)\phi_2(x_2)\rangle}{\langle \phi_1(x_1)\phi_2(x_2)\rangle} = -\xi f'(\xi). \tag{81}$$

We also need a contribution $-\Delta f(\xi)$ from the difference in the Casimir eigenvalue, so

$$\xi(\xi+1)f''(\xi) + \left(2\xi + \frac{3}{2}\right)f'(\xi) - \Delta(\Delta-1)f(\xi) = 0. \tag{82}$$

This is solved by the same block Philine found:

$$f(\xi) = \xi^{-\Delta}\,{}_2F_1\left(\Delta, \frac{1}{2}(2\Delta-1); 2\Delta; -\frac{1}{\xi}\right). \tag{83}$$

2.3.2 Bulk channel

The bosonic part and supersymmetric parts are the same as before, the only difference are the Ward identites we can use to rewrite $\langle Q\phi Q\phi\rangle$ in terms of derivatives of $\langle\phi\phi\rangle$. As before, we can only use the preserved supercharges, so the Ward identities are

$$\begin{aligned}
&\langle \{Q_1^+ + Q_2^-, [Q_2^-, \phi_1]\phi_2\}\rangle = 0, \\
&\langle \{Q_2^+ + Q_1^-, [Q_1^-, \phi_1]\phi_2\}\rangle = 0, \\
&\dots
\end{aligned} \tag{84}$$

All in all we get:

$$\frac{C_{\text{bulk,ferm}}\langle \phi_1(x_1)\phi_2(x_2)\rangle}{\langle \phi_1(x_1)\phi_2(x_2)\rangle} = 4\xi f'(\xi) + 4\Delta_\phi f(\xi). \tag{85}$$

Following the same arguments as before, we find the blocks

$$G(\xi) = \xi^{\Delta/2}\,{}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta + \frac{1}{2}; -\xi\right), \tag{86}$$

consistent with Philine's results once more!

2.4 $OSp(1|4)$ subalgebra for $d = 4$

We can obtain a subalgebra restricting to

$$D, \quad M_{12}, M_{13}, M_{23}, \quad P_1, P_2, P_3, \quad K_1, K_2, K_3, \quad Q_1^+ + Q_2^-, Q_2^+ - Q_1^-, \quad S^{1+} - S^{2-}, S^{2+} + S^{1-}. \tag{87}$$

This can be mapped to the $OSp(1|4)$ algebra in $3d$. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a, K^a\} - \frac{1}{2}M_{ab}M^{ab} + \frac{1}{4}[S^{2+} + S^{1-}, Q_1^+ + Q_2^-] - \frac{1}{4}[S^{1+} - S^{2-}, Q_2^+ - Q_1^-]. \tag{88}$$

We are using $a, b = 1, 2, 3$ and the boundary sits at $x^4 = 0$. Acting with it on an operator with quantum numbers Δ, ℓ gives the eigenvalue:

$$\lambda_C = \Delta(\Delta - 2) + \ell(\ell + 1). \tag{89}$$

2.4.1 Defect channel

The relevant part of the Casimir is

$$C_{\text{def,ferm}} = -\frac{1}{2}Q_2^- S^{1-} + \frac{1}{2}Q_1^- S^{2-} + \frac{3}{2}R + \ldots, \tag{90}$$

which acting at point 1 gives

$$[C_{\text{def,ferm}}, \phi_1(x_1)]|0\rangle = \left[x_1^3\{Q_1^-, [Q_2^-, \phi_1(x_1)]\} + \Delta_\phi \phi_1(x_1)\right]|0\rangle. \tag{91}$$

Using Ward identities we find

$$\frac{C_{\text{def,ferm}}\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = -\xi f'(\xi). \tag{92}$$

This is the same as the $\mathcal{N} = (1, 1)$ boundary in $3d$.

2.4.2 Bulk channel

It works as before, but the Ward identities are different. The result is

$$\frac{C_{\text{bulk,ferm}}\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = 4\xi f'(\xi) + 4\Delta_\phi f(\xi). \tag{93}$$

This is once again the same as for the $\mathcal{N} = (1, 1)$ boundary in $3d$.

2.5 Boundary across dimensions

We conjecture that the fermionic part of the Casimir gives the same contribution regardless of the dimension $2 \leq d \leq 4$. (We should still check the boundary in $2d$, but it probably also works there) Combining the bosonic and supersymmetric parts we get blocks "across dimensions".

2.5.1 Defect channel

The solution is

$$
\begin{aligned}
f_{\Delta}^{\mathrm{SUSY}}(\xi) &= \xi^{-\Delta}\,{}_2F_1\left(\Delta, \frac{1}{2}(2\Delta+2-d); 2\Delta+3-d; -\frac{1}{\xi}\right) \\
&= f_{\Delta}(\xi) + \frac{\Delta}{4\Delta-2d+6} f_{\Delta+1}(\xi).
\end{aligned}
\tag{94}
$$

2.5.2 Bulk channel

The solution is

$$
\begin{aligned}
G_{\Delta}^{\mathrm{SUSY}}(\xi) &= \xi^{\Delta/2}\,{}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta+2-\frac{d}{2}; -\xi\right) \\
&= G_{\Delta}(\xi) + \frac{\Delta^2}{(2\Delta-d+2)(2\Delta-d+4)} G_{\Delta+2}(\xi).
\end{aligned}
\tag{95}
$$

3 Codimension two objects

3.1 Non-supersymmetric

We can obtain a subalgebra restricting to

$$
D, \quad R, \quad P_a, \quad K_a, \quad M_{ab}, \quad M_{ij}. \tag{96}
$$

where $a, b = 1, \ldots, p$ live on the defect and $i, j = p+1, \ldots, d$ are orthogonal. Also p is the dimension of the defect and $q = d - p$ is the codimension. The defect Casimir is obtained from the full one restricting to the defect operators. It factorizes into two commuting pieces

$$
C_{\mathrm{def},1} = D^2 - \frac{1}{2}\{P_a, K^a\} - \frac{1}{2} M_{ab} M^{ab}, \qquad C_{\mathrm{def},2} = -\frac{1}{2} M_{ij} M^{ij}. \tag{97}
$$

The Ward identities fix the two-point function but now there are two cross-ratios

$$
\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{f(\xi, \eta)}{|x_1^{\perp}|^{\Delta_1} |x_2^{\perp}|^{\Delta_2}}, \qquad \xi = \frac{x_{12}^2}{|x_1^{\perp}||x_2^{\perp}|}, \qquad \eta = \frac{x_1^{\perp} \cdot x_2^{\perp}}{|x_1^{\perp}||x_2^{\perp}|}. \tag{98}
$$

For the defect channel we will also use

$$
\chi = \frac{x_{12}^2 + 2 x_1^{\perp} \cdot x_2^{\perp}}{|x_1^{\perp}||x_2^{\perp}|}. \tag{99}
$$

In the defect channel, using the normalization of Billo and friends, we get

$$
f(\chi, \eta) \to \chi^{-\Delta}\, 2^{-s} \binom{s + \frac{q}{2} - 2}{\frac{q}{2} - 2}^{-1} C_s^{q/2-1}(\eta) \qquad \text{as} \qquad \chi \to \infty. \tag{100}
$$

3.1.1 Defect channel

The action of the conformal group gives, with $\lambda = \Delta(\Delta - p)$:

$$\frac{(C_{\text{def},1} - \lambda)\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = (\chi^2 - 4)f''(\chi) + (p+1)\chi f'(\chi) - \Delta(\Delta - p)f(\chi) = 0. \tag{101}$$

The action of transverse rotations gives, with $\lambda = s(s + q - 2)$:

$$\frac{(C_{\text{def},2} - \lambda)\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = (\eta^2 - 1)f''(\eta) + (q-1)\eta f'(\eta) - s(s+q-2)f(\eta) = 0. \tag{102}$$

In total the block is

$$\hat{f}_{\Delta,s}(\chi,\eta) = \alpha_{s,q}\chi^{-\Delta}\,{}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta + 1 - \frac{p}{2}; \frac{4}{\chi^2}\right)\,{}_2F_1\left(-\frac{s}{2}, \frac{q+s-2}{2}; \frac{q-1}{2}; 1 - \eta^2\right), \tag{103}$$

with normalization

$$\alpha_{s,q} = 2^{-s}\frac{\Gamma(q+s-2)\Gamma(q/2-1)}{\Gamma(q/2+s-1)\Gamma(q-2)}. \tag{104}$$

3.1.2 Bulk channel

3.2 Non-supersymmetric with $U(1)$ mixing

Let's imagine we have an extra non-compact $U(1)_R$ symmetry with an Hermitian generator R. Then for codimension-two defect, the transverse rotations $SO(2) \simeq U(1)$ can mix with the $U(1)_R$ symmetry

$$M_{12} + iR \tag{105}$$

(We can always normalize the R generator such that the above equation holds). The second Casimir is modified to

$$C_{\text{def},2} = -(M_{12} + iR)^2. \tag{106}$$

3.2.1 Bulk-defect correlator in $3d$

Consider

$$\langle\phi(x_1)\hat{\mathcal{O}}(x_2)\rangle \tag{107}$$

Using P_3 it can depend on

$$x_1^1, \quad x_1^2, \quad x_{12}^3 \tag{108}$$

Using M_{12} we find that $r_1 = r_2 + s_2$ and

$$(x_1^{\perp})^2, \quad x_{12}^3 \tag{109}$$

Using dilatations and K_3 (and assuming $s_2 = 0$) we find the usual expression

$$\langle\phi(x_1)\hat{\mathcal{O}}(x_2)\rangle = \frac{1}{[(x_1^{\perp})^2 + (x_{12}^3)^2]^{\Delta_2}|x_1^{\perp}|^{\Delta_1 - \Delta_2}} \tag{110}$$

Could we have $s_2 \neq 0$?!?

3.2.2 Defect channel

Since we assume (from the previous discussion) that the exchanged operator has $r = r_1$ and $s = 0$ the first Casimir equation is the same as before, and the second gives

$$\frac{(C_{\text{def},2} - \lambda)\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = (\eta^2 - 1)f''(\eta) + (\eta + 2r\sqrt{\eta^2 - 1})f'(\eta) = 0. \tag{111}$$

The contribution from R^2 cancels from the eigenvalue part, and s does not contribute by assumption (is this correct?!). Here we assume $\eta \geq 1$, or in another set of coordinates

$$\eta = \frac{1}{2}\left(w + \frac{1}{w}\right), \qquad 0 < w < 1. \tag{112}$$

In these coordinates the block reads

$$f(w) = c_1 w^{2r} + c_2, \tag{113}$$

where the constants c_i should be fixed by consistency with the OPE.

3.3 Line defect in $3d$

We can obtain a subalgebra restricting to

$$D, \quad P_3, \quad K_3, \quad -iM_{12} + R, \quad Q_1^+, Q_1^- \quad S^{1+}, S^{1-}. \tag{114}$$

This can be mapped to the $\mathcal{N} = 2$ algebra in $1d$ (the left-moving part of the $\mathcal{N} = (2,0)$ algebra in $2d$). It is natural that M_{12} appears with an extra factor of i compared to R, since in our conventions M_{12} is anti-hermitian while R is hermitian. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_3, K^3\} - (-iM_{12} + R)^2 + \frac{1}{2}[S^{1+}, Q_1^-] + \frac{1}{2}[S^{1-}, Q_1^+]. \tag{115}$$

Acting with it on an operator with quantum numbers Δ, ℓ, where ℓ is the eigenvalue of $-iM_{12} + R$ gives the eigenvalue:

$$\lambda_C = \Delta^2 - \ell^2. \tag{116}$$

3.3.1 Defect channel

The new contribution from susy is

$$\begin{aligned} C_{\text{def, ferm}} &= -(-iM_{12} + R)^2 + \frac{1}{2}[S^{1+}, Q_1^-] + \frac{1}{2}[S^{1-}, Q_1^+] \\ &= -(-iM_{12} + R)^2 + (-iM_{12} + R) - Q_1^- S^{1+} + S^{1-} Q_1^+. \end{aligned} \tag{117}$$

Acting at point 1 only the contribution from M and R survives and S and Q drop:

$$[C_{\text{ferm}}, \phi_1(x_1)]|0\rangle = \left(M_{12}^2 + i(2\Delta_\phi - 1)M_{12} - \Delta_\phi(\Delta_\phi - 1)\right)\phi_1(x_1)|0\rangle. \tag{118}$$

The action of the generators is

$$iM_{12} \to \pm i\sqrt{1 - \eta^2}, \tag{119}$$

$$(M_{12})^2 \to -C_{\text{def},2} \tag{120}$$

Comments:

- In Lorentzian signature we could have $\eta \geq 1$, which eats the factor i (what about Euclidean signature?!?).
- The sign of the $(M_{12})^2$ term also looks off, I would expect it would be $(M_{12})^2 = +C_{\text{def},2}$.

In any case, the Casimir equations to be solved are

$$\begin{aligned}&(\chi^2-4)f^{(2,0)}(\chi,\eta)+2\chi f^{(1,0)}(\chi,\eta)\\&-(\eta^2-1)f^{(0,2)}(\chi,\eta)-\eta f^{(0,1)}(\chi,\eta)-(2\Delta_\phi-1)\sqrt{\eta^2-1}f^{(0,1)}(\chi,\eta)\\&-\left(\Delta_\phi(\Delta_\phi-1)+\mathcal{C}_2\right)f(\chi,\eta)=0\end{aligned}\tag{121}$$

Comments:

- A superspace calculation gives no nilpotent invariants and the same Casimir equation.
- The two sets of equations are decoupled so solutions are products of a function of χ and a function of η. The one for χ

$$(\chi^2-4)f''(\chi)+2\chi f'(\chi)-\left(\Delta+\frac{1}{2}\right)\left(\Delta-\frac{1}{2}\right)f(\chi)=0,\tag{122}$$

  can be easily solved into a shifted block: $f_{\Delta+1/2}(\chi)$.
- It would be nice to find whether the rest (or something similar up to some signs) has a solution in terms of a sum of non-supersymmetric blocks.

4 Superspace calculation $3d$

We translate as

$$\mathcal{O}(z)=e^{x^\mu P_\mu+\theta^{\alpha-}Q^+_\alpha+\theta^{\dot\alpha+}Q^-_{\dot\alpha}}\mathcal{O}(0),\tag{123}$$

where the adjoint action is implicit. Thus

$$[P_\mu,\mathcal{O}(z)]=\partial_\mu\mathcal{O}(z),\tag{124}$$

$$[Q^+_\alpha,\mathcal{O}(z)]=\left(\frac{\partial}{\partial\theta^{\alpha-}}-\frac{1}{2}\Sigma^\mu_{\alpha\dot\alpha}\theta^{+\dot\alpha}\partial_\mu\right)\mathcal{O}(z),\tag{125}$$

$$[Q^-_{\dot\alpha},\mathcal{O}(z)]=\left(\frac{\partial}{\partial\theta^{\dot\alpha+}}-\frac{1}{2}\Sigma^\mu_{\alpha\dot\alpha}\theta^{-\alpha}\partial_\mu\right)\mathcal{O}(z),\tag{126}$$

$$[D,\phi(x)]=\left(x^\mu\partial_\mu+\theta^{\dot\alpha+}\frac{\partial}{\partial\theta^{\dot\alpha+}}+\theta^{\alpha-}\frac{\partial}{\partial\theta^{\alpha-}}+\Delta\right)\phi(x),\tag{127}$$

$$\ldots\tag{128}$$

The covariant derivatives are

$$D^+_\alpha=\frac{\partial}{\partial\theta^{\alpha-}}+\frac{1}{2}\Sigma^\mu_{\alpha\dot\alpha}\theta^{+\dot\alpha}\partial_\mu,\tag{129}$$

$$D^-_{\dot\alpha}=\frac{\partial}{\partial\theta^{\dot\alpha+}}+\frac{1}{2}\Sigma^\mu_{\alpha\dot\alpha}\theta^{-\alpha}\partial_\mu,\tag{130}$$

The chiral and antichiral coordinates

$$y^\mu = x^\mu - \frac{1}{2}\theta^-\Sigma^\mu\theta^+, \qquad D^+_\alpha y^\mu = 0, \quad D^+_\alpha\theta^+ = 0, \tag{131}$$

$$\tilde{y}^\mu = x^\mu + \frac{1}{2}\theta^-\Sigma^\mu\theta^+, \qquad D^-_\alpha y^\mu = 0, \quad D^-_\alpha\theta^- = 0. \tag{132}$$

Imposing conservation of $Q_1^\pm$ and P_3 the invariant objects are

$$z_1^1 = y_1^1 - \theta_1^{2+}\theta_2^{1-}, \quad z_1^2 = y_1^2 + i\theta_1^{2+}\theta_2^{1-}, \quad \theta_1^{2+}, \tag{133}$$

$$\tilde{z}_2^1 = \tilde{y}_2^1 + \theta_1^{1+}\theta_2^{2-}, \quad \tilde{z}_2^2 = \tilde{y}_2^2 + i\theta_1^{1+}\theta_2^{2-}, \quad \theta_2^{2-}, \tag{134}$$

$$z_{12}^3 = y_1^3 - \tilde{y}_2^3 - \theta_1^{1+}\theta_2^{1-}. \tag{135}$$

A Useful identities

$$\{A, BC\} = \{A, B\}C - B[A, C], \tag{136}$$

$$\{A, [B, C]\} = -\{B, [A, C]\} + [\{A, B\}, C], \tag{137}$$

$$\tag{138}$$