

Defects across dimensions

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1 Superconformal algebra

We use the conventions from Bobev and friends 1503.02081, which we now summarize. We work in euclidean signature, so upper and lower indices do not matter. The conformal algebra is:

$$[D, P_\mu] = P_\mu, \quad (1)$$

$$[D, K_\mu] = -K_\mu, \quad (2)$$

$$[K_\mu, P_\nu] = 2(\delta_{\mu\nu} D - M_{\mu\nu}), \quad (3)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -\delta_{\mu\rho} M_{\nu\sigma} \pm \dots, \quad (4)$$

$$[M_{\mu\nu}, P_\rho] = -\delta_{\mu\rho} P_\nu + \delta_{\nu\rho} P_\mu, \quad (5)$$

$$[M_{\mu\nu}, K_\rho] = -\delta_{\mu\rho} K_\nu + \delta_{\nu\rho} K_\mu. \quad (6)$$

The supercharges transform as:

$$[D, Q_\alpha^+] = \frac{1}{2} Q_\alpha^+, \quad (7)$$

$$[D, Q_{\dot{\alpha}}^-] = \frac{1}{2} Q_{\dot{\alpha}}^-, \quad (8)$$

$$[R, Q_\alpha^+] = Q_\alpha^+, \quad (9)$$

$$[R, Q_{\dot{\alpha}}^-] = -Q_{\dot{\alpha}}^-, \quad (10)$$

$$[K^\mu, Q_\alpha^+] = \Sigma_{\alpha\dot{\alpha}}^\mu S^{\dot{\alpha}+}, \quad (11)$$

$$[K^\mu, Q_{\dot{\alpha}}^-] = \Sigma_{\alpha\dot{\alpha}}^\mu S^{\alpha-}, \quad (12)$$

$$[M_{\mu\nu}, Q_\alpha^+] = (m_{\mu\nu})_\alpha{}^\beta Q_\beta^+, \quad (13)$$

$$[M_{\mu\nu}, Q_{\dot{\alpha}}^-] = (\tilde{m}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} Q_{\dot{\beta}}^-, \quad (14)$$

and

$$[D, S^{\dot{\alpha}+}] = -\frac{1}{2} S^{\dot{\alpha}+}, \quad (15)$$

$$[D, S^{\alpha-}] = -\frac{1}{2} S^{\alpha-}, \quad (16)$$

$$[R, S^{\dot{\alpha}+}] = S^{\dot{\alpha}+}, \quad (17)$$

$$[R, S^{\alpha-}] = -S^{\alpha-}, \quad (18)$$

$$[P_\mu, S^{\dot{\alpha}+}] = -\bar{\Sigma}_\mu^{\dot{\alpha}\alpha} Q_\alpha^+, \quad (19)$$

$$[P_\mu, S^{\alpha-}] = -\bar{\Sigma}_\mu^{\dot{\alpha}\alpha} Q_{\dot{\alpha}}^-, \quad (20)$$

$$[M_{\mu\nu}, S^{\dot{\alpha}+}] = -(\tilde{m}_{\mu\nu})_{\dot{\beta}}{}^{\dot{\alpha}} S^{\dot{\beta}+}, \quad (21)$$

$$[M_{\mu\nu}, S^{\alpha-}] = -(m_{\mu\nu})_\beta{}^\alpha S^{\beta-}. \quad (22)$$

The anticommutators are:

$$\{Q_\alpha^+, Q_{\dot{\alpha}}^-\} = \Sigma_{\alpha\dot{\alpha}}^\mu P_\mu, \quad (23)$$

$$\{S^{\dot{\alpha}+}, S^{\alpha-}\} = \bar{\Sigma}_\mu^{\dot{\alpha}\alpha} K_\mu, \quad (24)$$

$$\{S^{\dot{\alpha}+}, Q_{\dot{\beta}}^-\} = \delta_{\dot{\beta}}^{\dot{\alpha}} \left(D + \frac{d-1}{2} R \right) - (\tilde{m}_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} M_{\mu\nu} + \dots, \quad (25)$$

$$\{S^{\alpha-}, Q_\beta^+\} = \delta_\beta^\alpha \left(D - \frac{d-1}{2} R \right) - (m_{\mu\nu})_\beta^\alpha M_{\mu\nu} + \dots \quad (26)$$

The gamma matrices are defined from

$$\Sigma_i \bar{\Sigma}_j + \Sigma_j \bar{\Sigma}_i = 2\delta_{ij}, \quad (27)$$

$$\bar{\Sigma}_i \Sigma_j + \bar{\Sigma}_j \Sigma_i = 2\delta_{ij}, \quad (28)$$

$$\bar{\Sigma}_\mu^{\dot{\alpha}\alpha} = (\Sigma_{\alpha\dot{\alpha}}^\mu)^*, \quad (29)$$

and then

$$m_{\mu\nu} = \frac{1}{4} (\Sigma_\nu \bar{\Sigma}_\mu - \Sigma_\mu \bar{\Sigma}_\nu), \quad (30)$$

$$\tilde{m}_{\mu\nu} = \frac{1}{4} (\bar{\Sigma}_\mu \Sigma_\nu - \bar{\Sigma}_\nu \Sigma_\mu). \quad (31)$$

The Casimir is

$$C_2 = D^2 - \frac{1}{2} \{P_\mu, K^\mu\} - \frac{1}{2} M_{\mu\nu} M^{\mu\nu} - \frac{1}{2} R^2 + \frac{1}{2} [S^{\dot{\alpha}+}, Q_{\dot{\alpha}}^-] + \frac{1}{2} [S^{\alpha-}, Q_\alpha^+]. \quad (32)$$

Acting with it on an operator with quantum numbers Δ, ℓ, r gives the eigenvalue

$$\lambda_C = \Delta(\Delta - d + 2) + \ell(\ell + d - 2) - \frac{r^2}{2}. \quad (33)$$

When we look at the particular case $d = 3$ we take indices $\mu, \nu = 1, 2, 3$ and build the gamma matrices from the Pauli matrices

$$\Sigma^1 = \sigma^1, \quad \Sigma^2 = \sigma^2, \quad \Sigma^3 = \sigma^3, \quad (34)$$

When we look at the particular case $d = 4$ we take indices $\mu, \nu = 1, 2, 3, 4$ and build the remaining matrix as

$$\Sigma^4 = i\mathbb{1}_2. \quad (35)$$

1.1 Action of generators on fields

We will consider two-point functions $\langle \phi_1(x_1) \phi_2(x_2) \rangle$, where ϕ_1 is chiral and ϕ_2 is antichiral

$$[Q_\alpha^+, \phi_1(0)] = 0, \quad (36)$$

$$[Q_{\dot{\alpha}}^-, \phi_2(0)] = 0. \quad (37)$$

Consistency with the superconformal algebra fixes the R -charge in terms of the dimension:

$$[D, \phi_1(0)] = \Delta_\phi, \quad (38)$$

$$[D, \phi_2(0)] = \Delta_\phi, \quad (39)$$

$$[R, \phi_1(0)] = \frac{2}{d-1} \Delta_\phi, \quad (40)$$

$$[R, \phi_2(0)] = \frac{2}{d-1} \Delta_\phi. \quad (41)$$

The operators are translated as

$$\phi(x) = e^{x \cdot P} \phi(0) e^{-x \cdot P}, \quad (42)$$

which implies the usual

$$[P_\mu, \phi(x)] = \partial_\mu \phi(x), \quad (43)$$

$$[D, \phi(x)] = (x^\mu \partial_\mu + \Delta) \phi(x), \quad (44)$$

$$[M_{\mu\nu}, \phi(x)] = (x_\nu \partial_\mu - x_\mu \partial_\nu) \phi(x), \quad (45)$$

$$\dots \quad (46)$$

The shortening condition implies

$$[Q_\alpha^+, \phi_1(x_1)] = 0, \quad (47)$$

$$[S^{\dot{\alpha}+}, \phi_1(x_1)] = 0, \quad (48)$$

$$[Q_{\dot{\alpha}}^-, \phi_2(x_2)] = 0, \quad (49)$$

$$[S^{\alpha-}, \phi_2(x_2)] = 0. \quad (50)$$

For the remaining S generators we get

$$[S^{\alpha-}, \phi_1(x_1)] = x_1^\mu \bar{\Sigma}_\mu^{\dot{\alpha}\alpha} [Q_{\dot{\alpha}}^-, \phi_1(x_1)], \quad (51)$$

$$[S^{\dot{\alpha}+}, \phi_2(x_2)] = x_2^\mu \bar{\Sigma}_\mu^{\dot{\alpha}\alpha} [Q_\alpha^+, \phi_2(x_2)]. \quad (52)$$

2 Half-BPS boundaries

2.1 Non-supersymmetric

We can obtain a subalgebra restricting to

$$D, \quad R, \quad M_{ab}, \quad P_a, \quad K_a, \quad (53)$$

where $a, b = 1, \dots, d-1$ are parallel to the boundary, which sits at $x^d = 0$. The defect Casimir is obtained from the full one restricting to the defect operators

$$C_{\text{def}} = D^2 - \frac{1}{2} \{P_a, K^a\} - \frac{1}{2} M_{ab} M^{ab} \quad (54)$$

The Ward identities fix the two-point function

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{f(\xi)}{(2x_1^d)^{\Delta_1} (2x_2^d)^{\Delta_2}}, \quad \xi = \frac{x_{12}^2}{4x_1^d x_2^d}. \quad (55)$$

2.1.1 Defect channel

Acting with the bosonic part of the defect Casimir and using the non-supersymmetric Casimir $\lambda = \Delta(\Delta - d + 1)$ we get in general dimension

$$\frac{(C_{\text{def,bos}} - \lambda) \langle \phi_1(x_1) \phi_2(x_2) \rangle}{\langle \phi_1(x_1) \phi_2(x_2) \rangle} = \xi(\xi + 1)f''(\xi) + \frac{d}{2}(2\xi + 1)f'(\xi) - \Delta(\Delta - d + 1)f(\xi) = 0, \quad (56)$$

so the defect block that solves this is

$$f_\Delta(\xi) = \xi^{-\Delta} {}_2F_1\left(\Delta, \Delta - \frac{d}{2} + 1; 2\Delta - d + 2; -\frac{1}{\xi}\right). \quad (57)$$

2.1.2 Bulk channel

We should act with the full Casimir. The bosonic differential equation is

$$\begin{aligned} \frac{(C_{\text{bulk,bos}} - \lambda) \langle \phi_1(x_1) \phi_2(x_2) \rangle}{\langle \phi_1(x_1) \phi_2(x_2) \rangle} &= 4\xi^2(\xi + 1)f''(\xi) + (4(2\Delta_\phi + 1)\xi(\xi + 1) - 2d\xi)f'(\xi) \\ &+ (-\Delta(\Delta - d) - 2d\Delta_\phi + 4\Delta_\phi^2(\xi + 1))f(\xi) = 0, \end{aligned} \quad (58)$$

where the bosonic eigenvalue is $\lambda = \Delta(\Delta - d)$. The block is usually written in terms of $G(\xi) = \xi^{\Delta_\phi} f(\xi)$ because the dependance on the external dimensions drops:

$$G_\Delta(\xi) = \xi^{\Delta/2} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta - \frac{d}{2} + 1; -\xi\right). \quad (59)$$

2.2 $\mathcal{N} = (2, 0)$ subalgebra for $d = 3$

We can obtain a subalgebra restricting to

$$D, \quad R, \quad M_{12}, \quad P_1, P_2, \quad K_1, K_2, \quad Q_1^+, Q_2^-, \quad S^{1-}, S^{2+}. \quad (60)$$

This can be mapped to the $(2, 0)$ algebra in $2d$. The defect Casimir is obtained from the full one restricting to the defect operators

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a, K^a\} - M_{12}^2 - \frac{1}{2}R^2 + \frac{1}{2}[S^{2+}, Q_2^-] + \frac{1}{2}[S^{1-}, Q_1^+]. \quad (61)$$

Acting with it on an operator with quantum numbers Δ, ℓ, r gives the eigenvalue:

$$\lambda_C = \Delta(\Delta - 1) + \ell(\ell - 1) - \frac{r^2}{2}. \quad (62)$$

2.2.1 Defect channel

We should act with the defect Casimir at position x_1 . The new contribution from susy is

$$C_{\text{ferm}} = -\frac{1}{2}R^2 + \frac{1}{2}[S^{2+}, Q_2^-] + \frac{1}{2}[S^{1-}, Q_1^+] = -Q_2^- S^{2+} + S^{1-} Q_1^+ - \frac{1}{2}R^2 + R. \quad (63)$$

Acting on one point and using the chirality of ϕ_1 we get

$$[C_{\text{ferm}}, \phi_1(x_1)]|0\rangle = \left(-\frac{1}{2}\Delta_\phi^2 + \Delta_\phi\right) \phi_1(x_1)|0\rangle. \quad (64)$$

This suggests that to obtain the supersymmetric block we should just add a term $(-\frac{1}{2}\Delta_\phi^2 + \Delta_\phi)f(\xi)$ in the Casimir equation. Consider a superprimary $\hat{\mathcal{O}}_p$ of dimension Δ , spin $\ell = 1/2$ and charge $r = r_\phi - 1$, which has Casimir

$$\lambda = \Delta(\Delta - 1) + \ell(\ell - 1) - \frac{r^2}{2} = \Delta(\Delta - 1) - \frac{1}{2}\Delta_\phi^2 + \Delta_\phi - \frac{3}{4}. \quad (65)$$

Then only the descendant $Q_1^+ \hat{\mathcal{O}}_p$ has the right quantum numbers to appear in the defect OPE $\phi \sim Q_1^+ \hat{\mathcal{O}}_p$, so the superconformal block should reduce to a single non-supersymmetric block. Indeed, in the differential equation all Δ_ϕ terms drop and we get

$$\xi(\xi + 1)f''(\xi) + \frac{3}{2}(2\xi + 1)f'(\xi) - \left(\Delta + \frac{1}{2}\right)\left(\Delta + \frac{1}{2} - 2\right)f(\xi) = 0, \quad (66)$$

which is solved by a shifted non-supersymmetric block $f_{\Delta+\frac{1}{2}}(\xi)$ as expected.

2.2.2 Bulk channel

We should act with the full Casimir. The only new contributions from susy are

$$C_{\text{ferm}} = \frac{1}{2}[S^{\dot{\alpha}+}, Q_{\dot{\alpha}}^-] + \frac{1}{2}[S^{\alpha-}, Q_\alpha^+] = -Q_{\dot{\alpha}}^- S^{\dot{\alpha}+} + S^{\dot{\alpha}-} Q_\alpha^+ + 2R. \quad (67)$$

There is a contribution $-\frac{1}{2}R^2$ that cancels between the two sides of the Casimir equation. Acting on the two points we get

$$\begin{aligned} & [C_{\text{ferm}}, \phi_1(x_1)\phi_2(x_2)]|0\rangle \\ &= ([S^{\alpha-}, \phi_1(x_1)][Q_\alpha^+, \phi_2(x_2)] - [Q_{\dot{\alpha}}^-, \phi_1(x_1)][S^{\dot{\alpha}+}, \phi_2(x_2)] + 4\Delta_\phi \phi_1(x_1)\phi_2(x_2))|0\rangle \\ &= (x_{12}^\mu \bar{\Sigma}_\mu^{\dot{\alpha}\alpha} [Q_{\dot{\alpha}}^-, \phi_1(x_1)][Q_\alpha^+, \phi_2(x_2)] + 4\Delta_\phi \phi_1(x_1)\phi_2(x_2))|0\rangle \end{aligned} \quad (68)$$

We use Ward identities to figure them out. We are only allowed to use Q_1^+ , Q_2^- , S^{1-} , S^{2+} to get Ward identities, because they are the only supercharges preserved by our defect. The first Ward identity

$$\langle \{Q_1^+, [Q_{\dot{\alpha}}^-, \phi_1(x_1)]\phi_2(x_2)\} \rangle = 0 \quad (69)$$

implies

$$\langle [Q_{\dot{\alpha}}^-, \phi_1(x_1)][Q_1^+, \phi_2(x_2)] \rangle = \langle [\{Q_1^+, Q_{\dot{\alpha}}^-\}, \phi_1(x_1)]\phi_2(x_2) \rangle \quad (70)$$

$$= \Sigma_{1\dot{\alpha}}^\mu \partial_\mu^{x_1} \langle \phi_1(x_1)\phi_2(x_2) \rangle. \quad (71)$$

We can play the same game with the other Q^\pm and S^\pm preserved by the defect. All in all, when we act with the fermionic part of the Casimir on the two-point function we get

$$\frac{C_{\text{bulk,ferm}} \langle \phi_1(x_1)\phi_2(x_2) \rangle}{\langle \phi_1(x_1)\phi_2(x_2) \rangle} = 4\xi(\xi + 1)f'(\xi) + 4\Delta_\phi(\xi + 1)f(\xi). \quad (72)$$

It is fairly non-trivial to get a differential operator that acting on the two-point function can be rewritten in terms of ξ . This is a sanity check for our calculation. The supersymmetric Casimir eigenvalue is $\lambda = \Delta(\Delta - 1) = \lambda_{bos} + 2\Delta$, so we should also add a term $-2\Delta f(\xi)$ to the LHS. Adding all the contributions we get

$$4\xi^2(\xi + 1)f''(\xi) + 2\xi(4\Delta_\phi(\xi + 1) + 4\xi + 1)f'(\xi) + (2\Delta_\phi(2\Delta_\phi(\xi + 1) + 2\xi - 1) - \Delta(\Delta - 1))f(\xi) = 0. \quad (73)$$

If we define $G(\xi) = \xi^{\Delta_\phi} f(\xi)$ the dependence on Δ_ϕ drops

$$4\xi^2(\xi + 1)G''(\xi) + 2\xi(4\xi + 1)G'(\xi) - \Delta(\Delta - 1)G(\xi) = 0, \quad (74)$$

and we find the solution

$$G(\xi) = \xi^{\Delta/2} {}_2F_1\left(\frac{\Delta}{2} + 1, \frac{\Delta}{2}; \Delta + \frac{1}{2}; -\xi\right), \quad (75)$$

which is exactly what Philine already found!

2.3 $\mathcal{N} = (1, 1)$ subalgebra for $d = 3$

We can obtain a subalgebra restricting to

$$D, \quad M_{12}, \quad P_1, P_2, \quad K_1, K_2, \quad Q_1^+ + Q_2^-, Q_2^+ + Q_1^-, \quad S^{1+} + S^{2-}, S^{2+} + S^{1-}. \quad (76)$$

This can be mapped to the $(1, 1)$ algebra in $2d$. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a, K^a\} - M_{12}^2 + \frac{1}{4}[S^{2+} + S^{1-}, Q_1^+ + Q_2^-] + \frac{1}{4}[S^{1+} + S^{2-}, Q_2^+ + Q_1^-]. \quad (77)$$

Acting with it on an operator with quantum numbers Δ, ℓ gives the eigenvalue:

$$\lambda_C = \Delta(\Delta - 1) + \ell^2. \quad (78)$$

2.3.1 Defect channel

The fermionic contribution to the Casimir equation is

$$\frac{1}{4}[S^{2+} + S^{1-}, Q_1^+ + Q_2^-] + \frac{1}{4}[S^{1+} + S^{2-}, Q_2^+ + Q_1^-] \rightarrow R - \frac{1}{2}(Q_2^- S^{1-} + Q_1^- S^{2-}) + \dots, \quad (79)$$

where we dropped all terms that vanish when acting on $\phi_1(x_1)$:

$$[C_{\text{def,ferm}}, \phi_1(x_1)]|0\rangle = (\Delta_\phi \phi_1(x_1) + x_1^3 \{Q_1^-, [Q_2^-, \phi_1(x_1)]\})|0\rangle. \quad (80)$$

Using Ward identities we express $Q_1^- Q_2^-$ as a differential operator acting at x_1 , and we get

$$\frac{C_{\text{bulk,ferm}} \langle \phi_1(x_1) \phi_2(x_2) \rangle}{\langle \phi_1(x_1) \phi_2(x_2) \rangle} = -\xi f'(\xi). \quad (81)$$

We also need a contribution $-\Delta f(\xi)$ from the difference in the Casimir eigenvalue, so

$$\xi(\xi + 1)f''(\xi) + \left(2\xi + \frac{3}{2}\right)f'(\xi) - \Delta(\Delta - 1)f(\xi) = 0. \quad (82)$$

This is solved by the same block Philine found:

$$f(\xi) = \xi^{-\Delta} {}_2F_1\left(\Delta, \frac{1}{2}(2\Delta - 1); 2\Delta; -\frac{1}{\xi}\right). \quad (83)$$

2.3.2 Bulk channel

The bosonic part and supersymmetric parts are the same as before, the only difference are the Ward identities we can use to rewrite $\langle Q\phi Q\phi \rangle$ in terms of derivatives of $\langle \phi\phi \rangle$. As before, we can only use the preserved supercharges, so the Ward identities are

$$\begin{aligned}\langle \{Q_1^+ + Q_2^-, [Q_2^-, \phi_1]\phi_2\} \rangle &= 0, \\ \langle \{Q_2^+ + Q_1^-, [Q_1^-, \phi_1]\phi_2\} \rangle &= 0, \\ &\dots\end{aligned}\tag{84}$$

All in all we get:

$$\frac{C_{\text{bulk,ferm}} \langle \phi_1(x_1)\phi_2(x_2) \rangle}{\langle \phi_1(x_1)\phi_2(x_2) \rangle} = 4\xi f'(\xi) + 4\Delta_\phi f(\xi).\tag{85}$$

Following the same arguments as before, we find the blocks

$$G(\xi) = \xi^{\Delta/2} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta + \frac{1}{2}; -\xi\right),\tag{86}$$

consistent with Philine's results once more!

2.4 $OSp(1|4)$ subalgebra for $d = 4$

We can obtain a subalgebra restricting to

$$D, \quad M_{12}, M_{13}, M_{23}, \quad P_1, P_2, P_3, \quad K_1, K_2, K_3, \quad Q_1^+ + Q_2^-, Q_2^+ - Q_1^-, \quad S^{1+} - S^{2-}, S^{2+} + S^{1-}.\tag{87}$$

This can be mapped to the $OSp(1|4)$ algebra in $3d$. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a, K^a\} - \frac{1}{2}M_{ab}M^{ab} + \frac{1}{4}[S^{2+} + S^{1-}, Q_1^+ + Q_2^-] - \frac{1}{4}[S^{1+} - S^{2-}, Q_2^+ - Q_1^-].\tag{88}$$

We are using $a, b = 1, 2, 3$ and the boundary sits at $x^4 = 0$. Acting with it on an operator with quantum numbers Δ, ℓ gives the eigenvalue:

$$\lambda_C = \Delta(\Delta - 2) + \ell(\ell + 1).\tag{89}$$

2.4.1 Defect channel

The relevant part of the Casimir is

$$C_{\text{def,ferm}} = -\frac{1}{2}Q_2^- S^{1-} + \frac{1}{2}Q_1^- S^{2-} + \frac{3}{2}R + \dots,\tag{90}$$

which acting at point 1 gives

$$[C_{\text{def,ferm}}, \phi_1(x_1)]|0\rangle = [x_1^3\{Q_1^-, [Q_2^-, \phi_1(x_1)]\} + \Delta_\phi \phi_1(x_1)]|0\rangle.\tag{91}$$

Using Ward identities we find

$$\frac{C_{\text{def,ferm}} \langle \phi_1(x_1)\phi_2(x_2) \rangle}{\langle \phi_1(x_1)\phi_2(x_2) \rangle} = -\xi f'(\xi).\tag{92}$$

This is the same as the $\mathcal{N} = (1, 1)$ boundary in $3d$.

2.4.2 Bulk channel

It works as before, but the Ward identities are different. The result is

$$\frac{C_{\text{bulk,ferm}} \langle \phi_1(x_1) \phi_2(x_2) \rangle}{\langle \phi_1(x_1) \phi_2(x_2) \rangle} = 4\xi f'(\xi) + 4\Delta_\phi f(\xi). \quad (93)$$

This is once again the same as for the $\mathcal{N} = (1, 1)$ boundary in $3d$.

2.5 Boundary across dimensions

We conjecture that the fermionic part of the Casimir gives the same contribution regardless of the dimension $2 \leq d \leq 4$. (We should still check the boundary in $2d$, but it probably also works there) Combining the bosonic and supersymmetric parts we get blocks “across dimensions”.

2.5.1 Defect channel

The solution is

$$\begin{aligned} f_\Delta^{\text{SUSY}}(\xi) &= \xi^{-\Delta} {}_2F_1\left(\Delta, \frac{1}{2}(2\Delta + 2 - d); 2\Delta + 3 - d; -\frac{1}{\xi}\right) \\ &= f_\Delta(\xi) + \frac{\Delta}{4\Delta - 2d + 6} f_{\Delta+1}(\xi). \end{aligned} \quad (94)$$

2.5.2 Bulk channel

The solution is

$$\begin{aligned} G_\Delta^{\text{SUSY}}(\xi) &= \xi^{\Delta/2} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta + 2 - \frac{d}{2}; -\xi\right) \\ &= G_\Delta(\xi) + \frac{\Delta^2}{(2\Delta - d + 2)(2\Delta - d + 4)} G_{\Delta+2}(\xi). \end{aligned} \quad (95)$$

3 Codimension two objects

3.1 Non-supersymmetric

We can obtain a subalgebra restricting to

$$D, \quad R, \quad P_a, \quad K_a, \quad M_{ab}, \quad M_{ij}. \quad (96)$$

where $a, b = 1, \dots, p$ live on the defect and $i, j = p+1, \dots, d$ are orthogonal. Also p is the dimension of the defect and $q = d - p$ is the codimension. The defect Casimir is obtained from the full one restricting to the defect operators. It factorizes into two commuting pieces

$$C_{\text{def},1} = D^2 - \frac{1}{2}\{P_a, K^a\} - \frac{1}{2}M_{ab}M^{ab}, \quad C_{\text{def},2} = -\frac{1}{2}M_{ij}M^{ij}. \quad (97)$$

The Ward identities fix the two-point function but now there are two cross-ratios

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{f(\xi, \eta)}{|x_1^\perp|^{\Delta_1} |x_2^\perp|^{\Delta_2}}, \quad \xi = \frac{x_{12}^2}{|x_1^\perp| |x_2^\perp|}, \quad \eta = \frac{x_1^\perp \cdot x_2^\perp}{|x_1^\perp| |x_2^\perp|}. \quad (98)$$

For the defect channel we will also use

$$\chi = \frac{x_{12}^2 + 2x_1^\perp \cdot x_2^\perp}{|x_1^\perp| |x_2^\perp|}. \quad (99)$$

In the defect channel, using the normalization of Billo and friends, we get

$$f(\chi, \eta) \rightarrow \chi^{-\Delta} 2^{-s} \left(s + \frac{q}{2} - 2 \right)^{-1} C_s^{q/2-1}(\eta) \quad \text{as} \quad \chi \rightarrow \infty. \quad (100)$$

3.1.1 Defect channel

The action of the conformal group gives, with $\lambda = \Delta(\Delta - p)$:

$$\frac{(C_{\text{def},1} - \lambda) \langle \phi_1(x_1) \phi_2(x_2) \rangle}{\langle \phi_1(x_1) \phi_2(x_2) \rangle} = (\chi^2 - 4) f''(\chi) + (p+1) \chi f'(\chi) - \Delta(\Delta - p) f(\chi) = 0. \quad (101)$$

The action of transverse rotations gives, with $\lambda = s(s + q - 2)$:

$$\frac{(C_{\text{def},2} - \lambda) \langle \phi_1(x_1) \phi_2(x_2) \rangle}{\langle \phi_1(x_1) \phi_2(x_2) \rangle} = (\eta^2 - 1) f''(\eta) + (q-1) \eta f'(\eta) - s(s + q - 2) f(\eta) = 0. \quad (102)$$

In total the block is

$$\hat{f}_{\Delta,s}(\chi, \eta) = \alpha_{s,q} \chi^{-\Delta} {}_2F_1 \left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta+1 - \frac{p}{2}; \frac{4}{\chi^2} \right) {}_2F_1 \left(-\frac{s}{2}, \frac{q+s-2}{2}; \frac{q-1}{2}; 1 - \eta^2 \right), \quad (103)$$

with normalization

$$\alpha_{s,q} = 2^{-s} \frac{\Gamma(q+s-2) \Gamma(q/2-1)}{\Gamma(q/2+s-1) \Gamma(q-2)}. \quad (104)$$

3.1.2 Bulk channel

If we take

$$\langle \phi \phi \rangle = \frac{\xi^{-(\Delta_1 + \Delta_2)/2} f(\xi, \eta)}{|x_1^i|^{\Delta_1} |x_2^i|^{\Delta_2}} \quad (105)$$

the Casimir equation is

$$\begin{aligned} & \xi^2 (2\eta^2 + \eta\xi + 2) f^{(2,0)}(\xi, \eta) + (1 - \eta^2) (2(1 - \eta^2) - \eta\xi) f^{(0,2)}(\xi, \eta) \\ & - 2\xi (1 - \eta^2) (2\eta + \xi) f^{(1,1)}(\xi, \eta) + \xi (2(\eta^2 + 1) - (2d - \eta\xi)) f^{(1,0)}(\xi, \eta) \\ & + (\xi(\eta^2 + q - 2) - 2\eta(1 - \eta^2)) f^{(0,1)}(\xi, \eta) - \left(\lambda_C + \frac{1}{4} \Delta_{12}^2 (2\eta^2 + \xi\eta - 2) \right) f(\xi, \eta) = 0 \end{aligned} \quad (106)$$

3.2 Non-supersymmetric with $U(1)$ mixing

Let's imagine we have an extra non-compact $U(1)_R$ symmetry with an Hermitian generator R . Then for codimension-two defect, the transverse rotations $SO(2) \simeq U(1)$ can mix with the $U(1)_R$ symmetry

$$M_{12} + iR \quad (107)$$

(We can always normalize the R generator such that the above equation holds). The second Casimir is modified to

$$C_{\text{def},2} = -(M_{12} + iR)^2. \quad (108)$$

In practice, any correlation function

$$\langle \phi(x_1) \hat{\mathcal{O}}^{i\dots j}(x_2) \rangle \quad (109)$$

that was invariant before will be invariant with the extra $U(1)$ provided $r_\phi = -r_{\hat{\mathcal{O}}}$. Thus the allowed values of this “twisted” transverse spin are

$$s = -r_\phi, -r_\phi + 1, \dots \quad (110)$$

When M_{12} acts on a two point function it gives

$$\frac{M_{12} \langle \phi_1(x_1) \phi_2(x_2) \rangle}{\langle \phi_1(x_1) \phi_2(x_2) \rangle} = \frac{x_1^2 x_2^1 - x_1^1 x_2^2}{|x_1^\perp| |x_2^\perp|} f'(\eta) = \sqrt{1 - \eta^2} f'(\eta) \quad (111)$$

When identifying the invariant there is a sign ambiguity, which we fix by assuming

$$x_1^2 x_2^1 - x_1^1 x_2^2 > 0. \quad (112)$$

What is the physical picture? What happens if we take the opposite sign?

3.2.1 Defect channel

We assume from the previous discussion that the exchanged operator has $s = r_\phi + \ell$. The first Casimir equation is the same as before, and the second gives

$$\frac{(C_{\text{def},2} - \lambda) \langle \phi_1(x_1) \phi_2(x_2) \rangle}{\langle \phi_1(x_1) \phi_2(x_2) \rangle} = (\eta^2 - 1) f''(\eta) + (\eta + 2r\sqrt{\eta^2 - 1}) f'(\eta) - \ell(\ell + 2r_\phi) f(\eta) = 0. \quad (113)$$

Here we assume $\eta \geq 1$, or in another set of coordinates

$$\eta = \frac{1}{2} \left(w + \frac{1}{w} \right), \quad 0 < w < 1. \quad (114)$$

In these coordinates the block reads

$$f(w) = w^{-\ell} (c_1 w^{2r+2\ell} + c_2), \quad (115)$$

where the constants c_i should be fixed by consistency with the OPE.

3.3 Line defect in 3d

We can obtain a subalgebra restricting to

$$D, \quad P_3, \quad K_3, \quad M_{12} + iR, \quad Q_1^+, Q_1^-, \quad S^{1+}, S^{1-}. \quad (116)$$

This can be mapped to the $\mathcal{N} = 2$ algebra in 1d (the left-moving part of the $\mathcal{N} = (2, 0)$ algebra in 2d). It is natural that M_{12} appears with an extra factor of i compared to R , since in our conventions M_{12} is anti-hermitian while R is hermitian. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_3, K^3\} + (M_{12} + iR)^2 + \frac{1}{2}[S^{1+}, Q_1^-] + \frac{1}{2}[S^{1-}, Q_1^+]. \quad (117)$$

Acting with it on an operator with quantum numbers Δ, s , where $(M_{12} + iR)\mathcal{O} = is\mathcal{O}$:

$$\lambda_C = \Delta^2 - s^2. \quad (118)$$

3.3.1 Defect channel

The new contribution from susy is

$$\begin{aligned} C_{\text{def, ferm}} &= (M_{12} + iR)^2 + \frac{1}{2}[S^{1+}, Q_1^-] + \frac{1}{2}[S^{1-}, Q_1^+] \\ &= (M_{12} + iR)^2 - i(M_{12} + iR) - Q_1^- S^{1+} + S^{1-} Q_1^+. \end{aligned} \quad (119)$$

Acting at point 1 only the contribution from M and R survives and S and Q drop:

$$[C_{\text{ferm}}, \phi_1(x_1)]|0\rangle = (M_{12}^2 + i(2\Delta_\phi - 1)M_{12} - \Delta_\phi(\Delta_\phi - 1))\phi_1(x_1)|0\rangle. \quad (120)$$

The action of the generators is (NOTE: it is ambiguous how to define $\sqrt{1 - \eta^2}$, depending of x_1, x_2 we should put take a minus or plus sign):

$$iM_{12} \rightarrow \sqrt{\eta^2 - 1}, \quad (121)$$

$$(M_{12})^2 \rightarrow -C_{\text{def}, 2} \quad (122)$$

The Casimir equations to be solved are

$$\begin{aligned} &(\chi^2 - 4)f^{(2,0)}(\chi, \eta) + 2\chi f^{(1,0)}(\chi, \eta) \\ &- (\eta^2 - 1)f^{(0,2)}(\chi, \eta) - \eta f^{(0,1)}(\chi, \eta) - (2\Delta_\phi - 1)\sqrt{\eta^2 - 1}f^{(0,1)}(\chi, \eta) \\ &- (\Delta_\phi(\Delta_\phi - 1) + \mathcal{C}_2)f(\chi, \eta) = 0 \end{aligned} \quad (123)$$

The solution is

$$f(\chi, \eta) = f_{\Delta+1/2}(\chi)f_{\Delta_\phi-1/2, \ell}(\eta), \quad \lambda_C = \Delta^2 - \left(\Delta_\phi - \frac{1}{2} + \ell\right)^2 \quad (124)$$

Here something is weird, it is hard to understand this solution as a level one descendant...
TODO: Double check signs and factors of 1/2 and see if it can make more sense.

3.3.2 Bulk channel

Up to some signs which are unclear to fix, the contribution from susy is

$$\frac{1}{\sqrt{\eta^2 - 1}} \left[(\eta^2 - 1) \left(\sqrt{\eta^2 - 1} \xi - 2\eta^2 + 2\sqrt{\eta^2 - 1} \eta - \eta \xi + 2 \right) f^{(0,1)}(\xi, \eta) \right. \quad (125)$$

$$\left. + \xi \left(-2\eta^3 + \eta^2 \left(2\sqrt{\eta^2 - 1} - \xi \right) + \eta \left(\sqrt{\eta^2 - 1} \xi + 2 \right) + 2\sqrt{\eta^2 - 1} + \xi \right) f^{(1,0)}(\xi, \eta) \right] = 0. \quad (126)$$

This looks a bit daunting. However, a smart change of coordinates

$$u = z\bar{z} = -\xi e^{-i\phi}, \quad v = (1 - z)(1 - \bar{z}) = e^{-2i\phi}, \quad (127)$$

the contribution simplifies dramatically

$$2 \left(z(1 - z)\partial + \bar{z}(1 - \bar{z})\bar{\partial} \right) f(z, \bar{z}) \quad (128)$$

3.4 Surface defect in 4d

We can obtain a subalgebra restricting to

$$D, \quad P_3, P_4 \quad K_3, K_4 \quad M_{34}, \quad M_{12} + \frac{3}{2}iR, \quad Q_1^+, Q_1^- \quad S^{1+}, S^{1-}. \quad (129)$$

This can be mapped to the $\mathcal{N} = (2, 0)$ algebra in 2d. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2} \{P_a, K^a\} - M_{34}^2 + \frac{1}{2} \left(M_{12} + \frac{3}{2}iR \right)^2 + \frac{1}{2} [S^{1+}, Q_1^-] + \frac{1}{2} [S^{1-}, Q_1^+]. \quad (130)$$

Acting with it on an operator with quantum numbers Δ, s , where $(M_{12} + iR)\mathcal{O} = is\mathcal{O}$:

$$\lambda_C = \Delta(\Delta - 1) + \ell(\ell + 1) - \frac{1}{2}s^2. \quad (131)$$

3.4.1 Defect channel

The solution is exactly the same one as before.

3.4.2 Bulk channel

Doing the usual calculation gives the same result as before in the right coordinates:

$$2 \left(z(1 - z)\partial + \bar{z}(1 - \bar{z})\bar{\partial} \right) f(z, \bar{z}) \quad (132)$$

3.5 Across dimensions

3.5.1 Bulk channel

We get the same equation that 1503.02081 across dimensions so the block is

$$G_{\Delta, \ell}(z, \bar{z}) = (z\bar{z})^{-1/2} g_{\Delta+1, \ell}^{-1, -1}(z, \bar{z}). \quad (133)$$

We could expand in non-susy blocks just as they do.

4 Superspace calculation 3d

4.1 Generalities

We translate as

$$\mathcal{O}(z) = e^{x^\mu P_\mu + \theta^{\alpha-} Q_\alpha^+ + \theta^{\dot{\alpha}+} Q_{\dot{\alpha}}^-} \mathcal{O}(0), \quad (134)$$

where the adjoint action is implicit. Thus

$$[P_\mu, \mathcal{O}(z)] = \partial_\mu \mathcal{O}(z), \quad (135)$$

$$[Q_\alpha^+, \mathcal{O}(z)] = \left(\frac{\partial}{\partial \theta^{\alpha-}} - \frac{1}{2} \Sigma_{\alpha\dot{\alpha}}^\mu \theta^{+\dot{\alpha}} \partial_\mu \right) \mathcal{O}(z), \quad (136)$$

$$[Q_{\dot{\alpha}}^-, \mathcal{O}(z)] = \left(\frac{\partial}{\partial \theta^{\dot{\alpha}+}} - \frac{1}{2} \Sigma_{\alpha\dot{\alpha}}^\mu \theta^{-\alpha} \partial_\mu \right) \mathcal{O}(z), \quad (137)$$

$$[D, \phi(x)] = \left(x^\mu \partial_\mu + \theta^{\dot{\alpha}+} \frac{\partial}{\partial \theta^{\dot{\alpha}+}} + \theta^{\alpha-} \frac{\partial}{\partial \theta^{\alpha-}} + \Delta \right) \phi(x), \quad (138)$$

$$\dots \quad (139)$$

The covariant derivatives are

$$D_\alpha^+ = \frac{\partial}{\partial \theta^{\alpha-}} + \frac{1}{2} \Sigma_{\alpha\dot{\alpha}}^\mu \theta^{+\dot{\alpha}} \partial_\mu, \quad (140)$$

$$D_{\dot{\alpha}}^- = \frac{\partial}{\partial \theta^{\dot{\alpha}+}} + \frac{1}{2} \Sigma_{\alpha\dot{\alpha}}^\mu \theta^{-\alpha} \partial_\mu, \quad (141)$$

The chiral and antichiral coordinates

$$y^\mu = x^\mu - \frac{1}{2} \theta^- \Sigma^\mu \theta^+, \quad D_\alpha^+ y^\mu = 0, \quad D_\alpha^+ \theta^+ = 0, \quad (142)$$

$$\tilde{y}^\mu = x^\mu + \frac{1}{2} \theta^- \Sigma^\mu \theta^+, \quad D_\alpha^- y^\mu = 0, \quad D_\alpha^- \theta^- = 0. \quad (143)$$

4.2 Line defect

Imposing conservation of Q_1^\pm and P_3 the invariant objects are

$$z_1^1 = y_1^1 - \theta_1^{2+} \theta_2^{1-}, \quad z_1^2 = y_1^2 + i \theta_1^{2+} \theta_2^{1-}, \quad \theta_1^{2+}, \quad (144)$$

$$\tilde{z}_2^1 = \tilde{y}_2^1 + \theta_1^{1+} \theta_2^{2-}, \quad \tilde{z}_2^2 = \tilde{y}_2^2 + i \theta_1^{1+} \theta_2^{2-}, \quad \theta_2^{2-}, \quad (145)$$

$$z_{12}^3 = y_1^3 - \tilde{y}_2^3 - \theta_1^{1+} \theta_2^{1-}. \quad (146)$$

4.3 Boundary $\mathcal{N} = (1, 1)$ $\langle \phi \hat{O} \hat{O} \rangle$

The parallel theta coordinates are

$$\theta^{1\parallel} \equiv \theta^{-1} + \theta^{+2}, \quad \theta^{2\parallel} \equiv \theta^{-2} + \theta^{+1}. \quad (147)$$

The perpendicular coordinates vanish at the boundary

$$\theta^{-1} - \theta^{+2} = 0, \quad \theta^{-2} - \theta^{+1} = 0. \quad (148)$$

A Useful identities

$$\{A, BC\} = \{A, B\}C - B[A, C], \quad (149)$$

$$\{A, [B, C]\} = -\{B, [A, C]\} + [\{A, B\}, C], \quad (150)$$

$$(151)$$