Defects across dimensions

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Contents

1 Superconformal algebra

We use the conventions from Bobev and friends `1503.02081`, which we now summarize. We work in euclidean signature, so upper and lower indices do not matter. The conformal algebra is:

$$[D,P_\mu]=P_\mu, \tag{1}$$
$$[D,K_\mu]=-K_\mu, \tag{2}$$
$$[K_\mu,P_\nu]=2\left(\delta_{\mu\nu}D-M_{\mu\nu}\right), \tag{3}$$
$$[M_{\mu\nu},M_{\rho\sigma}]=-\delta_{\mu\rho}M_{\nu\sigma}\pm\dots, \tag{4}$$
$$[M_{\mu\nu},P_\rho]=-\delta_{\mu\rho}P_\nu+\delta_{\nu\rho}P_\mu, \tag{5}$$
$$[M_{\mu\nu},K_\rho]=-\delta_{\mu\rho}K_\nu+\delta_{\nu\rho}K_\mu. \tag{6}$$

The supercharges transform as:

$$[D,Q^+_\alpha]=\tfrac{1}{2}Q^+_\alpha, \tag{7}$$
$$[D,Q^-_{\dot\alpha}]=\tfrac{1}{2}Q^-_{\dot\alpha}, \tag{8}$$
$$[R,Q^+_\alpha]=Q^+_\alpha, \tag{9}$$
$$[R,Q^-_{\dot\alpha}]=-Q^-_{\dot\alpha}, \tag{10}$$
$$[K^\mu,Q^+_\alpha]=\Sigma^\mu_{\alpha\dot\alpha}S^{\dot\alpha+}, \tag{11}$$
$$[K^\mu,Q^-_{\dot\alpha}]=\Sigma^\mu_{\alpha\dot\alpha}S^{\alpha-}, \tag{12}$$
$$[M_{\mu\nu},Q^+_\alpha]=(m_{\mu\nu})_\alpha{}^\beta Q^+_\beta, \tag{13}$$
$$[M_{\mu\nu},Q^-_{\dot\alpha}]=(\tilde m_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}Q^-_{\dot\beta}, \tag{14}$$

and

$$
\begin{aligned}
[D, S^{\dot\alpha +}] &= -\tfrac{1}{2} S^{\dot\alpha +}, && (15)\\
[D, S^{\alpha -}] &= -\tfrac{1}{2} S^{\alpha -}, && (16)\\
[R, S^{\dot\alpha +}] &= S^{\dot\alpha +}, && (17)\\
[R, S^{\alpha -}] &= -S^{\alpha -}, && (18)\\
[P_\mu, S^{\dot\alpha +}] &= -\bar\Sigma_\mu^{\dot\alpha\alpha} Q^+_\alpha, && (19)\\
[P_\mu, S^{\alpha -}] &= -\bar\Sigma_\mu^{\dot\alpha\alpha} Q^-_{\dot\alpha}, && (20)\\
[M_{\mu\nu}, S^{\dot\alpha +}] &= -(\tilde m_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} S^{\dot\beta +}, && (21)\\
[M_{\mu\nu}, S^{\alpha -}] &= -(m_{\mu\nu})_\beta{}^\alpha S^{\beta -}. && (22)
\end{aligned}
$$

The anticommutators are:

$$
\begin{aligned}
\{Q^+_\alpha, Q^-_{\dot\alpha}\} &= \Sigma^\mu_{\alpha\dot\alpha} P_\mu, && (23)\\
\{S^{\dot\alpha +}, S^{\alpha -}\} &= \bar\Sigma_\mu^{\dot\alpha\alpha} K_\mu, && (24)\\
\{S^{\dot\alpha +}, Q^-_{\dot\beta}\} &= \delta^{\dot\alpha}{}_{\dot\beta}\left(D + \frac{d-1}{2} R\right) - (\tilde m_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} M_{\mu\nu} + \ldots, && (25)\\
\{S^{\alpha -}, Q^+_\beta\} &= \delta^\alpha{}_\beta\left(D - \frac{d-1}{2} R\right) - (m_{\mu\nu})_\beta{}^\alpha M_{\mu\nu} + \ldots. && (26)
\end{aligned}
$$

The gamma matrices are defined from

$$
\begin{aligned}
\Sigma_i \bar\Sigma_j + \Sigma_j \bar\Sigma_i &= 2\delta_{ij}, && (27)\\
\bar\Sigma_i \Sigma_j + \bar\Sigma_j \Sigma_i &= 2\delta_{ij}, && (28)\\
\bar\Sigma_\mu^{\dot\alpha\alpha} &= \left(\Sigma^\mu_{\alpha\dot\alpha}\right)^*, && (29)
\end{aligned}
$$

and then

$$
\begin{aligned}
m_{\mu\nu} &= \frac{1}{4}\left(\Sigma_\nu \bar\Sigma_\mu - \Sigma_\mu \bar\Sigma_\nu\right), && (30)\\
\tilde m_{\mu\nu} &= \frac{1}{4}\left(\bar\Sigma_\mu \Sigma_\nu - \bar\Sigma_\nu \Sigma_\mu\right). && (31)
\end{aligned}
$$

The Casimir is

$$
C_2 = D^2 - \frac{1}{2}\{P_\mu, K^\mu\} - \frac{1}{2} M_{\mu\nu} M^{\mu\nu} - \frac{1}{2} R^2 + \frac{1}{2}[S^{\dot\alpha +}, Q^-_{\dot\alpha}] + \frac{1}{2}[S^{\alpha -}, Q^+_\alpha]. \tag{32}
$$

Acting with it on an operator with quantum numbers Δ, ℓ, r gives the eigenvalue

$$
\lambda_C = \Delta(\Delta - d + 2) + \ell(\ell + d - 2) - \frac{r^2}{2}. \tag{33}
$$

When we look at the particular case $d = 3$ we take indices $\mu, \nu = 1, 2, 3$ and build the gamma matrices from the Pauli matrices

$$
\Sigma^1 = \sigma^1, \quad \Sigma^2 = \sigma^2, \quad \Sigma^3 = \sigma^3, \tag{34}
$$

In this language raising/lowering indices is equivalent to adding/removing a dot

$$\psi_\alpha = \psi^{\dot\alpha}, \qquad \Sigma^\mu_{\alpha\dot\beta} = \bar\Sigma^{\dot\alpha\beta}_\mu, \qquad \dots \tag{35}$$

When we look at the particular case $d = 4$ we take indices $\mu, \nu = 1, 2, 3, 4$ and build the remaining matrix as

$$\Sigma^4 = i\mathbb{1}_2. \tag{36}$$

1.1 Action of generators on fields

We will consider two-point functions $\langle\phi_1(x_1)\phi_2(x_2)\rangle$, where ϕ_1 is chiral and ϕ_2 is antichiral

$$[Q^+_\alpha, \phi_1(0)] = 0, \tag{37}$$
$$[Q^-_{\dot\alpha}, \phi_2(0)] = 0. \tag{38}$$

Consistency with the superconformal algebra fixes the R-charge in terms of the dimension:

$$[D, \phi_1(0)] = \Delta_\phi, \tag{39}$$
$$[D, \phi_2(0)] = \Delta_\phi, \tag{40}$$
$$[R, \phi_1(0)] = \frac{2}{d-1}\Delta_\phi, \tag{41}$$
$$[R, \phi_2(0)] = \frac{2}{d-1}\Delta_\phi. \tag{42}$$

The operators are translated as

$$\phi(x) = e^{x\cdot P}\phi(0)e^{-x\cdot P}, \tag{43}$$

which implies the usual

$$[P_\mu, \phi(x)] = \partial_\mu\phi(x), \tag{44}$$
$$[D, \phi(x)] = (x^\mu\partial_\mu + \Delta)\phi(x), \tag{45}$$
$$[M_{\mu\nu}, \phi(x)] = (x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x), \tag{46}$$
$$\dots \tag{47}$$

The shortening condition implies

$$[Q^+_\alpha, \phi_1(x_1)] = 0, \tag{48}$$
$$[S^{\dot\alpha+}, \phi_1(x_1)] = 0, \tag{49}$$
$$[Q^-_{\dot\alpha}, \phi_2(x_2)] = 0, \tag{50}$$
$$[S^{\alpha-}, \phi_2(x_2)] = 0. \tag{51}$$

For the remaining S generators we get

$$[S^{\alpha-}, \phi_1(x_1)] = x_1^\mu\bar\Sigma^{\dot\alpha\alpha}_\mu[Q^-_{\dot\alpha}, \phi_1(x_1)], \tag{52}$$
$$[S^{\dot\alpha+}, \phi_2(x_2)] = x_2^\mu\bar\Sigma^{\dot\alpha\alpha}_\mu[Q^+_\alpha, \phi_2(x_2)]. \tag{53}$$

2 Half-BPS boundaries

2.1 Non-supersymmetric

We can obtain a subalgebra restricting to

$$D, \quad R, \quad M_{ab}, \quad P_a, \quad K_a, \tag{54}$$

where $a, b = 1, \ldots, d-1$ are parallel to the boundary, which sits at $x^d = 0$. The defect Casimir is obtained from the full one restricting to the defect operators

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a, K^a\} - \frac{1}{2}M_{ab}M^{ab} \tag{55}$$

The Ward identities fix the two-point function

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{f(\xi)}{(2x_1^d)^{\Delta_1}(2x_2^d)^{\Delta_2}}, \qquad \xi = \frac{x_{12}^2}{4x_1^d x_2^d}. \tag{56}$$

2.1.1 Defect channel

Acting with the bosonic part of the defect Casimir and using the non-supersymmetric Casimir $\lambda = \Delta(\Delta - d + 1)$ we get in general dimension

$$\frac{(C_{\text{def,bos}} - \lambda)\langle \phi_1(x_1)\phi_2(x_2)\rangle}{\langle \phi_1(x_1)\phi_2(x_2)\rangle} = \xi(\xi+1)f''(\xi) + \frac{d}{2}(2\xi+1)f'(\xi) - \Delta(\Delta - d + 1)f(\xi) = 0, \tag{57}$$

so the defect block that solves this is

$$f_\Delta(\xi) = \xi^{-\Delta}\, {}_2F_1\left(\Delta, \Delta - \frac{d}{2} + 1; 2\Delta - d + 2; -\frac{1}{\xi}\right). \tag{58}$$

2.1.2 Bulk channel

We should act with the full Casimir. The bosonic differential equation is

$$\begin{aligned}\frac{(C_{\text{bulk,bos}} - \lambda)\langle \phi_1(x_1)\phi_2(x_2)\rangle}{\langle \phi_1(x_1)\phi_2(x_2)\rangle} &= 4\xi^2(\xi+1)f''(\xi) + \big(4(2\Delta_\phi + 1)\xi(\xi+1) - 2d\xi\big)f'(\xi) \\ &\quad + \big(-\Delta(\Delta - d) - 2d\Delta_\phi + 4\Delta_\phi^2(\xi+1)\big)f(\xi) = 0,\end{aligned} \tag{59}$$

where the bosonic eigenvalue is $\lambda = \Delta(\Delta - d)$. The block is usually written in terms of $G(\xi) = \xi^{\Delta_\phi} f(\xi)$ because the dependance on the external dimensions drops:

$$G_\Delta(\xi) = \xi^{\Delta/2}\, {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta - \frac{d}{2} + 1; -\xi\right). \tag{60}$$

2.2 $\mathcal{N} = (2,0)$ subalgebra for $d = 3$

We can obtain a subalgebra restricting to

$$D, \quad R, \quad M_{12}, \quad P_1, P_2, \quad K_1, K_2, \quad Q_1^+, Q_2^-, \quad S^{1-}, S^{2+}. \tag{61}$$

This can be mapped to the $(2,0)$ algebra in $2d$. The defect Casimir is obtained from the full one restricting to the defect operators

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a, K^a\} - M_{12}^2 - \frac{1}{2}R^2 + \frac{1}{2}[S^{2+}, Q_2^-] + \frac{1}{2}[S^{1-}, Q_1^+]. \tag{62}$$

Acting with it on an operator with quantum numbers Δ, ℓ, r gives the eigenvalue:

$$\lambda_C = \Delta(\Delta - 1) + \ell(\ell - 1) - \frac{r^2}{2}. \tag{63}$$

Consider a chiral field in $3d$

$$\Phi_{3d}(y, \theta^+) = \phi(y) + \theta^{1+}\psi_1(y) + \theta^{2+}\psi_2(y) + \theta^{1+}\theta^{2+}\rho(y) \tag{64}$$

$$= \Phi(y, \theta^{2+}) + \theta^{1+}\Psi(y, \theta^{2+}). \tag{65}$$

Where we defined

$$\Phi(y, \theta^{2+}) = \phi(y) + \theta^{2+}\psi_2(y), \tag{66}$$

$$\Psi(y, \theta^{2+}) = \psi_1(y) + \theta^{2+}\rho(y). \tag{67}$$

2.2.1 Defect channel

We should act with the defect Casimir at position x_1. The new contribution from susy is

$$C_{\text{ferm}} = -\frac{1}{2}R^2 + \frac{1}{2}[S^{2+}, Q_2^-] + \frac{1}{2}[S^{1-}, Q_1^+] = -Q_2^- S^{2+} + S^{1-}Q_1^+ - \frac{1}{2}R^2 + R. \tag{68}$$

Acting on one point and using the chirality of ϕ_1 we get

$$[C_{\text{ferm}}, \phi_1(x_1)]|0\rangle = \left(-\frac{1}{2}\Delta_\phi^2 + \Delta_\phi\right)\phi_1(x_1)|0\rangle. \tag{69}$$

This suggests that to obtain the supersymmetric block we should just add a term $(-\frac{1}{2}\Delta_\phi^2 + \Delta_\phi)f(\xi)$ in the Casimir equation.

Fermi multiplet exchange Consider a superprimary $\hat{\mathcal{O}}_p$ of dimension Δ, spin $\ell = 1/2$ and charge $r = r_\phi - 1$, which has Casimir

$$\lambda = \Delta(\Delta - 1) + \ell(\ell - 1) - \frac{r^2}{2} = \Delta(\Delta - 1) - \frac{1}{2}\Delta_\phi^2 + \Delta_\phi - \frac{3}{4}. \tag{70}$$

Then only the descendant $Q_1^+\hat{\mathcal{O}}_p$ has the right quantum numbers to appear in the defect OPE $\phi \sim Q_1^+\hat{\mathcal{O}}_p$, so the superconformal block should reduce to a single non-supersymmetric block. Indeed, in the differential equation all Δ_ϕ terms drop and we get

$$\xi(\xi+1)f''(\xi) + \frac{3}{2}(2\xi+1)f'(\xi) - \left(\Delta + \frac{1}{2}\right)\left(\Delta + \frac{1}{2} - 2\right)f(\xi) = 0, \tag{71}$$

which is solved by a shifted non-supersymmetric block $f_{\Delta+\frac{1}{2}}(\xi)$ as expected. In terms of superspace, we should find $\langle\phi\hat{\Psi}\rangle$ with arbitrary $\hat{\Delta}$.

Chiral multiplet exchange Consider a superprimary $\hat{\Phi}$ with $\hat{\Delta} = \hat{r} = \Delta_\phi$ then the Casimir equation is

$$\xi(\xi+1)f''(\xi) + \frac{3}{2}(2\xi+1)f'(\xi) - \Delta_\phi\left(\Delta_\phi - 2\right)f(\xi) = 0. \tag{72}$$

Now the primary is exchanged, i.e. the solution of the Casimir equation is $f_{\Delta_\phi}(\xi)$. This agrees with the superspace analysis, where $\langle \phi\hat{\Phi}\rangle$ only exists for $\hat{\Delta} = \Delta_\phi$.

2.2.2 Bulk channel

We should act with the full Casimir. The only new contributions from susy are

$$C_{\text{ferm}} = \frac{1}{2}[S^{\dot{\alpha}+}, Q^-_{\dot{\alpha}}] + \frac{1}{2}[S^{\alpha-}, Q^+_\alpha] = -Q^-_{\dot{\alpha}}S^{\dot{\alpha}+} + S^{\dot{\alpha}-}Q^+_\alpha + 2R. \tag{73}$$

There is a contribution $-\frac{1}{2}R^2$ that cancels between the two sides of the Casimir equation. Acting on the two points we get

$$\begin{aligned}
[C_{\text{ferm}}, &\phi_1(x_1)\phi_2(x_2)]|0\rangle \\
&= \big([S^{\alpha-},\phi_1(x_1)][Q^+_\alpha,\phi_2(x_2)] - [Q^-_{\dot{\alpha}},\phi_1(x_1)][S^{\dot{\alpha}+},\phi_2(x_2)] + 4\Delta_\phi\,\phi_1(x_1)\phi_2(x_2)\big)\,|0\rangle \\
&= \big(x^\mu_{12}\bar{\Sigma}^{\dot{\alpha}\alpha}_\mu[Q^-_{\dot{\alpha}},\phi_1(x_1)][Q^+_\alpha,\phi_2(x_2)] + 4\Delta_\phi\,\phi_1(x_1)\phi_2(x_2)\big)\,|0\rangle
\end{aligned} \tag{74}$$

We use Ward identities to figure them out. We are only allowed to use Q^+_1, $Q^-_{\dot{2}}$, S^{1-}, $S^{\dot{2}+}$ to get Ward identities, because they are the only supercharges preserved by our defect. The first Ward identity

$$\langle\{Q^+_1, [Q^-_{\dot{\alpha}},\phi_1(x_1)]\phi_2(x_2)\}\rangle = 0 \tag{75}$$

implies

$$\begin{aligned}
\langle[Q^-_{\dot{\alpha}},\phi_1(x_1)][Q^+_1,\phi_2(x_2)]\rangle &= \langle[\{Q^+_1, Q^-_{\dot{\alpha}}\},\phi_1(x_1)]\phi_2(x_2)\rangle & (76)\\
&= \Sigma^\mu_{1\dot{\alpha}}\partial^{x_1}_\mu\langle\phi_1(x_1)\phi_2(x_2)\rangle. & (77)
\end{aligned}$$

We can play the same game with the other $Q^\pm$ and $S^\pm$ preserved by the defect. All in all, when we act with the fermionic part of the Casimir on the two-point function we get

$$\frac{C_{\text{bulk,ferm}}\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = 4\xi(\xi+1)f'(\xi) + 4\Delta_\phi(\xi+1)f(\xi). \tag{78}$$

It is fairly non-trivial to get a differential operator that acting on the two-point function can be rewritten in terms of ξ. This is a sanity check for our calculation. The supersymmetric Casimir eigenvalue is $\lambda = \Delta(\Delta-1) = \lambda_{bos} + 2\Delta$, so we should also add a term $-2\Delta f(\xi)$ to the LHS. Adding all the contributions we get

$$\begin{aligned}
4\xi^2(\xi+1)f''(\xi) &+ 2\xi(4\Delta_\phi(\xi+1) + 4\xi + 1)f'(\xi) \\
&+ \left(2\Delta_\phi(2\Delta_\phi(\xi+1) + 2\xi - 1) - \Delta(\Delta-1)\right)f(\xi) = 0.
\end{aligned} \tag{79}$$

If we define $G(\xi) = \xi^{\Delta_\phi} f(\xi)$ the dependence on Δ_ϕ drops

$$4\xi^2(\xi+1)G''(\xi) + 2\xi(4\xi+1)G'(\xi) - \Delta(\Delta-1)G(\xi) = 0, \tag{80}$$

and we find the solution

$$G(\xi) = \xi^{\Delta/2}\,{}_2F_1\left(\frac{\Delta}{2}+1, \frac{\Delta}{2}; \Delta+\frac{1}{2}; -\xi\right), \tag{81}$$

which is exactly what Philine already found!

2.3 $\mathcal{N} = (1,1)$ subalgebra for $d = 3$

We can obtain a subalgebra restricting to

$$D, \quad M_{12}, \quad P_1, P_2, \quad K_1, K_2, \quad Q_1^+ + Q_2^-, Q_2^+ + Q_1^-, \quad S^{1+} + S^{2-}, S^{2+} + S^{1-}. \tag{82}$$

This can be mapped to the $(1,1)$ algebra in $2d$. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a, K^a\} - M_{12}^2 + \frac{1}{4}[S^{2+} + S^{1-}, Q_1^+ + Q_2^-] + \frac{1}{4}[S^{1+} + S^{2-}, Q_2^+ + Q_1^-]. \tag{83}$$

Acting with it on an operator with quantum numbers Δ, ℓ gives the eigenvalue:

$$\lambda_C = \Delta(\Delta-1) + \ell^2. \tag{84}$$

2.3.1 Defect channel

The fermionic contribution to the Casimir equation is

$$\frac{1}{4}[S^{2+} + S^{1-}, Q_1^+ + Q_2^-] + \frac{1}{4}[S^{1+} + S^{2-}, Q_2^+ + Q_1^-] \to R - \frac{1}{2}\left(Q_2^- S^{1-} + Q_1^- S^{2-}\right) + \ldots, \tag{85}$$

where we dropped all terms that vanish when acting on $\phi_1(x_1)$:

$$[C_{\text{def,ferm}}, \phi_1(x_1)]|0\rangle = \left(\Delta_\phi \phi_1(x_1) + x_1^3\{Q_1^-, [Q_2^-, \phi_1(x_1)]\}\right)|0\rangle. \tag{86}$$

Using Ward identities we express $Q_1^- Q_2^-$ as a differential operator acting at x_1, and we get

$$\frac{C_{\text{bulk,ferm}}\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = -\xi f'(\xi). \tag{87}$$

We also need a contribution $-\Delta f(\xi)$ from the difference in the Casimir eigenvalue, so

$$\xi(\xi+1)f''(\xi) + \left(2\xi + \frac{3}{2}\right) f'(\xi) - \Delta(\Delta-1)f(\xi) = 0. \tag{88}$$

This is solved by the same block Philine found:

$$f(\xi) = \xi^{-\Delta}\,{}_2F_1\left(\Delta, \frac{1}{2}(2\Delta-1); 2\Delta; -\frac{1}{\xi}\right). \tag{89}$$

2.3.2 Bulk channel

The bosonic part and supersymmetric parts are the same as before, the only difference are the Ward identites we can use to rewrite $\langle Q\phi Q\phi\rangle$ in terms of derivatives of $\langle\phi\phi\rangle$. As before, we can only use the preserved supercharges, so the Ward identities are

$$\begin{aligned}\langle\{Q_1^+ + Q_2^-, [Q_2^-, \phi_1]\phi_2\}\rangle &= 0,\\ \langle\{Q_2^+ + Q_1^-, [Q_1^-, \phi_1]\phi_2\}\rangle &= 0,\\ &\dots\end{aligned} \tag{90}$$

All in all we get:

$$\frac{C_{\text{bulk,ferm}}\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = 4\xi f'(\xi) + 4\Delta_\phi f(\xi). \tag{91}$$

Following the same arguments as before, we find the blocks

$$G(\xi) = \xi^{\Delta/2}\,{}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta + \frac{1}{2}; -\xi\right), \tag{92}$$

consistent with Philine's results once more!

2.4 $OSp(1|4)$ subalgebra for $d = 4$

We can obtain a subalgebra restricting to

$$D, \quad M_{12}, M_{13}, M_{23}, \quad P_1, P_2, P_3, \quad K_1, K_2, K_3, \quad Q_1^+ + Q_2^-, Q_2^+ - Q_1^-, \quad S^{1+} - S^{2-}, S^{2+} + S^{1-}. \tag{93}$$

This can be mapped to the $OSp(1|4)$ algebra in $3d$. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_a, K^a\} - \frac{1}{2}M_{ab}M^{ab} + \frac{1}{4}[S^{2+} + S^{1-}, Q_1^+ + Q_2^-] - \frac{1}{4}[S^{1+} - S^{2-}, Q_2^+ - Q_1^-]. \tag{94}$$

We are using $a, b = 1, 2, 3$ and the boundary sits at $x^4 = 0$. Acting with it on an operator with quantum numbers Δ, ℓ gives the eigenvalue:

$$\lambda_C = \Delta(\Delta - 2) + \ell(\ell + 1). \tag{95}$$

2.4.1 Defect channel

The relevant part of the Casimir is

$$C_{\text{def,ferm}} = -\frac{1}{2}Q_2^- S^{1-} + \frac{1}{2}Q_1^- S^{2-} + \frac{3}{2}R + \dots, \tag{96}$$

which acting at point 1 gives

$$[C_{\text{def,ferm}}, \phi_1(x_1)]|0\rangle = \left[x_1^3\{Q_1^-, [Q_2^-, \phi_1(x_1)]\} + \Delta_\phi\phi_1(x_1)\right]|0\rangle. \tag{97}$$

Using Ward identities we find

$$\frac{C_{\text{def,ferm}}\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = -\xi f'(\xi). \tag{98}$$

This is the same as the $\mathcal{N} = (1, 1)$ boundary in $3d$.

2.4.2 Bulk channel

It works as before, but the Ward identities are different. The result is

$$\frac{C_{\text{bulk,ferm}}\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = 4\xi f'(\xi) + 4\Delta_\phi f(\xi). \tag{99}$$

This is once again the same as for the $\mathcal{N}=(1,1)$ boundary in $3d$.

2.5 Boundary across dimensions

We conjecture that the fermionic part of the Casimir gives the same contribution regardless of the dimension $2 \leq d \leq 4$. (We should still check the boundary in $2d$, but it probably also works there) Combining the bosonic and supersymmetric parts we get blocks "across dimensions".

2.5.1 Defect channel

The solution is

$$\begin{aligned} f_\Delta^{\text{SUSY}}(\xi) &= \xi^{-\Delta}\,{}_2F_1\left(\Delta, \frac{1}{2}(2\Delta+2-d); 2\Delta+3-d; -\frac{1}{\xi}\right) \\ &= f_\Delta(\xi) + \frac{\Delta}{4\Delta-2d+6} f_{\Delta+1}(\xi). \end{aligned} \tag{100}$$

2.5.2 Bulk channel

The solution is

$$\begin{aligned} G_\Delta^{\text{SUSY}}(\xi) &= \xi^{\Delta/2}\,{}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta+2-\frac{d}{2}; -\xi\right) \\ &= G_\Delta(\xi) + \frac{\Delta^2}{(2\Delta-d+2)(2\Delta-d+4)} G_{\Delta+2}(\xi). \end{aligned} \tag{101}$$

3 Codimension two objects

3.1 Non-supersymmetric

We can obtain a subalgebra restricting to

$$D, \quad R, \quad P_a, \quad K_a, \quad M_{ab}, \quad M_{ij}. \tag{102}$$

where $a, b = 1, \ldots, p$ live on the defect and $i, j = p+1, \ldots, d$ are orthogonal. Also p is the dimension of the defect and $q = d - p$ is the codimension. The defect Casimir is obtained from the full one restricting to the defect operators. It factorizes into two commuting pieces

$$C_{\text{def},1} = D^2 - \frac{1}{2}\{P_a, K^a\} - \frac{1}{2}M_{ab}M^{ab}, \qquad C_{\text{def},2} = -\frac{1}{2}M_{ij}M^{ij}. \tag{103}$$

The Ward identities fix the two-point function but now there are two cross-ratios

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{f(\xi,\eta)}{|x_1^\perp|^{\Delta_1}|x_2^\perp|^{\Delta_2}}, \qquad \xi = \frac{x_{12}^2}{|x_1^\perp||x_2^\perp|}, \qquad \eta = \frac{x_1^\perp \cdot x_2^\perp}{|x_1^\perp||x_2^\perp|}. \tag{104}$$

For the defect channel we will also use

$$\chi = \frac{x_{12}^2 + 2x_1^\perp \cdot x_2^\perp}{|x_1^\perp||x_2^\perp|}. \tag{105}$$

In the defect channel, using the normalization of Billo and friends, we get

$$f(\chi,\eta) \to \chi^{-\Delta}\, 2^{-s} \binom{s+\frac{q}{2}-2}{\frac{q}{2}-2}^{-1} C_s^{q/2-1}(\eta) \qquad \text{as} \qquad \chi\to\infty. \tag{106}$$

3.1.1 Defect channel

The action of the conformal group gives, with $\lambda = \Delta(\Delta-p)$:

$$\frac{(C_{\text{def},1}-\lambda)\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = (\chi^2-4)f''(\chi) + (p+1)\chi f'(\chi) - \Delta(\Delta-p)f(\chi) = 0. \tag{107}$$

The action of transverse rotations gives, with $\lambda = s(s+q-2)$:

$$\frac{(C_{\text{def},2}-\lambda)\langle\phi_1(x_1)\phi_2(x_2)\rangle}{\langle\phi_1(x_1)\phi_2(x_2)\rangle} = (\eta^2-1)f''(\eta) + (q-1)\eta f'(\eta) - s(s+q-2)f(\eta) = 0. \tag{108}$$

In total the block is

$$\hat{f}_{\Delta,s}(\chi,\eta) = \alpha_{s,q}\chi^{-\Delta}\,{}_2F_1\left(\frac{\Delta}{2},\frac{\Delta+1}{2};\Delta+1-\frac{p}{2};\frac{4}{\chi^2}\right)\,{}_2F_1\left(-\frac{s}{2},\frac{q+s-2}{2};\frac{q-1}{2};1-\eta^2\right), \tag{109}$$

with normalization

$$\alpha_{s,q} = 2^{-s}\frac{\Gamma(q+s-2)\Gamma(q/2-1)}{\Gamma(q/2+s-1)\Gamma(q-2)}. \tag{110}$$

3.1.2 Bulk channel

If we take

$$\langle\phi\phi\rangle = \frac{\xi^{-(\Delta_1+\Delta_2)/2}f(\xi,\eta)}{|x_1^i|^{\Delta_1}|x_2^i|^{\Delta_2}} \tag{111}$$

the Casimir equation is

$$\begin{aligned}
&\xi^2\left(2\eta^2+\eta\xi+2\right)f^{(2,0)}(\xi,\eta) + \left(1-\eta^2\right)\left(2\left(1-\eta^2\right)-\eta\xi\right)f^{(0,2)}(\xi,\eta)\\
&-2\xi\left(1-\eta^2\right)(2\eta+\xi)f^{(1,1)}(\xi,\eta) + \xi\left(2\left(\eta^2+1\right)-(2d-\eta\xi)\right)f^{(1,0)}(\xi,\eta)\\
&+\left(\xi\left(\eta^2+q-2\right)-2\eta\left(1-\eta^2\right)\right)f^{(0,1)}(\xi,\eta) - \left(\lambda_C + \frac{1}{4}\Delta_{12}^2\left(2\eta^2+\xi\eta-2\right)\right)f(\xi,\eta) = 0
\end{aligned} \tag{112}$$

3.2 Non-supersymmetric with $U(1)$ mixing

Let's imagine we have an extra non-compact $U(1)_R$ symmetry with an Hermitian generator R. Then for codimension-two defect, the transverse rotations $SO(2) \simeq U(1)$ can mix with the $U(1)_R$ symmetry

$$M_{12} + iR \tag{113}$$

(We can always normalize the R generator such that the above equation holds). The second Casimir is modified to

$$C_{\text{def},2} = -(M_{12} + iR)^2. \tag{114}$$

In practice, any correlation function

$$\langle \phi(x_1) \hat{\mathcal{O}}^{i\ldots j}(x_2) \rangle \tag{115}$$

that was invariant before will be invariant with the extra $U(1)$ provided $r_\phi = -r_{\hat{\mathcal{O}}}$. Thus the allowed values of this "twisted" transverse spin are

$$s = -r_\phi, -r_\phi + 1, \ldots \tag{116}$$

When M_{12} acts on a two point function it gives

$$\frac{M_{12}\langle \phi_1(x_1)\phi_2(x_2)\rangle}{\langle \phi_1(x_1)\phi_2(x_2)\rangle} = \frac{x_1^2 x_2^1 - x_1^1 x_2^2}{|x_1^\perp||x_2^\perp|} f'(\eta) = \sqrt{1-\eta^2} f'(\eta) \tag{117}$$

When identifying the invariant there is a sign ambiguity, which we fix by assuming

$$x_1^2 x_2^1 - x_1^1 x_2^2 > 0. \tag{118}$$

What is the physical picture? What happens if we take the oposite sign?

3.2.1 Defect channel

We assume from the previous discussion that the exchanged operator has $s = r_\phi + \ell$. The first Casimir equation is the same as before, and the second gives

$$\frac{(C_{\text{def},2} - \lambda)\langle \phi_1(x_1)\phi_2(x_2)\rangle}{\langle \phi_1(x_1)\phi_2(x_2)\rangle} = (\eta^2 - 1) f''(\eta) + (\eta + 2r\sqrt{\eta^2 - 1}) f'(\eta) - \ell(\ell + 2r_\phi) f(\eta) = 0. \tag{119}$$

Here we assume $\eta \geq 1$, or in another set of coordinates

$$\eta = \frac{1}{2}\left(w + \frac{1}{w}\right), \qquad 0 < w < 1. \tag{120}$$

In these coordinates the block reads

$$f(w) = w^{-\ell}\left(c_1 w^{2r+2\ell} + c_2\right), \tag{121}$$

where the constants c_i should be fixed by consistency with the OPE.

3.3 Line defect in $3d$

We can obtain a subalgebra restricting to

$$D, \quad P_3, \quad K_3, \quad M_{12}+iR, \quad Q_1^+, Q_1^- \quad S^{1+}, S^{1-}. \tag{122}$$

This can be mapped to the $\mathcal{N}=2$ algebra in $1d$ (the left-moving part of the $\mathcal{N}=(2,0)$ algebra in $2d$). It is natural that M_{12} appears with an extra factor of i compared to R, since in our conventions M_{12} is anti-hermitian while R is hermitian. The defect Casimir now changes a bit

$$C_{\text{def}} = D^2 - \frac{1}{2}\{P_3, K^3\} + (M_{12}+iR)^2 + \frac{1}{2}[S^{1+}, Q_1^-] + \frac{1}{2}[S^{1-}, Q_1^+]. \tag{123}$$

Acting with it on an operator with quantum numbers Δ, s, where $(M_{12}+iR)\mathcal{O} = is\mathcal{O}$:

$$\lambda_C = \Delta^2 - s^2. \tag{124}$$

3.3.1 Defect channel

The new contribution from susy is

$$\begin{aligned} C_{\text{def, ferm}} &= (M_{12}+iR)^2 + \frac{1}{2}[S^{1+}, Q_1^-] + \frac{1}{2}[S^{1-}, Q_1^+] \\ &= (M_{12}+iR)^2 - i(M_{12}+iR) - Q_1^- S^{1+} + S^{1-} Q_1^+. \end{aligned} \tag{125}$$

Acting at point 1 only the contribution from M and R survives and S and Q drop:

$$[C_{\text{ferm}}, \phi_1(x_1)]|0\rangle = \left(M_{12}^2 + i(2\Delta_\phi - 1)M_{12} - \Delta_\phi(\Delta_\phi - 1)\right)\right) \phi_1(x_1)|0\rangle. \tag{126}$$

The action of the generators is (NOTE: it is ambigous how to define $\sqrt{1-\eta^2}$, depending of x_1, x_2 we should put take a minus or plus sign):

$$iM_{12} \to \sqrt{\eta^2 - 1}, \tag{127}$$

$$(M_{12})^2 \to -C_{\text{def},2} \tag{128}$$

The Casimir equations to be solved are

$$\begin{aligned} &(\chi^2-4) f^{(2,0)}(\chi,\eta) + 2\chi f^{(1,0)}(\chi,\eta) \\ &-(\eta^2-1) f^{(0,2)}(\chi,\eta) - \eta f^{(0,1)}(\chi,\eta) - (2\Delta_\phi - 1)\sqrt{\eta^2-1} f^{(0,1)}(\chi,\eta) \\ &-\left(\Delta_\phi(\Delta_\phi - 1) + \mathcal{C}_2\right) f(\chi,\eta) = 0 \end{aligned} \tag{129}$$

The solution is

$$f(\chi,\eta) = f_{\Delta+1/2}(\chi) f_{\Delta_\phi - 1/2, \ell}(\eta), \qquad \lambda_C = \Delta^2 - \left(\Delta_\phi - \frac{1}{2} + \ell\right)^2 \tag{130}$$

Here something is weird, it is hard to understand this solution as a level one descendant... TODO: Double check signs and factors of 1/2 and see if it can make more sense.

3.3.2 Bulk channel

Up to some signs which are unclear to fix, the contribution from susy is

$$\frac{1}{\sqrt{\eta^2-1}}\Big[\left(\eta^2-1\right)\left(\sqrt{\eta^2-1}\xi-2\eta^2+2\sqrt{\eta^2-1}\eta-\eta\xi+2\right)f^{(0,1)}(\xi,\eta) \tag{131}$$

$$+\,\xi\left(-2\eta^3+\eta^2\left(2\sqrt{\eta^2-1}-\xi\right)+\eta\left(\sqrt{\eta^2-1}\xi+2\right)+2\sqrt{\eta^2-1}+\xi\right)f^{(1,0)}(\xi,\eta)\Big]=0. \tag{132}$$

This looks a bit dauting. However, a smart change of coordinates

$$u=z\bar{z}=-\xi e^{-i\phi},\qquad v=(1-z)(1-\bar{z})=e^{-2i\phi}, \tag{133}$$

the contribution simplifies dramatically

$$2\Big(z(1-z)\partial+\bar{z}(1-\bar{z})\bar{\partial}\Big)f(z,\bar{z}) \tag{134}$$

3.4 Surface defect in $4d$

We can obtain a subalgebra restricting to

$$D,\quad P_3,P_4\quad K_3,K_4\quad M_{34},\quad M_{12}+\frac{3}{2}iR,\quad Q_1^+,Q_1^-\quad S^{1+},S^{1-}. \tag{135}$$

This can be mapped to the $\mathcal{N}=(2,0)$ algebra in $2d$. The defect Casimir now changes a bit

$$C_{\text{def}}=D^2-\frac{1}{2}\{P_a,K^a\}-M_{34}^2+\frac{1}{2}\left(M_{12}+\frac{3}{2}iR\right)^2+\frac{1}{2}[S^{1+},Q_1^-]+\frac{1}{2}[S^{1-},Q_1^+]. \tag{136}$$

Acting with it on an operator with quantum numbers Δ,s, where $(M_{12}+iR)\mathcal{O}=is\mathcal{O}$:

$$\lambda_C=\Delta(\Delta-1)+\ell(\ell+1)-\frac{1}{2}s^2. \tag{137}$$

3.4.1 Defect channel

The solution is exactly the same one as before.

3.4.2 Bulk channel

Doing the usual calculation gives the same result as before in the right coordinates:

$$2\Big(z(1-z)\partial+\bar{z}(1-\bar{z})\bar{\partial}\Big)f(z,\bar{z}) \tag{138}$$

3.5 Across dimensions

3.5.1 Bulk channel

We get the same equation that `1503.02081` across dimensions so the block is

$$G_{\Delta,\ell}(z,\bar{z})=(z\bar{z})^{-1/2}g^{-1,-1}_{\Delta+1,\ell}(z,\bar{z}). \tag{139}$$

We could expand in non-susy blocks just as they do.

4 Superspace calculation $3d$

4.1 Generalities

We translate as

$$\mathcal{O}(z) = e^{x^\mu P_\mu + \theta^{\alpha-} Q^+_\alpha + \theta^{\dot\alpha+} Q^-_{\dot\alpha}} \mathcal{O}(0), \tag{140}$$

where the adjoint action is implicit. Thus

$$[P_\mu, \mathcal{O}(z)] = \partial_\mu \mathcal{O}(z), \tag{141}$$

$$[Q^+_\alpha, \mathcal{O}(z)] = \left(\frac{\partial}{\partial\theta^{\alpha-}} - \frac{1}{2}\Sigma^\mu_{\alpha\dot\alpha}\theta^{+\dot\alpha}\partial_\mu\right)\mathcal{O}(z), \tag{142}$$

$$[Q^-_{\dot\alpha}, \mathcal{O}(z)] = \left(\frac{\partial}{\partial\theta^{\dot\alpha+}} - \frac{1}{2}\Sigma^\mu_{\alpha\dot\alpha}\theta^{-\alpha}\partial_\mu\right)\mathcal{O}(z), \tag{143}$$

$$[D, \phi(x)] = \left(x^\mu\partial_\mu + \theta^{\dot\alpha+}\frac{\partial}{\partial\theta^{\dot\alpha+}} + \theta^{\alpha-}\frac{\partial}{\partial\theta^{\alpha-}} + \Delta\right)\phi(x), \tag{144}$$

$$\dots \tag{145}$$

The covariant derivatives are

$$D^+_\alpha = \frac{\partial}{\partial\theta^{\alpha-}} + \frac{1}{2}\Sigma^\mu_{\alpha\dot\alpha}\theta^{+\dot\alpha}\partial_\mu, \tag{146}$$

$$D^-_{\dot\alpha} = \frac{\partial}{\partial\theta^{\dot\alpha+}} + \frac{1}{2}\Sigma^\mu_{\alpha\dot\alpha}\theta^{-\alpha}\partial_\mu, \tag{147}$$

The chiral and antichiral coordinates

$$y^\mu = x^\mu - \frac{1}{2}\theta^-\Sigma^\mu\theta^+, \qquad D^+_\alpha y^\mu = 0, \quad D^+_\alpha\theta^+ = 0, \tag{148}$$

$$\tilde{y}^\mu = x^\mu + \frac{1}{2}\theta^-\Sigma^\mu\theta^+, \qquad D^-_\alpha y^\mu = 0, \quad D^-_\alpha\theta^- = 0. \tag{149}$$

4.2 Line defect

Imposing conservation of $Q^\pm_1$ and P_3 the invariant objects are

$$z^1_1 = y^1_1 - \theta^{2+}_1\theta^{1-}_2, \quad z^2_1 = y^2_1 + i\theta^{2+}_1\theta^{1-}_2, \quad \theta^{2+}_1, \tag{150}$$

$$\tilde{z}^1_2 = \tilde{y}^1_2 + \theta^{1+}_1\theta^{2-}_2, \quad \tilde{z}^2_2 = \tilde{y}^2_2 + i\theta^{1+}_1\theta^{2-}_2, \quad \theta^{2-}_2, \tag{151}$$

$$z^3_{12} = y^3_1 - \tilde{y}^3_2 - \theta^{1+}_1\theta^{1-}_2. \tag{152}$$

4.3 Boundary $\mathcal{N} = (1,1)$ $\langle\phi\hat{O}\hat{O}\rangle$

The parallel theta coordinates are

$$\theta^{1\|} \equiv \theta^{-1} + \theta^{+2}, \quad \theta^{2\|} \equiv \theta^{-2} + \theta^{+1}. \tag{153}$$

The perpendicular coordinates vanish at the boundary

$$\theta^{-1} - \theta^{+2} = 0, \quad \theta^{-2} - \theta^{+1} = 0. \tag{154}$$

Using P_a, Q_a, K_a and S_a (i.e. parallel to boundary) it is clear we can send the boundary points (x^a, θ^a) to $(0,0)$ and $(\infty, 0)$ and we are left with the bulk point. We need to build invariants respect to the remaining generators M_{12} and D. The bosonic one is clear, and taking same conventions as Lauria, Liendo and friends we get

$$\chi = \frac{(x_1^a)^2}{(x_1^i)^2} \to \frac{(y_1^a)^2}{(y_1^i)^2}. \tag{155}$$

(After the arrow we supersymmetrize the invariant requiring it is chiral at point 1). For the nilpotent ones we can use $\theta^{\dot{\alpha}+}$, but M_{12} invariance requires the free index is contracted, so there is only one invariant. All in all

$$\langle \phi(x,\theta) \hat{\mathcal{O}}^{(j)}(0,w) \hat{\mathcal{O}}(\infty) \rangle = \frac{(\hat{y}_a w^a)^j}{|y^3|^{\Delta_\phi + \Delta_2 - \Delta_3}} \left(f_1(\chi) + \frac{\theta^{1+}\theta^{2+}}{|y_1^i|} f_2(\chi) \right). \tag{156}$$

We defined $\hat{y}^a = y^a/|y^a|$ and w is null $w^2 = 0$ to enforce tracelessness. To get the contributions from descendants, notice that a chiral field is expanded as

$$\phi = \phi(y^\mu, \theta^{\dot{\alpha}+}) = \phi(y^\mu) + \theta^{\dot{\alpha}+}\psi_{\dot{\alpha}}(y^\mu) + \theta^{1+}\theta^{2+}\rho(y^\mu) \tag{157}$$

So clearly the functions $f_i(\hat{\chi})$ already capture the blocks of the descendants:

$$f_1(\hat{\chi}) \sim \langle \phi \hat{\mathcal{O}} \hat{\mathcal{O}} \rangle, \tag{158}$$

$$f_2(\hat{\chi}) \sim \langle \rho \hat{\mathcal{O}} \hat{\mathcal{O}} \rangle. \tag{159}$$

The Casimir equations read:

$$\begin{aligned}
&4\chi(\chi+1) f_1''(\chi) + (4\Delta_{23}\chi + 8\chi + 4) f_1'(\chi) - f_2(\chi) \\
&\qquad\qquad + \left(\Delta_1 + \Delta_{23}(\Delta_{23}+2) - j^2\chi^{-1} - c_2\right) f_1(\chi) = 0, \\
&4\chi(\chi+1) f_2''(\chi) + (4\Delta_{23}\chi + 8\chi + 4) f_2'(\chi) + (\Delta_1(\Delta_1 - 1) - c_2) f_1(\chi) \\
&\qquad\qquad + \left(\Delta_{23}(\Delta_{23}+2) - \Delta_1 + 1 - j^2\chi^{-1} - c_2\right) f_2(\chi) = 0.
\end{aligned} \tag{160}$$

The solution for $c_2 = \Delta(\Delta - 1)$ is

$$f_1(\chi) = f_\Delta(\chi) + a f_{\Delta+1}(\chi), \tag{161}$$

$$f_2(\chi) = (\Delta_\phi - \Delta) f_\Delta(\chi) + a(\Delta_\phi + \Delta - 1) f_{\Delta+1}(\chi). \tag{162}$$

where the bosonic blocks are

$$f_\Delta(\chi) = \chi^{-1/2(\Delta+\Delta_{23})} {}_2F_1\left(\frac{1}{2}(\Delta + \Delta_{23} - j), \frac{1}{2}(\Delta + \Delta_{23} + j); \Delta; -\frac{1}{\chi}\right). \tag{163}$$

4.3.1 Equations of motion

Now we impose the EOM of the chiral field. They will require

$$(D^-)^2\phi(z) = 0 \quad \Rightarrow \quad \partial^2\phi = 0, \qquad \partial^{\alpha\beta}\psi_\beta = 0, \qquad \rho = 0, \tag{164}$$

and $\Delta_\phi = 1/2$. Acting with the EOM on (156) we get:

$$\begin{aligned} -8\chi(\Delta_{23}\chi + 2\chi + 1)f_1'(\chi) + \frac{1}{2}f_1(\chi)\left(-4\Delta_{23}^2\chi - 8\Delta_{23}\chi + 4j^2 - 3\chi\right) - 8(\chi+1)\chi^2 f_1''(\chi) = 0, \\ f_2(\chi) = 0. \end{aligned} \tag{165}$$

We get that $f_2(\chi) = 0$, which is a consequence of $\rho = 0$ (see (158)). The differential equation can be solved in terms of two conformal blocks:

$$f_1(\chi) = \lambda_{\hat{\phi}\hat{\mathcal{O}}^{(j)}\hat{\mathcal{O}}} f_{\hat{\Delta}=1/2}(\chi) + \lambda_{\partial_\perp\hat{\phi}\hat{\mathcal{O}}^{(j)}\hat{\mathcal{O}}} f_{\hat{\Delta}=3/2}(\chi). \tag{166}$$

The exchanged operator can have $\Delta = 1/2, 3/2$, so $f_1(\chi)$ is determined up to two OPE coefficients, which correspond to the exchange of $\hat{\phi}$ and $\partial_\perp\hat{\phi}$.

4.4 Line $3d$ $\langle\phi\hat{O}\hat{O}\rangle$

The parallel and perpendicular coordinates are

$$\text{orthogonal}: \quad \theta^{1+},\ \theta^{2-}, \tag{167}$$

$$\text{parallel}: \quad \theta^{2+},\ \theta^{1-}. \tag{168}$$

The frame is the same as before, but now $\theta^{1+}\theta^{2+}$ is not invariant under $M_{12} + iR$, so there is no nilpotent invariant in $\langle\phi\hat{\mathcal{O}}\hat{\mathcal{O}}\rangle$.

4.5 Boundary $3d$ $\langle\phi\hat{O}\rangle$

4.5.1 $\mathcal{N} = (1,1)$

The correlator is

$$\langle\phi(y,\theta^+)\hat{O}(0)\rangle = \frac{1}{|y|^{2\hat{\Delta}}(y^3)^{\Delta-\hat{\Delta}}}\left(1 + \Delta_{12}\frac{\theta^{1+}\theta^{2+}}{y^3}\right). \tag{169}$$

The equations of motion

$$\epsilon^{\dot{\alpha}\dot{\beta}} D^-_{\dot{\alpha}} D^-_{\dot{\beta}} \langle\phi(y,\theta^+)\hat{O}(0)\rangle = 0 \tag{170}$$

are only solved for $\hat{\Delta} = \Delta = 1/2$.

4.5.2 $\mathcal{N} = (2, 0)$

The correlator is

$$\langle \phi(y, \theta^+) \hat{O}(0) \rangle = \frac{\delta_{\Delta, \hat{\Delta}}}{|y|^{2\Delta}}. \tag{171}$$

The equations of motion are only solved for $\hat{\Delta} = \Delta = 1/2$. This looks off, I would expect that $\hat{\Delta} = 1$ and $\Delta = 1/2$ should be a solution, corresponding to $\Phi|_\partial = 0$ and $\Psi|_\partial \neq 0$ in the notation of Gaiotto, and $\partial_\perp \phi$ is a level-one descendant:

$$\Psi = \psi + \theta \partial_\perp \phi + \dots \tag{172}$$

Perhaps the problem is that we are assuming that Ψ is bosonic, but assuming it's fermionic we could find another structure in the two-point function?

Consider the non-supersymmetric correlators first. Two bulk fermions without boundary give (recall that $\psi_{\dot{\beta}} = \psi^\beta$ and so on)

$$\langle \psi_\alpha(x_1) \psi_{\dot{\beta}}(x_2) \rangle = \frac{\delta_{\Delta_1, \Delta_2} \Sigma^\mu_{\alpha\dot{\beta}} x^\mu_{12}}{(x^2_{12})^{\Delta_1 + 1/2}}. \tag{173}$$

Similarly, it seems that a bulk to boundary fermion gives (is it the only tensor structure?!) (does it make sense to have a spinor index in the bdy?!)

$$\langle \psi_\alpha(x_1) \hat{\psi}_{\dot{\beta}}(\hat{x}_2) \rangle = \frac{\left(\Sigma^a_{\alpha\dot{\beta}} x^a_{12} + \Sigma^3_{\alpha\dot{\beta}} x^3_1 \right)}{(x^2_{12})^{\hat{\Delta} + 1/2} (x^3_1)^{\Delta - \hat{\Delta}}}. \tag{174}$$

Now we can consider the supersymmetric version, where we have a chiral in the bulk and a fermion in the boundary

$$\Phi_{3d}(y, \theta^+) = \phi(y) + \theta^{\alpha+} \psi_\alpha(y) + \theta^{1+} \theta^{2+} \rho(y) \tag{175}$$

$$\Psi(y, \theta^{2+}) = \psi_1(y) + \theta^{2+} \rho(y). \tag{176}$$

Imagine we translate the boundary point to the origin using supertranslations, then $\Psi(0) = \psi_1(0)$ and only the level-one descendant in the chiral field contributes to the two-point function

$$\langle \Phi_{3d}(y, \theta^+) \Psi(0) \rangle = \theta_1^{\alpha+} \langle \psi_\alpha(y) \psi_1(0) \rangle = \frac{\theta_1^{\alpha+} \Sigma^\mu_{\alpha 1} y^\mu_1}{|y_1|^{2\hat{\Delta}+1} (y^3_1)^{\Delta - \hat{\Delta} + 1/2}}, \tag{177}$$

where Δ and $\hat{\Delta}$ are the dimensions of the primaries in the Φ and Ψ multiplets. However, in my codes this seems to satisfy Ward identities for $D, K_{1,2}, M_{12}, R$ but not for S...

Why does it not work?

- A mistake in the calculation?
- The non-supersymmetric bulk to bdy correlator can be more general.
- The idea is conceptually wrong?

Any comments/ideas are appreciated!

A Useful identities

$$\{A, BC\} = \{A, B\}C - B[A, C], \tag{178}$$
$$\{A, [B, C]\} = -\{B, [A, C]\} + [\{A, B\}, C], \tag{179}$$
$$\tag{180}$$

(In conventions of DSD Tasi) If we use $e^{\frac{K\cdot x}{x^2}}$ we can send $\mathcal{O}(x) \to \mathcal{O}(\infty)$. Then we can get Ward identities using

$$[D, \mathcal{O}(\infty)] = -\Delta\mathcal{O}(\infty), \tag{181}$$
$$[P_\mu, \mathcal{O}(\infty)] = 0, \tag{182}$$
$$[M_{\mu\nu}, [e^{K\cdot x/x^2}, \mathcal{O}(x)]] = \left[e^{K\cdot x/x^2}, \left(S_{\mu\nu} - 2\frac{x^\nu x^\rho}{x^2} S_{\mu\rho} + 2\frac{x^\mu x^\rho}{x^2} S_{\nu\rho}\right) \mathcal{O}(x)\right]. \tag{183}$$

The last one is hard to understand unless $\mathcal{O}(x)$ is a scalar and then it just vanishes (as expected!).

Also consider a transformation with $a^\mu = (1-\epsilon)\frac{x_2^\mu}{x_2^2}$, we have at first order in ϵ that $x_2' = x_2/\epsilon$ and

$$\Omega(x_1') = \frac{x_2^2}{x_{12}^2}, \qquad \Omega(x_2') = \epsilon^{-2}. \tag{184}$$

From this we can for instance recover the two point function

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \Omega(x_1')^{\Delta_1}\Omega(x_2')^{\Delta_2}\langle \mathcal{O}(x_1')\mathcal{O}(x_2')\rangle \tag{185}$$
$$= \frac{1}{x_{12}^{2\Delta_1}}\langle \mathcal{O}(x_1') \lim_{\epsilon\to 0}\left(\frac{x_2}{\epsilon}\right)^{2\Delta_2} \mathcal{O}(x_2/\epsilon)\rangle \tag{186}$$
$$= \frac{1}{x_{12}^{2\Delta_1}}\langle \mathcal{O}(x_1')\mathcal{O}(\infty)\rangle \tag{187}$$
$$= \frac{1}{x_{12}^{2\Delta_1}}. \tag{188}$$

Noting that $R^\mu{}_\nu(0) = \delta^\mu_\nu$ we can also reconstruct the three-point function from

$$\langle \mathcal{O}_1^{\mu_1\cdots\mu_\ell}(0)\mathcal{O}_2(x)\mathcal{O}_3(\infty)\rangle = \frac{x^{\mu_1}\ldots x^{\mu_\ell}}{|x|^{\Delta_1+\Delta_2-\Delta_3+\ell}} \tag{189}$$

The case when $I^\mu{}_\nu(x)$ appears seems more subtle, because it is not clear how to send this factor to infinity.