In this chapter we will study the properties of several types of transmission lines and waveguides that are in common use. As we know from Chapter 2, a transmission line is characterized by a propagation constant, an attenuation constant, and a characteristic impedance. These quantities will be derived by field theory analysis for the various lines and waveguides treated here.

We begin with a discussion of the different types of wave propagation and modes that can exist on general transmission lines and waveguides. Transmission lines that consist of two or more conductors may support *transverse electromagnetic* (TEM) waves, characterized by the lack of longitudinal field components. Such lines have a uniquely defined voltage, current, and characteristic impedance. Waveguides, often consisting of a single conductor, support *transverse electric* (TE) and/or *transverse magnetic* (TM) waves, characterized by the presence of longitudinal magnetic or electric field components. As we will see in Chapter 4, a unique definition of characteristic impedance is not possible for such waves, although definitions can be chosen so that the characteristic impedance concept can be extended to waveguides with meaningful results.

3.1 GENERAL SOLUTIONS FOR TEM, TE, AND TM WAVES

In this section we will find general solutions to Maxwell's equations for the specific cases of TEM, TE, and TM wave propagation in cylindrical transmission lines or waveguides. The geometry of an arbitrary transmission line or waveguide is shown in Figure 3.1 and is characterized by conductor boundaries that are parallel to the z-axis. These structures are assumed to be uniform in shape and dimension in the z direction and infinitely long. The conductors will initially be assumed to be perfectly conducting, but attenuation can be found by the perturbation method discussed in Chapter 2.

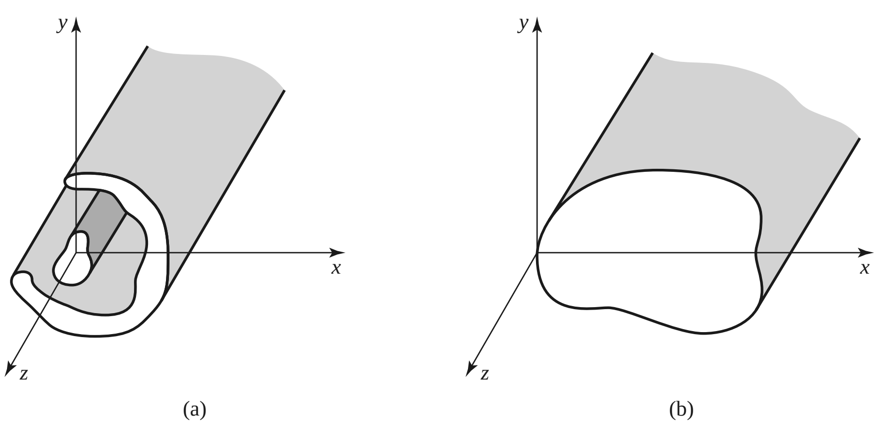

FIGURE 3.1 (a) General two-conductor transmission line and (b) closed waveguide.

We assume time-harmonic fields with an $e^{j\omega t}$ dependence and wave propagation along the z-axis. The electric and magnetic fields can then be written as

$$\bar{E}(x, y, z) = [\bar{e}(x, y) + \hat{z}e_z(x, y)]e^{-j\beta z}, \tag{3.1a}$$

$$\bar{H}(x, y, z) = [\bar{h}(x, y) + \hat{z}h_z(x, y)]e^{-j\beta z}, \tag{3.1b}$$

where $\bar{e}(x, y)$ and $\bar{h}(x, y)$ represent the transverse $(\hat{x}, \hat{y})$ electric and magnetic field components, and e_z and h_z are the longitudinal electric and magnetic field components. In (3.1) the wave is propagating in the $+z$ direction; $-z$ propagation can be obtained by replacing β with $-\beta$. In addition, if conductor or dielectric loss is present, the propagation constant will be complex; $j\beta$ should then be replaced with $\gamma = \alpha + j\beta$.

Assuming that the transmission line or waveguide region is source free, we can write Maxwell's equations as

$$\nabla \times \bar{E} = -j\omega\mu\bar{H}, \tag{3.2a}$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E}. \tag{3.2b}$$

With an $e^{-j\beta z}$ z dependence, the three components of each of these vector equations can be reduced to the following:

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x, \tag{3.3a}$$

$$-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y, \tag{3.3b}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z, \tag{3.3c}$$

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x, \tag{3.4a}$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y, \tag{3.4b}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z. \tag{3.4c}$$

These six equations can be solved for the four transverse field components in terms of E_z and H_z [e.g., H_x can be derived by eliminating E_y from (3.3a) and (3.4b)] as follows:

$$H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right), \tag{3.5a}$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right), \tag{3.5b}$$

$$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right), \tag{3.5c}$$

$$E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right), \tag{3.5d}$$

where

$$k_c^2 = k^2 - \beta^2 \tag{3.6}$$

is defined as the *cutoff wave number*; the reason for this terminology will become clear later. As in previous chapters,

$$k = \omega\sqrt{\mu\epsilon} = 2\pi/\lambda \tag{3.7}$$

is the wave number of the material filling the transmission line or waveguide region. If dielectric loss is present, ϵ can be made complex by using $\epsilon = \epsilon_o\epsilon_r(1 - j\tan\delta)$, where $\tan\delta$ is the loss tangent of the material.

Equations (3.5a)–(3.5d) are general results that can be applied to a variety of waveguiding systems. We will now specialize these results to specific wave types.

TEM Waves

Transverse electromagnetic (TEM) waves are characterized by $E_z = H_z = 0$. Observe from (3.5) that if $E_z = H_z = 0$, then the transverse fields are also all zero, unless $k_c^2 = 0 (k^2 = \beta^2)$, in which case we have an indeterminate result. However, we can return to (3.3)–(3.4) and apply the condition that $E_z = H_z = 0$. Then from (3.3a) and (3.4b), we can eliminate H_x to obtain

$$\beta^2 E_y = \omega^2\mu\epsilon E_y,$$

or

$$\beta = \omega\sqrt{\mu\epsilon} = k, \tag{3.8}$$

as noted earlier. [This result can also be obtained from (3.3b) and (3.4a).] The cutoff wave number, $k_c = \sqrt{k^2 - \beta^2}$, is thus zero for TEM waves.

The Helmholtz wave equation for E_x is, from (1.42),

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0, \tag{3.9}$$

but for $e^{-j\beta z}$ dependence, $(\partial^2/\partial z^2)E_x = -\beta^2 E_x = -k^2 E_x$, so (3.9) reduces to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_x = 0. \tag{3.10}$$

A similar result also applies to E_y, so using the form of $\bar{E}$ assumed in (3.1a), we can write

$$\nabla_t^2 \bar{e}(x, y) = 0, \tag{3.11}$$

where $\nabla_t^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator in the two transverse dimensions.

The result of (3.11) shows that the transverse electric fields, $\bar{e}(x, y)$, of a TEM wave satisfy Laplace's equation. It is easy to show in the same way that the transverse magnetic fields also satisfy Laplace's equation:

$$\nabla_t^2 \bar{h}(x, y) = 0. \tag{3.12}$$

The transverse fields of a TEM wave are thus the same as the static fields that can exist between the conductors. In the electrostatic case, we know that the electric field can be expressed as the gradient of a scalar potential, $\Phi(x, y)$:

$$\bar{e}(x, y) = -\nabla_t \Phi(x, y), \tag{3.13}$$

where $\nabla_t = \hat{x}(\partial/\partial x) + \hat{y}(\partial/\partial y)$ is the transverse gradient operator in two dimensions. For the relation in (3.13) to be valid, the curl of $\bar{e}$ must vanish, and this is indeed the case here since

$$\nabla_t \times \bar{e} = -j\omega\mu h_z \hat{z} = 0.$$

Using the fact that $\nabla \cdot \bar{D} = \epsilon \nabla_t \cdot \bar{e} = 0$ with (3.13) shows that $\Phi(x, y)$ also satisfies Laplace's equation,

$$\nabla_t^2 \Phi(x, y) = 0, \tag{3.14}$$

as expected from electrostatics. The voltage between two conductors can be found as

$$V_{12} = \Phi_1 - \Phi_2 = \int_1^2 \bar{E} \cdot d\bar{\ell}, \tag{3.15}$$

where Φ_1 and Φ_2 represent the potential at conductors 1 and 2, respectively. The current flow on a given conductor can be found from Ampere's law as

$$I = \oint_C \bar{H} \cdot d\bar{\ell}, \tag{3.16}$$

where C is the cross-sectional contour of the conductor.

TEM waves can exist when two or more conductors are present. Plane waves are also examples of TEM waves since there are no field components in the direction of propagation; in this case the transmission line conductors may be considered to be two infinitely large plates separated to infinity. The above results show that a closed conductor (such as a rectangular waveguide) cannot support TEM waves since the corresponding static potential in such a region would be zero (or possibly a constant), leading to $\bar{e} = 0$.

The wave impedance of a TEM mode can be found as the ratio of the transverse electric and magnetic fields:

$$Z_{\text{TEM}} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta, \tag{3.17a}$$

where (3.4a) was used. The other pair of transverse field components, from (3.3a), gives

$$Z_{\text{TEM}} = \frac{-E_y}{H_x} = \sqrt{\frac{\mu}{\epsilon}} = \eta. \tag{3.17b}$$

Combining the results of (3.17a) and (3.17b) gives a general expression for the transverse fields as

$$\bar{h}(x, y) = \frac{1}{Z_{\text{TEM}}} \hat{z} \times \bar{e}(x, y). \tag{3.18}$$

Note that the wave impedance is the same as that for a plane wave in a lossless medium, as derived in Chapter 1; the reader should not confuse this impedance with the characteristic impedance, Z_0, of a transmission line. The latter relates traveling voltage and current and is a function of the line geometry as well as the material filling the line, while the wave impedance relates transverse field components and is dependent only on the material constants. From (2.32), the characteristic impedance of the TEM line is $Z_0 = V/I$, where V and I are the amplitudes of the incident voltage and current waves.

The procedure for analyzing a TEM line can be summarized as follows:

1. Solve Laplace's equation, (3.14), for $\Phi(x, y)$. The solution will contain several unknown constants.

2. Find these constants by applying the boundary conditions for the known voltages on the conductors.
3. Compute $\bar{e}$ and $\bar{E}$ from (3.13) and (3.1a). Compute $\bar{h}$ and $\bar{H}$ from (3.18) and (3.1b).
4. Compute V from (3.15) and I from (3.16).
5. The propagation constant is given by (3.8), and the characteristic impedance is given by $Z_0 = V/I$.

TE Waves

Transverse electric (TE) waves, (also referred to as H-waves) are characterized by $E_z = 0$ and $H_z \neq 0$. Equations (3.5) then reduce to

$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x}, \tag{3.19a}$$

$$H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y}, \tag{3.19b}$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}, \tag{3.19c}$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}. \tag{3.19d}$$

In this case $k_c \neq 0$, and the propagation constant $\beta = \sqrt{k^2 - k_c^2}$ is generally a function of frequency and the geometry of the line or guide. To apply (3.19), one must first find H_z from the Helmholtz wave equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_z = 0, \tag{3.20}$$

which, since $H_z(x, y, z) = h_z(x, y)e^{-j\beta z}$, can be reduced to a two-dimensional wave equation for h_z:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z = 0, \tag{3.21}$$

since $k_c^2 = k^2 - \beta^2$. This equation must be solved subject to the boundary conditions of the specific guide geometry.

The TE wave impedance can be found as

$$Z_{\mathrm{TE}} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta}, \tag{3.22}$$

which is seen to be frequency dependent. TE waves can be supported inside closed conductors, as well as between two or more conductors.

TM Waves

Transverse magnetic (TM) waves (also referred to as E-waves) are characterized by $E_z \neq 0$ and $H_z = 0$. Equations (3.5) then reduce to

$$H_x = \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial y}, \tag{3.23a}$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial x}, \tag{3.23b}$$

$$E_x = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x}, \tag{3.23c}$$

$$E_y = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial y}. \tag{3.23d}$$

As in the TE case, $k_c \neq 0$, and the propagation constant $\beta = \sqrt{k^2 - k_c^2}$ is a function of frequency and the geometry of the line or guide. E_z is found from the Helmholtz wave equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_z = 0, \tag{3.24}$$

which, since $E_z(x, y, z) = e_z(x, y)e^{-j\beta z}$, can be reduced to a two-dimensional wave equation for e_z:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z = 0, \tag{3.25}$$

since $k_c^2 = k^2 - \beta^2$. This equation must be solved subject to the boundary conditions of the specific guide geometry.

The TM wave impedance can be found as

$$Z_{\text{TM}} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \frac{\beta\eta}{k}, \tag{3.26}$$

which is frequency dependent. As for TE waves, TM waves can be supported inside closed conductors, as well as between two or more conductors.

The procedure for analyzing TE and TM waveguides can be summarized as follows:

1. Solve the reduced Helmholtz equation, (3.21) or (3.25), for h_z or e_z. The solution will contain several unknown constants and the unknown cutoff wave number, k_c.
2. Use (3.19) or (3.23) to find the transverse fields from h_z or e_z.
3. Apply the boundary conditions to the appropriate field components to find the unknown constants and k_c.
4. The propagation constant is given by (3.6) and the wave impedance by (3.22) or (3.26).

Attenuation Due to Dielectric Loss

Attenuation in a transmission line or waveguide can be caused by either dielectric loss or conductor loss. If α_d is the attenuation constant due to dielectric loss and α_c is the attenuation constant due to conductor loss, then the total attenuation constant is $\alpha = \alpha_d + \alpha_c$.

Attenuation caused by conductor loss can be calculated using the perturbation method of Section 2.7; this loss depends on the field distribution in the guide and so must be evaluated separately for each type of transmission line or waveguide. However, if the line or guide is completely filled with a homogeneous dielectric, the attenuation due to a lossy dielectric material can be calculated from the propagation constant, and this result will apply to any guide or line with a homogeneous dielectric filling.

Thus, use of the complex permittivity allows the complex propagation constant to be written as

$$\begin{aligned}\gamma = \alpha_d + j\beta &= \sqrt{k_c^2 - k^2} \\ &= \sqrt{k_c^2 - \omega^2\mu_0\epsilon_0\epsilon_r(1 - j\tan\delta)}.\end{aligned} \tag{3.27}$$