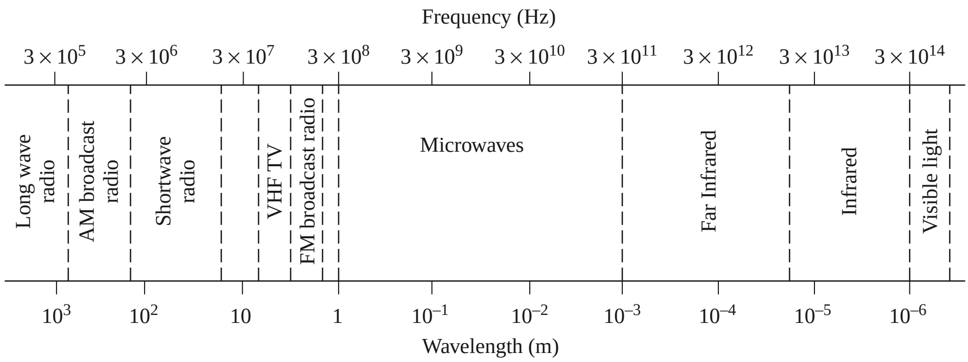
Chapter One

Electromagnetic Theory

We begin our study of microwave engineering with a brief overview of the history and major applications of microwave technology, followed by a review of some of the fundamental topics in electromagnetic theory that we will need throughout the book. Further discussion of these topics may be found in references [1–8].

1.1 INTRODUCTION TO MICROWAVE ENGINEERING

The field of radio frequency (RF) and microwave engineering generally covers the behavior of alternating current signals with frequencies in the range of 100 MHz (1 MHz = 10^6 Hz) to 1000 GHz (1 GHz = 10^9 Hz). RF frequencies range from very high frequency (VHF) (30–300 MHz) to ultra high frequency (UHF) (300–3000 MHz), while the term *microwave* is typically used for frequencies between 3 and 300 GHz, with a corresponding electrical wavelength between $\lambda = c/f = 10$ cm and $\lambda = 1$ mm, respectively. Signals with wavelengths on the order of millimeters are often referred to as *millimeter waves*. Figure 1.1 shows the location of the RF and microwave frequency bands in the electromagnetic spectrum. Because of the high frequencies (and short wavelengths), standard circuit theory often cannot be used directly to solve microwave network problems. In a sense, standard circuit theory is an approximation, or special case, of the broader theory of electromagnetics as described by Maxwell's equations. This is due to the fact that, in general, the lumped circuit element approximations of circuit theory may not be valid at high RF and microwave frequencies. Microwave components often act as *distributed elements*, where the phase of the voltage or current changes significantly over the physical extent of the device because the device dimensions are on the order of the electrical wavelength. At much lower frequencies the wavelength is large enough that there is insignificant phase variation across the dimensions of a component. The other extreme of frequency can be identified as optical engineering, in which the wavelength is much shorter than the dimensions of the component. In this case Maxwell's equations can be simplified to the geometrical optics regime, and optical systems can be designed with the theory of geometrical optics. Such



Typical Frequencies	
AM broadcast band	535–1605 kHz
Short wave radio band	3–30 MHz
FM broadcast band	88–108 MHz
VHF TV (2–4)	54–72 MHz
VHF TV (5–6)	76–88 MHz
UHF TV (7–13)	174–216 MHz
UHF TV (14–83)	470–890 MHz
US cellular telephone	824–849 MHz
	869–894 MHz
European GSM cellular	880–915 MHz
	925–960 MHz
GPS	1575.42 MHz
	1227.60 MHz
Microwave ovens	2.45 GHz
US DBS	11.7–12.5 GHz
US ISM bands	902–928 MHz
	2.400–2.484 GHz
	5.725–5.850 GHz
US UWB radio	3.1–10.6 GHz

Approximate Band Designations	
Medium frequency	300 kHz–3 MHz
High frequency (HF)	3 MHz–30 MHz
Very high frequency (VHF)	30 MHz–300 MHz
Ultra high frequency (UHF)	300 MHz–3 GHz
L band	1–2 GHz
S band	2–4 GHz
C band	4–8 GHz
X band	8–12 GHz
Ku band	12–18 GHz
K band	18–26 GHz
Ka band	26–40 GHz
U band	40–60 GHz
V band	50–75 GHz
E band	60–90 GHz
W band	75–110 GHz
F band	90–140 GHz

FIGURE 1.1 The electromagnetic spectrum.

techniques are sometimes applicable to millimeter wave systems, where they are referred to as *quasi-optical*.

In RF and microwave engineering, then, one must often work with Maxwell's equations and their solutions. It is in the nature of these equations that mathematical complexity arises since Maxwell's equations involve vector differential or integral operations on vector field quantities, and these fields are functions of spatial coordinates. One of the goals of this book is to try to reduce the complexity of a field theory solution to a result that can be expressed in terms of simpler circuit theory, perhaps extended to include distributed elements (such as transmission lines) and concepts (such as reflection coefficients and scattering parameters). A field theory solution generally provides a complete description of the electromagnetic field at every point in space, which is usually much more information than we need for most practical purposes. We are typically more interested in terminal quantities such as power, impedance, voltage, and current, which can often be expressed in terms of these extended circuit theory concepts. It is this complexity that adds to the challenge, as well as the rewards, of microwave engineering.

Applications of Microwave Engineering

Just as the high frequencies and short wavelengths of microwave energy make for difficulties in the analysis and design of microwave devices and systems, these same aspects provide unique opportunities for the application of microwave systems. The following considerations can be useful in practice:

- Antenna gain is proportional to the electrical size of the antenna. At higher frequencies, more antenna gain can be obtained for a given physical antenna size, and this has important consequences when implementing microwave systems.
- More bandwidth (directly related to data rate) can be realized at higher frequencies. A 1% bandwidth at 600 MHz is 6 MHz, which (with binary phase shift keying modulation) can provide a data rate of about 6 Mbps (megabits per second), while at 60 GHz a 1% bandwidth is 600 MHz, allowing a 600 Mbps data rate.
- Microwave signals travel by line of sight and are not bent by the ionosphere as are lower frequency signals. Satellite and terrestrial communication links with very high capacities are therefore possible, with frequency reuse at minimally distant locations.
- The effective reflection area (radar cross section) of a radar target is usually proportional to the target's electrical size. This fact, coupled with the frequency characteristics of antenna gain, generally makes microwave frequencies preferred for radar systems.
- Various molecular, atomic, and nuclear resonances occur at microwave frequencies, creating a variety of unique applications in the areas of basic science, remote sensing, medical diagnostics and treatment, and heating methods.

The majority of today's applications of RF and microwave technology are to wireless networking and communications systems, wireless security systems, radar systems, environmental remote sensing, and medical systems. As the frequency allocations listed in Figure 1.1 show, RF and microwave communications systems are pervasive, especially today when wireless connectivity promises to provide voice and data access to "anyone, anywhere, at any time."

Modern wireless telephony is based on the concept of *cellular frequency reuse*, a technique first proposed by Bell Labs in 1947 but not practically implemented until the 1970s. By this time advances in miniaturization, as well as increasing demand for wireless communications, drove the introduction of several early cellular telephone systems in Europe, the United States, and Japan. The *Nordic Mobile Telephone* (NMT) system was deployed in 1981 in the Nordic countries, the *Advanced Mobile Phone System* (AMPS) was introduced in the United States in 1983 by AT&T, and NTT in Japan introduced its first mobile phone service in 1988. All of these early systems used analog FM modulation, with their allocated frequency bands divided into several hundred narrow band voice channels. These early systems are usually referred to now as *first-generation* cellular systems, or 1G.

Second-generation (2G) cellular systems achieved improved performance by using various digital modulation schemes, with systems such as GSM, CDMA, DAMPS, PCS, and PHS being some of the major standards introduced in the 1990s in the United States, Europe, and Japan. These systems can handle digitized voice, as well as some limited data, with data rates typically in the 8 to 14 kbps range. In recent years there has been a wide variety of new and modified standards to transition to handheld services that include voice, texting, data networking, positioning, and Internet access. These standards are variously known as 2.5G, 3G, 3.5G, 3.75G, and 4G, with current plans to provide data rates up to at least 100 Mbps. The number of subscribers to wireless services seems to be keeping pace with the growing power and access provided by modern handheld wireless devices; as of 2010 there were more than five billion cell phone users worldwide.

Satellite systems also depend on RF and microwave technology, and satellites have been developed to provide cellular (voice), video, and data connections worldwide. Two large satellite constellations, Iridium and Globalstar, were deployed in the late 1990s to provide worldwide telephony service. Unfortunately, these systems suffered from both technical drawbacks and weak business models and have led to multibillion dollar financial failures. However, smaller satellite systems, such as the Global Positioning Satellite (GPS) system and the Direct Broadcast Satellite (DBS) system, have been extremely successful.

Wireless local area networks (WLANs) provide high-speed networking between computers over short distances, and the demand for this capability is expected to remain strong. One of the newer examples of wireless communications technology is *ultra wide band* (UWB) radio, where the broadcast signal occupies a very wide frequency band but with a very low power level (typically below the ambient radio noise level) to avoid interference with other systems.

Radar systems find application in military, commercial, and scientific fields. Radar is used for detecting and locating air, ground, and seagoing targets, as well as for missile guidance and fire control. In the commercial sector, radar technology is used for air traffic control, motion detectors (door openers and security alarms), vehicle collision avoidance, and distance measurement. Scientific applications of radar include weather prediction, remote sensing of the atmosphere, the oceans, and the ground, as well as medical diagnostics and therapy. Microwave radiometry, which is the passive sensing of microwave energy emitted by an object, is used for remote sensing of the atmosphere and the earth, as well as in medical diagnostics and imaging for security applications.

A Short History of Microwave Engineering

Microwave engineering is often considered a fairly mature discipline because the fundamental concepts were developed more than 50 years ago, and probably because radar, the first major application of microwave technology, was intensively developed as far back as World War II. However, recent years have brought substantial and continuing developments in high-frequency solid-state devices, microwave integrated circuits, and computer-aided design techniques, and the ever-widening applications of RF and microwave technology to wireless communications, networking, sensing, and security have kept the field active and vibrant.

The foundations of modern electromagnetic theory were formulated in 1873 by James Clerk Maxwell, who hypothesized, solely from mathematical considerations, electromagnetic wave propagation and the idea that light was a form of electromagnetic energy. Maxwell's formulation was cast in its modern form by Oliver Heaviside during the period from 1885 to 1887. Heaviside was a reclusive genius whose efforts removed many of the mathematical complexities of Maxwell's theory, introduced vector notation, and provided a foundation for practical applications of guided waves and transmission lines. Heinrich Hertz, a German professor of physics and a gifted experimentalist who understood the theory published by Maxwell, carried out a set of experiments during the period 1887–1891 that validated Maxwell's theory of electromagnetic waves. Figure 1.2 is a photograph of the original equipment used by Hertz in his experiments. It is interesting to observe that this is an instance of a discovery occurring after a prediction has been made on theoretical grounds—a characteristic of many of the major discoveries throughout the history of science. All of the practical applications of electromagnetic theory—radio, television, radar, cellular telephones, and wireless networking—owe their existence to the theoretical work of Maxwell.

Because of the lack of reliable microwave sources and other components, the rapid growth of radio technology in the early 1900s occurred primarily in the HF to VHF range. It was not until the 1940s and the advent of radar development during World War II that microwave theory and technology received substantial interest. In the United States, the Radiation Laboratory was established at the Massachusetts Institute of Technology to develop radar theory and practice. A number of talented scientists, including N. Marcuvitz,

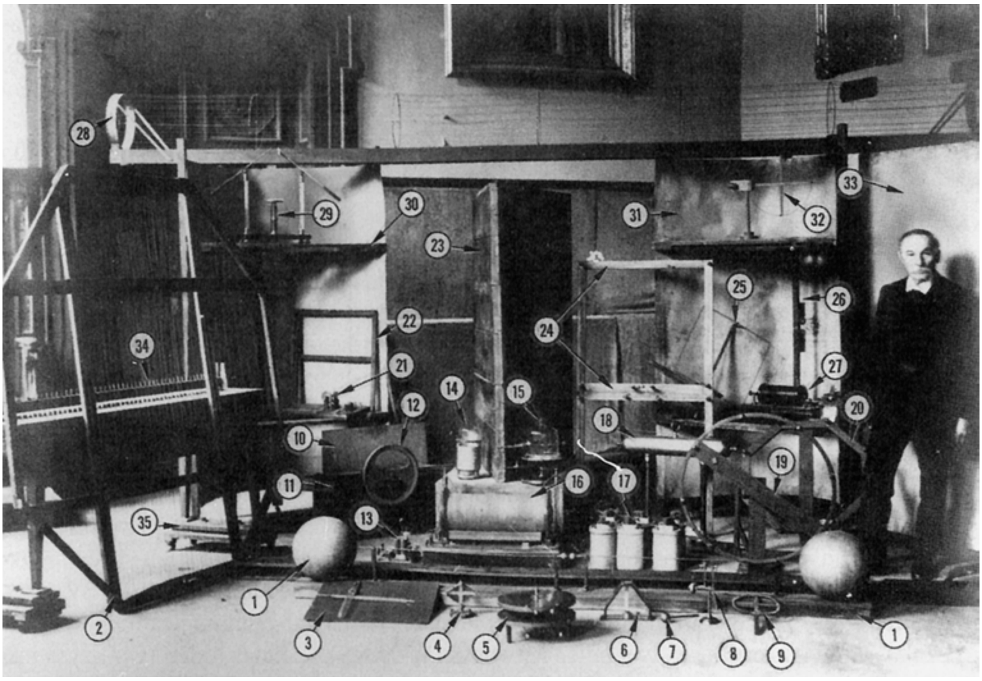


FIGURE 1.2 Original apparatus used by Hertz for his electromagnetics experiments. (1) 50 MHz transmitter spark gap and loaded dipole antenna. (2) Wire grid for polarization experiments. (3) Vacuum apparatus for cathode ray experiments. (4) Hot-wire galvanometer. (5) Reiss or Knochenhauer spirals. (6) Rolled-paper galvanometer. (7) Metal sphere probe. (8) Reiss spark micrometer. (9) Coaxial line. (10–12) Equipment to demonstrate dielectric polarization effects. (13) Mercury induction coil interrupter. (14) Meidinger cell. (15) Bell jar. (16) Induction coil. (17) Bunsen cells. (18) Large-area conductor for charge storage. (19) Circular loop receiving antenna. (20) Eight-sided receiver detector. (21) Rotating mirror and mercury interrupter. (22) Square loop receiving antenna. (23) Equipment for refraction and dielectric constant measurement. (24) Two square loop receiving antennas. (25) Square loop receiving antenna. (26) Transmitter dipole. (27) Induction coil. (28) Coaxial line. (29) High-voltage discharger. (30) Cylindrical parabolic reflector/receiver. (31) Cylindrical parabolic reflector/transmitter. (32) Circular loop receiving antenna. (33) Planar reflector. (34, 35) Battery of accumulators. Photographed on October 1, 1913, at the Bavarian Academy of Science, Munich, Germany, with Hertz's assistant, Julius Amman.

Photograph and identification courtesy of J. H. Bryant.

I. I. Rabi, J. S. Schwinger, H. A. Bethe, E. M. Purcell, C. G. Montgomery, and R. H. Dicke, among others, gathered for a very intensive period of development in the microwave field. Their work included the theoretical and experimental treatment of waveguide components, microwave antennas, small-aperture coupling theory, and the beginnings of microwave network theory. Many of these researchers were physicists who returned to physics research after the war, but their microwave work is summarized in the classic 28-volume Radiation Laboratory Series of books that still finds application today.

Communications systems using microwave technology began to be developed soon after the birth of radar, benefiting from much of the work that was originally done for radar systems. The advantages offered by microwave systems, including wide bandwidths and line-of-sight propagation, have proved to be critical for both terrestrial and satellite communications systems and have thus provided an impetus for the continuing development of low-cost miniaturized microwave components. We refer the interested reader to references [1] and [2] for further historical perspectives on the fields of wireless communications and microwave engineering.

1.2 MAXWELL'S EQUATIONS

Electric and magnetic phenomena at the macroscopic level are described by Maxwell's equations, as published by Maxwell in 1873. This work summarized the state of electromagnetic science at that time and hypothesized from theoretical considerations the existence of the electrical displacement current, which led to the experimental discovery by Hertz of electromagnetic wave propagation. Maxwell's work was based on a large body of empirical and theoretical knowledge developed by Gauss, Ampere, Faraday, and others. A first course in electromagnetics usually follows this historical (or deductive) approach, and it is assumed that the reader has had such a course as a prerequisite to the present material. Several references are available [3–7] that provide a good treatment of electromagnetic theory at the undergraduate or graduate level.

This chapter will outline the fundamental concepts of electromagnetic theory that we will require later in the book. Maxwell's equations will be presented, and boundary conditions and the effect of dielectric and magnetic materials will be discussed. Wave phenomena are of essential importance in microwave engineering, and thus much of the chapter is spent on topics related to plane waves. Plane waves are the simplest form of electromagnetic waves and so serve to illustrate a number of basic properties associated with wave propagation. Although it is assumed that the reader has studied plane waves before, the present material should help to reinforce the basic principles in the reader's mind and perhaps to introduce some concepts that the reader has not seen previously. This material will also serve as a useful reference for later chapters.

With an awareness of the historical perspective, it is usually advantageous from a pedagogical point of view to present electromagnetic theory from the "inductive," or axiomatic, approach by beginning with Maxwell's equations. The general form of time-varying Maxwell equations, then, can be written in "point," or differential, form as

$$\nabla \times \bar{\mathcal{E}} = \frac{-\partial \bar{\mathcal{B}}}{\partial t} - \bar{\mathcal{M}}, \tag{1.1a}$$

$$\nabla \times \bar{\mathcal{H}} = \frac{\partial \bar{\mathcal{D}}}{\partial t} + \bar{\mathcal{J}}, \tag{1.1b}$$

$$\nabla \cdot \bar{\mathcal{D}} = \rho, \tag{1.1c}$$

$$\nabla \cdot \bar{\mathcal{B}} = 0. \tag{1.1d}$$

The MKS system of units is used throughout this book. The script quantities represent time-varying vector fields and are real functions of spatial coordinates x, y, z, and the time variable t. These quantities are defined as follows:

$\bar{\mathcal{E}}$ is the electric field, in volts per meter (V/m).[1]

$\bar{\mathcal{H}}$ is the magnetic field, in amperes per meter (A/m).

[1] As recommended by the *IEEE Standard Definitions of Terms for Radio Wave Propagation, IEEE Standard 211-1997,* the terms "electric field" and "magnetic field" are used in place of the older terminology of "electric field intensity" and "magnetic field intensity."

$\bar{\mathcal{D}}$ is the electric flux density, in coulombs per meter squared (Coul/m^2).
$\bar{\mathcal{B}}$ is the magnetic flux density, in webers per meter squared (Wb/m^2).
$\bar{\mathcal{M}}$ is the (fictitious) magnetic current density, in volts per meter (V/m^2).
$\bar{\mathcal{J}}$ is the electric current density, in amperes per meter squared (A/m^2).
ρ is the electric charge density, in coulombs per meter cubed (Coul/m^3).

The sources of the electromagnetic field are the currents $\bar{\mathcal{M}}$ and $\bar{\mathcal{J}}$ and the electric charge density ρ. The magnetic current $\bar{\mathcal{M}}$ is a fictitious source in the sense that it is only a mathematical convenience: the real source of a magnetic current is always a loop of electric current or some similar type of magnetic dipole, as opposed to the flow of an actual magnetic charge (magnetic monopole charges are not known to exist). The magnetic current is included here for completeness, as we will have occasion to use it in Chapter 4 when dealing with apertures. Since electric current is really the flow of charge, it can be said that the electric charge density ρ is the ultimate source of the electromagnetic field.

In free-space, the following simple relations hold between the electric and magnetic field intensities and flux densities:

$$\bar{\mathcal{B}} = \mu_0 \bar{\mathcal{H}}, \tag{1.2a}$$

$$\bar{\mathcal{D}} = \epsilon_0 \bar{\mathcal{E}}, \tag{1.2b}$$

where $\mu_0 = 4\pi \times 10^{-7}$ henry/m is the permeability of free-space, and $\epsilon_0 = 8.854 \times 10^{-12}$ farad/m is the permittivity of free-space. We will see in the next section how media other than free-space affect these constitutive relations.

Equations (1.1a)–(1.1d) are linear but are not independent of each other. For instance, consider the divergence of (1.1a). Since the divergence of the curl of any vector is zero [vector identity (B.12), from Appendix B], we have

$$\nabla \cdot \nabla \times \bar{\mathcal{E}} = 0 = -\frac{\partial}{\partial t}(\nabla \cdot \bar{\mathcal{B}}) - \nabla \cdot \bar{\mathcal{M}}.$$

Since there is no free magnetic charge, $\nabla \cdot \bar{\mathcal{M}} = 0$, which leads to $\nabla \cdot \bar{\mathcal{B}} = 0$, or (1.1d). The *continuity equation* can be similarly derived by taking the divergence of (1.1b), giving

$$\nabla \cdot \bar{\mathcal{J}} + \frac{\partial \rho}{\partial t} = 0, \tag{1.3}$$

where (1.1c) was used. This equation states that charge is conserved, or that current is continuous, since $\nabla \cdot \bar{\mathcal{J}}$ represents the outflow of current at a point, and $\partial\rho/\partial t$ represents the charge buildup with time at the same point. It is this result that led Maxwell to the conclusion that the displacement current density $\partial\bar{\mathcal{D}}/\partial t$ was necessary in (1.1b), which can be seen by taking the divergence of this equation.

The above differential equations can be converted to integral form through the use of various vector integral theorems. Thus, applying the divergence theorem (B.15) to (1.1c) and (1.1d) yields

$$\oint_S \bar{\mathcal{D}} \cdot d\bar{s} = \int_V \rho \, dv = Q, \tag{1.4}$$

$$\oint_S \bar{\mathcal{B}} \cdot d\bar{s} = 0, \tag{1.5}$$

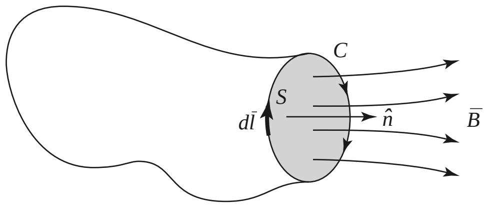


FIGURE 1.3 The closed contour C and surface S associated with Faraday's law.

where Q in (1.4) represents the total charge contained in the closed volume V (enclosed by a closed surface S). Applying Stokes' theorem (B.16) to (1.1a) gives

$$\oint_C \bar{\mathcal{E}} \cdot d\bar{l} = -\frac{\partial}{\partial t}\int_S \bar{\mathcal{B}} \cdot d\bar{s} - \int_S \bar{\mathcal{M}} \cdot d\bar{s}, \tag{1.6}$$

which, without the $\bar{\mathcal{M}}$ term, is the usual form of *Faraday's law* and forms the basis for *Kirchhoff's voltage law*. In (1.6), C represents a closed contour around the surface S, as shown in Figure 1.3. *Ampere's law* can be derived by applying Stokes' theorem to (1.1b):

$$\oint_C \bar{\mathcal{H}} \cdot d\bar{l} = \frac{\partial}{\partial t}\int_S \bar{\mathcal{D}} \cdot d\bar{s} + \int_S \bar{\mathcal{J}} \cdot d\bar{s} = \frac{\partial}{\partial t}\int_S \bar{\mathcal{D}} \cdot d\bar{s} + \mathcal{I}, \tag{1.7}$$

where $\mathcal{I} = \int_S \bar{\mathcal{J}} \cdot d\bar{s}$ is the total electric current flow through the surface S. Equations (1.4)–(1.7) constitute the integral forms of Maxwell's equations.

The above equations are valid for arbitrary time dependence, but most of our work will be involved with fields having a sinusoidal, or harmonic, time dependence, with steady-state conditions assumed. In this case phasor notation is very convenient, and so all field quantities will be assumed to be complex vectors with an implied $e^{j\omega t}$ time dependence and written with roman (rather than script) letters. Thus, a sinusoidal electric field polarized in the $\hat{x}$ direction of the form

$$\bar{\mathcal{E}}(x, y, z, t) = \hat{x} A\,(x, y, z) \cos\,(\omega t + \phi), \tag{1.8}$$

where A is the (real) amplitude, ω is the radian frequency, and ϕ is the phase reference of the wave at $t = 0$, has the phasor for

$$\bar{E}(x,\ y,\ z) = \hat{x} A(x,\ y,\ z) e^{j\phi}. \tag{1.9}$$

We will assume cosine-based phasors in this book, so the conversion from phasor quantities to real time-varying quantities is accomplished by multiplying the phasor by $e^{j\omega t}$ and taking the real part:

$$\bar{\mathcal{E}}(x,\ y,\ z,\ t) = \mathrm{Re}\{\bar{E}(x,\ y,\ z) e^{j\omega t}\}, \tag{1.10}$$

as substituting (1.9) into (1.10) to obtain (1.8) demonstrates. When working in phasor notation, it is customary to suppress the factor $e^{j\omega t}$ that is common to all terms.

When dealing with power and energy we will often be interested in the time average of a quadratic quantity. This can be found very easily for time harmonic fields. For example, the average of the square of the magnitude of an electric field, given as

$$\bar{\mathcal{E}} = \hat{x} E_1 \cos(\omega t + \phi_1) + \hat{y} E_2 \cos(\omega t + \phi_2) + \hat{z} E_2 \cos(\omega t + \phi_3), \tag{1.11}$$

has the phasor form

$$\bar{E} = \hat{x} E_1 e^{j\phi_1} + \hat{y} E_2 e^{j\phi_2} + \hat{z} E_3 e^{j\phi_3}, \tag{1.12}$$

can be calculated as

$$
\begin{aligned}
|\bar{\mathcal{E}}|^2_{\text{avg}} &= \frac{1}{T}\int_0^T \bar{\mathcal{E}} \cdot \bar{\mathcal{E}}\, dt \\
&= \frac{1}{T}\int_0^T \left[E_1^2 \cos^2(\omega t + \phi_1) + E_2^2 \cos^2(\omega t + \phi_2) + E_3^2 \cos^2(\omega t + \phi_3)\right] dt \\
&= \frac{1}{2}\left(E_1^2 + E_2^2 + E_3^2\right) = \frac{1}{2}|\bar{E}|^2 = \frac{1}{2}\bar{E} \cdot \bar{E}^*.
\end{aligned}
\tag{1.13}
$$

Then the root-mean-square (rms) value is $|\bar{E}|_{\text{rms}} = |\bar{E}|/\sqrt{2}$.

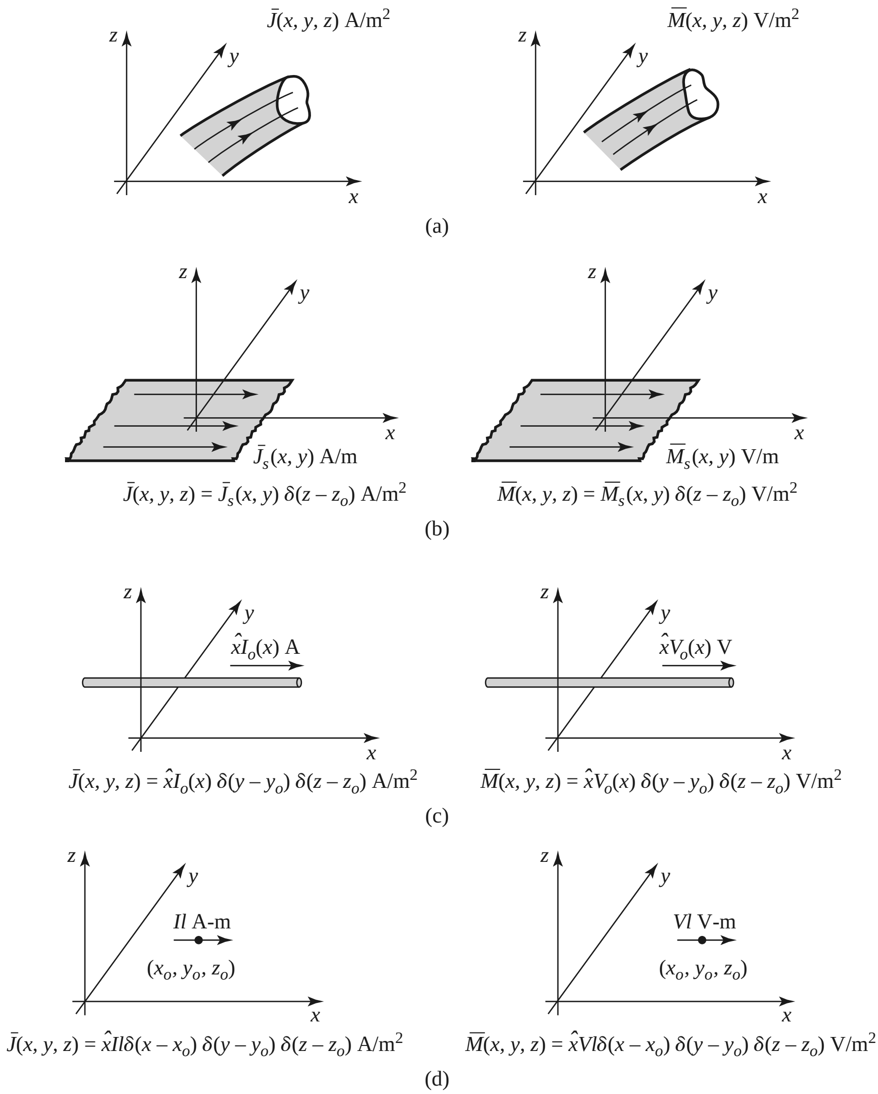


FIGURE 1.4 Arbitrary volume, surface, and line currents. (a) Arbitrary electric and magnetic volume current densities. (b) Arbitrary electric and magnetic surface current densities in the $z = z_0$ plane. (c) Arbitrary electric and magnetic line currents. (d) Infinitesimal electric and magnetic dipoles parallel to the x-axis.

Assuming an $e^{j\omega t}$ time dependence, we can replace the time derivatives in (1.1a)–(1.1d) with $j\omega$. Maxwell's equations in phasor form then become

$$\nabla \times \bar{E} = -j\omega\bar{B} - \bar{M}, \tag{1.14a}$$

$$\nabla \times \bar{H} = j\omega\bar{D} + \bar{J}, \tag{1.14b}$$

$$\nabla \cdot \bar{D} = \rho, \tag{1.14c}$$

$$\nabla \cdot \bar{B} = 0. \tag{1.14d}$$

The Fourier transform can be used to convert a solution to Maxwell's equations for an arbitrary frequency ω into a solution for arbitrary time dependence.

The electric and magnetic current sources, $\bar{J}$ and $\bar{M}$, in (1.14) are volume current densities with units A/m^2 and V/m^2. In many cases, however, the actual currents will be in the form of a current sheet, a line current, or an infinitesimal dipole current. These special types of current distributions can always be written as volume current densities through the use of delta functions. Figure 1.4 shows examples of this procedure for electric and magnetic currents.

1.3 FIELDS IN MEDIA AND BOUNDARY CONDITIONS

In the preceding section it was assumed that the electric and magnetic fields were in free-space, with no material bodies present. In practice, material bodies are often present; this complicates the analysis but also allows the useful application of material properties to microwave components. When electromagnetic fields exist in material media, the field vectors are related to each other by the constitutive relations.

For a dielectric material, an applied electric field $\bar{E}$ causes the polarization of the atoms or molecules of the material to create electric dipole moments that augment the total displacement flux, $\bar{D}$. This additional polarization vector is called $\bar{P}_e$, the *electric polarization*, where

$$\bar{D} = \epsilon_0\bar{E} + \bar{P}_e. \tag{1.15}$$

In a linear medium the electric polarization is linearly related to the applied electric field as

$$\bar{P}_e = \epsilon_0\chi_e\bar{E}, \tag{1.16}$$

where χ_e, which may be complex, is called the *electric susceptibility*. Then,

$$\bar{D} = \epsilon_0\bar{E} + \bar{P}_e = \epsilon_0(1+\chi_e)\bar{E} = \epsilon\bar{E}, \tag{1.17}$$

where

$$\epsilon = \epsilon' - j\epsilon'' = \epsilon_0(1+\chi_e) \tag{1.18}$$

is the complex permittivity of the medium. The imaginary part of ϵ accounts for loss in the medium (heat) due to damping of the vibrating dipole moments. (Free-space, having a real ϵ, is lossless.) Due to energy conservation, as we will see in Section 1.6, the imaginary part of ϵ must be negative (ϵ'' positive). The loss of a dielectric material may also be considered as an equivalent conductor loss. In a material with conductivity σ, a conduction current density will exist:

$$\bar{J} = \sigma\bar{E}, \tag{1.19}$$

which is *Ohm's law* from an electromagnetic field point of view. Maxwell's curl equation for $\bar{H}$ in (1.14b) then becomes

$$\begin{aligned}\nabla \times \bar{H} &= j\omega\bar{D} + \bar{J}\\ &= j\omega\epsilon\bar{E} + \sigma\bar{E}\\ &= j\omega\epsilon'\bar{E} + (\omega\epsilon'' + \sigma)\bar{E}\\ &= j\omega\left(\epsilon' - j\epsilon'' - j\frac{\sigma}{\omega}\right)\bar{E},\end{aligned} \tag{1.20}$$

where it is seen that loss due to dielectric damping ($\omega\epsilon''$) is indistinguishable from conductivity loss (σ). The term $\omega\epsilon'' + \sigma$ can then be considered as the total effective conductivity. A related quantity of interest is the *loss tangent,* defined as

$$\tan\delta = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'}, \tag{1.21}$$

which is seen to be the ratio of the real to the imaginary part of the total displacement current. Microwave materials are usually characterized by specifying the real relative permittivity (the *dielectric constant*),[2] ϵ_r, with $\epsilon' = \epsilon_r\epsilon_0$, and the loss tangent at a certain frequency. These properties are listed in Appendix G for several types of materials. It is useful to note that, after a problem has been solved assuming a lossless dielectric, loss can easily be introduced by replacing the real ϵ with a complex $\epsilon = \epsilon' - j\epsilon'' = \epsilon'(1 - j\tan\delta) = \epsilon_0\epsilon_r(1 - j\tan\delta)$.

In the preceding discussion it was assumed that $\bar{P}_e$ was a vector in the same direction as $\bar{E}$. Such materials are called *isotropic* materials, but not all materials have this property. Some materials are *anisotropic* and are characterized by a more complicated relation between $\bar{P}_e$ and $\bar{E}$, or $\bar{D}$ and $\bar{E}$. The most general linear relation between these vectors takes the form of a tensor of rank two (a dyad), which can be written in matrix form as

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = [\epsilon] \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{1.22}$$

It is thus seen that a given vector component of $\bar{E}$ gives rise, in general, to three components of $\bar{D}$. Crystal structures and ionized gases are examples of anisotropic dielectrics. For a linear isotropic material, the matrix of (1.22) reduces to a diagonal matrix with elements ϵ.

An analogous situation occurs for magnetic materials. An applied magnetic field may align magnetic dipole moments in a magnetic material to produce a *magnetic polarization* (or magnetization) $\bar{P}_m$. Then,

$$\bar{B} = \mu_0(\bar{H} + \bar{P}_m). \tag{1.23}$$

For a linear magnetic material, $\bar{P}_m$ is linearly related to $\bar{H}$ as

$$\bar{P}_m = \chi_m\bar{H}, \tag{1.24}$$

where χ_m is a complex *magnetic susceptibility*. From (1.23) and (1.24),

$$\bar{B} = \mu_0(1 + \chi_m)\bar{H} = \mu\bar{H}, \tag{1.25}$$

[2] The *IEEE Standard Definitions of Terms for Radio Wave Propagation, IEEE Standard 211-1997,* suggests that the term "relative permittivity" be used instead of "dielectric constant." The *IEEE Standard Definitions of Terms for Antennas, IEEE Standard 145-1993,* however, still recognizes "dielectric constant." Since this term is commonly used in microwave engineering work, it will occasionally be used in this book.

where $\mu = \mu_0(1 + \chi_m) = \mu' - j\mu''$ is the complex permeability of the medium. Again, the imaginary part of χ_m or μ accounts for loss due to damping forces; there is no magnetic conductivity because there is no real magnetic current. As in the electric case, magnetic materials may be anisotropic, in which case a tensor permeability can be written as

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = [\mu] \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}. \tag{1.26}$$

An important example of anisotropic magnetic materials in microwave engineering is the class of ferrimagnetic materials known as *ferrites*; these materials and their applications will be discussed further in Chapter 9.

If linear media are assumed (ϵ, μ not depending on $\bar{E}$ or $\bar{H}$), then Maxwell's equations can be written in phasor form as

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} - \bar{M}, \tag{1.27a}$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E} + \bar{J}, \tag{1.27b}$$

$$\nabla \cdot \bar{D} = \rho, \tag{1.27c}$$

$$\nabla \cdot \bar{B} = 0. \tag{1.27d}$$

The constitutive relations are

$$\bar{D} = \epsilon\bar{E}, \tag{1.28a}$$

$$\bar{B} = \mu\bar{H}, \tag{1.28b}$$

where ϵ and μ may be complex and may be tensors. Note that relations like (1.28a) and (1.28b) generally cannot be written in time domain form, even for linear media, because of the possible phase shift between $\bar{D}$ and $\bar{E}$, or $\bar{B}$ and $\bar{H}$. The phasor representation accounts for this phase shift by the complex form of ϵ and μ.

Maxwell's equations (1.27a)–(1.27d) in differential form require known boundary values for a complete and unique solution. A general method used throughout this book is to solve the source-free Maxwell equations in a certain region to obtain solutions with unknown coefficients and then apply boundary conditions to solve for these coefficients. A number of specific cases of boundary conditions arise, as discussed in what follows.

Fields at a General Material Interface

Consider a plane interface between two media, as shown in Figure 1.5. Maxwell's equations in integral form can be used to deduce conditions involving the normal and tangential

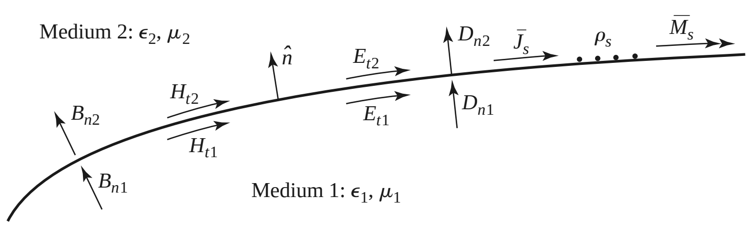


FIGURE 1.5 Fields, currents, and surface charge at a general interface between two media.

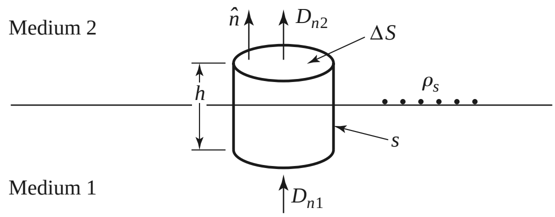


FIGURE 1.6 Closed surface S for equation (1.29).

fields at this interface. The time-harmonic version of (1.4), where S is the closed "pillbox"-shaped surface shown in Figure 1.6, can be written as

$$\oint_S \bar{D} \cdot d\bar{s} = \int_V \rho \, dv. \tag{1.29}$$

In the limit as $h \to 0$, the contribution of $D_{\tan}$ through the sidewalls goes to zero, so (1.29) reduces to

$$\Delta S D_{2n} - \Delta S D_{1n} = \Delta S \rho_s,$$

or

$$D_{2n} - D_{1n} = \rho_s, \tag{1.30}$$

where ρ_s is the surface charge density on the interface. In vector form, we can write

$$\hat{n} \cdot (\bar{D}_2 - \bar{D}_1) = \rho_s. \tag{1.31}$$

A similar argument for $\bar{B}$ leads to the result that

$$\hat{n} \cdot \bar{B}_2 = \hat{n} \cdot \bar{B}_1, \tag{1.32}$$

because there is no free magnetic charge.

For the tangential components of the electric field we use the phasor form of (1.6),

$$\oint_C \bar{E} \cdot d\bar{l} = -j\omega \int_S \bar{B} \cdot d\bar{s} - \int_S \bar{M} \cdot d\bar{s}, \tag{1.33}$$

in connection with the closed contour C shown in Figure 1.7. In the limit as $h \to 0$, the surface integral of $\bar{B}$ vanishes (because $S = h\Delta\ell$ vanishes). The contribution from the surface integral of $\bar{M}$, however, may be nonzero if a magnetic surface current density $\bar{M}_s$ exists on the surface. The Dirac delta function can then be used to write

$$\bar{M} = \bar{M}_s \delta(h), \tag{1.34}$$

where h is a coordinate measured normal from the interface. Equation (1.33) then gives

$$\Delta\ell E_{t1} - \Delta\ell E_{t2} = -\Delta\ell M_s,$$

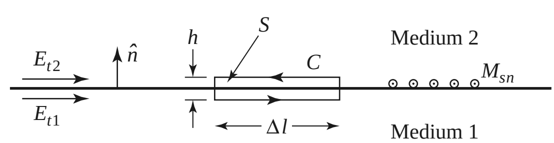


FIGURE 1.7 Closed contour C for equation (1.33).

or

$$E_{t1} - E_{t2} = -M_s, \tag{1.35}$$

which can be generalized in vector form as

$$(\bar{E}_2 - \bar{E}_1) \times \hat{n} = \bar{M}_s. \tag{1.36}$$

A similar argument for the magnetic field leads to

$$\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s, \tag{1.37}$$

where $\bar{J}_s$ is an electric surface current density that may exist at the interface. Equations (1.31), (1.32), (1.36), and (1.37) are the most general expressions for the boundary conditions at an arbitrary interface of materials and/or surface currents.

Fields at a Dielectric Interface

At an interface between two lossless dielectric materials, no charge or surface current densities will ordinarily exist. Equations (1.31), (1.32), (1.36), and (1.37) then reduce to

$$\hat{n} \cdot \bar{D}_1 = \hat{n} \cdot \bar{D}_2, \tag{1.38a}$$

$$\hat{n} \cdot \bar{B}_1 = \hat{n} \cdot \bar{B}_2, \tag{1.38b}$$

$$\hat{n} \times \bar{E}_1 = \hat{n} \times \bar{E}_2, \tag{1.38c}$$

$$\hat{n} \times \bar{H}_1 = \hat{n} \times \bar{H}_2. \tag{1.38d}$$

In words, these equations state that the normal components of $\bar{D}$ and $\bar{B}$ are continuous across the interface, and the tangential components of $\bar{E}$ and $\bar{H}$ are continuous across the interface. Because Maxwell's equations are not all linearly independent, the six boundary conditions contained in the above equations are not all linearly independent. Thus, the enforcement of (1.38c) and (1.38d) for the four tangential field components, for example, will automatically force the satisfaction of the equations for the continuity of the normal components.

Fields at the Interface with a Perfect Conductor (Electric Wall)

Many problems in microwave engineering involve boundaries with good conductors (e.g., metals), which can often be assumed as lossless ($\sigma \to \infty$). In this case of a perfect conductor, all field components must be zero inside the conducting region. This result can be seen by considering a conductor with finite conductivity ($\sigma < \infty$) and noting that the skin depth (the depth to which most of the microwave power penetrates) goes to zero as $\sigma \to \infty$. (Such an analysis will be performed in Section 1.7.) If we also assume here that $\bar{M}_s = 0$, which would be the case if the perfect conductor filled all the space on one side of the boundary, then (1.31), (1.32), (1.36), and (1.37) reduce to the following:

$$\hat{n} \cdot \bar{D} = \rho_s, \tag{1.39a}$$

$$\hat{n} \cdot \bar{B} = 0, \tag{1.39b}$$

$$\hat{n} \times \bar{E} = 0, \tag{1.39c}$$

$$\hat{n} \times \bar{H} = \bar{J}_s, \tag{1.39d}$$

where ρ_s and $\bar{J}_s$ are the electric surface charge density and current density, respectively, on the interface, and $\hat{n}$ is the normal unit vector pointing out of the perfect conductor. Such a boundary is also known as an *electric wall* because the tangential components of $\bar{E}$ are "shorted out," as seen from (1.39c), and must vanish at the surface of the conductor.

The Magnetic Wall Boundary Condition

Dual to the preceding boundary condition is the *magnetic wall* boundary condition, where the tangential components of $\bar{H}$ must vanish. Such a boundary does not really exist in practice but may be approximated by a corrugated surface or in certain planar transmission line problems. In addition, the idealization that $\hat{n} \times \bar{H} = 0$ at an interface is often a convenient simplification, as we will see in later chapters. We will also see that the magnetic wall boundary condition is analogous to the relations between the voltage and current at the end of an open-circuited transmission line, while the electric wall boundary condition is analogous to the voltage and current at the end of a short-circuited transmission line. The magnetic wall condition, then, provides a degree of completeness in our formulation of boundary conditions and is a useful approximation in several cases of practical interest.

The fields at a magnetic wall satisfy the following conditions:

$$\hat{n} \cdot \bar{D} = 0, \tag{1.40a}$$

$$\hat{n} \cdot \bar{B} = 0, \tag{1.40b}$$

$$\hat{n} \times \bar{E} = -\bar{M}_s, \tag{1.40c}$$

$$\hat{n} \times \bar{H} = 0, \tag{1.40d}$$

where $\hat{n}$ is the normal unit vector pointing out of the magnetic wall region.

The Radiation Condition

When dealing with problems that have one or more infinite boundaries, such as plane waves in an infinite medium, or infinitely long transmission lines, a condition on the fields at infinity must be enforced. This boundary condition is known as the *radiation condition* and is essentially a statement of energy conservation. It states that, at an infinite distance from a source, the fields must either be vanishingly small (i.e., zero) or propagating in an outward direction. This result can easily be seen by allowing the infinite medium to contain a small loss factor (as any physical medium would have). Incoming waves (from infinity) of finite amplitude would then require an infinite source at infinity and so are disallowed.

1.4 THE WAVE EQUATION AND BASIC PLANE WAVE SOLUTIONS

The Helmholtz Equation

In a source-free, linear, isotropic, homogeneous region, Maxwell's curl equations in phasor form are

$$\nabla \times \bar{E} = -j\omega\mu\bar{H}, \tag{1.41a}$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E}, \tag{1.41b}$$

and constitute two equations for the two unknowns, $\bar{E}$ and $\bar{H}$. As such, they can be solved for either $\bar{E}$ or $\bar{H}$. Taking the curl of (1.41a) and using (1.41b) gives

$$\nabla \times \nabla \times \bar{E} = -j\omega\mu\nabla \times \bar{H} = \omega^2\mu\epsilon\bar{E},$$

which is an equation for $\bar{E}$. This result can be simplified through the use of vector identity (B.14), $\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$, which is valid for the rectangular components of an arbitrary vector $\bar{A}$. Then,

$$\nabla^2 \bar{E} + \omega^2 \mu \epsilon \bar{E} = 0, \tag{1.42}$$

because $\nabla \cdot \bar{E} = 0$ in a source-free region. Equation (1.42) is the *wave equation*, or *Helmholtz equation*, for $\bar{E}$. An identical equation for $\bar{H}$ can be derived in the same manner:

$$\nabla^2 \bar{H} + \omega^2 \mu \epsilon \bar{H} = 0. \tag{1.43}$$

A constant $k = \omega\sqrt{\mu\epsilon}$ is defined and called the *propagation constant* (also known as the *phase constant*, or *wave number*), of the medium; its units are 1/m.

As a way of introducing wave behavior, we will next study the solutions to the above wave equations in their simplest forms, first for a lossless medium and then for a lossy (conducting) medium.

Plane Waves in a Lossless Medium

In a lossless medium, ϵ and μ are real numbers, and so k is real. A basic plane wave solution to the above wave equations can be found by considering an electric field with only an $\hat{x}$ component and uniform (no variation) in the x and y directions. Then, $\partial/\partial x = \partial/\partial y = 0$, and the Helmholtz equation of (1.42) reduces to

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0. \tag{1.44}$$

The two independent solutions to this equation are easily seen, by substitution, to be of the form

$$E_x(z) = E^+ e^{-jkz} + E^- e^{jkz}, \tag{1.45}$$

where E^+ and E^- are arbitrary amplitude constants.

The above solution is for the time harmonic case at frequency ω. In the time domain, this result is written as

$$\mathcal{E}_x(z, t) = E^+ \cos(\omega t - kz) + E^- \cos(\omega t + kz), \tag{1.46}$$

where we have assumed that E^+ and E^- are real constants. Consider the first term in (1.46). This term represents a wave traveling in the $+z$ direction because, to maintain a fixed point on the wave ($\omega t - kz =$ constant), one must move in the $+z$ direction as time increases. Similarly, the second term in (1.46) represents a wave traveling in the negative z direction—hence the notation E^+ and E^- for these wave amplitudes. The velocity of the wave in this sense is called the *phase velocity* because it is the velocity at which a fixed phase point on the wave travels, and it is given by

$$v_p = \frac{dz}{dt} = \frac{d}{dt}\left(\frac{\omega t - \text{constant}}{k}\right) = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \tag{1.47}$$

In free-space, we have $v_p = 1/\sqrt{\mu_0\epsilon_0} = c = 2.998 \times 10^8$ m/sec, which is the speed of light.

The *wavelength*, λ, is defined as the distance between two successive maxima (or minima, or any other reference points) on the wave at a fixed instant of time. Thus,

$$(\omega t - kz) - [\omega t - k(z + \lambda)] = 2\pi,$$

so

$$\lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}. \tag{1.48}$$

A complete specification of the plane wave electromagnetic field should include the magnetic field. In general, whenever $\bar{E}$ or $\bar{H}$ is known, the other field vector can be readily found by using one of Maxwell's curl equations. Thus, applying (1.41a) to the electric field of (1.45) gives $H_x = H_z = 0$, and

$$H_y = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z} = \frac{1}{\eta}(E^+ e^{-jkz} - E^- e^{jkz}), \tag{1.49}$$

where $\eta = \omega\mu/k = \sqrt{\mu/\epsilon}$ is known as the *intrinsic impedance* of the medium. The ratio of the $\bar{E}$ and $\bar{H}$ field components is seen to have units of impedance, known as the *wave impedance*; for planes waves the wave impedance is equal to the intrinsic impedance of the medium. In free-space the intrinsic impedance is $\eta_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$. Note that the $\bar{E}$ and $\bar{H}$ vectors are orthogonal to each other and orthogonal to the direction of propagation ($\pm\hat{z}$); this is a characteristic of transverse electromagnetic (TEM) waves.

EXAMPLE 1.1 BASIC PLANE WAVE PARAMETERS

A plane wave propagating in a lossless dielectric medium has an electric field given as $\mathcal{E}_x = E_0 \cos(\omega t - \beta z)$ with a frequency of 5.0 GHz and a wavelength in the material of 3.0 cm. Determine the propagation constant, the phase velocity, the relative permittivity of the medium, and the wave impedance.

Solution

From (1.48) the propagation constant is $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{0.03} = 209.4\ \text{m}^{-1}$, and from (1.47) the phase velocity is

$$v_p = \frac{\omega}{k} = \frac{2\pi f}{k} = \lambda f = (0.03)\,(5 \times 10^9) = 1.5 \times 10^8\ \text{m/sec}.$$

This is slower than the speed of light by a factor of 2.0. The relative permittivity of the medium can be found from (1.47) as

$$\epsilon_r = \left(\frac{c}{v_p}\right)^2 = \left(\frac{3.0 \times 10^8}{1.5 \times 10^8}\right)^2 = 4.0$$

The wave impedance is

$$\eta = \eta_0/\sqrt{\epsilon_r} = \frac{377}{\sqrt{4.0}} = 188.5\ \Omega \qquad \blacksquare$$

Plane Waves in a General Lossy Medium

Now consider the effect of a lossy medium. If the medium is conductive, with a conductivity σ, Maxwell's curl equations can be written, from (1.41a) and (1.20) as

$$\nabla \times \bar{E} = -j\omega\mu\bar{H}, \tag{1.50a}$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E} + \sigma\bar{E}. \tag{1.50b}$$

The resulting wave equation for $\bar{E}$ then becomes

$$\nabla^2 \bar{E} + \omega^2 \mu\epsilon \left(1 - j\frac{\sigma}{\omega\epsilon}\right) \bar{E} = 0, \tag{1.51}$$

where we see a similarity with (1.42), the wave equation for $\bar{E}$ in the lossless case. The difference is that the quantity $k^2 = \omega^2\mu\epsilon$ of (1.42) is replaced by $\omega^2\mu\epsilon[1 - j(\sigma/\omega\epsilon)]$ in (1.51). We then define a *complex propagation constant* for the medium as

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\sqrt{1 - j\frac{\sigma}{\omega\epsilon}} \tag{1.52}$$

where α is the *attenuation constant* and β is the *phase constant*. If we again assume an electric field with only an $\hat{x}$ component and uniform in x and y, the wave equation of (1.51) reduces to

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0, \tag{1.53}$$

which has solutions

$$E_x(z) = E^+ e^{-\gamma z} + E^- e^{\gamma z}. \tag{1.54}$$

The positive traveling wave then has a propagation factor of the form

$$e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z},$$

which in the time domain is of the form

$$e^{-\alpha z} \cos(\omega t - \beta z).$$

We see that this represents a wave traveling in the $+z$ direction with a phase velocity $v_p = \omega/\beta$, a wavelength $\lambda = 2\pi/\beta$, and an exponential damping factor. The rate of decay with distance is given by the attenuation constant, α. The negative traveling wave term of (1.54) is similarly damped along the $-z$ axis. If the loss is removed, $\sigma = 0$, and we have $\gamma = jk$ and $\alpha = 0$, $\beta = k$.

As discussed in Section 1.3, loss can also be treated through the use of a complex permittivity. From (1.52) and (1.20) with $\sigma = 0$ but $\epsilon = \epsilon' - j\epsilon''$ complex, we have that

$$\gamma = j\omega\sqrt{\mu\epsilon} = jk = j\omega\sqrt{\mu\epsilon'(1 - j\tan\delta)}, \tag{1.55}$$

where $\tan\delta = \epsilon''/\epsilon'$ is the loss tangent of the material.

The associated magnetic field can be calculated as

$$H_y = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z} = \frac{-j\gamma}{\omega\mu}(E^+ e^{-\gamma z} - E^- e^{\gamma z}). \tag{1.56}$$

The intrinsic impedance of the conducting medium is now complex,

$$\eta = \frac{j\omega\mu}{\gamma}, \tag{1.57}$$

but is still identified as the wave impedance, which expresses the ratio of electric to magnetic field components. This allows (1.56) to be rewritten as

$$H_y = \frac{1}{\eta}(E^+ e^{-\gamma z} - E^- e^{\gamma z}). \tag{1.58}$$

Note that although η of (1.57) is, in general, complex, it reduces to the lossless case of $\eta = \sqrt{\mu/\epsilon}$ when $\gamma = jk = j\omega\sqrt{\mu\epsilon}$.

Plane Waves in a Good Conductor

Many problems of practical interest involve loss or attenuation due to good (but not perfect) conductors. A good conductor is a special case of the preceding analysis, where the conductive current is much greater than the displacement current, which means that $\sigma \gg \omega\epsilon$. Most metals can be categorized as good conductors. In terms of a complex ϵ, rather than conductivity, this condition is equivalent to $\epsilon'' \gg \epsilon'$. The propagation constant of (1.52) can then be adequately approximated by ignoring the displacement current term, to give

$$\gamma = \alpha + j\beta \simeq j\omega\sqrt{\mu\epsilon}\sqrt{\frac{\sigma}{j\omega\epsilon}} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}. \tag{1.59}$$

The *skin depth*, or characteristic depth of penetration, is defined as

$$\delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}. \tag{1.60}$$

Thus the amplitude of the fields in the conductor will decay by an amount $1/e$, or 36.8%, after traveling a distance of one skin depth, because $e^{-\alpha z} = e^{-\alpha\delta_s} = e^{-1}$. At microwave frequencies, for a good conductor, this distance is very small. The practical importance of this result is that only a thin plating of a good conductor (e.g., silver or gold) is necessary for low-loss microwave components.

EXAMPLE 1.2 SKIN DEPTH AT MICROWAVE FREQUENCIES

Compute the skin depth of aluminum, copper, gold, and silver at a frequency of 10 GHz.

Solution

The conductivities for these metals are listed in Appendix F. Equation (1.60) gives the skin depths as

$$\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f \mu_0 \sigma}} = \sqrt{\frac{1}{\pi(10^{10})(4\pi\times 10^{-7})}}\sqrt{\frac{1}{\sigma}}$$
$$= 5.03\times 10^{-3}\sqrt{\frac{1}{\sigma}}.$$

For aluminum: $\delta_s = 5.03\times 10^{-3}\sqrt{\dfrac{1}{3.816\times 10^7}} = 8.14\times 10^{-7}\text{m}.$

For copper: $\delta_s = 5.03\times 10^{-3}\sqrt{\dfrac{1}{5.813\times 10^7}} = 6.60\times 10^{-7}\text{m}.$

For gold: $\delta_s = 5.03\times 10^{-3}\sqrt{\dfrac{1}{4.098\times 10^7}} = 7.86\times 10^{-7}\text{m}.$

For silver: $\delta_s = 5.03\times 10^{-3}\sqrt{\dfrac{1}{6.173\times 10^7}} = 6.40\times 10^{-7}\text{m}.$

These results show that most of the current flow in a good conductor occurs in an extremely thin region near the surface of the conductor. ■

TABLE 1.1 Summary of Results for Plane Wave Propagation in Various Media

	Type of Medium		
Quantity	Lossless ($\epsilon'' = \sigma = 0$)	General Lossy	Good Conductor ($\epsilon'' \gg \epsilon'$ or $\sigma \gg \omega\epsilon'$)
Complex propagation constant	$\gamma = j\omega\sqrt{\mu\epsilon}$	$\gamma = j\omega\sqrt{\mu\epsilon}$ $= j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\dfrac{\sigma}{\omega\epsilon'}}$	$\gamma = (1+j)\sqrt{\omega\mu\sigma/2}$
Phase constant (wave number)	$\beta = k = \omega\sqrt{\mu\epsilon}$	$\beta = \text{Im}\{\gamma\}$	$\beta = \text{Im}\{\gamma\} = \sqrt{\omega\mu\sigma/2}$
Attenuation constant	$\alpha = 0$	$\alpha = \text{Re}\{\gamma\}$	$\alpha = \text{Re}\{\gamma\} = \sqrt{\omega\mu\sigma/2}$
Impedance	$\eta = \sqrt{\mu/\epsilon} = \omega\mu/k$	$\eta = j\omega\mu/\gamma$	$\eta = (1+j)\sqrt{\omega\mu/2\sigma}$
Skin depth	$\delta_s = \infty$	$\delta_s = 1/\alpha$	$\delta_s = \sqrt{2/\omega\mu\sigma}$
Wavelength	$\lambda = 2\pi/\beta$	$\lambda = 2\pi/\beta$	$\lambda = 2\pi/\beta$
Phase velocity	$v_p = \omega/\beta$	$v_p = \omega/\beta$	$v_p = \omega/\beta$

The intrinsic impedance inside a good conductor can be obtained from (1.57) and (1.59). The result is

$$\eta = \frac{j\omega\mu}{\gamma} \simeq (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\frac{1}{\sigma\delta_s}. \tag{1.61}$$

Notice that the phase angle of this impedance is 45°, a characteristic of good conductors. The phase angle of the impedance for a lossless material is 0°, and the phase angle of the impedance of an arbitrary lossy medium is somewhere between 0° and 45°.

Table 1.1 summarizes the results for plane wave propagation in lossless and lossy homogeneous media.

1.5 GENERAL PLANE WAVE SOLUTIONS

Some specific features of plane waves were discussed in Section 1.4, but we will now look at plane waves from a more general point of view and solve the wave equation by the method of separation of variables. This technique will find application in succeeding chapters. We will also discuss circularly polarized plane waves, which will be important for the discussion of ferrites in Chapter 9.

In free-space, the Helmholtz equation for $\bar{E}$ can be written as

$$\nabla^2\bar{E} + k_0^2\bar{E} = \frac{\partial^2\bar{E}}{\partial x^2} + \frac{\partial^2\bar{E}}{\partial y^2} + \frac{\partial^2\bar{E}}{\partial z^2} + k_0^2\bar{E} = 0, \tag{1.62}$$

and this vector wave equation holds for each rectangular component of $\bar{E}$:

$$\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} + k_0^2 E_i = 0, \tag{1.63}$$

where the index $i = x, y$, or z. This equation can be solved by the method of *separation of variables*, a standard technique for treating such partial differential equations. The method begins by assuming that the solution to (1.63) for, say, E_x, can be written as a product of three functions for each of the three coordinates:

$$E_x(x, y, z) = f(x)g(y)h(z). \tag{1.64}$$

Substituting this form into (1.63) and dividing by fgh gives

$$\frac{f''}{f}+\frac{g''}{g}+\frac{h''}{h}+k_0^2=0, \tag{1.65}$$

where the double primes denote the second derivative. The key step in the argument is to recognize that each of the terms in (1.65) must be equal to a constant because they are independent of each other. That is, f''/f is only a function of x, and the remaining terms in (1.65) do not depend on x, so f''/f must be a constant, and similarly for the other terms in (1.65). Thus, we define three separation constants, k_x, k_y, and k_z, such that

$$f''/f=-k_x^2;\quad g''/g=-k_y^2;\quad h''/h=-k_z^2;$$

or

$$\frac{d^2f}{dx^2}+k_x^2f=0;\quad \frac{d^2g}{dy^2}+k_y^2g=0;\quad \frac{d^2h}{dz^2}+k_z^2h=0. \tag{1.66}$$

Combining (1.65) and (1.66) shows that

$$k_x^2+k_y^2+k_z^2=k_0^2. \tag{1.67}$$

The partial differential equation of (1.63) has now been reduced to three separate ordinary differential equations in (1.66). Solutions to these equations have the forms $e^{\pm jk_x x}$, $e^{\pm jk_y y}$, and $e^{\pm jk_z z}$, respectively. As we saw in the previous section, the terms with $+$ signs result in waves traveling in the negative x, y, or z direction, while the terms with $-$ signs result in waves traveling in the positive direction. Both solutions are possible and are valid; the amount to which these various terms are excited is dependent on the source of the fields and the boundary conditions. For our present discussion we will select a plane wave traveling in the positive direction for each coordinate and write the complete solution for E_x as

$$E_x(x,\ y,\ z)=Ae^{-j(k_x x+k_y y+k_z z)}, \tag{1.68}$$

where A is an arbitrary amplitude constant. Now define a wave number vector $\bar{k}$ as

$$\bar{k}=k_x\hat{x}+k_y\hat{y}+k_z\hat{z}=k_0\hat{n}. \tag{1.69}$$

Then from (1.67), $|\bar{k}|=k_0$, and so $\hat{n}$ is a unit vector in the direction of propagation. Also define a position vector as

$$\bar{r}=x\hat{x}+y\hat{y}+z\hat{z}; \tag{1.70}$$

then (1.68) can be written as

$$E_x(x,\ y,\ z)=Ae^{-j\bar{k}\cdot\bar{r}}. \tag{1.71}$$

Solutions to (1.63) for E_y and E_z are, of course, similar in form to E_x of (1.71), but with different amplitude constants:

$$E_y(x,y,z)=Be^{-j\bar{k}\cdot\bar{r}}, \tag{1.72}$$

$$E_z(x,y,z)=Ce^{-j\bar{k}\cdot\bar{r}}. \tag{1.73}$$

The x, y, and z dependences of the three components of $\bar{E}$ in (1.71)–(1.73) must be the same (same k_x, k_y, k_z), because the divergence condition that

$$\nabla\cdot\bar{E}=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0$$

must also be applied in order to satisfy Maxwell's equations, and this implies that E_x, E_y, and E_z must each have the same variation in x, y, and z. (Note that the solutions in the preceding section automatically satisfied the divergence condition because E_x was the only component of $\bar{E}$, and E_x did not vary with x.) This condition also imposes a constraint on the amplitudes A, B, and C because if

$$\bar{E}_0 = A\hat{x} + B\hat{y} + C\hat{z},$$

we have

$$\bar{E} = \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}},$$

and

$$\nabla \cdot \bar{E} = \nabla \cdot (\bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}) = \bar{E}_0 \cdot \nabla e^{-j\bar{k}\cdot\bar{r}} = -j\bar{k} \cdot \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}} = 0,$$

where vector identity (B.7) was used. Thus, we must have

$$\bar{k} \cdot \bar{E}_0 = 0, \tag{1.74}$$

which means that the electric field amplitude vector $\bar{E}_0$ must be perpendicular to the direction of propagation, $\bar{k}$. This condition is a general result for plane waves and implies that only two of the three amplitude constants, A, B, and C, can be chosen independently.

The magnetic field can be found from Maxwell's equation,

$$\nabla \times \bar{E} = -j\omega\mu_0 \bar{H}, \tag{1.75}$$

to give

$$\begin{aligned}
\bar{H} &= \frac{j}{\omega\mu_0} \nabla \times \bar{E} = \frac{j}{\omega\mu_0} \nabla \times (\bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}) \\
&= \frac{-j}{\omega\mu_0} \bar{E}_0 \times \nabla e^{-j\bar{k}\cdot\bar{r}} \\
&= \frac{-j}{\omega\mu_0} \bar{E}_0 \times (-j\bar{k}) e^{-j\bar{k}\cdot\bar{r}} \\
&= \frac{k_0}{\omega\mu_0} \hat{n} \times \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}} \\
&= \frac{1}{\eta_0} \hat{n} \times \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}} \\
&= \frac{1}{\eta_0} \hat{n} \times \bar{E},
\end{aligned} \tag{1.76}$$

where vector identity (B.9) was used in obtaining the second line. This result shows that the magnetic field vector $\bar{H}$ lies in a plane normal to $\bar{k}$, the direction of propagation, and that $\bar{H}$ is perpendicular to $\bar{E}$. See Figure 1.8 for an illustration of these vector relations. The quantity $\eta_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$ in (1.76) is the intrinsic impedance of free-space.

The time domain expression for the electric field can be found as

$$\begin{aligned}
\bar{\mathcal{E}}(x, y, z, t) &= \text{Re}\left\{\bar{E}(x, y, z) e^{j\omega t}\right\} \\
&= \text{Re}\left\{\bar{E}_0 e^{-j\bar{k}\cdot\bar{r}} e^{j\omega t}\right\} \\
&= \bar{E}_0 \cos(\bar{k} \cdot \bar{r} - \omega t),
\end{aligned} \tag{1.77}$$

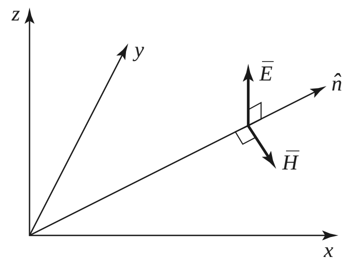
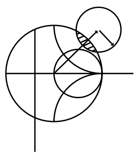


FIGURE 1.8 Orientation of the $\bar{E}$, $\bar{H}$, and $\bar{k} = k_0\hat{n}$ vectors for a general plane wave.

assuming that the amplitude constants A, B, and C contained in $\bar{E}_0$ are real. If these constants are not real, their phases should be included inside the cosine term of (1.77). It is easy to show that the wavelength and phase velocity for this solution are the same as obtained in Section 1.4.



EXAMPLE 1.3 CURRENT SHEETS AS SOURCES OF PLANE WAVES

An infinite sheet of surface current can be considered as a source for plane waves. If an electric surface current density $\bar{J}_s = J_0\hat{x}$ exists on the $z = 0$ plane in free-space, find the resulting fields by assuming plane waves on either side of the current sheet and enforcing boundary conditions.

Solution

Since the source does not vary with x or y, the fields will not vary with x or y but will propagate away from the source in the $\pm z$ direction. The boundary conditions to be satisfied at $z = 0$ are

$$\hat{n} \times (\bar{E}_2 - \bar{E}_1) = \hat{z} \times (\bar{E}_2 - \bar{E}_1) = 0,$$
$$\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \hat{z} \times (\bar{H}_2 - \bar{H}_1) = J_0\hat{x},$$

where $\bar{E}_1$, $\bar{H}_1$ are the fields for $z < 0$, and $\bar{E}_2$, $\bar{H}_2$ are the fields for $z > 0$. To satisfy the second condition, $\bar{H}$ must have a $\hat{y}$ component. Then for $\bar{E}$ to be orthogonal to $\bar{H}$ and $\hat{z}$, $\bar{E}$ must have an $\hat{x}$ component. Thus the fields will have the following form:

$$\text{for } z < 0, \qquad \bar{E}_1 = \hat{x}A\eta_0 e^{jk_0 z},$$
$$\bar{H}_1 = -\hat{y}Ae^{jk_0 z},$$
$$\text{for } z > 0, \qquad \bar{E}_2 = \hat{x}B\eta_0 e^{-jk_0 z},$$
$$\bar{H}_2 = \hat{y}Be^{-jk_0 z},$$

where A and B are arbitrary amplitude constants. The first boundary condition, that E_x is continuous at $z = 0$, yields $A = B$, while the boundary condition for $\bar{H}$ yields the equation

$$-B - A = J_0.$$

Solving for A, B gives

$$A = B = -J_0/2,$$

which completes the solution. ■

Circularly Polarized Plane Waves

The plane waves discussed previously all had their electric field vector pointing in a fixed direction and so are called *linearly polarized* waves. In general, the *polarization* of a plane wave refers to the orientation of the electric field vector, which may be in a fixed direction or may change with time.

Consider the superposition of an $\hat{x}$ linearly polarized wave with amplitude E_1 and a $\hat{y}$ linearly polarized wave with amplitude E_2, both traveling in the positive $\hat{z}$ direction. The total electric field can be written as

$$\bar{E} = (E_1\hat{x} + E_2\hat{y})e^{-jk_0z}. \tag{1.78}$$

A number of possibilities now arise. If $E_1 \neq 0$ and $E_2 = 0$, we have a plane wave linearly polarized in the $\hat{x}$ direction. Similarly, if $E_1 = 0$ and $E_2 \neq 0$, we have a plane wave linearly polarized in the $\hat{y}$ direction. If E_1 and E_2 are both real and nonzero, we have a plane wave linearly polarized at the angle

$$\phi = \tan^{-1}\frac{E_2}{E_1}.$$

For example, if $E_1 = E_2 = E_0$, we have

$$\bar{E} = E_0(\hat{x} + \hat{y})e^{-jk_0z},$$

which represents an electric field vector at a 45° angle from the x-axis.

Now consider the case in which $E_1 = jE_2 = E_0$, where E_0 is real, so that

$$\bar{E} = E_0(\hat{x} - j\hat{y})e^{-jk_0z}. \tag{1.79}$$

The time domain form of this field is

$$\bar{\mathcal{E}}(z,\ t) = E_0[\hat{x}\cos(\omega t - k_0z) + \hat{y}\cos(\omega t - k_0z - \pi/2)]. \tag{1.80}$$

This expression shows that the electric field vector changes with time or, equivalently, with distance along the z-axis. To see this, pick a fixed position, say $z = 0$. Equation (1.80) then reduces to

$$\bar{\mathcal{E}}(0,\ t) = E_0[\hat{x}\cos\omega t + \hat{y}\sin\omega t], \tag{1.81}$$

so as ωt increases from zero, the electric field vector rotates counterclockwise from the x-axis. The resulting angle from the x-axis of the electric field vector at time t, at $z = 0$, is then

$$\phi = \tan^{-1}\left(\frac{\sin\omega t}{\cos\omega t}\right) = \omega t,$$

which shows that the polarization rotates at the uniform angular velocity ω. Since the fingers of the right hand point in the direction of rotation of the electric field vector when the thumb points in the direction of propagation, this type of wave is referred to as a *right-hand circularly polarized* (RHCP) wave. Similarly, a field of the form

$$\bar{E} = E_0(\hat{x} + j\hat{y})e^{-jk_0z} \tag{1.82}$$

constitutes a *left-hand circularly polarized* (LHCP) wave, where the electric field vector rotates in the opposite direction. See Figure 1.9 for a sketch of the polarization vectors for RHCP and LHCP plane waves.

The magnetic field associated with a circularly polarized wave may be found from Maxwell's equations or by using the wave impedance applied to each component of the

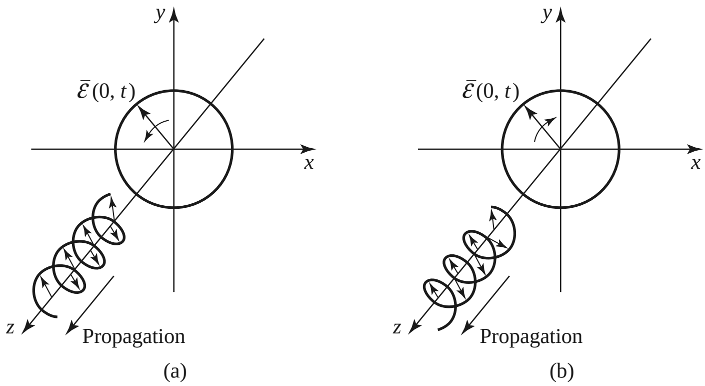


FIGURE 1.9 Electric field polarization for (a) RHCP and (b) LHCP plane waves.

electric field. For example, applying (1.76) to the electric field of a RHCP wave as given in (1.79) yields

$$\bar{H} = \frac{E_0}{\eta_0}\hat{z} \times (\hat{x} - j\hat{y})e^{-jk_0z} = \frac{E_0}{\eta_0}(\hat{y} + j\hat{x})e^{-jk_0z} = \frac{jE_0}{\eta_0}(\hat{x} - j\hat{y})e^{-jk_0z},$$

which is also seen to represent a vector rotating in the RHCP sense.

1.6 ENERGY AND POWER

In general, a source of electromagnetic energy sets up fields that store electric and magnetic energy and carry power that may be transmitted or dissipated as loss. In the sinusoidal steady-state case, the time-average stored electric energy in a volume V is given by

$$W_e = \frac{1}{4}\text{Re}\int_V \bar{E} \cdot \bar{D}^* \, dv, \tag{1.83}$$

which in the case of simple lossless isotropic, homogeneous, linear media, where ϵ is a real scalar constant, reduces to

$$W_e = \frac{\epsilon}{4}\int_V \bar{E} \cdot \bar{E}^* \, dv. \tag{1.84}$$

Similarly, the time-average magnetic energy stored in the volume V is

$$W_m = \frac{1}{4}\text{Re}\int_V \bar{H} \cdot \bar{B}^* \, dv, \tag{1.85}$$

which becomes

$$W_m = \frac{\mu}{4}\int_V \bar{H} \cdot \bar{H}^* \, dv, \tag{1.86}$$

for a real, constant, scalar μ.

We can now derive Poynting's theorem, which leads to energy conservation for electromagnetic fields and sources. If we have an electric source current $\bar{J}_s$ and a conduction current $\sigma\bar{E}$ as defined in (1.19), then the total electric current density is $\bar{J} = \bar{J}_s + \sigma\bar{E}$. Multiplying (1.27a) by $\bar{H}^*$ and multiplying the conjugate of (1.27b) by $\bar{E}$ yields

$$\bar{H}^* \cdot (\nabla \times \bar{E}) = -j\omega\mu|\bar{H}|^2 - \bar{H}^* \cdot \bar{M}_s,$$
$$\bar{E} \cdot (\nabla \times \bar{H}^*) = \bar{E} \cdot \bar{J}^* - j\omega\epsilon^*|\bar{E}|^2 = \bar{E} \cdot \bar{J}_s^* + \sigma|\bar{E}|^2 - j\omega\epsilon^*|\bar{E}|^2,$$

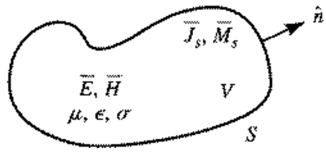


FIGURE 1.10 A volume V, enclosed by the closed surface S, containing fields $\bar{E}$, $\bar{H}$, and current sources $\bar{J}_s$, $\bar{M}_s$.

where $\bar{M}_s$ is the magnetic source current. Using these two results in vector identity (B.8) gives

$$\begin{aligned}\nabla \cdot (\bar{E} \times \bar{H}^*) &= \bar{H}^* \cdot (\nabla \times \bar{E}) - \bar{E} \cdot (\nabla \times \bar{H}^*) \\ &= -\sigma|\bar{E}|^2 + j\omega(\epsilon^*|\bar{E}|^2 - \mu|\bar{H}|^2) - (\bar{E} \cdot \bar{J}_s^* + \bar{H}^* \cdot \bar{M}_s).\end{aligned}$$

Now integrate over a volume V and use the divergence theorem:

$$\begin{aligned}\int_V \nabla \cdot (\bar{E} \times \bar{H}^*)\, dv &= \oint_S \bar{E} \times \bar{H}^* \cdot d\bar{s} \\ &= -\sigma \int_V |\bar{E}|^2\, dv + j\omega \int_V (\epsilon^*|\bar{E}|^2 - \mu|\bar{H}|^2)\, dv - \int_V (\bar{E} \cdot \bar{J}_s^* + \bar{H}^* \cdot \bar{M}_s)\, dv, \quad (1.87)\end{aligned}$$

where S is a closed surface enclosing the volume V, as shown in Figure 1.10. Allowing $\epsilon = \epsilon' - j\epsilon''$ and $\mu = \mu' - j\mu''$ to be complex to allow for loss, and rewriting (1.87), gives

$$\begin{aligned}-\frac{1}{2}\int_V (\bar{E} \cdot \bar{J}_s^* + \bar{H}^* \cdot \bar{M}_s)\, dv &= \frac{1}{2}\oint_S \bar{E} \times \bar{H}^* \cdot d\bar{s} + \frac{\sigma}{2}\int_V |\bar{E}|^2\, dv \\ &+ \frac{\omega}{2}\int_V (\epsilon''|\bar{E}|^2 + \mu''|\bar{H}|^2)\, dv + j\frac{\omega}{2}\int_V (\mu'|\bar{H}|^2 - \epsilon'|\bar{E}|^2)\, dv. \quad (1.88)\end{aligned}$$

This result is known as *Poynting's theorem*, after the physicist J. H. Poynting (1852–1914), and is basically a power balance equation. Thus, the integral on the left-hand side represents the complex power P_s delivered by the sources $\bar{J}_s$ and $\bar{M}_s$ inside S:

$$P_s = -\frac{1}{2}\int_V (\bar{E} \cdot \bar{J}_s^* + \bar{H}^* \cdot \bar{M}_s)\, dv. \quad (1.89)$$

The first integral on the right-hand side of (1.88) represents complex power flow out of the closed surface S. If we define a quantity $\bar{S}$, called the *Poynting vector*, as

$$\bar{S} = \bar{E} \times \bar{H}^*, \quad (1.90)$$

then this power can be expressed as

$$P_o = \frac{1}{2}\oint_S \bar{E} \times \bar{H}^* \cdot d\bar{s} = \frac{1}{2}\oint_S \bar{S} \cdot d\bar{s}. \quad (1.91)$$

The surface S in (1.91) must be a closed surface for this interpretation to be valid. The real parts of P_s and P_o in (1.89) and (1.91) represent time-average powers.

The second and third integrals in (1.88) are real quantities representing the time-average power dissipated in the volume V due to conductivity, dielectric, and magnetic losses. If we define this power as P_ℓ we have

$$P_\ell = \frac{\sigma}{2}\int_V |\bar{E}|^2\, dv + \frac{\omega}{2}\int_V (\epsilon''|\bar{E}|^2 + \mu''|\bar{H}|^2)\, dv, \quad (1.92)$$

which is sometimes referred to as *Joule's law*. The last integral in (1.88) can be seen to be related to the stored electric and magnetic energies, as defined in (1.84) and (1.86).

With the above definitions, Poynting's theorem can be rewritten as

$$P_s = P_o + P_\ell + 2j\omega(W_m - W_e). \tag{1.93}$$

In words, this complex power balance equation states that the power delivered by the sources (P_s) is equal to the sum of the power transmitted through the surface (P_o), the power lost to heat in the volume (P_ℓ), and 2ω times the net reactive energy stored in the volume.

Power Absorbed by a Good Conductor

Practical transmission lines involve imperfect conductors, leading to attenuation and power losses, as well as the generation of noise. To calculate loss and attenuation due to an imperfect conductor we must find the power dissipated in the conductor. We will show that this can be accomplished using only the fields at the surface of the conductor, which is a very helpful simplification when calculating attenuation.

Consider the geometry of Figure 1.11, which shows the interface between a lossless medium and a good conductor. A field is incident from $z < 0$, and the field penetrates into the conducting region, $z > 0$. The real average power entering the conductor volume defined by the cross-sectional area S_0 at the interface and the surface S is given from (1.91) as

$$P_{\text{avg}} = \frac{1}{2}\text{Re}\int_{S_0+S} \bar{E} \times \bar{H}^* \cdot \hat{n}\, ds, \tag{1.94}$$

where $\hat{n}$ is a unit normal vector pointing into the closed surface $S_0 + S$, and $\bar{E}$, $\bar{H}$ are the fields over this surface. The contribution to the integral in (1.94) from the surface S can be made zero by proper selection of this surface. For example, if the field is a normally incident plane wave, the Poynting vector $\bar{S} = \bar{E} \times \bar{H}^*$ will be in the $\hat{z}$ direction, and so tangential to the top, bottom, front, and back of S, if these walls are made parallel to the z-axis. If the wave is obliquely incident, these walls can be slanted to obtain the same result. If the conductor is good, the decay of the fields away from the interface at $z = 0$ will be very rapid, so the right-hand end of S can be made far enough away from $z = 0$ such that there is negligible contribution to the integral from this part of the surface S. The

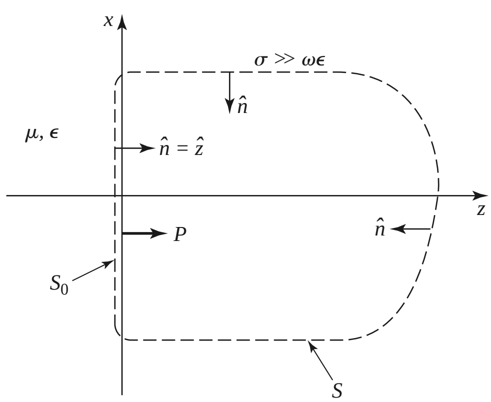


FIGURE 1.11 An interface between a lossless medium and a good conductor with a closed surface $S_0 + S$ for computing the power dissipated in the conductor.

time-average power entering the conductor through S_0 can then be written as

$$P_{avg} = \frac{1}{2}\mathrm{Re}\int_{S_0} \bar{E} \times \bar{H}^* \cdot \hat{z}\, ds. \qquad (1.95)$$

From vector identity (B.3) we have

$$\hat{z} \cdot (\bar{E} \times \bar{H}^*) = (\hat{z} \times \bar{E}) \cdot \bar{H}^* = \eta \bar{H} \cdot \bar{H}^*, \qquad (1.96)$$

since $\bar{H} = \hat{n} \times \bar{E}/\eta$, as generalized from (1.76) for conductive media, where η is the intrinsic impedance (complex) of the conductor. Equation (1.95) can then be written as

$$P_{avg} = \frac{R_s}{2}\int_{S_0} |\bar{H}|^2\, ds, \qquad (1.97)$$

where

$$R_s = \mathrm{Re}\{\eta\} = \mathrm{Re}\left\{(1+j)\sqrt{\frac{\omega\mu}{2\sigma}}\right\} = \sqrt{\frac{\omega\mu}{2\sigma}} = \frac{1}{\sigma\delta_s} \qquad (1.98)$$

is defined as the *surface resistance* of the conductor. The magnetic field $\bar{H}$ in (1.97) is tangential to the conductor surface and needs only to be evaluated at the surface of the conductor; since H_t is continuous at $z = 0$, it does not matter whether this field is evaluated just outside the conductor or just inside the conductor. In the next section we will show how (1.97) can be evaluated in terms of a surface current density flowing on the surface of the conductor, where the conductor can be approximated as perfect.

1.7 PLANE WAVE REFLECTION FROM A MEDIA INTERFACE

A number of problems to be considered in later chapters involve the behavior of electromagnetic fields at the interface of various types of media, including lossless media, lossy media, a good conductor, or a perfect conductor, and so it is beneficial at this time to study the reflection of a plane wave normally incident from free-space onto a half-space of an arbitrary material. The geometry is shown in Figure 1.12, where the material half-space $z > 0$ is characterized by the parameters ϵ, μ, and σ.

General Medium

With no loss of generality we can assume that the incident plane wave has an electric field vector oriented along the x-axis and is propagating along the positive z-axis. The incident

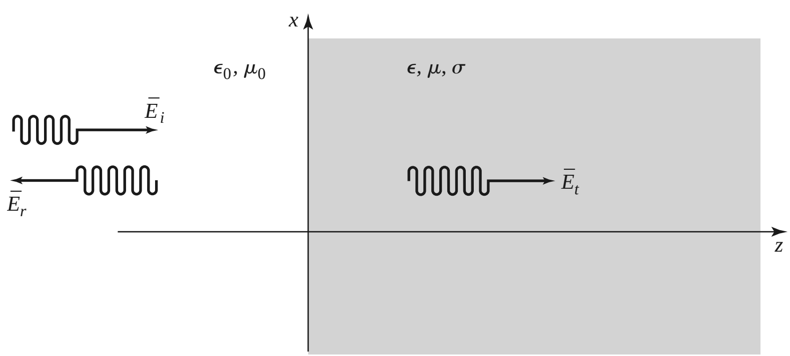


FIGURE 1.12 Plane wave reflection from an arbitrary medium; normal incidence.

fields can then be written, for $z < 0$, as

$$\bar{E}_i = \hat{x} E_0 e^{-jk_0 z}, \tag{1.99a}$$

$$\bar{H}_i = \hat{y} \frac{1}{\eta_0} E_0 e^{-jk_0 z}, \tag{1.99b}$$

where η_0 is the impedance of free-space and E_0 is an arbitrary amplitude. Also in the region $z < 0$, a reflected wave may exist with the form

$$\bar{E}_r = \hat{x} \Gamma E_0 e^{+jk_0 z}, \tag{1.100a}$$

$$\bar{H}_r = -\hat{y} \frac{\Gamma}{\eta_0} E_0 e^{+jk_0 z}, \tag{1.100b}$$

where Γ is the unknown *reflection coefficient* of the reflected electric field. Note that in (1.100), the sign in the exponential terms has been chosen as positive, to represent waves traveling in the $-\hat{z}$ direction of propagation, as derived in (1.46). This is also consistent with the Poynting vector $\bar{S}_r = \bar{E}_r \times \bar{H}_r^* = -|\Gamma|^2 |E_0|^2 \hat{z}/\eta_0$, which shows power to be traveling in the $-\hat{z}$ direction for the reflected wave.

As shown in Section 1.4, from equations (1.54) and (1.58), the transmitted fields for $z > 0$ in the lossy medium can be written as

$$\bar{E}_t = \hat{x} T E_0 e^{-\gamma z}, \tag{1.101a}$$

$$\bar{H}_t = \frac{\hat{y} T E_0}{\eta} e^{-\gamma z}, \tag{1.101b}$$

where T is the *transmission coefficient* of the transmitted electric field and η is the intrinsic impedance (complex) of the lossy medium in the region $z > 0$. From (1.57) and (1.52) the intrinsic impedance is

$$\eta = \frac{j\omega\mu}{\gamma}, \tag{1.102}$$

and the propagation constant is

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\sqrt{1 - j\sigma/\omega\epsilon}. \tag{1.103}$$

We now have a boundary value problem where the general form of the fields are known via (1.99)–(1.101) on either side of the material discontinuity at $z = 0$. The two unknown constants Γ and T are found by applying boundary conditions for E_x and H_y at $z = 0$. Since these tangential field components must be continuous at $z = 0$, we arrive at the following two equations:

$$1 + \Gamma = T, \tag{1.104a}$$

$$\frac{1 - \Gamma}{\eta_0} = \frac{T}{\eta}. \tag{1.104b}$$

Solving these equations for the reflection and transmission coefficients gives

$$\Gamma = \frac{\eta - \eta_0}{\eta + \eta_0}, \tag{1.105a}$$

$$T = 1 + \Gamma = \frac{2\eta}{\eta + \eta_0}. \tag{1.105b}$$

This is a general solution for reflection and transmission of a normally incident wave at the interface of an arbitrary material, where η is the intrinsic impedance of the material. We now consider three special cases of this result.

Lossless Medium

If the region for $z > 0$ is a lossless dielectric, then $\sigma = 0$, and μ and ϵ are real quantities. The propagation constant in this case is purely imaginary and can be written as

$$\gamma = j\beta = j\omega\sqrt{\mu\epsilon} = jk_0\sqrt{\mu_r\epsilon_r}, \tag{1.106}$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the propagation constant (wave number) of a plane wave in free-space. The wavelength in the dielectric is

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\epsilon}} = \frac{\lambda_0}{\sqrt{\mu_r\epsilon_r}}, \tag{1.107}$$

the phase velocity is

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu_r\epsilon_r}}, \tag{1.108}$$

(slower than the speed of light in free-space) and the intrinsic impedance of the dielectric is

$$\eta = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{\mu}{\epsilon}} = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}}. \tag{1.109}$$

For this lossless case, η is real, so both Γ and T from (1.105) are real, and $\bar{E}$ and $\bar{H}$ are in phase with each other in both regions.

Power conservation for the incident, reflected, and transmitted waves can be demonstrated by computing the Poynting vectors in the two regions. Thus, for $z < 0$, the complex Poynting vector is found from the total fields in this region as

$$\begin{aligned}
\bar{S}^- &= \bar{E} \times \bar{H}^* = (\bar{E}_i + \bar{E}_r) \times (\bar{H}_i + \bar{H}_r)^* \\
&= \hat{z}|E_0|^2 \frac{1}{\eta_0}(e^{-jk_0z} + \Gamma e^{jk_0z})(e^{-jk_0z} - \Gamma e^{jk_0z})^* \\
&= \hat{z}|E_0|^2 \frac{1}{\eta_0}(1 - |\Gamma|^2 + \Gamma e^{2jk_0z} - \Gamma^* e^{-2jk_0z}) \\
&= \hat{z}|E_0|^2 \frac{1}{\eta_0}(1 - |\Gamma|^2 + 2j\Gamma \sin 2k_0z),
\end{aligned} \tag{1.110a}$$

since Γ is real. For $z > 0$ the complex Poynting vector is

$$\bar{S}^+ = \bar{E}_t \times \bar{H}_t^* = \hat{z}\frac{|E_0|^2|T|^2}{\eta},$$

which can be rewritten, using (1.105), as

$$\bar{S}^+ = \hat{z}|E_0|^2 \frac{4\eta}{(\eta + \eta_0)^2} = \hat{z}|E_0|^2 \frac{1}{\eta_0}(1 - |\Gamma|^2). \tag{1.110b}$$

Now observe that at $z = 0$, $\bar{S}^- = \bar{S}^+$, so that complex power flow is conserved across the interface. Next consider the time-average power flow in the two regions. For $z < 0$ the time-average power flow through a 1 m^2 cross section is

$$P^- = \frac{1}{2}\text{Re}\left\{\bar{S}^- \cdot \hat{z}\right\} = \frac{1}{2}|E_0|^2 \frac{1}{\eta_0}(1 - |\Gamma|^2). \tag{1.111a}$$

and for $z > 0$, the time-average power flow through a 1 m^2 cross section is

$$P^+ = \frac{1}{2}\text{Re}\left\{\bar{S}^+ \cdot \hat{z}\right\} = \frac{1}{2}|E_0|^2\frac{1}{\eta_0}(1 - |\Gamma|^2) = P^-, \tag{1.111b}$$

so real power flow is conserved.

We now note a subtle point. When computing the complex Poynting vector for $z < 0$ in (1.110a), we used the total $\bar{E}$ and $\bar{H}$ fields. If we compute separately the Poynting vectors for the incident and reflected waves, we obtain

$$\bar{S}_i = \bar{E}_i \times \bar{H}_i^* = \hat{z}\frac{|E_0|^2}{\eta_0}, \tag{1.112a}$$

$$\bar{S}_r = \bar{E}_r \times \bar{H}_r^* = -\hat{z}\frac{|E_0|^2|\Gamma|^2}{\eta_0}, \tag{1.112b}$$

and we see that $\bar{S}_i + \bar{S}_r \neq \bar{S}^-$ of (1.110a). The missing cross-product terms account for stored reactive energy in the standing wave in the $z < 0$ region. Thus, the decomposition of a Poynting vector into incident and reflected components is not, in general, meaningful. It is possible to define a time-average Poynting vector as $(1/2)\text{Re}\{\bar{E} \times \bar{H}^*\}$, and in this case such a definition applied to the individual incident and reflected components will give the correct result since $P_i = (1/2)|\bar{E}_0|^2/\eta_0$ and $P_r = (-1/2)|E_0|^2|\Gamma|^2/\eta_0$, so $P_i + P_r = P^-$. However, this definition will fail to provide meaningful results when the medium for $z < 0$ is lossy.

Good Conductor

If the region for $z > 0$ is a good (but not perfect) conductor, the propagation constant can be written as discussed in Section 1.4:

$$\gamma = \alpha + j\beta = (1 + j)\sqrt{\frac{\omega\mu\sigma}{2}} = (1 + j)\frac{1}{\delta_s}. \tag{1.113}$$

Similarly, the intrinsic impedance of the conductor simplifies to

$$\eta = (1 + j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1 + j)\frac{1}{\sigma\delta_s}. \tag{1.114}$$

Now the impedance is complex, with a phase angle of 45°, so $\bar{E}$ and $\bar{H}$ will be 45° out of phase, and Γ and T will be complex. In (1.113) and (1.114), $\delta_s = 1/\alpha$ is the skin depth, as defined in (1.60).

For $z < 0$ the complex Poynting vector can be evaluated at $z = 0$ to give

$$\bar{S}^-(z = 0) = \hat{z}|E_0|^2\frac{1}{\eta_0}(1 - |\Gamma|^2 + \Gamma - \Gamma^*). \tag{1.115a}$$

For $z > 0$ the complex Poynting vector is

$$\bar{S}^+ = \bar{E}_t \times \bar{H}_t^* = \hat{z}|E_0|^2|T|^2\frac{1}{\eta^*}e^{-2\alpha z},$$

and using (1.105) for T and Γ gives

$$\bar{S}^+ = \hat{z}|E_0|^2\frac{4\eta}{|\eta + \eta_0|^2}e^{-2\alpha z} = \hat{z}|E_0|^2\frac{1}{\eta_0}(1 - |\Gamma|^2 + \Gamma - \Gamma^*)e^{-2\alpha z}. \tag{1.115b}$$

So at the interface at $z = 0$, $\bar{S}^- = \bar{S}^+$, and complex power is conserved.

Observe that if we were to compute the separate incident and reflected Poynting vectors for $z < 0$ as

$$\bar{S}_i = \bar{E}_i \times \bar{H}_i^* = \hat{z}\frac{|E_0|^2}{\eta_0}, \tag{1.116a}$$

$$\bar{S}_r = \bar{E}_r \times \bar{H}_r^* = -\hat{z}\frac{|E_0|^2|\Gamma|^2}{\eta_0}, \tag{1.116b}$$

we would not obtain $\bar{S}_i + \bar{S}_r = \bar{S}^-$ of (1.115a), even for $z = 0$. It is possible, however, to consider real power flow in terms of the individual traveling wave components. Thus, the time-average power flows through a 1 m^2 cross section are

$$P^- = \frac{1}{2}\text{Re}(\bar{S}^- \cdot \hat{z}) = \frac{1}{2}|E_0|^2\frac{1}{\eta_0}(1 - |\Gamma|^2), \tag{1.117a}$$

$$P^+ = \frac{1}{2}\text{Re}(\bar{S}^- \cdot \hat{z}) = \frac{1}{2}|E_0|^2\frac{1}{\eta_0}(1 - |\Gamma|^2)e^{-2\alpha z}, \tag{1.117b}$$

which shows power balance at $z = 0$. In addition, $P_i = |E_0|^2/2\eta_0$ and $P_r = -|E_0|^2\,|\Gamma|^2\,/\,2\eta_0$, so that $P_i + P_r = P^-$, showing that the real power flow for $z < 0$ can be decomposed into incident and reflected wave components.

Notice that $\bar{S}^+$, the power density in the lossy conductor, decays exponentially according to the $e^{-2\alpha z}$ attenuation factor. This means that power is being dissipated in the lossy material as the wave propagates into the medium in the $+z$ direction. The power, and also the fields, decay to a negligibly small value within a few skin depths of the material, which for a reasonably good conductor is an extremely small distance at microwave frequencies.

The electric volume current density flowing in the conducting region is given as

$$\bar{J}_t = \sigma\bar{E}_t = \hat{x}\sigma E_0 T e^{-\gamma z}\,\text{A/m}^2, \tag{1.118}$$

and so the average power dissipated in (or transmitted into) a 1 m^2 cross-sectional volume of the conductor can be calculated from the conductor loss term of (1.92) (Joule's law) as

$$\begin{aligned} P^t &= \frac{1}{2}\int_V \bar{E}_t \cdot \bar{J}_t^*\,dv = \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{\infty}(\hat{x}E_0Te^{-\gamma z})\cdot(\hat{x}\sigma E_0Te^{-\gamma z})^*\,dz\,dy\,dx \\ &= \frac{1}{2}\sigma|E_0|^2|T|^2\int_{z=0}^{\infty}e^{-2\alpha z}\,dz = \frac{\sigma|E_0|^2|T|^2}{4\alpha}. \end{aligned} \tag{1.119}$$

Since $1/\eta = \sigma\delta_s/(1 + j) = (\sigma/2\alpha)(1 - j)$, the real power entering the conductor through a 1 m^2 cross section [as given by $(1/2)\text{Re}\{\bar{S}^+ \cdot \hat{z}\}$ at $z = 0$] can be expressed using (1.115b) as $P^t = |E_0|^2|T|^2(\sigma/4\alpha)$, which is in agreement with (1.119).

Perfect Conductor

Now assume that the region $z > 0$ contains a perfect conductor. The above results can be specialized to this case by allowing $\sigma \to \infty$. Then, from (1.113), $\alpha \to \infty$; from (1.114), $\eta \to 0$; from (1.60), $\delta_s \to 0$; and from (1.105a, b), $T \to 0$ and $\Gamma \to -1$. The fields for $z > 0$ thus decay infinitely fast and are identically zero in the perfect conductor. The perfect conductor can be thought of as "shorting out" the incident electric field. For $z < 0$, from (1.99) and (1.100), the total $\bar{E}$ and $\bar{H}$ fields are, since $\Gamma = -1$,

$$\bar{E} = \bar{E}_i + \bar{E}_r = \hat{x}E_0(e^{-jk_0z} - e^{jk_0z}) = -\hat{x}2jE_0\sin k_0z, \tag{1.120a}$$

$$\bar{H} = \bar{H}_i + \bar{H}_r = \hat{y}\frac{1}{\eta_0}E_0(e^{-jk_0z} + e^{jk_0z}) = \hat{y}\frac{2}{\eta_0}E_0\cos k_0z. \tag{1.120b}$$

Observe that at $z = 0$, $\bar{E} = 0$ and $\bar{H} = \hat{y}(2/\eta_0)E_0$. The Poynting vector for $z < 0$ is

$$\bar{S}^- = \bar{E} \times \bar{H}^* = -\hat{z} j \frac{4}{\eta_0}|E_0|^2 \sin k_0 z \cos k_0 z, \tag{1.121}$$

which has a zero real part and thus indicates that no real power is delivered to the perfect conductor.

The volume current density of (1.118) for the lossy conductor reduces to an infinitely thin sheet of surface current in the limit of infinite conductivity:

$$\bar{J}_s = \hat{n} \times \bar{H} = -\hat{z} \times \left(\hat{y} \frac{2}{\eta_0} E_0 \cos k_0 z \right) \Big|_{z=0} = \hat{x} \frac{2}{\eta_0} E_0 \text{ A/m}. \tag{1.122}$$

The Surface Impedance Concept

In many problems, particularly those in which the effect of attenuation or conductor loss is needed, the presence of an imperfect conductor must be taken into account. The surface impedance concept allows us to do this in an approximate, but very convenient and accurate, manner. We will develop this method from the theory presented in the previous sections.

Consider a good conductor in the region $z > 0$. As we have seen, a plane wave normally incident on this conductor is mostly reflected, and the power that is transmitted into the conductor is dissipated as heat within a very short distance from the surface. There are three ways to compute this power.

First, we can use Joule's law, as in (1.119). For a 1 m^2 area of conductor surface, the power transmitted through this surface and dissipated as heat is given by (1.119). Using (1.105b) for T, (1.114) for η, and the fact that $\alpha = 1/\delta_s$ gives the following result:

$$\frac{\sigma |T|^2}{\alpha} = \frac{\sigma \delta_s 4|\eta|^2}{|\eta + \eta_0|^2} \simeq \frac{8}{\sigma \delta_s \eta_0^2}, \tag{1.123}$$

where we have assumed $\eta \ll \eta_0$, which is true for a good conductor. Then the power of (1.119) can be written as

$$P^t = \frac{\sigma |E_0|^2 |T|^2}{4\alpha} = \frac{2\,|E_0|^2}{\sigma \delta_s \eta_0^2} = \frac{2|E_0|^2 R_s}{\eta_0^2}, \tag{1.124}$$

where

$$R_s = \text{Re}\{\eta\} = \text{Re}\left\{ \frac{1+j}{\sigma \delta_s} \right\} = \frac{1}{\sigma \delta_s} = \sqrt{\frac{\omega \mu}{2\sigma}} \tag{1.125}$$

is the surface resistance of the metal.

Another way to find the power loss is to compute the power flow into the conductor using the Poynting vector since all power entering the conductor at $z = 0$ is dissipated. As in (1.115b), we have

$$P^t = \frac{1}{2}\text{Re}\left\{\bar{S}^+ \cdot \hat{z}\right\}\Big|_{z=0} = \frac{2|E_0|^2 \text{Re}\,\{\eta\}}{|\eta + \eta_0|^2},$$

which for large conductivity becomes, since $\eta \ll \eta_0$,

$$P^t = \frac{2|E_0|^2 R_s}{\eta_0^2}, \tag{1.126}$$

which agrees with (1.124).

A third method uses an effective surface current density and the surface impedance, without the need for knowing the fields inside the conductor. From (1.118), the volume current density in the conductor is

$$\bar{J}_t = \hat{x}\sigma T E_0 e^{-\gamma z} \text{ A/m}^2, \tag{1.127}$$

so the total (surface) current flow per unit width in the x direction is

$$\bar{J}_s = \int_0^\infty \bar{J}_t \, dz = \hat{x}\sigma T E_0 \int_0^\infty e^{-\gamma z} \, dz = \frac{\hat{x}\sigma T E_0}{\gamma} \text{A/m.}$$

Approximating $\sigma T/\gamma$ for large σ and using (1.113), (1.105b), and (1.114) gives

$$\frac{\sigma T}{\gamma} = \frac{\sigma \delta_s}{(1+j)} \frac{2\eta}{(\eta+\eta_0)} \simeq \frac{\sigma \delta_s}{(1+j)} \frac{2(1+j)}{\sigma \delta_s \eta_0} = \frac{2}{\eta_0},$$

so

$$\bar{J}_s \simeq \hat{x}\frac{2E_0}{\eta_0} \text{ A/m.} \tag{1.128}$$

If the conductivity were infinite, then $\Gamma = -1$ and a true surface current density of

$$\bar{J}_s = \hat{n} \times \bar{H}|_{z=0} = -\hat{z} \times (\bar{H}_i + \bar{H}_r)|_{z=0} = \hat{x}E_0 \frac{1}{\eta_0}(1-\Gamma) = \hat{x}\frac{2E_0}{\eta_0} \text{ A/m}$$

would flow, which is identical to the total current in (1.128).

Now replace the exponentially decaying volume current of (1.127) with a uniform volume current extending a distance of one skin depth. Thus, let

$$\bar{J}_t = \begin{cases} \bar{J}_s/\delta_s & \text{for } 0 < z < \delta_s \\ 0 & \text{for } z > \delta_s, \end{cases} \tag{1.129}$$

so that the total current flow is the same. Then Joule's law gives the power lost:

$$P^t = \frac{1}{2\sigma} \int_S \int_{z=0}^{\delta_s} \frac{|\bar{J}_s|^2}{\delta_s^2} \, dz \, ds = \frac{R_s}{2} \int_S |\bar{J}_s|^2 \, ds = \frac{2|E_0|^2 R_s}{\eta_0^2}, \tag{1.130}$$

where $\int_S$ denotes a surface integral over the conductor surface, in this case chosen as 1 m^2. The result of (1.130) agrees with our previous results for P^t in (1.126) and (1.124) and shows that the power loss in a good conductor can be accurately and simply calculated as

$$P^t = \frac{R_s}{2} \int_S |\bar{J}_s|^2 \, ds = \frac{R_s}{2} \int_S |\bar{H}_t|^2 \, ds, \tag{1.131}$$

in terms of the surface resistance R_s and the surface current $\bar{J}_s$, or tangential magnetic field $\bar{H}_t$. It is important to realize that the surface current can be found from $\bar{J}_s = \hat{n} \times \bar{H}$, as if the metal were a perfect conductor. This method is very general, applying to fields other than plane waves and to conductors of arbitrary shape, as long as bends or corners have radii on the order of a skin depth or larger. The method is also quite accurate, as the only approximation was that $\eta \ll \eta_0$, which is a good approximation. As an example, copper at 1 GHz has $|\eta| = 0.012\ \Omega$, which is indeed much less than $\eta_0 = 377\ \Omega$.

EXAMPLE 1.4 PLANE WAVE REFLECTION FROM A CONDUCTOR

Consider a plane wave normally incident on a half-space of copper. If $f = 1$ GHz, compute the propagation constant, intrinsic impedance, and skin depth for the conductor. Also compute the reflection and transmission coefficients.

Solution
For copper, $\sigma = 5.813 \times 10^7$ S/m, so from (1.60) the skin depth is

$$\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}} = 2.088 \times 10^{-6}\text{m},$$

and the propagation constant is, from (1.113),

$$\gamma = \frac{1+j}{\delta_s} = (4.789 + j4.789) \times 10^5 \text{ m}^{-1}.$$

The intrinsic impedance is, from (1.114),

$$\eta = \frac{1+j}{\sigma\delta_s} = (8.239 + j8.239) \times 10^{-3}\Omega,$$

which is quite small relative to the impedance of free-space ($\eta_0 = 377\ \Omega$). The reflection coefficient is, from (1.105a),

$$\Gamma = \frac{\eta - \eta_0}{\eta + \eta_0} = 1.0\angle 179.99^\circ$$

(practically that of an ideal short circuit), and the transmission coefficient is

$$T = \frac{2\eta}{\eta + \eta_0} = 6.181 \times 10^{-5}\angle 45^\circ. \qquad \blacksquare$$

1.8 OBLIQUE INCIDENCE AT A DIELECTRIC INTERFACE

We continue our discussion of plane waves by considering the problem of a plane wave obliquely incident on a plane interface between two lossless dielectric regions, as shown in Figure 1.13. There are two canonical cases of this problem: the electric field is either in the xz plane (parallel polarization) or normal to the xz plane (perpendicular polarization). An arbitrary incident plane wave, of course, may have a polarization that is neither of these, but it can be expressed as a linear combination of these two individual cases.

The general method of solution is similar to the problem of normal incidence: we will write expressions for the incident, reflected, and transmitted fields in each region and match boundary conditions to find the unknown amplitude coefficients and angles.

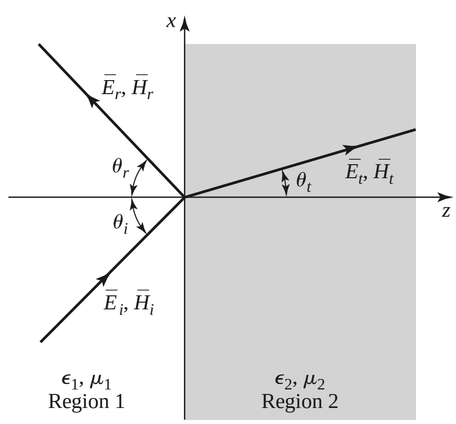


FIGURE 1.13 Geometry for a plane wave obliquely incident at the interface between two dielectric regions.

Parallel Polarization

In this case the electric field vector lies in the xz plane, and the incident fields can be written as

$$\bar{E}_i = E_0(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}, \tag{1.132a}$$

$$\bar{H}_i = \frac{E_0}{\eta_1}\hat{y}e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}, \tag{1.132b}$$

where $k_1 = \omega\sqrt{\mu_0\epsilon_1}$ and $\eta_1 = \sqrt{\mu_0/\epsilon_1}$ are the propagation constant and impedance of region 1. The reflected and transmitted fields can be written as

$$\bar{E}_r = E_0\Gamma(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}, \tag{1.133a}$$

$$\bar{H}_r = \frac{-E_0\Gamma}{\eta_1}\hat{y}e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}, \tag{1.133b}$$

$$\bar{E}_t = E_0T(\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}, \tag{1.134a}$$

$$\bar{H}_t = \frac{E_0T}{\eta_2}\hat{y}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}. \tag{1.134b}$$

Here, Γ and T are the reflection and transmission coefficients, and k_2 and η_2 are the propagation constant and impedance of region 2, defined as

$$k_2 = \omega\sqrt{\mu_0\epsilon_2}, \qquad \eta_2 = \sqrt{\mu_0/\epsilon_2}.$$

At this point we have $\Gamma,\ T,\ \theta_r$, and θ_t as unknowns.

We can obtain two complex equations for these unknowns by enforcing the continuity of E_x and H_y, the tangential field components, at the interface between the two regions at $z = 0$. We then obtain

$$\cos\theta_i\, e^{-jk_1x\sin\theta_i} + \Gamma\cos\theta_r e^{-jk_1x\sin\theta_r} = T\cos\theta_t e^{-jk_2x\sin\theta_t}, \tag{1.135a}$$

$$\frac{1}{\eta_1}e^{-jk_1x\sin\theta_i} - \frac{\Gamma}{\eta_1}e^{-jk_1x\sin\theta_r} = \frac{T}{\eta_2}e^{-jk_2x\sin\theta_t}. \tag{1.135b}$$

Both sides of (1.135a) and (1.135b) are functions of the coordinate x. If E_x and H_y are to be continuous at the interface $z = 0$ for all x, then this x variation must be the same on both sides of the equations, leading to the following condition:

$$k_1\sin\theta_i = k_1\sin\theta_r = k_2\sin\theta_t.$$

This results in the well-known *Snell's laws* of reflection and refraction:

$$\theta_i = \theta_r, \tag{1.136a}$$

$$k_1\sin\theta_i = k_2\sin\theta_t. \tag{1.136b}$$

The above argument ensures that the phase terms in (1.135) vary with x at the same rate on both sides of the interface, and so is often called the *phase matching condition*.

Using (1.136) in (1.135) allows us to solve for the reflection and transmission coefficients as

$$\Gamma = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}, \tag{1.137a}$$

$$T = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}. \tag{1.137b}$$

Observe that for normal incidence $\theta_i = 0$, we have $\theta_r = \theta_t = 0$, so then

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{and} \quad T = \frac{2\eta_2}{\eta_2 + \eta_1},$$

which is in agreement with the results of Section 1.7.

For this polarization a special angle of incidence, θ_b, called the *Brewster angle*, exists where $\Gamma = 0$. This occurs when the numerator of (1.137a) goes to zero ($\theta_i = \theta_b$): $\eta_2 \cos\theta_t = \eta_1 \cos\theta_b$, which can be rewritten using

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{k_1^2}{k_2^2}\sin^2\theta_b},$$

to give

$$\sin\theta_b = \frac{1}{\sqrt{1 + \epsilon_1/\epsilon_2}}. \tag{1.138}$$

Perpendicular Polarization

In this case the electric field vector is perpendicular to the xz plane. The incident field can be written as

$$\bar{E}_i = E_0\hat{y}e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}, \tag{1.139a}$$

$$\bar{H}_i = \frac{E_0}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}, \tag{1.139b}$$

where $k_1 = \omega\sqrt{\mu_0\epsilon_1}$ and $\eta_1 = \sqrt{\mu_0/\epsilon_1}$ are the propagation constant and impedance for region 1, as before. The reflected and transmitted fields can be expressed as

$$\bar{E}_r = E_0\Gamma\hat{y}e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}, \tag{1.140a}$$

$$\bar{H}_r = \frac{E_0\Gamma}{\eta_1}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}, \tag{1.140b}$$

$$\bar{E}_t = E_0T\hat{y}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}, \tag{1.141a}$$

$$\bar{H}_t = \frac{E_0T}{\eta_2}(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}, \tag{1.141b}$$

with $k_2 = \omega\sqrt{\mu_0\epsilon_2}$ and $\eta_2 = \sqrt{\mu_0/\epsilon_2}$ being the propagation constant and impedance in region 2.

Equating the tangential field components E_y and H_x at $z = 0$ gives

$$e^{-jk_1x\sin\theta_i} + \Gamma e^{-jk_1x\sin\theta_r} = Te^{-jk_2x\sin\theta_t}, \tag{1.142a}$$

$$\frac{-1}{\eta_1}\cos\theta_i e^{-jk_1x\sin\theta_i} + \frac{\Gamma}{\eta_1}\cos\theta_r e^{-jk_2x\sin\theta_r} = \frac{-T}{\eta_2}\cos\theta_t e^{-jk_2x\sin\theta_t}. \tag{1.142b}$$

By the same phase matching argument that was used in the parallel case, we obtain Snell's laws

$$k_1\sin\theta_i = k_1\sin\theta_r = k_2\sin\theta_t$$

identical to (1.136).

Using (1.136) in (1.142) allows us to solve for the reflection and transmission coefficients as

$$\Gamma = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}, \tag{1.143a}$$

$$T = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}. \tag{1.143b}$$

Again, for the normally incident case, these results reduce to those of Section 1.7.

For this polarization no Brewster angle exists where $\Gamma = 0$, as we can see by examining the possibility that the numerator of (1.143a) could be zero:

$$\eta_2 \cos\theta_i = \eta_1 \cos\theta_t,$$

and using Snell's law to give

$$k_2^2\left(\eta_2^2 - \eta_1^2\right) = \left(k_2^2\eta_2^2 - k_1^2\eta_1^2\right)\sin^2\theta_i.$$

This leads to a contradiction since the term in parentheses on the right-hand side is identically zero for dielectric media. Thus, no Brewster angle exists for perpendicular polarization for dielectric media.

EXAMPLE 1.5 OBLIQUE REFLECTION FROM A DIELECTRIC INTERFACE

Plot the reflection coefficients versus incidence angle for parallel and perpendicular polarized plane waves incident from free-space onto a dielectric region with $\epsilon_r = 2.55$.

Solution

The impedances for the two regions are

$$\begin{aligned} \eta_1 &= 377\Omega, \\ \eta_2 &= \frac{\eta_0}{\sqrt{\epsilon_r}} = \frac{377}{\sqrt{2.55}} = 236\Omega. \end{aligned}$$

We then evaluate (1.137a) and (1.143a) versus incidence angle; the results are shown in Figure 1.14. ■

Total Reflectio and Surface Waves

Snell's law of (1.136b) can be rewritten as

$$\sin\theta_t = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i. \tag{1.144}$$

Consider the case (for either parallel or perpendicular polarization) where $\epsilon_1 > \epsilon_2$. As θ_i increases, the refraction angle θ_t will increase, but at a faster rate than θ_i increases. The incidence angle θ_i for which $\theta_t = 90^\circ$ is called the *critical angle*, θ_c, where

$$\sin\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}}. \tag{1.145}$$

At this angle and beyond, the incident wave will be totally reflected, as the transmitted wave will not propagate into region 2. Let us look at this situation more closely for the case of $\theta_i > \theta_c$ with parallel polarization.

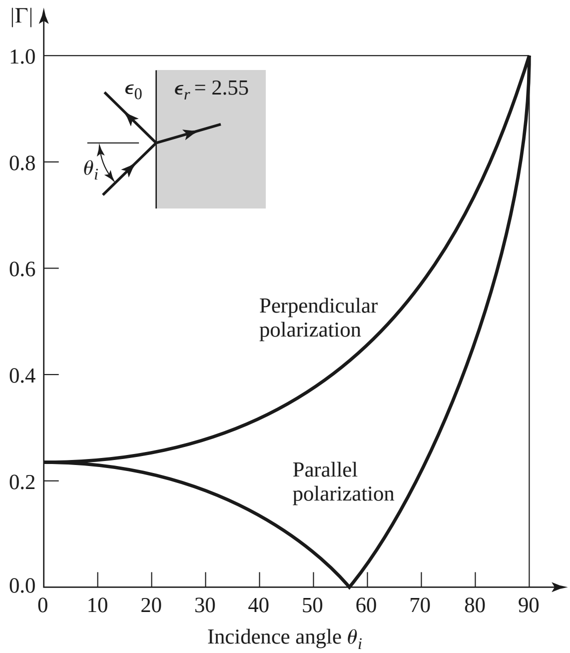


FIGURE 1.14 Reflection coefficient magnitude for parallel and perpendicular polarizations of a plane wave obliquely incident on a dielectric half-space.

When $\theta_i > \theta_c$ (1.144) shows that $\sin\theta_t > 1$, so that $\cos\theta_t = \sqrt{1 - \sin^2\theta_t}$ must be imaginary, and the angle θ_t loses its physical significance. At this point, it is better to replace the expressions for the transmitted fields in region 2 with the following:

$$\bar{E}_t = E_0 T \left(\frac{-j\alpha}{k_2}\hat{x} - \frac{\beta}{k_2}\hat{z} \right) e^{-j\beta x} e^{-\alpha z}, \tag{1.146a}$$

$$\bar{H}_t = \frac{E_0 T}{\eta_2} \hat{y} e^{-j\beta x} e^{-\alpha z}. \tag{1.146b}$$

The form of these fields is derived from (1.134) after noting that $-jk_2 \sin\theta_t$ is still imaginary for $\sin\theta_t > 1$ but $-jk_2\cos\theta_t$ is real, so we can replace $\sin\theta_t$ by β/k_2 and $\cos\theta_t$ by $-j\alpha/k_2$. Substituting (1.146b) into the Helmholtz wave equation for $\bar{H}$ gives

$$-\beta^2 + \alpha^2 + k_2^2 = 0. \tag{1.147}$$

Matching E_x and H_y of (1.146) with the $\hat{x}$ and $\hat{y}$ components of the incident and reflected fields of (1.132) and (1.133) at $z = 0$ gives

$$\cos\theta_i e^{-jk_1 x \sin\theta_i} + \Gamma \cos\theta_r e^{-jk_1 x \sin\theta_r} = \frac{-j\alpha}{k_2} T e^{-j\beta x}, \tag{1.148a}$$

$$\frac{1}{\eta_1} e^{-jk_1 x \sin\theta_i} - \frac{\Gamma}{\eta_1} e^{-jk_1 x \sin\theta_r} = \frac{T}{\eta_2} e^{-j\beta x}. \tag{1.148b}$$

To obtain phase matching at the $z = 0$ boundary, we must have

$$k_1 \sin\theta_i = k_1 \sin\theta_r = \beta,$$

which leads again to Snell's law for reflection, $\theta_i = \theta_r$, and to $\beta = k_1 \sin\theta_i$. Then α is determined from (1.147) as

$$\alpha = \sqrt{\beta^2 - k_2^2} = \sqrt{k_1^2 \sin^2\theta_i - k_2^2}, \tag{1.149}$$

which is seen to be a positive real number since $\sin^2\theta_i > \epsilon_2/\epsilon_1$. The reflection and transmission coefficients can be obtained from (1.148) as

$$\Gamma = \frac{(-j\alpha/k_2)\eta_2 - \eta_1\cos\theta_i}{(-j\alpha/k_2)\eta_2 + \eta_1\cos\theta_i}, \tag{1.150a}$$

$$T = \frac{2\eta_2\cos\theta_i}{(-j\alpha/k_2)\eta_2 + \eta_1\cos\theta_i}. \tag{1.150b}$$

Since Γ is of the form $(ja - b)/(ja + b)$, its magnitude is unity, indicating that all incident power is reflected.

The transmitted fields of (1.146) show propagation in the x direction, along the interface, but exponential decay in the z direction. Such a field is known as a *surface wave*[3] since it is tightly bound to the interface. A surface wave is an example of a nonuniform plane wave, so called because it has an amplitude variation in the z direction, apart from the propagation factor in the x direction.

Finally, it is of interest to calculate the complex Poynting vector for the surface wave fields of (1.146):

$$\bar{S}_t = \bar{E}_t \times \bar{H}_t^* = \frac{|E_0|^2|T|^2}{\eta_2}\left(\hat{z}\frac{-j\alpha}{k_2} + \hat{x}\frac{\beta}{k_2}\right)e^{-2\alpha z}. \tag{1.151}$$

This shows that no real power flow occurs in the z direction. The real power flow in the x direction is that of the surface wave field, and it decays exponentially with distance into region 2. So even though no real power is transmitted into region 2, a nonzero field does exist there, in order to satisfy the boundary conditions at the interface.

1.9 SOME USEFUL THEOREMS

Finally, we discuss several theorems in electromagnetics that we will find useful for later discussions.

The Reciprocity Theorem

Reciprocity is a general concept that occurs in many areas of physics and engineering, and the reader may already be familiar with the reciprocity theorem of circuit theory. Here we will derive the Lorentz reciprocity theorem for electromagnetic fields in two different forms. This theorem will be used later in the book to obtain general properties of network matrices representing microwave circuits and to evaluate the coupling of waveguides from current probes and loops, as well as the coupling of waveguides through apertures. There are a number of other important uses of this powerful concept.

[3] Some authors argue that the term "surface wave" should not be used for a field of this type since it exists only when plane wave fields exist in the $z < 0$ region, and so prefer the term "surface wave–like" field, or a "forced surface wave."

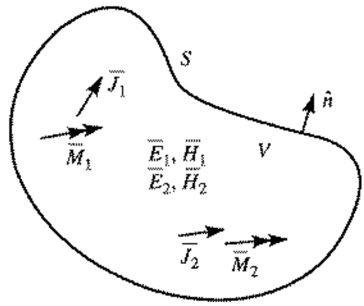


FIGURE 1.15 Geometry for the Lorentz reciprocity theorem.

Consider the two separate sets of sources, $\bar{J}_1,\ \bar{M}_1$ and $\bar{J}_2,\ \bar{M}_2$, which generate the fields $\bar{E}_1,\ \bar{H}_1$, and $\bar{E}_2,\ \bar{H}_2$, respectively, in the volume V enclosed by the closed surface S, as shown in Figure 1.15. Maxwell's equations are satisfied individually for these two sets of sources and fields, so we can write

$$\nabla \times \bar{E}_1 = -j\omega\mu\bar{H}_1 - \bar{M}_1, \tag{1.152a}$$

$$\nabla \times \bar{H}_1 = j\omega\epsilon\bar{E}_1 + \bar{J}_1, \tag{1.152b}$$

$$\nabla \times \bar{E}_2 = -j\omega\mu\bar{H}_2 - \bar{M}_2, \tag{1.153a}$$

$$\nabla \times \bar{H}_2 = j\omega\epsilon\bar{E}_2 + \bar{J}_2. \tag{1.153b}$$

Now consider the quantity $\nabla \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1)$, which can be expanded using vector identity (B.8) to give

$$\nabla \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) = \bar{J}_1 \cdot \bar{E}_2 - \bar{J}_2 \cdot \bar{E}_1 + \bar{M}_2 \cdot \bar{H}_1 - \bar{M}_1 \cdot \bar{H}_2. \tag{1.154}$$

Integrating over the volume V and applying the divergence theorem (B.15), gives

$$\begin{aligned}\int_V \nabla \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1)\, dv &= \oint_S (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) \cdot ds \\ &= \int_V (\bar{E}_2 \cdot \bar{J}_1 - \bar{E}_1 \cdot \bar{J}_2 + \bar{H}_1 \cdot \bar{M}_2 - \bar{H}_2 \cdot \bar{M}_1)\, dv\end{aligned} \tag{1.155}$$

Equation (1.155) represents a general form of the *reciprocity theorem*, but in practice a number of special situations often occur leading to some simplification. We will consider three cases.

S encloses no sources: Then $\bar{J}_1 = \bar{J}_2 = \bar{M}_1 = \bar{M}_2 = 0$, and the fields $\bar{E}_1,\ \bar{H}_1$ and $\bar{E}_2,\ \bar{H}_2$ are source-free fields. In this case, the right-hand side of (1.155) vanishes, with the result that

$$\oint_S \bar{E}_1 \times \bar{H}_2 \cdot d\bar{s} = \oint_S \bar{E}_2 \times \bar{H}_1 \cdot d\bar{s}. \tag{1.156}$$

This result will be used in Chapter 4 when we demonstrate the symmetry of the impedance matrix for a reciprocal microwave network.

S bounds a perfect conductor: For example, S may be the inner surface of a perfectly conducting closed cavity. Then the surface integral of (1.155) vanishes since $\bar{E}_1 \times \bar{H}_2 \cdot \hat{n} = (\hat{n} \times \bar{E}_1) \cdot \bar{H}_2$ [by vector identity (B.3)], and $\hat{n} \times \bar{E}_1$ is zero on the surface of a perfect conductor (similarly for $\bar{E}_2$). The result is

$$\int_V (\bar{E}_1 \cdot \bar{J}_2 - \bar{H}_1 \cdot \bar{M}_2)\, dv = \int_V (\bar{E}_2 \cdot \bar{J}_1 - \bar{H}_2 \cdot \bar{M}_1)\, dv. \tag{1.157}$$

This result is analogous to the reciprocity theorem of circuit theory. In words, this result states that the system response $\bar{E}_1$ or $\bar{E}_2$ is not changed when the source and observation points are interchanged. That is, $\bar{E}_2$ (caused by $\bar{J}_2$) at $\bar{J}_1$ is the same as $\bar{E}_1$ (caused by $\bar{J}_1$) at $\bar{J}_2$.

S is a sphere at infinity: In this case the fields evaluated on S are very far from the sources and so can be considered locally as plane waves. Then the wave impedance relation $\bar{H} = \hat{n} \times \bar{E}/\eta$ applies to (1.155) to give

$$\begin{aligned}(\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) \cdot \hat{n} &= (\hat{n} \times \bar{E}_1) \cdot \bar{H}_2 - (\hat{n} \times \bar{E}_2) \cdot \bar{H}_1 \\ &= \frac{1}{\eta}\bar{H}_1 \cdot \bar{H}_2 - \frac{1}{\eta}\bar{H}_2 \cdot \bar{H}_1 = 0,\end{aligned}$$

so that the result of (1.157) is again obtained. This result can also be obtained for the case of a closed surface S where the surface impedance boundary condition applies.

Image Theory

In many problems a current source (electric or magnetic) is located in the vicinity of a conducting ground plane. Image theory permits the removal of the ground plane by placing a virtual *image source* on the other side of the ground plane. The reader should be familiar with this concept from electrostatics, so we will prove the result for an infinite current sheet next to an infinite ground plane and then summarize other possible cases.

Consider the surface current density $\bar{J}_s = J_{s0}\hat{x}$ parallel to a ground plane, as shown in Figure 1.16a. Because the current source is of infinite extent and is uniform in the x, y directions, it will excite plane waves traveling outward from it. The negatively traveling

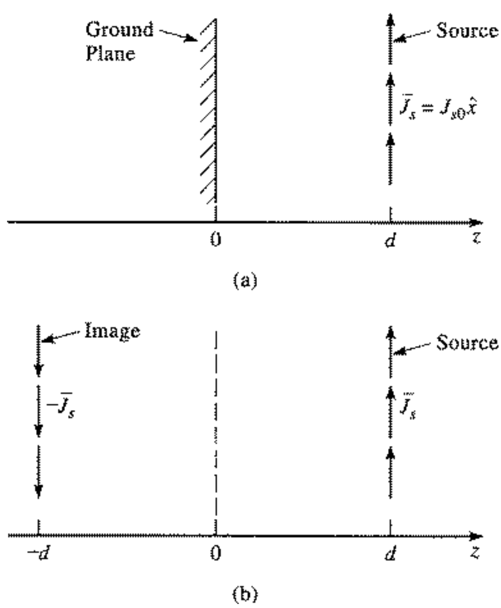


FIGURE 1.16 Illustration of image theory as applied to an electric current source next to a ground plane. (a) An electric surface current density parallel to a ground plane. (b) The ground plane of (a) is replaced with image current at $z = -d$.

wave will reflect from the ground plane at $z = 0$ and then travel in the positive direction. Thus, there will be a standing wave field in the region $0 < z < d$ and a positively traveling wave for $z > d$. The forms of the fields in these two regions can thus be written as

$$E_x^s = A(e^{jk_0 z} - e^{-jk_0 z}), \qquad \text{for } 0 < z < d, \tag{1.158a}$$

$$H_y^s = \frac{-A}{\eta_0}(e^{jk_0 z} + e^{-jk_0 z}), \qquad \text{for } 0 < z < d, \tag{1.158b}$$

$$E_x^+ = Be^{-jk_0 z}, \qquad \text{for } z > d, \tag{1.159a}$$

$$H_y^+ = \frac{B}{\eta_0}e^{-jk_0 z}, \qquad \text{for } z > d, \tag{1.159b}$$

where η_0 is the impedance of free-space. Note that the standing wave fields of (1.158) have been constructed to satisfy the boundary condition that $E_x = 0$ at $z = 0$. The remaining boundary conditions to satisfy are the continuity of $\bar{E}$ at $z = d$ and the discontinuity in the $\bar{H}$ field at $z = d$ due to the current sheet. From (1.36), since $\bar{M}_s = 0$,

$$E_x^s = E_x^+|_{z=d}, \tag{1.160a}$$

while from (1.37) we have

$$\bar{J}_s = \hat{z} \times \hat{y}(H_y^+ - H_y^s)|_{z=d}. \tag{1.160b}$$

Using (1.158) and (1.159) then gives

$$2jA \sin k_0 d = Be^{-jk_0 d}$$

$$\text{and} \quad J_{s0} = -\frac{B}{\eta_0}e^{-jk_0 d} - \frac{2A}{\eta_0}\cos k_0 d,$$

which can be solved for A and B:

$$A = \frac{-J_{s0}\eta_0}{2}e^{-jk_0 d},$$

$$B = -jJ_{s0}\eta_0 \sin k_0 d.$$

So the total fields are

$$E_x^s = -jJ_{s0}\eta_0 e^{-jk_0 d}\sin k_0 z, \qquad \text{for } 0 < z < d, \tag{1.161a}$$

$$H_y^s = J_{s0}e^{-jk_0 d}\cos k_0 z, \qquad \text{for } 0 < z < d, \tag{1.161b}$$

$$E_x^+ = -jJ_{s0}\eta_0 \sin k_0 d e^{-jk_0 z,} \qquad \text{for } z > d, \tag{1.162a}$$

$$H_y^+ = -jJ_{s0}\sin k_0 d e^{-jk_0 z,} \qquad \text{for } z > d. \tag{1.162b}$$

Now consider the application of image theory to this problem. As shown in Figure 1.16b, the ground plane is removed and an image source of $-\bar{J}_s$ is placed at $z = -d$. By superposition, the total fields for $z > 0$ can be found by combining the fields from the two sources individually. These fields can be derived by a procedure similar to that in the above, with the following results:

Fields due to source at $z = d$:

$$E_x = \begin{cases} \dfrac{-J_{s0}\eta_0}{2}e^{-jk_0(z-d)} & \text{for } z > d \\ \dfrac{-J_{s0}\eta_0}{2}e^{jk_0(z-d)} & \text{for } z < d, \end{cases} \tag{1.163a}$$

$$H_y = \begin{cases} \dfrac{-J_{s0}}{2} e^{-jk_0(z-d)} & \text{for } z > d \\ \dfrac{J_{s0}}{2} e^{jk_0(z-d)} & \text{for } z < d. \end{cases} \tag{1.163b}$$

Fields due to source at $z = -d$:

$$E_x = \begin{cases} \dfrac{J_{s0}\eta_0}{2} e^{-jk_0(z+d)} & \text{for } z > -d \\ \dfrac{J_{s0}\eta_0}{2} e^{jk_0(z+d)} & \text{for } z < -d, \end{cases} \tag{1.164a}$$

$$H_y = \begin{cases} \dfrac{J_{s0}}{2} e^{-jk_0(z+d)} & \text{for } z > -d \\ \dfrac{-J_{s0}}{2} e^{jk_0(z+d)} & \text{for } z < -d. \end{cases} \tag{1.164b}$$

The reader can verify that this solution is identical to that of (1.161) for $0 < z < d$ and to that of (1.162) for $z > d$, thus verifying the validity of the image theory solution. Note that image theory only gives the correct fields to the right of the conducting plane. Figure 1.17 shows more general image theory results for electric and magnetic dipoles.

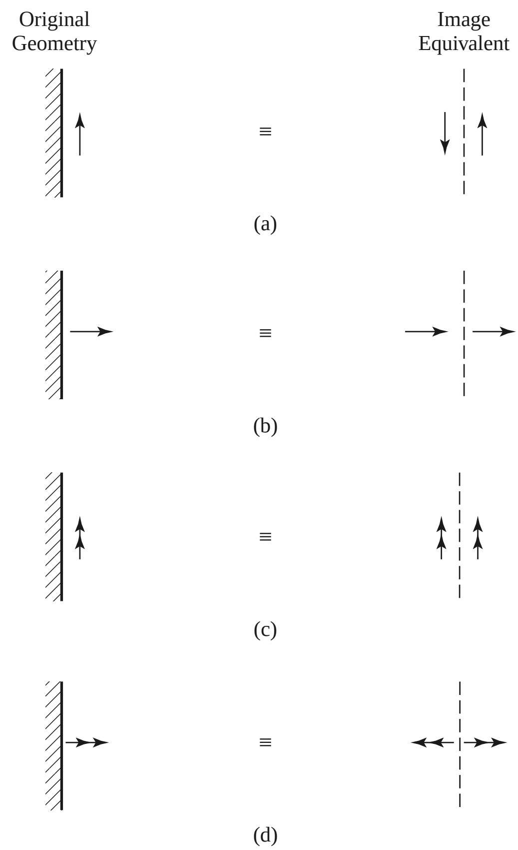


FIGURE 1.17 Electric and magnetic current images. (a) An electric current parallel to a ground plane. (b) An electric current normal to a ground plane. (c) A magnetic current parallel to a ground plane. (d) A magnetic current normal to a ground plane.

REFERENCES

[1] T. S. Sarkar, R. J. Mailloux, A. A. Oliner, M. Salazar-Palma, and D. Sengupta, *History of Wireless*, John Wiley & Sons, Hoboken, N.J., 2006.
[2] A. A. Oliner, "Historical Perspectives on Microwave Field Theory," *IEEE Transactions on Microwave Theory and Techniques*, vol. MTT-32, pp. 1022–1045, September 1984 [this special issue contains other articles on the history of microwave engineering].
[3] F. Ulaby, *Fundamentals of Applied Electromagnetics*, 6th edition, Prentice-Hall, Upper Saddle River, N.J., 2010.
[4] J. D. Kraus and D. A. Fleisch, *Electromagnetics*, 5th edition, McGraw-Hill, New York, 1999.
[5] S. Ramo, T. R. Whinnery, and T. van Duzer, *Fields and Waves in Communication Electronics*, 3rd edition, John Wiley & Sons, New York, 1994.
[6] R. E. Collin, *Foundations for Microwave Engineering*, 2nd edition, Wiley-IEEE Press, Hoboken, N.J., 2001.
[7] C. A. Balanis, *Advanced Engineering Electromagnetics*, John Wiley & Sons, New York, 1989.
[8] D. M. Pozar, *Microwave and RF Design of Wireless Systems*, John Wiley & Sons, Hoboken N.J., 2001.

PROBLEMS

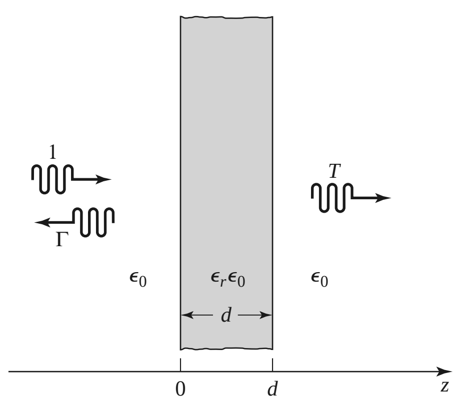
1.1 Who invented radio? Guglielmo Marconi often receives credit for the invention of modern radio, but there were several important developments by other workers before Marconi. Write a brief summary of the early work in wireless during the period of 1865–1900, particularly the work by Mahlon Loomis, Oliver Lodge, Nikola Tesla, and Marconi. Explain the difference between inductive communication schemes and wireless methods that involve wave propagation. Can the development of radio be attributed to a single individual? Reference [1] may be a good starting point.

1.2 A plane wave traveling along the x-axis in a polystyrene-filled region with $\epsilon_r = 2.54$ has an electric field given by $E_y = E_0 \cos(\omega t - kx)$. The frequency is 2.4 GHz, and $E_0 = 5.0$ V/m. Find the following: (a) the amplitude and direction of the magnetic field, (b) the phase velocity, (c) the wavelength, and (d) the phase shift between the positions $x_1 = 0.1$ m and $x_2 = 0.15$ m.

1.3 Show that a linearly polarized plane wave of the form $\bar{E} = E_0(a\hat{x} + b\hat{y})e^{-jk_0 z}$, where a and b are real numbers, can be represented as the sum of an RHCP and an LHCP wave.

1.4 Compute the Poynting vector for the general plane wave field of (1.76).

1.5 A plane wave is normally incident on a dielectric slab of permittivity ϵ_r and thickness d, where $d = \lambda_0/(4\sqrt{\epsilon_r})$ and λ_0 is the free-space wavelength of the incident wave, as shown in the accompanying figure. If free-space exists on both sides of the slab, find the reflection coefficient of the wave reflected from the front of the slab.



1.6 Consider an RHCP plane wave normally incident from free-space ($z < 0$) onto a half-space ($z > 0$) consisting of a good conductor. Let the incident electric field be of the form

$$\bar{E}_i = E_0(\hat{x} - j\hat{y})e^{-jk_0 z},$$

and find the electric and magnetic fields in the region $z > 0$. Compute the Poynting vectors for $z < 0$ and $z > 0$ and show that complex power is conserved. What is the polarization of the reflected wave?

1.7 Consider a plane wave propagating in a lossy dielectric medium for $z < 0$, with a perfectly conducting plate at $z = 0$. Assume that the lossy medium is characterized by $\epsilon = (5 - j2)\epsilon_0$, $\mu = \mu_0$, and that the frequency of the plane wave is 1.0 GHz, and let the amplitude of the incident electric field be 4 V/m at $z = 0$. Find the reflected electric field for $z < 0$ and plot the magnitude of the total electric field for $-0.5 \le z \le 0$.

1.8 A plane wave at 1 GHz is normally incident on a thin copper sheet of thickness t. (a) Compute the transmission losses, in dB, of the wave at the air–copper and the copper–air interfaces. (b) If the sheet is to be used as a shield to reduce the level of the transmitted wave by 150 dB, what is the minimum sheet thickness?

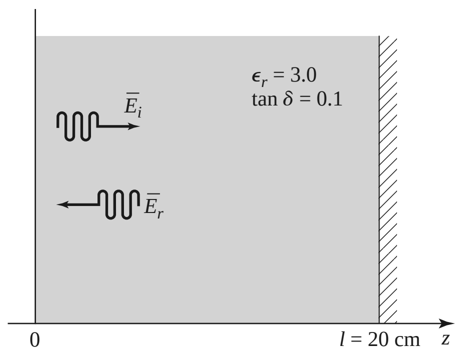
1.9 A uniform lossy medium with $\epsilon_r = 3.0$, $\tan\delta = 0.1$, and $\mu = \mu_0$ fills the region between $z = 0$ and $z = 20$ cm, with a ground plane at $z = 20$ cm, as shown in the accompanying figure. An incident plane wave with an electric field

$$\bar{E}_i = \hat{x}100e^{-\gamma z} \text{ V/m}$$

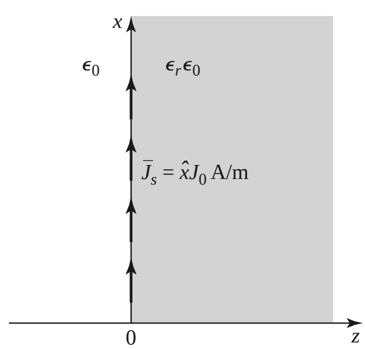
is present at $z = 0$ and propagates in the $+z$ direction. The frequency is 3.0 GHz.

(a) Compute S_i, the power density of the incident wave, and S_r, the power density of the reflected wave, at $z = 0$.

(b) Compute the input power density, S_{in}, at $z = 0$ from the total fields at $z = 0$. Does $S_{\text{in}} = S_i - S_r$?



1.10 Assume that an infinite sheet of electric surface current density $\bar{J}_s = J_0\hat{x}$ A/m is placed on the $z = 0$ plane between free-space for $z < 0$ and a dielectric with $\epsilon = \epsilon_r\epsilon_0$ for $z > 0$, as in the accompanying figure. Find the resulting $\bar{E}$ and $\bar{H}$ fields in the two regions. HINT: Assume plane wave solutions propagating away from the current sheet, and match boundary conditions to find the amplitudes, as in Example 1.3.



1.11 Redo Problem 1.10, but with an electric surface current density of $\bar{J}_s = J_0\hat{x}e^{-j\beta x}$ A/m, where $\beta < k_0$.

1.12 A parallel polarized plane wave is obliquely incident from free-space onto a magnetic material with permittivity ϵ_0 and permeability $\mu_0\mu_r$. Find the reflection and transmission coefficients. Does a Brewster angle exist for this case where the reflection coefficient vanishes for a particular angle of incidence?

1.13 Repeat Problem 1.12 for the perpendicularly polarized case.

1.14 An artificial anisotropic dielectric material has the tensor permittivity $[\epsilon]$ given as follows:

$$[\epsilon] = \epsilon_0 \begin{bmatrix} 1 & 3j & 0 \\ -3j & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

At a certain point in the material the electric field is known to be $\bar{E} = 3\hat{x} - 2\hat{y} + 5\hat{z}$. What is $\bar{D}$ at this point?

1.15 The permittivity tensor for a gyrotropic dielectric material is

$$[\epsilon] = \epsilon_0 \begin{bmatrix} \epsilon_r & j\kappa & 0 \\ -j\kappa & \epsilon_r & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Show that the transformations

$$\begin{aligned} E_+ &= E_x - jE_y, \qquad & D_+ &= D_x - jD_y, \\ E_- &= E_x + jE_y, \qquad & D_- &= D_x + jD_y, \end{aligned}$$

allow the relation between $\bar{E}$ and $\bar{D}$ to be written as

$$\begin{bmatrix} D_+ \\ D_- \\ D_z \end{bmatrix} = [\epsilon'] \begin{bmatrix} E_+ \\ E_- \\ E_z \end{bmatrix},$$

where $[\epsilon']$ is now a diagonal matrix. What are the elements of $[\epsilon']$? Using this result, derive wave equations for E_+ and E_- and find the resulting propagation constants.

1.16 Show that the reciprocity theorem expressed in (1.157) also applies to a region enclosed by a closed surface S, where a surface impedance boundary condition applies.

1.17 Consider an electric surface current density of $\bar{J}_s = \hat{y} J_0 e^{-\beta x}$ A/m located on the $z = d$ plane. If a perfectly conducting ground plane is located at $z = 0$, use image theory to find the total fields for $z > 0$.

1.18 Let $\bar{E} = E_\rho \hat{\rho} + E_\phi \hat{\phi} + E_z \hat{z}$ be an electric field vector in cylindrical coordinates. Demonstrate that it is incorrect to interpret the expression $\nabla^2 \bar{E}$ in cylindrical coordinates as $\hat{\rho}\nabla^2 E_\rho + \hat{\phi}\nabla^2 E_\phi + \hat{z}\nabla^2 E_z$ by evaluating both sides of the vector identity $\nabla \times \nabla \times \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E}$ for the given electric field.