EJERCICIOS VARIOS

1) a) combinació lineal : un vector u es combinació lineal de v_1 i v_2, sent v_1 i v_2 una base vectorial, si es pot expressar com $u = \lambda v_1 + \beta v_2$

b) i) v_1, v_2, v_3 suposo que són la base canónica de $\mathbb{R}^3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ que v_1, v_2 i v_3 siguin linealment independents no implica que v_4 tmb no sigui. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ $v_4 = v_2 + v_3$

2) S_a $\mathbb{R}^4$ $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ a \end{pmatrix} \right\}$

a) $\dim S_a = \text{Rang } S_a$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & -2 & a \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & a \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & a+1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

si $a = -1$ $\text{Rang } A = 2 \Rightarrow \dim S_{-1} = 2$

si $a \neq -1$ $\text{Rang } A = 3 \Rightarrow \dim S_{a \neq -1} = 3$

b) Base de S_{-1} i S_{-1} a $\mathbb{R}^4$

$$S^{-1} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \quad \text{i} \quad \mathbb{R}^4 \Rightarrow S^{-1} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

c) $$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 1 & z \\ -1 & 0 & t \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 1 & z \\ 0 & 0 & t+x \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z-y \\ 0 & 0 & t+x \end{pmatrix}$$ $\text{Rang } A = 2 \Rightarrow$ $z - y = 0$ i $t + x = 0$.

d) $\begin{cases} z - y = 0 \\ t + x = 0 \end{cases} \Rightarrow \begin{cases} -12 - 12 \neq 0 \\ \end{cases}$ el vector $u \notin S_{-1}$

$\rightarrow \begin{cases} 14 - 14 = 0 \\ -31 + 31 = 0 \end{cases}$ el vector $v \in S_{-1}$

3. $f : \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}^3$

$$f\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ f\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ f\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \ f\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

i)

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = f\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = f\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - f\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = f\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - f\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

ii)

$$\begin{pmatrix} 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$\dim \text{Im} f = \text{Rang } A = 3$ $\quad\quad$ $\dim \text{Ker} f = \dim \mathcal{M}_2 - \text{Rang } A = 4 - 3 = 1$

Base de Imf = $\left\langle \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\-1\end{pmatrix} \right\rangle$

Base de kerf :

$\begin{pmatrix}1&0&0&0\\0&1&0&-2\\0&0&1&-1\end{pmatrix}$ $\quad a=0 \quad d=c$
$\quad b-2d=0 \;;\; b=2d \quad\Rightarrow\quad$ kerf = $\begin{pmatrix}0&2d\\d&d\end{pmatrix}$

Rang $M = 3 \neq 4 = \dim \mathcal{M}_2(\mathbb{R})$

Rang $M = 3 = 3 = \dim(\mathbb{R}^3) \Rightarrow f$: exhaustiva

1. $f: E \to F$

a) u serà del subespai $\langle v_1, \dots, v_k \rangle$ si $\exists\, \alpha_1, \dots, \alpha_k \in \mathbb{K}$ tq $u = \alpha v$

b) f és lineal si compleix :

- $f(u+v) = f(u) + f(v)$
- $f(\lambda u) = \lambda \cdot f(u)$

On u, v són vectors qualsevols que $\in E$ i λ és un escalar qualsevol

c) Sabem que $u \in \langle v_1, \dots, v_k \rangle \Rightarrow u = \alpha v \Rightarrow f(u) = f(\alpha v) = \alpha \cdot f(v)$

2. $F = \left\langle \begin{pmatrix}2\\3\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}5\\8\\2\\-1\end{pmatrix}, \begin{pmatrix}-2\\-4\\0\\2\end{pmatrix} \right\rangle$

a) $\begin{pmatrix}2&1&5&-2\\3&1&8&-4\\1&1&2&0\\0&1&-1&2\end{pmatrix} \sim \begin{pmatrix}1&1&2&0\\0&-2&2&-4\\0&0&0&0\\0&1&-1&2\end{pmatrix} \sim \begin{pmatrix}1&1&2&0\\0&-1&1&-2\\0&0&0&0\\0&0&0&0\end{pmatrix}$ Rang 2 $\Rightarrow \dim F = \text{Rang } F = 2$

$B_{(F)} = \left\langle \begin{pmatrix}2\\3\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \right\rangle$

b) $\begin{pmatrix}2&1&x\\3&1&y\\1&1&z\\0&1&t\end{pmatrix} \sim \begin{pmatrix}1&1&z\\0&1&t\\2&1&x\\3&1&y\end{pmatrix} \sim \begin{pmatrix}1&1&z\\0&1&t\\0&-1&x-2z\\0&-2&y-3z\end{pmatrix} \sim \begin{pmatrix}1&1&z\\0&1&t\\0&0&x-2z+t\\0&0&y-3z+2t\end{pmatrix}$

Com que Rg = 2 ha de complir $x-2z+t=0$ i $y-3z+2t=0$

$F = \left\{ \begin{pmatrix}x\\y\\z\\t\end{pmatrix} : x-2z+t=0,\ y-3z+2t=0 \right\}$

c) $u = \begin{pmatrix}4\\6\\1\\-1\end{pmatrix}$, $v = \begin{pmatrix}0\\-1\\1\\2\end{pmatrix}$

$u: \begin{cases} 4-2\cdot1+(-1)=0 \;;\; 1 \neq 0 \\ \end{cases}$ $\quad u \notin F$

$v: \begin{cases} 0-2\cdot1+2=0 \\ -1-3\cdot1+2\cdot2=0 \end{cases}$ $\quad v \in F$

$\begin{pmatrix}0\\-1\\1\\2\end{pmatrix} = \alpha \cdot \begin{pmatrix}2\\3\\1\\0\end{pmatrix} + \beta \cdot \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ $\quad \begin{cases} 0 = 2\alpha+\beta \\ -1 = 3\alpha+\beta \\ 1 = \alpha+\beta \\ 2 = \beta \end{cases}$ $\quad \beta = 2 \Rightarrow 1 = \alpha + 2 \;;\; \alpha = -1$

$v = (-1)\cdot \begin{pmatrix}2\\3\\1\\0\end{pmatrix} + (2)\cdot \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \Rightarrow$ coord. $v_B = \begin{pmatrix}-1\\2\end{pmatrix}$

d) $B(\mathbb{R}^4) = \left\langle \begin{pmatrix}2\\3\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle$ ja que tots 4 són linealment independents .

3. $f: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3$

$$f\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix}, \quad f\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}, \quad f\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -14 \end{pmatrix}, \quad f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -7 \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \cdot f\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \qquad f\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = f\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} - f\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 2f\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = f\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ -2 & -3 & 3 & -7 \end{pmatrix}$$

f no pot ser injectiva ja que $\dim M_{2\times 2} > \dim \mathbb{R}^3$.

f serà exhaustiva si $Rg(A) = \dim \mathbb{R}^3 = 3$

$$\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad Rg(A) = 2 \neq \dim \mathbb{R}^3 = 3$$

Per tant f no és exhaustiva ni injectiva ni bijectiva

b) $$M = \begin{pmatrix} -1 & 0 & -3 \\ 3 & 2 & 3 \\ -3 & 0 & -1 \end{pmatrix}$$

$$\det(M - xI_3) = \begin{vmatrix} -1-x & 0 & -3 \\ 3 & 2-x & 3 \\ -3 & 0 & -1-x \end{vmatrix} = (2-x)\cdot[(-1-x)^2 - (-3)^2] = (2-x)\cdot(x^2+2x-8)$$

$x = 2$ i $x = -4$ $\Rightarrow$ Els valors propis de f són 2 i -4

subespai E_2 i E_{-4}

$$E_2 \Rightarrow \quad M - 2I_3 \quad \begin{pmatrix} -3 & 0 & -3 \\ 3 & 0 & 3 \\ -3 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} -3 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \begin{aligned} x + z &= 0 \\ x &= -z \end{aligned}$$

$$E_2 = \left\{ \begin{pmatrix} x \\ y \\ -x \end{pmatrix} : x, y \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$E_{-4} \Rightarrow \quad M - (-4)\cdot I_3 \quad \begin{pmatrix} 3 & 0 & -3 \\ 3 & 6 & 3 \\ -3 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x - z = 0 &;\ x = z \\ y + z = 0 &;\ y = z \end{aligned}$$

$$E_{-4} = \left\{ \begin{pmatrix} z \\ -z \\ z \end{pmatrix} : z \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

ii) Com que $\dim E_2 = 2$ i $\dim E_{-4} = 1$ com que la multipicitat del valor propi 2 es 2 i la de -4 és 1 f diagonalitza.

$$B = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

1. a) Que a de complir f per que sigui una aplicació lineal
   - $\forall u, v$ parella de vectors si $u, v \in E \Rightarrow f(u+v) = f(u) + f(v)$
   - $\forall u$ vector si $u \in E$ i $\lambda \in \mathbb{R} \to f(\lambda u) = \lambda f(u)$

   b) S subespai de E, $f: E \to F$ lineal $\Rightarrow$ $f(S)$ subespai de F
   - $f(S) \neq \emptyset$
   - $f(0_E) = 0_F$

   $\Rightarrow$ si $x, y \in f(S) \Rightarrow x = f(u)$ i $y = f(v)$ on $u, v \in S$
   $x + y = f(v) + f(u) = f(v + u)$

   $\Rightarrow$ si $x \in f(S)$ i $\lambda \in \mathbb{R} \Rightarrow x = f(u)$ $u \in S$
   $\lambda x = \lambda f(u) = f(\lambda u)$

2. $\mathbb{R}^4$ $a \in \mathbb{R}$

$$E = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \mid x - 2z = 0,\ y - 2z + 2t = 0,\ x - 4y + 2z = 0,\ y - 2t = 0 \right\}$$

$$F_a = \left\langle \begin{pmatrix} 1 \\ 3 \\ a \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ a \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

a) $$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 2 \\ 1 & -4 & 2 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -12 & 12 \\ 0 & 0 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\dim E$ = nº incognites - $Rg\, M = 4 - 3 = 1$. $E = \left\langle \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

b) $$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 0 & a \\ a & 0 & 0 \\ 0 & a & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & a-3 \\ 0 & a & -a \\ 0 & a & -1 \end{pmatrix}$$

$\cancel{3} + a - \cancel{3} = 0$

Tenim per $a = 0$ i $a \neq 0$

* $a = 0$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

1. a) $\{v_1, \ldots, v_k\} \in E$

Una combinació lineal és qualsevol vector $u \in E$ tq $u = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_k v_k$ per qualsevol λ.

$\{v_1, \ldots, v_k\} \in E$ seran linealment independents si $\lambda_1 v_1 + \ldots + \lambda_k v_k = 0_E \Rightarrow \lambda_1 = \ldots = \lambda_k = 0$

b) Suposem $\{v_1, \ldots, v_k\} \in E$ una família linealment independent $\Rightarrow \exists\, \lambda_1 \ldots \lambda_k \neq 0$ tq $\lambda_1 v_1 + \ldots + \lambda_2 v_2 = 0_E \Rightarrow v_1 = \frac{1}{\lambda_1}(-\lambda_2 v_2 - \ldots - \lambda_k v_k) \Rightarrow v_1$ combinació lineal dels altres. Recíprocament si $v_1 = \lambda_2 v_2 + \ldots + \lambda_k v_k \Rightarrow v_1 - \lambda_2 v_2 - \ldots - \lambda_k v_k = 0_E$.

2. $\lambda \in \mathbb{R}$ i $F_\lambda \in \mathbb{R}^4$

$$F_\lambda = \left\{ \begin{pmatrix} x \\ y \\ t \\ t \end{pmatrix} \in \mathbb{R}^4 : x + y + 2z + t = 0,\ 3x + 2y - 3z + (2+\lambda)t = 0,\ y - 3z + (-2\lambda - 1)t = 0 \right\}$$

a) $\dim F_y = 4 - Rg$

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 3 & 2 & -3 & 2+\lambda \\ 0 & 1 & -3 & -2\lambda-1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -3 & -2\lambda-1 \\ 3 & 2 & -3 & 2+\lambda \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -3 & -2\lambda-1 \\ 0 & -1 & 3 & \lambda-1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -3 & -2\lambda-1 \\ 0 & 0 & 0 & +\lambda+2 \end{pmatrix}$$

* si $\lambda \neq -2$ $Rg = 3 \Rightarrow \dim F_\lambda = 1$

* si $\lambda = -2$

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $Rg = 2 \Rightarrow \dim F_\lambda = 2$

b) B de F_{-2}, completela fins $\mathbb{R}^4$.

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -3 & 3 \end{pmatrix}$$

$x + y - 2z + t = 0 \Rightarrow x + (3z - 3t) - 2z + t = 0 \Rightarrow x + z - 2t = 0 \Rightarrow x = -z + 2t$

$y - 3z + 3t = 0 \Rightarrow y = 3z - 3t$

$$B = \left\{ \begin{pmatrix} -z-6t \\ 3z+5t \\ z \\ t \end{pmatrix} \right\} \Rightarrow B = \left\langle \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$W = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

c) $$v = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} 0 = -\lambda + 2\beta & \Rightarrow -2 + 2 = 0 ✓ \\ 3 = 3\lambda - 3\beta & \Rightarrow 3 \cdot 2 - 3 = 3 ✓ \\ 2 = \lambda \\ 1 = \beta \end{cases}$$

$$v_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

d) $\begin{pmatrix} 1 & 1 & x \\ 2 & 1 & y \\ 0 & 0 & z \\ 3 & 1 & t \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & x \\ 0 & -1 & y-2x \\ 0 & 0 & z \\ 0 & -2 & t-3x \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & x \\ 0 & 1 & 2x-y \\ 0 & 0 & z \\ 0 & 0 & t-3x-2y+4x \end{pmatrix}$ $\begin{cases} z=0 \\ t+x-2y=0 \end{cases}$

$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in G \Leftrightarrow z=0,\ t+x-2y=0$

3. $f: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y+2z \\ x+2y-z \\ -x+y+7z \end{pmatrix}$

a) $f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $f\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $f\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 7 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \dim \operatorname{Im} f = 2 = Rg\,M$

$B\operatorname{Im} f = < \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} >$

Per que f sigui → injectiva $Rg\,M = \dim(\mathbb{R}^3) \Rightarrow 2 \neq 3$ no és injectiva
└→ exhaustiva $Rg\,M = \dim(\mathbb{R}^3) \Rightarrow 2 \neq 3$ no és exhaustiva

c) matriu asociada a f a $B = < \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} >$

$P_C^B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow P_B^C = (P_C^B)^{-1} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$

$P_B^C = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ $\quad$ volem $M_B^B(f) = P_B^C \cdot M \cdot P_C^B = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 4 & 2 & 2 \\ -4 & 0 & 1 \end{pmatrix}$

d) valors propis $f = \det(M - xI_3)$

$\begin{vmatrix} -x & 1 & 2 \\ 1 & 2-x & -1 \\ -1 & 1 & 7-x \end{vmatrix} = (-x\cdot(2-x)\cdot(7-x)) + 3 - (-2(2-x) + (-x) + 7 - x) =$
$= -14x + 2x^2 + 7x^2 - x^3 + 3 - (-4 + 2x - x + 7 - x) = -x^3 + 9x^2 - 16x = -x(x^2 - 9x + 16)$

$x = 0$, $x = \dfrac{9+\sqrt{17}}{2}$, $x = \dfrac{9-\sqrt{17}}{2}$ Espai $\dim(\mathbb{R}^3) = 3$ i endomorfisme té 3 valors propis f diagonalitza.

* VALOR PROPIO DE f: Es un escalar λ para el cual $\exists$ un vector $v \in E$ tq $f(v) = \lambda v$.
* VECTOR PROPIO DE VALOR PROPIO λ: Qualquiera de los vectores $v \in E$ $\neq 0_E$ tq $f(v) = \lambda v$.
* E_λ SUBESPACIO DE E:
  - $E_\lambda \neq \emptyset$
  - $u, v \in E_\lambda \Rightarrow f(u) = \lambda u \wedge f(v) = \lambda v \Rightarrow f(u+v) = f(u) + f(v) = \lambda u + \lambda v = \lambda(u+v)$
  - $u \in E_\lambda$ i $\mu \in K$ (escalar) $\Rightarrow f(u) = \lambda u \wedge f(\mu u) = \mu f(u) = \mu(\lambda u) = (\mu\lambda)u = \lambda(\mu u)$

  E_λ tancat per sumes i commutatiu $\Rightarrow$ subespai de E
* COMBINACIÓN LINEAL: $u \in <v_1, \dots, v_k> \Leftrightarrow \exists \alpha_1, \dots, \alpha_k \in K$ tq $u = \sum_{i=1}^k \alpha_i v_i$
* LINEALMENTE INDEPENDIENTES: $\{v_1, \dots, v_k\} \in E$ seràn linealment independents si $\lambda_1 v_1 + \dots + \lambda_k v_k = 0_E \Rightarrow \lambda_1 = \dots = \lambda_k = 0$
* f PQ SIGUI UNA APLICACIÓ LINEAL: $\forall u, v \in E$ i $\lambda \in K$
  - $f(u+v) = f(u) + f(v)$
  - $f(\lambda u) = \lambda f(u)$

①

a) El valor propi de f es un escalar λ per el qual $\exists$ un vector $v \neq 0$ tq $f(v) = \lambda v$.
un vector de valor propi λ es qualsevol vector $v \neq 0$ que s'expressa tq $f(v) = \lambda v$

b) $E_\lambda = \{ u \in E : f(u) = \lambda u \}$ subespai de E.

$E_\lambda \neq 0$

- $\forall u, v \in E_\lambda \Rightarrow f(u) = \lambda u$ i $f(v) = \lambda v \Rightarrow f(u+v) = f(u) + f(v) = \lambda u + \lambda v = \lambda(u+v)$
- $\forall u \in E_\lambda$ i $\mu \in K \Rightarrow f(u) = \lambda u \wedge f(\mu u) = \mu f(u) = \mu(\lambda u) = (\mu\lambda)u = \lambda(\mu u)$

E_λ tancat per suma i commutatiu $\Rightarrow E_\lambda$ subespai de E

2. $F = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} \in \mathbb{R}^5 \mid x = 0,\ y + t = z + u \right\} \subset \mathbb{R}^5$

a) F es un subespai de $\mathbb{R}^5$ ja que:
F es un sistema homogeni $x = 0$ i $y - z + t - u = 0$ i es subespai de $\mathbb{R}^n$ sempre que n = num. incognites de F.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \end{pmatrix} \quad Rg = 2 \qquad \dim F = 5 - 2 = 3$$

$x = 0$
$y = z - t + u$

$$F\begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = z\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow F = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

b) $F' = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ \lambda \end{pmatrix} \right\rangle \quad F \cap F' \neq 0,\ \dim F \cap F'$

Trovem el sistema homogeni de F':

$$\begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ \lambda \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & x \\ 1 & -1 & y \\ 0 & 1 & z \\ 0 & -1 & t \\ 1 & \lambda & u \end{array}\right) \sim \begin{pmatrix} 1 & 0 & x \\ 0 & -1 & y-x \\ 0 & 0 & z+y-x \\ 0 & 0 & t-y+x \\ 0 & \lambda & u-x \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & x \\ 0 & -1 & y-x \\ 0 & 0 & z+y-x \\ 0 & 0 & t-y+x \\ 0 & 0 & u-x+\lambda(y-x) \end{pmatrix}$$

$$\begin{cases} x = 0 \\ y + t - z - u = 0 \\ z + y - x = 0 \\ t - y + x = 0 \\ u + \lambda y - (1+\lambda)x = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ -(1+\lambda) & \lambda & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -(1+\lambda) & \lambda & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & \lambda & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & \lambda & -\lambda & 1+\lambda \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -2 & 1 & -1 \\ 0 & 0 & \lambda & -\lambda(1+\lambda) & \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & \lambda & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 1+\lambda/3 \end{pmatrix}$$

$F \cap F' \neq \{0\} \Rightarrow SCI$
$\lambda = -3 \quad Rg = 4$
$\dim F \cap F' = 5 - 4 = 1$

3 $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad f: \mathbb{R}^3 \to \mathbb{R}^3$

$$f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix},\ f\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix},\ f\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$$

i) $f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - f\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \qquad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

$f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = f\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + f\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$

$$M_C^C(f) = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -2/3 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -2/3 & 0 \\ 0 & 5/3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -2/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$Rg = 3 = \dim(\mathbb{R}^3)$ f és exhaustiva i injectiva

$$M_B^B(f) = P_B^C \cdot N_C^B$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) = P_B^C$$

$$M_B^B(f) = M_C^B \cdot P_B^C = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ 1 & -2 & 3 \\ 5 & -5 & 5 \end{pmatrix} = \begin{pmatrix} 1-1+5 & 3+2-5 & -2-3+5 \\ 0+1-5 & 0-2+5 & 0+3-5 \\ -1+2-5 & -3-4+5 & 2+6-5 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ -4 & 3 & -2 \\ -4 & -2 & 3 \end{pmatrix}$$

b) $M = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ $\quad \det(M - xI_3)$ $\quad \begin{pmatrix} 3-x & -2 & 0 \\ -2 & 3-x & 0 \\ 0 & 0 & 5-x \end{pmatrix} = (5-x) \cdot \begin{pmatrix} 3-x & -2 \\ -2 & 3-x \end{pmatrix} = (5-x) \cdot [(3-x)^2 - 4]$

$$= (5-x) \cdot [(x-3)^2 - 4] = (5-x) \cdot (x^2 - 6x + 9 - 4) = (5-x) \cdot (x^2 - 6x + 5) = (5-x)(x-5)(x-1)$$

$$= -(x-5)^2 (x-1)$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 5}}{2} = \frac{+6 \pm \sqrt{16}}{2} \begin{cases} \frac{6+4}{2} \Rightarrow x = 5 \\ \frac{6-4}{2} \Rightarrow x = 1 \end{cases}$$

Tenim el valor propi $x=5$ de multiplicitat 2 i $x=1$ de multiplicitat 1, $2+1=3=\dim(\mathbb{R}^3)$, f diagonalitza.

Mirem que la $\dim(E_5) = 2 =$ multiplicitat

$$\begin{pmatrix} 3-5 & -2 & 0 \\ -2 & 3-5 & 0 \\ 0 & 0 & 5-5 \end{pmatrix} \sim \begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\dim(E_5) = \mathbb{R}^3 - Rg = 3 - 1 = 2 \Rightarrow$ diagonalitza