Projective and Injective Modules

MATH 511: Algebra III

Alejandro Adames

University of Calgary

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Definitions

Definition (Hom Functor)

If R is a commutative ring, then $\mathrm{Hom}_R(A, -)$ (where A is an R-module) is a covariant functor that maps R-modules M in R-**Mod** to R-**Mod** via

$$M \mapsto \mathrm{Hom}_R(A, M)$$

and maps $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(\mathrm{Hom}_R(A, M), \mathrm{Hom}_R(A, N))$ via

$$f \mapsto \varphi \circ f$$

$$\begin{tikzcd} A \arrow[d, "\varphi"'] \arrow[dr, "f\circ\varphi"] & \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Definitions

Definition (Exact Functor)

A covariant functor $\mathscr{F}$ from R-**Mod** to R-**Mod** is exact if whenever

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$$

is an exact sequence, then

$$0 \longrightarrow \mathscr{F}(M) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(N) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(L) \longrightarrow 0$$

is an exact sequence.

Definitions

Split Epimorphisms

An epimorphism $\varphi : M \to N$ of $R-$modules is said to split if and only if it has a right inverse.

$$\exists \psi : N \to M, \quad \varphi \circ \psi = \mathrm{id}_N$$

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Split Epimorphisms

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$$\exists \psi : N \to M, \quad \varphi \circ \psi = \mathrm{id}_N$$

Proposition

This is equivalent to the sequence

$$0 \longrightarrow \ker \varphi \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0$$

splitting.

Definitions

Definition (Projective Module)

A module P is projective if the functor $\mathrm{Hom}_R(P, -)$ is exact.

Definition (Injective Module)

A module Q is projective if the functor $\mathrm{Hom}_R(-, Q)$ is exact.

Definitions

Definition (Pullack)

If M, N, Z are objects in a category and $\mu : M \to Z$ and $\nu : N \to Z$ are morphisms,

$$\begin{tikzcd} & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"] & Z \end{tikzcd}$$

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If M, N, Z are objects in a category and $\mu : M \to Z$ and $\nu : N \to Z$ are morphisms, a pullback is an object $M \times_Z N$ together with morphisms $\pi_N : M \times_Z N \to N$ and $\pi_M : M \times_Z N \to M$

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$$\begin{array}{ccc} M \times_Z N & \xrightarrow{\pi_N} & N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\mu} & Z \end{array}$$

Furthermore, it is universal, meaning that for every object L, and morphisms $\varphi_M : L \to M$ and $\varphi_N : L \to M$ such that $\mu \circ \varphi_M = \nu \circ \varphi_N$,

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\begin{tikzcd}
L \arrow[dr, "\exists!\sigma"] \arrow[drr, bend left, "\varphi_N"] \arrow[ddr, bend right, "\varphi_M"'] & & \\
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\end{tikzcd}
$$

Furthermore, it is universal, meaning that for every object L, and morphisms $\varphi_M : L \to M$ and $\varphi_N : L \to M$ such that $\mu \circ \varphi_M = \nu \circ \varphi_N$, there is a unique $\sigma : L \to M \times_Z N$ that makes the diagram commute.

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So, do pullbacks exists in R-**Mod**?

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For R-modules M, N, Z, and R-module homomorphisms $\mu : M \to Z$, $\nu : N \to Z$, define $M \times_Z N$ as

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$$M \times_Z N \doteq \{(m, n) \in M \times N : \mu(m) = \nu(n)\}$$

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Exercise for you: show that this is an R-module.

Do we have them in R-**Mod**?

Proof (Cont.)

What are the maps $\pi_N : M \times_Z N \to N$ and $\pi_M : M \times_Z N \to M$?

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What are the maps $\pi_N : M \times_Z N \to N$ and $\pi_M : M \times_Z N \to M$?

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They make the diagram

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commute since for all $(m,n) \in M \times_Z N$ we have

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Now we have to check that $M \times_Z N$ is universal with respect to this property.

Do we have them in R-**Mod**?

Proof (Cont.)

Suppose we had another R-module L and R-module homomorphisms $\varphi_N : L \to N$ and $\varphi_M : L \to M$ such that $\mu \circ \varphi_N = \nu \circ \varphi_M$.

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Suppose we had another R-module L and R-module homomorphisms $\varphi_N : L \to N$ and $\varphi_M : L \to M$ such that $\mu \circ \varphi_N = \nu \circ \varphi_M$. Define

$$\sigma(\ell) = (\varphi_M(\ell), \varphi_N(\ell))$$

Check: Does this map go to $M \times_Z N$? Is this really an R-module homomorphism?

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$$\pi_M(\sigma(\ell)) = \pi_M(\varphi_M(\ell), \varphi_N(\ell)) = \varphi_M(\ell)$$

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\end{tikzcd}$$

Commutes

Do we have them in R-**Mod**?

Proof (Cont.)

Now we need to check that σ is unique. Suppose that we had an R-module homomorphism $\alpha : L \to M \times_Z N$ such that

$$\pi_M(\alpha(\ell)) = \varphi_M(\ell) \qquad \pi_N(\alpha(\ell)) = \varphi_N(\ell)$$

and suppose $\alpha(\ell) = (m, n)$. Hence

$$\varphi_M(\ell) = \pi_M(\alpha(\ell)) = \pi_M(m, n) = m$$

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Therefore $\alpha(\ell) = \sigma(\ell)$ showing that σ is unique! Therefore pullbacks exist in R-mod.

$\square$

A useful Lemma

Lemma (If μ is surjective, then π_N is surjective)

Consider the following diagram

$$\begin{tikzcd}
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Suppose that μ is surjective.

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For any $n \in N$, $\nu(n) \in Z$.

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There exists some $m \in M$ such that $\mu(m) = \nu(n)$.

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There exists some $m \in M$ such that $\mu(m) = \nu(n)$.
But this means that $(m, n) \in M \times_Z N$.
Hence, $\pi_N(m, n) = n$.
This shows that π_N is surjective.

□

Characterization of Projective Modules

Theorem 1

An R-module P is projective if and only if for all epimorphisms of R-modules $\mu : M \to N$ every R-linear map $f : P \to N$ lifts to an R-linear map $\hat{f} : P \to M$.

Characterization of Projective Modules

Theorem 1

An R-module P is projective if and only if for all epimorphisms of R-modules $\mu : M \to N$ every R-linear map $f : P \to N$ lifts to an R-linear map $\hat{f} : P \to M$.

$$M \xrightarrow[\mu]{} N \longrightarrow 0$$

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$$\begin{tikzcd} & P \arrow[dl, "\exists \hat{f}"'] \arrow[d, "f"] & \\ M \arrow[r, "\mu"'] & N \arrow[r] & 0 \end{tikzcd}$$

Characterization of Projective Modules

Proof

($\Rightarrow$) Suppose that P is projective. If we have an epimorphism

$$M \xrightarrow{\mu} N \longrightarrow 0$$

we can extend it to a short exact sequence where K is the kernel of μ

$$0 \longrightarrow K \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}_R(P,K) \xrightarrow{\lambda\circ} \mathrm{Hom}_R(P,M) \xrightarrow{\mu\circ} \mathrm{Hom}_R(P,N) \longrightarrow 0$$

is also exact.

Characterization of Projective Modules

Proof (Cont.)

By exactness of

$$\mathrm{Hom}_R(P, M) \xrightarrow{\mu\circ} \mathrm{Hom}_R(P, N) \longrightarrow 0$$

Characterization of Projective Modules

Proof (Cont.)

By exactness of

$$\mathrm{Hom}_R(P, M) \xrightarrow{\mu\circ} \mathrm{Hom}_R(P, N) \longrightarrow 0$$

For each $f \in \mathrm{Hom}_R(P, N)$,

$$\begin{tikzcd} P \arrow[d, "f"] \\ N \end{tikzcd}$$

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Proof (Cont.)

By exactness of

$$\mathrm{Hom}_R(P, M) \xrightarrow{\mu\circ} \mathrm{Hom}_R(P, N) \longrightarrow 0$$

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there is some $\hat{f} \in \mathrm{Hom}_R(P, M)$,

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there is some $\hat{f} \in \mathrm{Hom}_R(P, M)$,
such that $\mu \circ \hat{f} = f$.

$$\begin{tikzcd} & P \arrow[dl, "\exists \hat{f}"'] \arrow[d, "f"] & \\ M \arrow[r, "\mu"'] & N \arrow[r] & 0 \end{tikzcd}$$

Characterization of Projective Modules

Proof (Cont.)

($\Leftarrow$) Now suppose

$$\begin{tikzcd} & P \arrow[dl, "\exists g"'] \arrow[d, "f"] & \\ M \arrow[r, "\mu"'] & N \arrow[r] & 0 \end{tikzcd}$$

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Since $\mathrm{Hom}_R(P-)$ is left exact we only have to show that

$$\mathrm{Hom}_R(P, M) \xrightarrow{\mu\circ} \mathrm{Hom}_R(P, N) \longrightarrow 0$$

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is exact. But we can do this since if we have $f \in \mathrm{Hom}(P, N)$, there is some $g \in \mathrm{Hom}_R(P, M)$ such that $\mu \circ g = f$.

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Theorem 2

P is projective if and only if every epimorphism $M \to P$ splits.

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Proof

($\Rightarrow$) If P is projective then using the characterization from Theorem 1, we have that there exists a g such that the following diagram commutes

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i.e $\mu \circ g = \mathrm{id}_P$ therefore μ splits.

Characterization of Projective Modules

Proof (Cont.)

($\Leftarrow$) Suppose every epimorphism to P splits. Here's where we get to use pullbacks! Consider an epimorphism

$$M \xrightarrow{\mu} Z \longrightarrow 0$$

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$$\begin{tikzcd}
M \times_Z P \arrow[r, "\pi_P"] \arrow[d, "\pi_M"] & P \arrow[d, "f"] & \\
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By our lemma, since π_P is surjective, (i.e. an epimorphism) it must split.

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By our lemma, since π_P is surjective, (i.e. an epimorphism) it must split. There exists $\psi : P \to M \times_Z P$ such that $\pi_P \circ \psi = \mathrm{id}_P$.

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$$f \circ \pi_P \circ \psi = \mu \circ \pi_M \circ \psi$$

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$$f \circ \pi_P \circ \psi = \mu \circ \pi_M \circ \psi$$

Therefore

$$f = \mu \circ g.$$

$\square$

Characterization of Projective Modules

Theorem 3

An R-module P is projective if and only if there exists a free module F, an R-module K, and an isomorphism $F \cong K \oplus P$.

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and the inclusion

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Since P is projective the sequence

$$0 \longrightarrow \ker \varphi \longrightarrow F \longrightarrow P \longrightarrow 0$$

splits so $F \cong \ker \varphi \oplus P$.

Characterization of Projective Modules

Proof (Cont.)

($\Leftarrow$) Suppose that $F \cong K \oplus P$.

$$\begin{tikzcd} & F \arrow[d, "f"] & \\ M \arrow[r, "\mu"'] & N \arrow[r] & 0 \end{tikzcd}$$

Characterization of Projective Modules

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\begin{tikzcd}
 & S \arrow[d, "j"] & \\
 & F \arrow[d, "f"] & \\
M \arrow[r, "\mu"'] & N \arrow[r] & 0
\end{tikzcd}
$$

$j(s) \in F$ so $f(j(s)) = \mu(m)$ for some $m \in M$.

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$j(s) \in F$ so $f(j(s)) = \mu(m)$ for some $m \in M$.

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Universal property of free modules gives us $\varphi : F \to M$.
Since $\mu \circ \overline{f} = f \circ j$, $\qquad \mu \circ \varphi = f$

Characterization of Projective Modules

Proof (Cont.)

We want to show that P is projective.

$$\begin{tikzcd}
P \arrow[r, hook, "\iota_P"] \arrow[rr, bend left, "\mathrm{id}_P"] \arrow[dr, "\psi=\varphi\circ\iota_P"'] & K \oplus P \arrow[r, two heads, "\pi_P"] \arrow[d, "\varphi"] \arrow[dr, "g\circ\pi_P"] & P \arrow[d, "g"] & \\
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P is projective. □

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Taking $R = \mathbb{Z}/6\mathbb{Z}$. The module $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is free as an R-module and by our previous theorem, $\mathbb{Z}/2\mathbb{Z}$ is projective as an R-module.

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What About Injective Modules?

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Theorem 4

A module Q is injective if and only if for all monomorphisms of R-modules $\lambda : L \to M$, every R-linear map $q : L \to Q$ extends to an R-linear map $\hat{q} : M \to Q$.

$$\begin{tikzcd} & Q & \\ 0 \arrow[r] & L \arrow[u, "q"] \arrow[r, "\lambda"'] & M \arrow[ul, dotted, "\exists \hat{q}"'] \end{tikzcd}$$

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The proof of this is essentially the same as the one for projective modules.

Theorem 5

A module Q is injective if and only if every monomorphism $Q \to M$ splits.

Pushouts (or fibered coproduct) are used to prove this. They're the dual of pullbacks.

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Remark

However, there is no theorem for injective modules similar to the one of a projective module being a direct summand of a free module.

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Theorem (Baer's Criterion)

An R-module Q is injective if and only if every R-linear map $f : I \to Q$, where I is an ideal of R, extends to an R-linear map $\hat{f} : R \to Q$.

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An R-module Q is injective if and only if every R-linear map $f : I \to Q$, where I is an ideal of R, extends to an R-linear map $\hat{f} : R \to Q$.

Proof.

The proof uses a clever application of Zorn's lemma. See Aluffi for full proof. □

Summarizing our results

	Lifting/Extending	Splitting	Summands of free modules
Projective Modules	Yes	Yes	Yes
Injective Modules	Yes	Yes	No

Table: Comparison between injective and projective module characterizations

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Thank you!

Thank you!
Any Questions?