Projective and Injective Modules

MATH 511: Algebra III

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December 4, 2020

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Definition (Hom Functor)

If R is a commutative ring, then $\operatorname{Hom}_R(A,-)$ (where A is an R-module) is a covariant functor that maps R-modules M in R-**Mod** to R-**Mod** via

$$M \mapsto \operatorname{\mathsf{Hom}}_R(A,M)$$

and maps $\operatorname{\mathsf{Hom}}_R(M,N) \to \operatorname{\mathsf{Hom}}_R(\operatorname{\mathsf{Hom}}_R(A,M),\operatorname{\mathsf{Hom}}_R(A,N))$ via

$$f\mapsto \varphi\circ f$$



Definition (Exact Functor)

A covariant functor \mathscr{F} from R-**Mod** to R-**Mod** is exact if whenever

$$0 \longrightarrow M \stackrel{f}{\longrightarrow} N \stackrel{g}{\longrightarrow} L \longrightarrow 0$$

is an exact sequence, then

$$0 \longrightarrow \mathscr{F}(M) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(N) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(L) \longrightarrow 0$$

is an exact sequence.

Split Epimorphisms

An epimorphism $\varphi:M\to N$ of R-modules is said to split if and only if it has a right inverse.

$$\exists \psi : N \to M, \quad \varphi \circ \psi = \mathrm{id}_N$$

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Proposition

This is equivalent to the sequence

$$0 \longrightarrow \ker \varphi \longrightarrow M \stackrel{\varphi}{\longrightarrow} N \longrightarrow 0$$

splitting.

Definition (Projective Module)

A module P is projective if the functor $Hom_R(P, -)$ is exact.

Definition (Injective Module)

A module Q is projective if the functor $Hom_R(-, Q)$ is exact.

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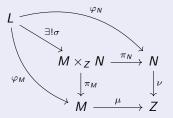
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Furthermore, it is universal, meaning that for every object L, and morphisms $\varphi_M: L \to M$ and $\varphi_N: L \to M$ such that $\mu \circ \varphi_M = \nu \circ \varphi_N$, there is a unique $\sigma: L \to M \times_Z N$ that makes the diagram commute.

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Exercise for you: show that this is an *R*-module.

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M \times_{Z} N & \xrightarrow{\pi_{N}} & N \\
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M & \xrightarrow{\mu} & Z
\end{array}$$

commute since for all $(m, n) \in M \times_Z N$ we have

$$\mu(\pi_M(m,n)) = \mu(m) = \nu(n) = \nu(\pi_N(m,n))$$

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Now we have to check that $M \times_Z N$ is universal with respect to this property.

Proof (Cont.)

Suppose we had another R-module L and R-module homomorphisms $\varphi_N:L\to N$ and $\varphi_M:L\to M$ such that $\mu\circ\varphi_N=\nu\circ\varphi_M$.

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$$\sigma(\ell) = (\varphi_M(\ell), \varphi_N(\ell))$$

Check: Does this map go to $M \times_Z N$? Is this really an R-module homomorphism?

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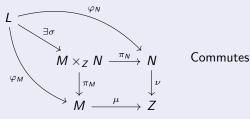
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Proof (Cont.)

Now we need to check that σ is unique. Suppose that we had an R-module homomorphism $\alpha:L\to M\times_Z N$ such that

$$\pi_{\mathcal{M}}(\alpha(\ell)) = \varphi_{\mathcal{M}}(\ell)$$
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and suppose $\alpha(\ell) = (m, n)$. Hence

$$\varphi_M(\ell) = \pi_M(\alpha(\ell)) = \pi_M(m, n) = m$$

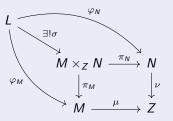
$$\varphi_N(\ell) = \pi_N(\alpha(\ell)) = \pi_M(m, n) = n$$

Therefore $\alpha(\ell) = \sigma(\ell)$ showing that σ is unique! Therefore pullbacks exist in R-mod.



Lemma (If μ is surjective, then π_N is surjective)

Consider the following diagram



If μ is surjective, then π_N is surjective.

Proof

Suppose that μ is surjective.

$$\begin{array}{ccc} M \times_{Z} N & \xrightarrow{\pi_{N}} & N \\ \downarrow^{\pi_{M}} & & \downarrow^{\nu} \\ M & \xrightarrow{\mu} & Z \end{array}$$

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For any $n \in N$, $\nu(n) \in Z$.

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This shows that π_N is surjective.



Theorem 1

An R-module P is projective if and only if for all epimorphisms of R-modules $\mu:M\to N$ every R-linear map $\hat f:P\to N$ lifts to an R-linear map $\hat f:P\to M$.

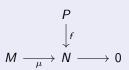
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$$M \xrightarrow{\mu} N \longrightarrow 0$$

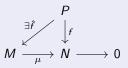
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Proof

 (\Rightarrow) Suppose that P is projective. If we have an epimorphism

$$M \xrightarrow{\mu} N \longrightarrow 0$$

we can extend it to a short exact sequence where ${\it K}$ is the kernel of μ

$$0 \longrightarrow K \stackrel{\lambda}{\longrightarrow} M \stackrel{\mu}{\longrightarrow} N \longrightarrow 0$$

 $0 \longrightarrow \operatorname{Hom}_R(P,K) \xrightarrow{\lambda \circ} \operatorname{Hom}_R(P,M) \xrightarrow{\mu \circ} \operatorname{Hom}_R(P,N) \longrightarrow 0$ is also exact.

Proof (Cont.)

By exactness of

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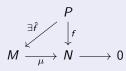


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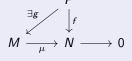
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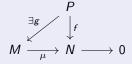
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is exact. But we can do this since if we have $f \in \text{Hom}(P, N)$, there is some $g \in \text{Hom}_R(P, M)$ such that $\mu \circ g = f$.

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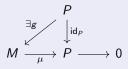
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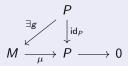


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i.e $\mu \circ g = id_P$ therefore μ splits.

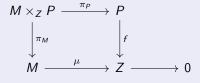
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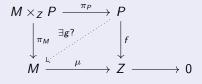
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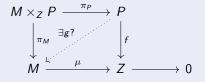


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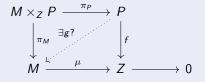
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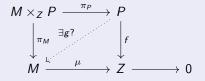
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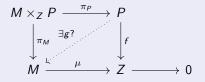
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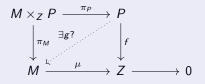


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Therefore

$$f = \mu \circ g$$
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Since *P* is projective the sequence

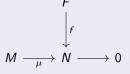
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splits so $F \cong \ker \varphi \oplus P$.



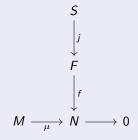
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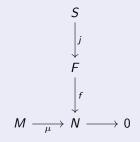
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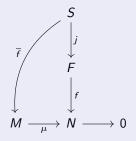


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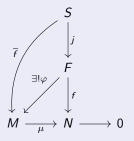
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Universal property of free modules gives us $\varphi : F \to M$.

Proof (Cont.)

 (\Leftarrow) Suppose that $F \cong K \oplus P$.



$$j(s) \in F$$
 so $f(j(s)) = \mu(m)$ for some $m \in M$.

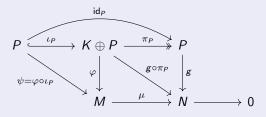
Define $\overline{f}:S\to M$ via $\overline{f}(s)=m$

Universal property of free modules gives us $\varphi : F \to M$.

Since $\mu \circ \overline{f} = f \circ j$, $\mu \circ \varphi = f$

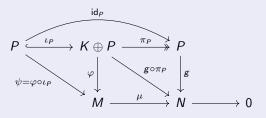
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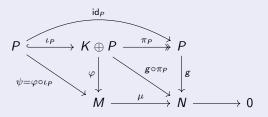


So there exists φ such that $g \circ \pi_P = \mu \circ \varphi$

Characterization of Projective Modules

Proof (Cont.)

We want to show that P is projective.



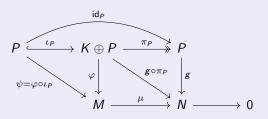
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Characterization of Projective Modules

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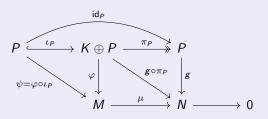
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Characterization of Projective Modules

Proof (Cont.)

We want to show that P is projective.



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P is projective.



Corollary

Free modules are projective.

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Taking $R = \mathbb{Z}/6\mathbb{Z}$.

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Taking $R = \mathbb{Z}/6\mathbb{Z}$. The module $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is free as an R-module and by our previous theorem, $\mathbb{Z}/2\mathbb{Z}$ is projective as an R-module.

Corollary

Free modules are projective.

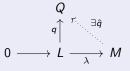
We incidentally proved this in the proof of the previous theorem. Question: Are there projective modules that are not free?

Example

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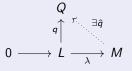
Theorem 4

A module Q is injective if and only if for all monomorphisms of R-modules $\lambda:L\to M$, every R-linear map $q:L\to Q$ extends to an R-linear map $\hat{q}:M\to Q$.



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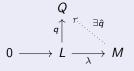
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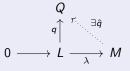
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Theorem 5

A module Q is injective if and only if every monomorphism $Q \to M$ splits.

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The proof of this is essentially the same as the one for projective modules.

Theorem 5

A module Q is injective if and only if every monomorphism $Q \to M$ splits.

Pushouts (or fibered coproduct) are used to prove this. They're the dual of pullbacks.

Remark

However, there is no theorem for injective modules similar to the one of a projective module being a direct summand of a free module.

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Theorem (Baer's Criterion)

An R-module Q is injective if and only if every R-linear map $f: I \to Q$, where I is an ideal of R, extends to an R-linear map $\hat{f}: R \to Q$.

Remark

However, there is no theorem for injective modules similar to the one of a projective module being a direct summand of a free module.

Theorem (Baer's Criterion)

An *R*-module *Q* is injective if and only if every *R*-linear map $f: I \to Q$, where *I* is an ideal of *R*, extends to an *R*-linear map $\hat{f}: R \to Q$.

Proof.

The proof uses a clever application of Zorn's lemma. See Aluffi for full proof.

Summarizing our results

	Lifting/Extending	Splitting	Summands of free modules
Projective	Yes	Yes	Yes
Modules			
Injective	Yes	Yes	No
Modules			

Table: Comparison between injective and projective module characterizations

References



P. Aluffi

Algebra: Chapter 0

Graduate studies in mathematics. American Mathematical Society,

2009.



Abstract Algebra: The Basic Graduate Year

Thank you!

Thank you! Any Questions?