Classification of the Irreducible Representations of the Symmetric Group

MATH 518: Honours Thesis

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Contents

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Abstract

In this thesis we classify all irreducible representations of symmetric groups over finite dimensional $\mathbb{C}$-vector spaces up to isomorphism. To do this, we use Young diagrams and Young tableaux which have many interesting combinatorial properties which not only let us determine all of the irreducible representations of the symmetric group, but also help us determine induced representations of the symmetric group.

Introduction

Representation theory, broadly speaking, is the mathematical branch that uses linear algebra to study groups by studying the way that groups act on vector spaces. It was pioneered mostly by Ferdinand Frobenius and Issai Schur in the early 1900's. Representation theory studied the way groups acted on invertible matrices and it was Emmy Noether who first considered the more general notion of a representation as a group homomorphism to the group of invertible linear transformations from a vector space to itself [Kna96]. The interested reader can learn more about Frobenius' contributions to representation theory in [Kna96] or [Haw13].

In this thesis, our main goal will be to classify the irreducible representations of symmetric groups over finite dimensional $\mathbb{C}$-vector spaces. Section 1 will cover the basic terminology of representation theory and show that any representation can be decomposed into irreducible representations. Section 2 will cover character theory which is the study of invariant properties of representations and gives us useful techniques for determining if two representations are isomorphic. In section 3, we will cover induced representations. If H is a subgroup of a group G, an induced representation is a way of extending a representation of H to a representation of G. Finally in section 4, we will systematically categorize all of the irreducible representations of any symmetric group S_n up to isomorphism by using Young diagrams. This technique of classifying irreducible representations of the symmetric group was first done by the British mathematician Alfred Young [You77].

1 Representations

Unless otherwise stated V denotes a $\mathbb{C}$-vector space and G denotes a finite group. Given a vector space V over the field $\mathbb{C}$, $\mathrm{GL}(V)$ is the set of all vector space isomorphisms from V to itself. This set can be made into a group with the binary operation being given by function composition. All of the theorems that we will prove hold over an algebraically closed field with characteristic 0. Some of these results do not necessarily hold for vector spaces over finite fields. We will not be considering the case of a vector space over a field with positive characteristic.

Let n be a positive integer, the symmetric group S_n is the set of all bijective functions from the set $\{1, 2, \ldots, n\}$ to itself with the group operation being function composition. If

$\sigma, \tau \in S_n$, then $\sigma\tau$ means the permutation defined by

$$\sigma\tau(k) = \sigma(\tau(k)),$$

for all $k \in \{1, \ldots, n\}$.

A cycle in S_n, denoted

$$(a_1\ a_2 \ldots a_k),$$

is the permutation in S_n that maps a_1 to a_2, a_2 to a_3, $\ldots$, a_{k-1} to a_k and a_k to a_1.

Example. Let σ and τ be elements of S_3 such that

$$\sigma(1) = 1, \sigma(2) = 3,\ \sigma(3) = 2,$$

and

$$\tau(1) = 2,\ \tau(2) = 3,\ \tau(3) = 1.$$

Then, in cycle notation we have $\sigma = (1)(2\ 3) = (2\ 3)$ and we write $\tau = (1\ 2\ 3)$. We also have $\sigma\tau = (2\ 3)(1\ 2\ 3) = (1\ 3)$.

The identity permutation of the symmetric group will always be denoted ε.

Every permutation in S_n can be written uniquely as a product of disjoint cycles up to ordering of the cycles [Alu09]. Because of this fact, we can introduce the notion of the cycle type of a permutation. If σ decomposes into a product of k cycles where each cycle has length t_i, we say that σ has cycle type $t_1, t_2, \ldots t_k$. By convention, we write the $t_i's$ in non-decreasing order.

Example. Consider the permutation

$$\sigma = (2\ 7\ 8\ 6)(1\ 3\ 4)(5) \in S_8.$$

σ has cycle type $4, 3, 1$.

A transposition is a 2-cycle i.e. a cycle of the form $(a_1\ a_2)$. The sign of a permutation, denoted $\text{sgn}(\sigma)$ is the function

$$\text{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ can be written as an even number of transpositions} \\ -1 & \text{if } \sigma \text{ can be written as an odd number of transpositions.} \end{cases}$$

This is a well defined function that is group homomorphism from S_n to $\{1, -1\}$ under multiplication [Nic12].

Definition 1.1 (Representation)**.** Let V be a $\mathbb{C}$-vector space and G a finite group. We say that (π, V) is a representation of G if $\pi : G \to \text{GL}(V)$ is a group homomorphism. Hence, for each $g \in G$, $\pi(g)$ is an invertible linear map from V to itself. We will often denote $\pi(g)$ as π_g since it is nicer to write $\pi_g(v)$ than $\pi(g)(v)$.

Definition 1.2 (Degree of a representation)**.** The degree of a representation (π, V) is the dimension of the vector space V.

Before we give examples of important representations, we first need to define a group algebra.

Definition 1.3 (Group algebra)**.** The group algebra of a finite group G, denoted $\mathbb{C}[G]$ is the $\mathbb{C}$-vector space with basis indexed by elements of G and with multiplication on basis elements given by $e_g \cdot e_h = e_{gh}$ and extended by distributivity to any $x, y \in \mathbb{C}[G]$. For the sake of notational clarity we will often write e_g as g in which case the multiplication of basis elements is denoted $g \cdot h = gh$ where $\cdot$ denotes multiplication in the group algebra and gh denotes the group operation of G.

Example. If $G = S_3$, then an arbitrary element x in $\mathbb{C}[S_3]$ looks like

$$x = c_1\varepsilon + c_2(1\ 2) + c_3(2\ 3) + c_4(1\ 3) + c_5(1\ 2\ 3) + c_6(1\ 3\ 2)$$

with $c_i \in \mathbb{C}$ for $1 \le i \le 6$.

If we take $2\varepsilon - (2\ 3)$ and $(1\ 2\ 3) + 2(1\ 2)$ in $\mathbb{C}[S_3]$ then

$$\begin{aligned}(2\varepsilon - (2\ 3)) \cdot ((1\ 2\ 3) + 2(1\ 2)) &= 2\varepsilon \cdot (1\ 2\ 3) + 4\varepsilon \cdot (1\ 2) \\ &\quad - (2\ 3) \cdot (1\ 2\ 3) - 2(2\ 3) \cdot (1\ 2) \\ &= 2(1\ 2\ 3) + 4(1\ 2) - (1\ 3) - 2(1\ 3\ 2)\end{aligned}$$

Now we list some important representations.

1. For any group G we have the trivial representation $(\mathbb{1}, \mathbb{C})$ which is defined by $\mathbb{1}_g(c) = 1c$ for all $g \in G$ and all $c \in \mathbb{C}$.

2. For any finite group G, we have the regular representation $(r, \mathbb{C}[G])$. The group homomorphism r is defined on basis elements $\{e_g : g \in G\}$ as $r_x(e_g) = e_{xg}$ and we extend r_x by linearity to any $v \in \mathbb{C}[G]$.

3. For the symmetric group S_n the representation $(\mathrm{sgn}, \mathbb{C})$ defined by $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma)c$ for all $\sigma \in S_n$ and $c \in \mathbb{C}$.

4. If $\sigma \in S_n$ then S_n acts on $\{1, 2, \ldots, n\}$ by permuting the elements. For example if $\sigma = (2\ 3\ 4) \in S_4$, the we can show how σ acts on $\{1, 2, 3, 4\}$ as by writing an array with $n \in \{1, 2, 3, 4\}$ in the first row and $\sigma(n)$ on the bottom row.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

   If V is an n-dimensional vector space with basis $\{e_1, e_2, \ldots, e_n\}$ then the permutation representation (ϕ, V) is given by $\phi_\sigma(e_i) = e_{\sigma \cdot i}$ for basis elements and is extended by linearity to any $v \in V$. We will always use ϕ to represent the permutation representation.

Example. Let's look at an example that will motivate the next definition. Let (ϕ, V) be the permutation representation of S_3. Suppose $\{e_1, e_2, e_3\}$ is a basis of V and consider the subspace $W = \text{span}\{e_1 + e_2 + e_3\}$. If $v \in W$, then $v = ae_1 + ae_2 + ae_3$ for some $a \in \mathbb{C}$ and thus for any $\sigma \in S_3$ we have

$$\begin{aligned}
\phi_\sigma(v) &= \phi_\sigma(ae_1 + ae_2 + ae_3) \\
&= a\phi_\sigma(e_1) + a\phi_\sigma(e_2) + a\phi_\sigma(e_3) \\
&= ae_{\sigma\cdot 1} + ae_{\sigma\cdot 2} + ae_{\sigma\cdot 3} \\
&= ae_1 + ae_2 + ae_3.
\end{aligned}$$

With the last equality following from the fact that σ is a permutation of $\{1, 2, 3\}$. This shows that W is an S_3-invariant subspace of V.

Contrast W with the subspace $U = \text{span}\{e_1, e_2\}$. If we take $\sigma = (2\ 3)$ then for any $v \in U$ we have

$$\begin{aligned}
\phi_{(2\ 3)}(v) &= \phi_{(2\ 3)}(ae_1 + be_2) \\
&= a\phi_{(2\ 3)}(e_1) + b\phi_{(2\ 3)}(e_2) \\
&= ae_1 + be_3.
\end{aligned}$$

Hence $\phi_{(2\ 3)}(v) \notin U$. This tells us that there is something special about the subspace W. The point is there exists $v \in U$ and $\sigma \in S_3$ such that $\phi_\sigma(v) \notin U$. This motivates the following definition.

Definition 1.4 (Subrepresentation)**.** Let (π, V) be a representation of G and W a vector subspace of V. Define $\pi'_g = \pi_g|_W$ for all $g \in G$ where $\pi_g|_W$ denotes the restriction of π_g to W. (π', W) is a subrepresentation of (π, V) if for all $g \in G$ and $w \in W$, $\pi'_g(w) \in W$. In this case, we say that W is a G-invariant subspace of V.

Remark. We routinely abuse notation and write π for both (π, V) and its subrepresentations instead of writing $\pi_g|_W$. Sometimes we will even omit reference to π altogether and say that W is a subrepresentation.

Definition 1.5. Let V be a finite dimensional $\mathbb{C}$-vector space and let W and U be two vector subspaces of V. V is equal to the direct sum of W and U if every $v \in V$ can be written uniquely as $v = w + u$ with $w \in W$ and $u \in U$. We denote the direct sum as

$$V = U \oplus W.$$

We say that U is a complementary subspace of W and W is a complementary subspace of U.

In our previous example W was a subrepresentation of the permutation representation. A natural question to ask, is whether the complementary subspace

$$W' = \{a_1e_1 + a_2e_2 + a_3e_3 \in V \mid a_1 + a_2 + a_3 = 0\}$$

is also a subrepresentation of the permutation representation. Suppose $v \in W'$, then for any $\sigma \in S_3$ we see that

$$\pi_\sigma(v) = \pi_\sigma(a_1e_1 + a_2e_2 + a_3e_3) = a_1e_{\sigma\cdot 1} + a_2e_{\sigma\cdot 2} + a_3e_{\sigma\cdot 3}.$$

Since σ permutes $\{1, 2, 3\}$, $\pi_g(v)$ will still be a linear combination of basis elements whose coefficients sum to 0, therefore W' is also a subrepresentation of (ϕ, V).

Definition 1.6. If (ϕ, V) is the permutation representation of S_n, then the complement of the subspace $W = \operatorname{span}\{e_1 + \cdots + e_n\}$, is called the standard representation.

Definition 1.7 (Irreducible representation)**.** A representation (π, V) of G is irreducible if the only subrepresentations are $\{0_V\}$ and V itself.

Later, using character theory, we will develop a criterion for determining whether a representation is irreducible or not.

Definition 1.8 (Morphism of Representations)**.** A morphism of two representations (π, V) and (ρ, W) of G is a linear map of vector spaces $T : V \to W$ such that $T \circ \pi_g = \rho_g \circ T$ for all $g \in G$. This means that the diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow{\scriptstyle \pi_g} & & \downarrow{\scriptstyle \rho_g} \\ V & \xrightarrow[T]{} & W \end{array}$$

commutes for all $g \in G$. The set of morphisms of representations from (π, V) to (ρ, W) is denoted $\operatorname{Hom}_G(V, W)$.

Remark. $\operatorname{Hom}_G(V, W)$ can be given a $\mathbb{C}$-vector space structure by defining, for all $T_1, T_2 \in \operatorname{Hom}_G(V, W)$, and $c \in \mathbb{C}$,

$$(T_1 + T_2)(v) \doteq T_1(v) + T_2(v)$$
$$(cT_1)(v) \doteq c(T_1(v))$$

for all $v \in V$.

Definition 1.9 (Isomorphisms of representations)**.** An isomorphism of representations (π, V) and (ρ, W) of a group G is a morphism T that is invertible. Two representations are isomorphic if there is a isomorphism between them.

Theorem 1.1 (Schur's Lemma)**.** *Let (π, V) and (ρ, W) be two irreducible representations of G and let T be a morphism of representations $T : (\pi, V) \to (\rho, W)$, then*

1. *Either T is an isomorphism or the 0 map.*
2. *If $(\pi, V) = (\rho, W)$, then $T = \lambda\, \mathrm{id}$ for some constant $\lambda \in \mathbb{C}$ and* id *is the identity map on V.*

Proof. Suppose $T \neq 0$. We know that $\ker(T) = \{v \in V : T(v) = 0_V\}$ is a vector subspace of V so we only need to show that it is a subrepresentation of V. Let $g \in G$ and let $v \in \ker(T)$, then $T(\pi_g(v)) = \rho_g(T(v)) = \rho_g(0_V) = 0_V$. This proves that $\pi_g(v) \in \ker(T)$. Since (π, V) is an irreducible representation either $\ker(T) = \{0_V\}$ or $\ker(T) = V$. But T is not the 0 map hence $\ker(T) \neq V$ hence $\ker(T) = \{0_V\}$ which implies that T is injective.

We now prove that $\mathrm{im}(T) = \{y \in W : \exists v \in V, T(v) = y\}$ is a subrepresentation of W. If $y \in \mathrm{im}(T)$ then there exists a $v \in V$ such that $T(v) = y$. It follows that $\rho_g(y) = \rho_g(T(v)) = T(\pi_g(v))$ hence $\rho_g(y) \in \mathrm{im}(T)$. This proves that $\mathrm{im}(T)$ is a subrepresentation of (ρ, W). Since W is irreducible and since $T \neq 0$ it follows that $\mathrm{im}(T) = W$ which implies that T is surjective.

Therefore T is a bijective morphism hence it is an isomorphism. Therefore (π, V) and (ρ, W) are isomorphic.

Now to prove the second part. Suppose T is a morphism from (π, V) to itself. Since $\mathbb{C}$ is algebraically closed there is an eigenvalue $\lambda \in \mathbb{C}$ of T. Define a new linear map $\overline{T} = T - \lambda\,\mathrm{id}$. By definition of an eigenvalue there is nonzero $x \in V$ such that $T(x) = \lambda x$ hence $\overline{T}(x) = 0$. This shows that $\overline{T}$ is not an isomorphism so by part 1 of this Theorem, it follows that $\overline{T} = 0_V$. Therefore $T = \lambda\,\mathrm{id}$.

□

We can use Schur's lemma to prove an interesting fact about abelian groups, namely that every irreducible representation of an abelian group has degree 1.

Corollary 1.2. If G is a finite abelian group and (π, V) is an irreducible representation then the degree of (π, V) is one.

Proof. If (π, V) is a representation of an abelian group G, then since G is abelian $\pi_g \pi_h = \pi_h \pi_g$ for all $g, h \in G$ so π_g is a morphism from (π, V) to itself.

By Schur's Lemma it follows that $\pi_g = \lambda$ id for all $g \in G$. Note that this λ depends on g. Since π_g acts by scalars, for all $g \in G$, it follows that every subspace of V is G-invariant. Since (π, V) is irreducible, V must have dimension 1. □

Definition 1.10. Suppose V is a vector space with basis $\{v_1, \ldots, v_n\}$ and W is a vector space with basis $\{w_1, \ldots w_m\}$, then the tensor product of V and W denoted $V \otimes W$ is the vector space with basis $\{v_i \otimes w_j\}_{1 \leq i \leq n, 1 \leq j \leq m}$ and $\otimes$ is bilinear meaning that for all $v, v' \in V$, for all $w, w' \in W$ and for all $c \in \mathbb{C}$,

1. $v \otimes w + v \otimes w' = v \otimes (w + w')$
2. $v \otimes w + v' \otimes w = (v + v') \otimes w$
3. $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$

Definition 1.11 (Tensor Product of Representations)**.** The tensor product of two representations (π, V) and (ρ, W) of a group G, is $(\pi \otimes \rho, V \otimes W)$ where

$$(\pi \otimes \rho)_g(v \otimes w) \doteq \pi_g(v) \otimes \rho_g(w)$$

for all $g \in G$ and $v \otimes w \in V \otimes W$ and the map is extended by linearity to all of $V \otimes W$.

Definition 1.12. A Hermitian inner product on a vector space is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all $v_1, v_2, v \in V$, for all $w \in W$ and for all $c \in \mathbb{C}$,

(i) $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$

(ii) $\langle v, w \rangle = \overline{\langle w, v \rangle}$

(iii) $\langle cv, w \rangle = c \langle v, w \rangle$

(iv) $\langle v, cw \rangle = \overline{c} \langle v, w \rangle$

(v) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$

The following concepts will be useful in showing that any representation can be written as a direct sum of irreducible representations.

Definition 1.13 (Unitary representation)**.** A representation (π, V) of a group G is unitary if there exists a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V such that for all $g \in G$ and for all $v, w \in V$,

$$\langle \pi_g(v), \pi_g(w) \rangle = \langle v, w \rangle.$$

Proposition 1.3. For every representation (π, V) there is a Hermitian inner product such that π is a unitary representation with respect to that inner product.

Proof. Given any Hermitian inner product $(\cdot, \cdot)$ on V (such an inner product always exists, for example take any basis of V $\{e_1, \dots e_n\}$ and define $(v, w) = \sum_{i=1}^{n} a_i \overline{b_i}$ where $v = a_1 e_1 + \cdots a_n e_n$ and $w = b_1 e_1 + \cdots b_n e_n$) we can define a new inner product as follows:

$$\langle v, w \rangle = \sum_{g \in G} (\pi_g(v), \pi_g(w)).$$

(This summation is valid because G is finite, hence the hypothesis that G be finite is necessary). The proof that $\langle \cdot, \cdot \rangle$ is an inner product can be found in [Ste12, p. 21].

To check that (π, V) is unitary with respect to $\langle \cdot, \cdot \rangle$, we compute

$$\begin{aligned}
\langle \pi_x(v), \pi_x(w) \rangle &= \sum_{g \in G} (\pi_g(\pi_x(v)), \pi_g(\pi_x(w))) \\
&= \sum_{g \in G} (\pi_{gx}(v), \pi_{gx}(w)) \\
&= \sum_{g' \in G} (\pi_{g'}(v), \pi_{g'}(w)) \\
&= \langle v, w \rangle.
\end{aligned}$$

The second last equality comes from the fact that for a fixed $x \in G$, as g ranges over all of G so will $g' = gx$. Hence (π, V) is unitary with respect to $\langle \cdot, \cdot \rangle$

$\square$

Theorem 1.4. *If (π, V) is a representation of a group G with a subrepresentation W. Then there exists a subspace U of V such that U is a subrepresentation (i.e. it is G-invariant) and $V = W \oplus U$.*

Proof. Let (π, V) be a representation of a finite group G and suppose W is a subspace of V that is a subrepresentation. Then by proposition (1.3) there exists a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V. Recall that the orthogonal complement of W with respect to an inner product $\langle \cdot, \cdot \rangle$ is defined as follows:

$$W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}$$

The orthogonal complement of W is also a subspace of V. We will show that the orthogonal complement is also G-invariant. Suppose $w' \in W^{\perp}$, then for any $g \in G$ and $w \in W$ we have

$$\begin{aligned} \langle \pi_g(w'), w \rangle &= \langle \pi_{g^{-1}}(\pi_g(w')), \pi_{g^{-1}}(w) \rangle && \text{(since } \pi \text{ is unitary)} \\ &= \langle w', \pi_{g^{-1}}(w) \rangle \\ &= 0. \end{aligned}$$

With the last equality holding because $\pi_{g^{-1}}(w) \in W$ and the fact that w' is in $W^{\perp}$. This shows that $\pi_g(w') \in W^{\perp}$ thus proving that $W^{\perp}$ is G-invariant. Finally we have $V = W \oplus W^{\perp}$ [Axl15, p. 194].

□

Now we come to the big theorem of this first section. Namely that any representation can be written as a direct sum of irreducible representations. This is an important theorem because it reduced the problem of understanding any finite dimensional representation of a finite group to understanding the irreducible representations and how any representation decomposes into irreducible representations. The goal for this thesis will be to classify all of the irreducible representations of the symmetric group S_n. We will not explore in detail the problem of understanding how any representation decomposes into a direct sum of irreducible representations.

Theorem 1.5. *If (π, V) is a representation then V decomposes into a direct sum of irreducible representations.*

Proof. We proceed by induction on the dimension of V. Let $\dim(V) = 1$. There is nothing to show here since V is already irreducible.

Suppose n is some positive integer such that $n > 1$, and that the theorem is true for all positive integers k such that $k < n$. Suppose $\dim(V) = n$. If (π, V) is irreducible, then we are done. Otherwise V has a proper subrepresentation W which implies $1 \leq \dim(W) < n$. By Theorem 1.4 $V = W \oplus U$ for some G-invariant subspace U. (π, U) is a representation since U is G invariant and since $\dim(V) = \dim(W) + \dim(U)$, and $1 \leq \dim(W) < n$ it follows that $1 \leq \dim(U) < n$ and thus U can be written as a direct sum of irreducible representations by the induction hypothesis. By the same argument, W is a direct sum of irreducibles since the induction hypothesis applies to W as well. This proves that V is also the direct sum of irreducible representations.

$\square$

We end this section by presenting a converse to Schur's lemma that will aid us later in classifying all of the irreducible representations up to isomorphism of the symmetric group.

Theorem 1.6 (Converse to Schur's lemma)**.** *If (π, V) is a representation of a finite group G such that $\dim(\mathrm{Hom}_G(V, V)) = 1$, then (π, V) is irreducible.*

Proof. Suppose (π, V) is a representation of a finite group G such that $\dim(\mathrm{Hom}_G(V, V)) = 1$. Since $\mathrm{id}_V \in \mathrm{Hom}_G(V, V)$, it follows that if $T \in \mathrm{Hom}_G(V, V)$, then T is a scalar identity of the identity map on V. Let $U \neq \{0_V\}$ be a subrepresentation of (π, V). By Theorem 1.4, there exists a G-invariant subspace W such that $V = U \oplus W$. For every $v \in V$, there exist a unique $u \in U$ and $w \in W$ such that $v = u + w$, $u \in U$ and $w \in W$ by definition of a direct sum of vector spaces. This gives a well defined linear transformation $P_U : V \to V$ defined by

$$P_U(v) = P_U(u + w) = u.$$

Now we show that $P_U \in \mathrm{Hom}_G(V, V)$. Suppose $g \in G$, and $v \in V$. It follows that

$$\pi_g(P_U(v)) = \pi_g(P_U(u + w)) = \pi_g(P_U(u)).$$

Since W is G-invariant $\pi_g(w) \in W$ and for all $w \in W$ we have $P_U(w) = P_U(0_V + w) = 0_V$ hence $P_U(\pi_g(w)) = 0_V$. Similarly, since U is G-invariant, we have $\pi_g(u) \in U$ hence $P_U(\pi_g(u)) = P_U(\pi_g(u) + 0_V) = \pi_g(u)$. So we have

$$\pi_g(P_U(v)) = \pi_g(u) = P_U(\pi_g(u) + \pi_g(w)) = P_U(\pi_g(u + w)) = P_U(\pi_g(v)).$$

Since $P_U \in \mathrm{Hom}_G(V, W)$, it follows that $P_U = c\,\mathrm{id}_V$ for some $c \in \mathbb{C}$. If $c = 0 \in \mathbb{C}$, then for all $u \in U$, we have

$$P_U(u) = u = 0\,\mathrm{id}_V(u) = 0_V$$

which implies that $U = \{0_V\}$ which is a contradiction.

Therefore $c \neq 0$. If we take any $w \in W$, then we have

$$P_U(w) = P_U(0_V + w) = 0_V = c\,\mathrm{id}_V(w) = cw$$

where, since $c \neq 0$, it follows that $w = 0_V$.

This implies that $W = \{0_V\}$ hence $U = V$. Therefore (π, V) is irreducible. $\square$

2 Characters of Representations

In this section, we introduce the idea of character theory. The character of a representation is an invariant that will give us some key information about representations and how they decompose into irreducible representations. Additionally, character theory will give us a useful criterion for determining whether a representation is irreducible and will be incredibly useful in determining all the irreducible representations of a group up to isomorphism. Finally we will see how to use character theory to find all of the irreducible representations of S_4.

In this section we will need some results from linear algebra which we will recall here. If T is a linear map from a vector space V to itself, then we can pick a basis of V, say $\{e_1, \dots, e_n\}$ and write down the matrix of the representation (which we denote $[T]$). Then the trace of T is the sum of diagonal entries of $[T]$. An important result from linear algebra is that the trace of a linear map does not depend on the choice of basis, so the trace of T is well defined. For any linear transformation $T : V \to V$ we denote the trace by $\mathrm{Tr}(T)$. When $T_1 : V \to V$ and $T_2 : V \to V$ are two linear maps we have $\mathrm{Tr}(T_1T_2) = \mathrm{Tr}(T_2T_1)$ as well as $\mathrm{Tr}(T_1 + T_2) = \mathrm{Tr}(T_1) + \mathrm{Tr}(T_2)$.

Definition 2.1 (Character of a Representation)**.** Let (π, V) be a representation of a group G where V is a $\mathbb{C}$-vector space. The character of (π, V) is the function $\chi_\pi : G \to \mathbb{C}$ defined by $\chi_\pi(g) = \mathrm{Tr}(\pi_g)$ for all $g \in G$. When there is no ambiguity about the representation whose character we're computing we will simply write χ.

Example.

- Consider the permutation representation (ϕ, V) of S_3 where V is a 3 dimensional vector space with basis $B = \{e_1, e_2, e_3\}$. For $(1\ 2) \in S_3$ we have

$$\begin{aligned} \pi_{(1\ 2)}(e_1) &= e_2, \\ \pi_{(1\ 2)}(e_2) &= e_1, \\ \pi_{(1\ 2)}(e_3) &= e_3. \end{aligned}$$

  Hence the matrix of $\pi_{(1\ 2)}$ with respect to B is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  hence $\chi((1\ 2)) = 1$.

  The matrix of $\pi_{(1\ 2\ 3)}$ with respect to B is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

  hence $\chi((1\ 2\ 3)) = 0$.

- For any representation (π, V), we know π_1 is the identity map hence its matrix with respect to any basis consists of 1s on the main diagonal and thus $\chi(1) = n$ where $n = \dim(V)$.

Proposition 2.1. If (π, V) and (ρ, W) are isomorphic representations of G then $\chi_\pi = \chi_\rho$

Proof. Suppose that (π, V) is isomorphic to (ρ, W). Thus there exists an invertible linear map $T : V \to W$ such that $\rho_g T = T\pi_g$ for all $g \in G$. Thus for any $g \in G$ we have

$$\chi_\pi(g) = \mathrm{Tr}(\pi_g) = \mathrm{Tr}(T^{-1}\rho_g T) = \mathrm{Tr}(TT^{-1}\rho_g) = \mathrm{Tr}(\rho_g) = \chi_\rho(g).$$

□

Definition 2.2 (Class function)**.** A function $f : G \to \mathbb{C}$ is a class function if $f(hgh^{-1}) = f(g)$ for all $g, h \in G$.

Proposition 2.2 (Characters are class functions)**.** For any representation (π, V) and for all $g, h \in G$ we have $\chi(hgh^{-1}) = \chi(g)$ where χ denotes the character of the representation (π, V).

Proof. Let (π, V) be a representation of G. To prove this we will show that $\chi(hg) = \chi(gh)$. Since π is a homomorphism we know that $\pi(gh) = \pi(g)\pi(h)$ and thus it follows that

$$\mathrm{Tr}(\pi(gh)) = \mathrm{Tr}(\pi(g)\pi(h)) = \mathrm{Tr}(\pi(h)\pi(g)) = \mathrm{Tr}(\pi(hg)).$$

Therefore, $\chi_\pi(gh) = \mathrm{Tr}(\pi(gh)) = \mathrm{Tr}(\pi(hg)) = \chi_\pi(hg)$. The result follows from substituting g with gh^{-1}. □

Definition 2.3. We define the following inner product on class functions. If f_1 and f_2 are class functions of G then define an inner product as follows:

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

Proposition 2.3. The function defined in Definition 2.3 is an inner product.

Let f_1, f_2, and f_3 be class functions and let $g \in G$.

Proof.

$$\begin{aligned}
\langle f_1 + f_2, f_3 \rangle &= \frac{1}{|G|} \sum_{g \in G} (f_1 + f_2)(g)\overline{f_3(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} (f_1(g) + f_2(g))\overline{f_3(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} (f_1(g)\overline{f_3(g)} + f_2(g)\overline{f_3(g)}) \\
&= \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_3(g)} + \frac{1}{|G|} \sum_{g \in G} f_2(g)\overline{f_3(g)} \\
&= \langle f_1, f_3 \rangle + \langle f_2, f_3 \rangle
\end{aligned}$$

$$\begin{aligned}
\langle f_1, f_2\rangle &= \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)} \\
&= \frac{1}{|G|}\sum_{g\in G} \overline{f_2(g)}\ \overline{\overline{f_1(g)}} \\
&= \frac{1}{|G|}\sum_{g\in G} \overline{f_2(g)}\ \overline{\overline{f_1(g)}} \\
&= \frac{1}{|G|}\sum_{g\in G} \overline{f_2(g)\overline{f_1(g)}} \\
&= \overline{\frac{1}{|G|}\sum_{g\in G} f_2(g)\overline{f_1(g)}} \\
&= \overline{\langle f_2, f_1\rangle}
\end{aligned}$$

We also have

$$\langle f_1, f_1\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_1(g)} = \frac{1}{|G|}\sum_{g\in G} |f_1(g)|^2$$

Hence $\langle f_1, f_1\rangle \geq 0$ and if $\langle f_1, f_1\rangle = 0$, then the only way this can happen is if $f_1(g) = 0$ for all $g \in G$ thus f_1 must be identically 0.

Let $c \in \mathbb{C}$, then we have

$$\langle cf_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G}(cf_1)(g)\overline{f_2(g)} = c\frac{1}{|G|}\sum_{g\in G}(f_1(g))\overline{f_2(g)} = c\langle f_1, f_2\rangle$$

and

$$\langle f_1, cf_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{(cf_2)(g)} = \frac{1}{|G|}\sum_{g\in G}\overline{c}f_1(g)\overline{f_2(g)} = \overline{c}\langle f_1, f_2\rangle.$$

Thus $\langle\cdot,\cdot\rangle$ is a Hermitian inner product. □

Proposition 2.4 (Character of direct sum)**.** Let (π, V) and (ρ, W) be two representations of a group G. If $V' = V \oplus W$, then

$$\chi_{\pi\oplus\rho} = \chi_\pi + \chi_\rho.$$

Proof. See [Ser77]. □

Proposition 2.5 (Character of tensor product)**.** Let (π, V) and (ρ, W) be two representations of a group G. The character of the tensor product $(\pi \otimes \rho, V \times W)$ of the two representations is

$$\chi_{\pi\otimes\rho} = \chi_\pi \cdot \chi_\rho.$$

Proof. See [Ser77]. □

Theorem 2.6.

1. *If χ is the character of an irreducible representation then* $\langle \chi, \chi \rangle = 1$.
2. *If χ_1 and χ_2 are two characters of non isomorphic irreducible representations then* $\langle \chi_1, \chi_2 \rangle = 0$.

Proof. See [Ser77] $\square$

From section 1, we know that every representation (π, V) decomposes into a direct sum of irreducible representations

$$V = W_1 \oplus \cdots \oplus W_k.$$

A natural question that arises is, if (ρ, V') is isomorphic to (π, V) where (ρ, V') decomposes into a direct sum of irreducible representations

$$V' = U_1 \oplus \cdots \oplus U_\ell,$$

are the representations that appear in the direct sum decomposition also isomorphic to each other? The answer will be yes, and this is where character theory can help us answer the question.

Theorem 2.7. *If (π, V) is a representation the decomposes into irreducible representations*

$$V = W_1 \oplus \cdots \oplus W_k$$

and (ρ, W) is an irreducible representation with character χ, then $\langle \chi_i, \chi \rangle$ is the number of W_i that are isomorphic to W.

Proof. Let χ_i be the character of (π, W_i). By proposition 2.4, we have

$$\chi_\pi = \chi_1 + \cdots + \chi_k.$$

Since $\langle \cdot, \cdot \rangle$ is a Hermitian inner product, we have

$$\langle \chi_\pi, \chi \rangle = \langle \chi_1 + \cdots + \chi_k, \chi \rangle = \langle \chi_1, \chi \rangle + \cdots + \langle \chi_k, \chi \rangle.$$

But by Theorem 2.6 $\langle \chi_i, \chi \rangle = 1$ if (π, W_i) is isomorphic to (ρ, W) and 0 otherwise. Therefore $\langle \chi_i, \chi \rangle$ is equal to the number of W_i that are isomorphic to (ρ, W). $\square$

Remark. Let $m_i W_i$ denote the direct sum $W_i \oplus \cdots \oplus W_i$ where W_i occurs m_i times. The theorem above shows that number of isomorphic (π_i, W_i) in the direct sum decomposition of (π, V) is constant. Thus any representation decomposes into a direct sum of irreducibles

$$W = m_1 W_1 \oplus \cdots \oplus m_k W_k$$

where W_i are W_j are not isomorphic if $i \neq j$. We call m_i, the *multiplicity* of W_i in the representation W.

Now we prove the converse to theorem 2.6.

Theorem 2.8. *If (π, V) is a representation such that $\langle \chi_\pi, \chi_\pi \rangle = 1$, then it is irreducible.*

Proof. Suppose (π, V) is a representation the decomposes into the irreducible representations

$$V = m_1 W_1 \oplus \cdots \oplus m_k W_k$$

where W_i is not isomorphic to W_j for $i \neq j$. By proposition 2.4, $\chi_\pi = m_1\chi_1 + \cdots + m_k\chi_k$ and by theorem 2.7, $m_i = \langle \chi_\pi, \chi_i \rangle$ hence

$$\langle \chi_\pi, \chi_\pi \rangle = \langle m_1\chi_i + \cdots + m_k\chi_k, \chi_\pi \rangle = m_1\langle \chi_1, \chi_\pi \rangle + \cdots + m_k\langle \chi_k, \chi_\pi \rangle.$$

For each $1 \leq i \leq k$ we have

$$m_i\langle \chi_i, \chi_\pi \rangle = m_i\langle \chi_i, m_1\chi_i + \cdots + m_k\chi_k \rangle = m_i\overline{m_1}\langle \chi_i, \chi_1 \rangle + \cdots + \overline{m_k}\langle \chi_i, \chi_k \rangle.$$

By Theorem 2.6, $\langle \chi_i, \chi_i \rangle = 1$ and $\langle \chi_i, \chi_j \rangle = 0$ for $i \neq j$. Hence

$$m_i\langle \chi_i, \chi_i \rangle = m_i\overline{m_i} = m_i^2$$

with the last equality following from the fact that m_i is an integer.

Therefore $\langle \chi_\pi, \chi_\pi \rangle = m_1^2 + \cdots m_k^2$.

If $\langle \chi_\pi, \chi_\pi \rangle = 1$, this implies that $m_i = 1$ for some i and $m_j = 0$ for all $i \neq j$ therefore (π, V) is isomorphic to W_i which is irreducible. □

Corollary 2.9. If (π, V) decomposes into irreducible representations $W_1, \ldots, W_k$

$$m_1 W_1 \oplus \cdots \oplus m_k V_k$$

where W_i and W_j are not isomorphic representations for all i and j such that $i \neq j$, then we have

$$\langle \chi, \chi \rangle = \sum_{i=1}^{k} m_i^2.$$

Theorem 2.10. *If $(r_G, \mathbb{C}[G])$ is the regular representation of a group G, then every irreducible representation of G (up to isomorphism) occurs in the direct sum decomposition of $(r_G, \mathbb{C}[G])$ and it's degree is equal to its multiplicity in the decomposition.*

Proof. If $(r, \mathbb{C}[G])$ is the regular representation of G. Letting 1, denote the identity element of G we have $\mathrm{Tr}(1) = |G|$ and for any $g \in G$ and $t \in G$ such that $t \neq 1$ we have

$$r_t(g) = tg \neq g$$

hence the matrix of r_t with respect to the basis $\{g \mid g \in G\}$ of the regular representation consists of 0's on the diagonal. Therefore

$$\chi_r(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}$$

If (π, W_i) is an irreducible representation of G, then we have

$$\langle \chi_r, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_r(g)\overline{\chi_i(g)} = \frac{1}{|G|}\chi_r(1)\overline{\chi_i(1)} = \frac{1}{|G|}\,|G| \dim W_i = \deg((\pi, W_i)).$$

$\square$

Corollary 2.11. If $n_1, n_2, \ldots, n_k$ are the degrees of all the irreducible representations of G up to isomorphism, then the degrees satisfy

$$|G| = n_1^2 + \cdots + n_k^2.$$

Proof. By Theorem 2.7 and Theorem 2.10, we have

$$\chi_r = n_1\chi_1 + \cdots + n_1\chi_k$$

hence

$$\chi_r(1) = n_1\chi_1(1) + \cdots + n_1\chi_k(1) = n_1^2 + \cdots + n_k^2.$$

$\square$

One of the reasons character theory is so interesting is that it gives us a way of determining the number of non-isomorphic irreducible representations of groups.

Theorem 2.12. *The number of non-isomorphic irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. See [Ser77]. $\square$

Example. Now we can use the results from this section to determine the irreducible representations of S_4. First we determine the number of conjugacy classes in S_4. There are 5 cycle types in S_4: corresponding to the identity permutation, two cycles, three cycles, four cycles, and two disjoint two cycles. Two permutations are conjugate if and only if they have the same cycle type [Alu09]. This gives 5 conjugacy classes represented by the elements $\varepsilon, (1\ 2), (1\ 2\ 3)$, $(1\ 2\ 3\ 4)$, and $(1\ 2)(3\ 4)$. Below, we give a table of conjugacy classes and the number of elements in each class.

conjugacy class representative	ε	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$	$(1\ 2)(3\ 4)$
number of elements in conjugacy class	1	6	9	6	3

Table 1: Number of elements in each conjugacy class class of S_4.

By Theorem 2.12 there are 5 irreducible representations of S_4 up to isomorphism. We know that the trivial and sign representation are always irreducible representations of a symmetric group. The let n_i denote the degree of each representation. By Corollary 2.11, we have

$$1 + 1 + n_3^2 + n_4^2 + n_5^2 = |S_4| = 4! = 24.$$

and this has the solution $n_3 = 3$, $n_4 = 3$, $n_5 = 2$ which is the only positive-integer solution up to relabelling. So we are looking for three representations with degrees 3, 3, and 2 respectively. We know that the permutation representation of S_4 decomposes into the direct sum of representations $V = W' \oplus W$ where W is isomorphic to the trivial representation. Let (γ, W') be the standard representation. The character of the permutation representation is given as follows:

$$\begin{aligned} \chi_\phi(\varepsilon) &= 4 \\ \chi_\phi((1\ 2)) &= 2 \\ \chi_\phi((1\ 2\ 3)) &= 1 \\ \chi_\phi((1\ 2\ 3\ 4)) &= 0 \\ \chi_\phi((1\ 2)(1\ 2)) &= 0. \end{aligned}$$

Therefore by Proposition 2.4

$$\begin{aligned} &\chi_\gamma(\varepsilon) = \chi_\phi(\varepsilon) - \chi_{\mathbb{1}}(\varepsilon) = 3 \\ &\chi_\gamma((1\ 2)) = \chi_\phi((1\ 2)) - \chi_{\mathbb{1}}((1\ 2)) = 1 \\ &\chi_\gamma((1\ 2\ 3)) = \chi_\phi((1\ 2\ 3)) - \chi_{\mathbb{1}} - \chi_{\mathbb{1}}((1\ 2\ 3)) = 0 \\ &\chi_\gamma((1\ 2\ 3\ 4)) = \chi_\phi((1\ 2\ 3\ 4)) - \chi_{\mathbb{1}} - \chi_{\mathbb{1}}((1\ 2\ 3\ 4)) = -1 \\ &\chi_\gamma((1\ 2)(3\ 4)) = \chi_\phi((1\ 2)(1\ 2)) - \chi_{\mathbb{1}} - \chi_{\mathbb{1}}((1\ 2)(3\ 4)) = -1. \end{aligned}$$

Now it just remains to check that this representation is irreducible. By Theorem 2.8, it suffices to show that the inner product $\langle \chi_\gamma, \chi_\gamma \rangle = 1$.

$$\langle \chi_\gamma, \chi_\gamma \rangle = \frac{1}{24} \sum_{\sigma \in S_4} \chi_\gamma(\sigma)\overline{\chi_\gamma(\sigma)} = \frac{1}{24}(9 + 6 \cdot 1 + 9 \cdot 0 + 6 \cdot 1 + 3 \cdot 1) = 1.$$

Hence we have found an irreducible representation of order 3.

We need another representation of order 3 so we conjecture that the representation $(\gamma \otimes \mathrm{sgn}, W' \otimes \mathbb{C})$ is also an irreducible representation that is not isomorphic to the standard representation. Indeed letting ψ denote $\gamma \otimes \mathrm{sgn}$ it follows by Proposition 2.5 that

$$\begin{aligned} \chi_\psi(\varepsilon) &= 3 \times 1 = 3 \\ \chi_\psi((1\ 2)) &= 2 \times -1 = -2 \\ \chi_\psi((1\ 2\ 3)) &= 0 \times -1 = 0 \\ \chi_\psi((1\ 2\ 3\ 4)) &= -1 \times -1 = 1 \\ \chi_\psi((1\ 2)(1\ 2)) &= -1 \times 1 = -1. \end{aligned}$$

We check that this representation is irreducible by computing

$$\langle \chi_\psi, \chi_\psi \rangle = \frac{1}{24} \sum_{\sigma \in S_4} \chi_\psi(\sigma)\overline{\chi_\psi(\sigma)} = \frac{1}{24}(9 + 6 \cdot (-1)^2 + 0 + 6 \cdot 1^1 + 3 \cdot (-1)^2) = 1.$$

Hence, by Proposition 2.1, it follows that $(\psi, W' \otimes \mathbb{C})$ is not isomorphic to (γ, W'). Thus we only have one irreducible representation left. Call this representation (η, U). We can compute the character of this representation by using the fact that the character of the regular representation is given by

$$\chi_{r_{S_4}} = \chi_{\mathbb{1}} + \chi_{\text{sgn}} + 3\chi_\gamma + 3\chi_\psi + 2\chi_\eta.$$

Thus

$$2\chi_\eta = \chi_{r_{S_4}} - \chi_{\mathbb{1}} - \chi_{\text{sgn}} - 3\chi_\gamma - 3\chi_\psi$$

which gives us

$$\begin{aligned}
\chi_\eta((1\ 2)) &= \frac{1}{2}(0 - 1 + 1 - 1 + 1) = 0 \\
\chi_\eta((1\ 2\ 3)) &= \frac{1}{2}(0 - 1 - 1 + 0 + 0) = -1 \\
\chi_\eta((1\ 2\ 3\ 4)) &= \frac{1}{2}(0 - 1 + 1 + 3 - 3) = 0 \\
\chi_\eta((1\ 2)(3\ 4)) &= \frac{1}{2}(0 - 1 - 1 + 3 + 3) = 2
\end{aligned}$$

This gives us the character table of the irreducible representations of S_4 the columns denote the conjugacy classes and the rows denote the character of elements.

	ε	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$	$(1\ 2)(3\ 4)$
$\chi_{\mathbb{1}}$	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_γ	3	1	0	-1	-1
χ_ψ	3	-1	0	1	-1
χ_η	2	0	-1	0	2

Table 2: Character Table for S_4

3 Induced Representations

Definition 3.1 (Restricted Representation)**.** Let H of be a subgroup of a group G, and (π, V) be a representation of G. The restricted representation of H is $(\pi|_H, V)$ where $\pi|_H$ is the restriction of the map π to H.

Definition 3.2 (Induced Representation)**.** Let (ρ, W) be a representation of H where H is a subgroup of a group G. The representation of G induced from the representation (ρ, W) is the vector space

$$\text{Ind}_H^G(W) = \{f : G \to \mathbb{C} \mid f(hg) = \rho_h(f(g)) \text{ for all } g \in G, h \in H\}$$

and the homomorphism $\mathrm{Ind}_H^G(\rho)$ from G to $\mathrm{GL}(\mathrm{Ind}_H^G(W))$is defined as

$$\mathrm{Ind}_H^G(\rho)_g(f)(x) = f(xg) \tag{1}$$

for all $x, g \in G$. We use R as shorthand for $\mathrm{Ind}_H^G(\rho)$ since this is the action by right translation of functions hence equation (1) becomes $R_g(f)(x) = f(xg)$.

Proving that $\mathrm{Ind}_H^G(W)$ is a vector space where addition is defined as $(f_1 + f_2)(x) \doteq f_1(x) + f_2(x)$ and scalar multiplication is defined as $(cf_1)(x) \doteq cf_1(x)$ for all $x \in G$ and all $f_1, f_2 \in \mathrm{Ind}_H^G(W)$ is a routine but tedious task and not very insightful so it will be omitted here. Proving that R defines a homomorphism from G to $\mathrm{GL}(\mathrm{Ind}_H^G(W))$ will also be omitted.

How can we better understand what induced representations look like? If G is a group and H is a subgroup of G, then, given any $g \in G$ a *right coset* of H is the set $Hg = \{hg : h \in H\}$. In other words, it is the set obtained by multiplying g on the right by every element in H. The set of all right cosets is denoted $H\backslash G$. Every $g \in G$, belongs to exactly one right coset. The index of H in G is the number of distinct right cosets of H and is denoted by the symbol $[G : H]$. If we take the set of right cosets and $C_1, \ldots, C_d$ we can write each coset as $Hg_1, \ldots Hg_d$ for $g_i \in C_i$. We call the set $\{g_1, \ldots, g_d\}$ a complete set of coset representatives.

Proposition 3.1. An element in $\mathrm{Ind}_H^G(W)$ is completely determined by where it sends representatives of a right cosets of H.

Proof. Let H be subgroup of a group G and $C_1, \ldots C_d$ be the right cosets of G. Let $\{g_1, \ldots, g_d\}$ be a complete set of coset representatives. Let g be in G. Since the right cosets of H partition the group G, g is in exactly one coset C_i. Thus it can be written uniquely as $g = hg_i$ for some $h \in H$. Now if $f \in \mathrm{Ind}_H^G(W)$, it follows that $f(g) = \rho_h(f(g_i))$ hence the value of $f(g)$ is determined by $f(g_i)$ and ρ_h. □

Definition 3.3. Let G be a group, H a subgroup and (ρ, W) a representation of H. Let C be a right coset H. We define the following subset of $\mathrm{Ind}_H^G(W)$

$$W_C = \{f \in \mathrm{Ind}_H^G(W) \mid f(x) = 0 \text{ for all } x \notin C\}.$$

That is, W_C is the set of all functions that are equal to 0 for any x outside of the coset C. It can be shown that W_C is a vector subspace of $\mathrm{Ind}_H^G(W)$ but restricting $\mathrm{Ind}_H^G(\rho)$ to W_C will, in general, not yield a subrepresentation of $\mathrm{Ind}_H^G(W)$.

Proposition 3.2. W_C is isomorphic to W as vector spaces.

Proof. Choose a representative g of the coset $C = Hg$. We claim that the map $\Psi(f) = f(g)$ defined for $f \in W_C$ is a bijective linear transformation from W_C to W. If $f_1, f_2 \in W_C$, then for any

$$\Psi(f_1 + f_2) = (f_1 + f_2)(g) = f_1(g) + f_2(g) = \Psi(f_1) + \Psi(f_2)$$

and for any $c \in C$ we have

$$\Psi(cf_1) = (cf_1)(g) = cf_1(g) = c\Psi(f_1).$$

Now to prove it is an isomorphism, if $\Psi(f) = 0$ then this means $f(g) = 0$ and thus for any $x \in Hg$, we have $x = hg$ for some $h \in H$, hence

$$f(x) = f(hg) = \rho_h(f(g)) = \rho_h(0) = 0.$$

Therefore, f is identically 0. This means Ψ is injective. To prove it is surjective, suppose $w \in W$, define $f : W_C \to W$ by setting $f(hg) = \rho_h(w)$ then

$$\Psi(f) = f(g) = f(1g) = \rho_1(w) = w$$

thus proving that Ψ is surjective. This proves that Ψ is an isomorphism and hence W_C is isomorphic to W. □

Remark. We have

$$\mathrm{Ind}_H^G(V) = \bigoplus_{C \in H\backslash G} W_C.$$

Proposition 3.3. $\dim(\mathrm{Ind}_H^G(W)) = [G : H]\dim(W)$

Proof. By the previous remark we know that $\dim(\bigoplus_{C\in H\backslash G} W_C) = \dim(\mathrm{Ind}_H^G(W))$. From linear algebra we know that

$$\dim\left(\bigoplus_{C\in H\backslash G} W_C\right) = \sum_{C\in H\backslash G} \dim(W_C).$$

Furthermore we know that $\dim(W_C) = \dim(W)$ Hence the result becomes

$$\dim(\mathrm{Ind}_H^G(W)) = \sum_{C\in H\backslash G} \dim(W) = [G : H]\dim(W)$$

since the sum is being taken over all right cosets of H and there are exactly $[G : H]$ right cosets of H. □

Theorem 3.4 (Mackey's Formula)**.** *Let G be a finite group and H be a subgroup of G. Let (π, W) be a representation of H. For each coset $C \in H\backslash G$, let g_C be a representative of that coset. The character of the induced representation $\big(R, Ind_H^G(W)\big)$ is*

$$\chi_R(g) = \sum_{\substack{C\in H\backslash G: \\ g_C g g_C^{-1} \in H}} \chi_\pi(g_i g g_i^{-1}).$$

Proof. See [Eti11]. □

We can use Proposition 3.3, Mackey's formula, and the results from character theory to determine what induced representations look like.

Example. Let $(\mathbb{1}, \mathbb{C})$ be the trivial representation of S_3. We can identify S_2 as a subgroup of S_3 by inclusion. We can determine $\mathrm{Ind}_{S_2}^{S_3}(\mathbb{C})$ by determining its character. For notational clarity let $H = S_2$. The right cosets of H are

$$\begin{aligned} H\varepsilon &= \{\varepsilon, (1\ 2)\}, \\ H(1\ 3) &= \{(1\ 3), (1\ 3\ 2)\}, \text{ and} \\ H(2\ 3) &= \{(2\ 3), (1\ 2\ 3)\}. \end{aligned}$$

Let $\varepsilon, (1\ 3), (2\ 3)$ be the coset representatives of $H\varepsilon, H(1\ 3)$, and $H(2\ 3)$ respectively. By Proposition 3.3, $\dim(\mathrm{Ind}_H^G(\mathbb{1})) = 3$ and the fact that $\chi_R(\varepsilon)$ is the dimension of $\mathrm{Ind}_H^G(\mathbb{1})$ we know that

$$\chi_R(\varepsilon) = 3.$$

To compute the character of $(1\ 2)$ we will use Mackey's formula so need to compute $g_C(1.2)g_C^{-1}$ for each coset representative g_C and determine whether $g_C(1.2)g_C^{-1} \in H$. To this end we compute

$$\begin{aligned} \varepsilon(1\ 2)\varepsilon &= (1\ 2) \in H \\ (1\ 3)(1\ 2)(1\ 3) &= (2\ 3) \notin H \\ (2\ 3)(1\ 2)(2\ 3) &= (1\ 3) \notin H. \end{aligned}$$

Hence

$$\chi_R((1\ 2)) = \chi_{\mathbb{1}}((1\ 2)) = 1.$$

To compute the character of $(1\ 2\ 3)$ we need compute $g_C(1.2\ 3)g_C^{-1}$ for each coset representative g_C and determine whether $g_C(1.2\ 3)g_C^{-1} \in H$.

$$\begin{aligned} \varepsilon(1\ 2\ 3)\varepsilon &= (1\ 2\ 3) \notin H \\ (1\ 3)(1\ 2\ 3)(1\ 3) &= (1\ 3\ 2) \notin H \\ (2\ 3)(1\ 2\ 3)(2\ 3) &= (1\ 3\ 2) \notin H. \end{aligned}$$

Hence

$$\chi_R((1\ 2\ 3)) = 0$$

Therefore the character of the induced representation is $\chi_R(\varepsilon) = 3, \chi_R((1\ 2)) = 1$ and $\chi_R((1\ 2\ 3)) = 0$. The induced representation has the same character as the permutation representation thus $\mathrm{Ind}_H^G(\mathbb{1})$ is isomorphic to the permutation representation. We have shown that the permutation decomposes into the direct sum of the trivial and standard representation therefore $\mathrm{Ind}_{S_2}^{S_3}(\mathbb{1})$ is isomorphic to the trivial and standard representation.

Example. What if we now consider the sign representation $(\mathrm{sgn}, \mathbb{C})$ of S_2.

Just like the last example we can compute $\dim(\mathrm{Ind}_H^G(\mathbb{C})) = 3$ hence $\chi_R(\varepsilon) = 3$ $\chi_R((1\ 2)) = \chi_{\mathrm{sgn}}((1\ 2)) = -1$, and $\chi_R((1\ 2)) = 0$.

This gives us that $\chi(\varepsilon) = 3, \chi((1\ 2)) = -1$, $\chi((1\ 2\ 3)) = 0$ which is equal to the sum of the characters of the standard and sign representations. This means that $\mathrm{Ind}_{S_2}^{S_3}(G)$ is isomorphic to the direct sum of the standard and sign representations.

Later, when we get into systematically classifying the representations of the symmetric group up to isomorphism, we will see that there will be a nice theorem that will allow us to determine what the induced representations of the symmetric group are simply by a combinatorial process.

4 Representations of S_n

We know that for any positive integer n, the the trivial representation of S_n and the sign representation of S_n are irreducible. Can we find another irreducible representation of that S_n has for any n? Well it turns out that the standard representation is irreducible.

4.1 Irreducibility of the Standard Representation

Theorem 4.1. *The standard representation of S_n is irreducible.*

Proof. Let (π, V) be the permutation representation of S_n for $n \geq 3$. We know that if $B = \{e_1, \ldots, e_n\}$ is a basis of V, then $W = \operatorname{span}\{e_1 + \cdots + e_n\}$ is S_n invariant by a similar argument to that of the specific case for S_3. The complementary space is

$$W' = \left\{ v = \sum_{i=1}^{n} a_i e_i \in V \mid \sum_{i=1}^{n} a_i = 0 \right\}$$

To show that W' is irreducible we proceed by assuming that U is some G irreducible subspace of W' and then prove that it must be equal to W'. Consider the basis $D = \{e_1 - e_2, e_2 - e_3, \ldots e_{n-1} - e_n\}$ of W'. We will show that W' is irreducible by assuming that U is an S_n-invariant vector subspace of W' and prove that that $D \subseteq U$, hence $\operatorname{span} D \subseteq W'$ but this implies that $U = W'$. Suppose $U \neq \{0\}$ is an S_n invariant subspace of W. Let v be a nonzero vector in U. Since $U \subseteq W$, v can be written as

$$v = \sum_{i=1}^{n} a_i e_i$$

such that $a_1 + \cdots + a_n = 0$. Since v is nonzero, there is some $1 \leq i < j \leq n$ such that $a_i \neq a_j$. Notice that

$$v' = \phi_{(1\ i)(2\ j)}(v) = \sum_{i=1}^{n} b_i e_i$$

must also be in U since U is S_n-invariant. Furthermore, $b_1 \neq b_2$. Now notice that

$$v' - \phi_{(1\ 2)}(v') = b_1 e_1 + b_2 e_2 + \cdots b_n e_n - b_1 e_2 - b_2 e_1 - \cdots - b_n e_n = (b_1 - b_2)(e_1 - e_2)$$

Since $b_1 - b_2 \neq 0$ it follows that $e_1 - e_2 \in U$ since U is a subspace of W'. By the subspace invariance of U, it follows that $e_{k-1} - e_k \in U$.

Therefore $W' = U$, hence W' is irreducible.

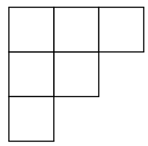
∎

4.2 Young Diagrams and Young Tableaux

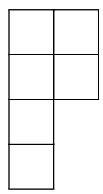
Definition 4.1 (Partition)**.** A partition λ of a natural number n is a tuple of non-decreasing positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

Definition 4.2 (Young Diagram)**.** The Young diagram of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n is a diagram consisting of n boxes with k rows where the i'th row has λ_k boxes.

Example. The partition $\lambda = (3, 2, 1)$ of 6 has the Young diagram



and the partition $\mu = (2, 2, 1, 1)$ of 6 has the corresponding Young diagram



Definition 4.3 (Young Tableau)**.** If λ is a partition of n, a Young tableau of the Young diagram of λ is obtained by placing the numbers 1 to n without repetition into the boxes of the Young diagram.

Remark. Since the word "tableau" comes from French, we will be consistent with French orthography and use "tableau" when it is singular and "tableaux" when it is plural.

In this thesis, we will consider the *canonical* Young tableau which consists of placing the numbers 1 to n in increasing order from left to right, top to bottom.

Example. If we have the partition $\lambda = (3, 2)$ of 5, then

1	2	3
4	5	

5	2	4
3	1	

are two Young tableaux of λ but only the first tableau is the canonical tableau.

We will need the following definitions:

Definition 4.4. We define an action $\cdot$ of S_n on the canonical Young tableau of a partition of n where σ acts on T_λ by applying σ to the entries of the canonical Young tableau. We say that σ *stabilizes* a row, if $\sigma(i)$ is in the same row as i for each i in the row. Similarly, we say that σ *stabilizes* a column, if $\sigma(i)$ is in the same column as i for each i in the column.

Definition 4.5 (Row and column stabilizers)**.** Let n be a positive integer, $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n, and T_λ be the canonical Young tableau of λ. We define two subgroups of S_n.

- The row stabilizer which is defined as

$$\mathcal{R}(T_\lambda) = \{\sigma \in S_n \mid \sigma \text{ stabilizes every row in } T_\lambda\}.$$

- The column stabilizer which is defined as

$$\mathcal{C}(T_\lambda) = \{\sigma \in S_n \mid \sigma \text{ stabilizes every column in } T_\lambda\}.$$

Remark. $\mathcal{R}(T_\lambda) \cap \mathcal{C}(T_\lambda) = \{\varepsilon\}$ because if $\sigma \in \mathcal{R}(T_\lambda) \cap \mathcal{C}(T_\lambda)$, then σ stabilizes every column and every row. So for any $k \in \{1, \ldots n\}$, $\sigma(k)$ is in the same column and the same row in $\sigma \cdot T_\lambda$ as it was in T_λ, hence $\sigma(k) = k$. Therefore $\sigma = \varepsilon$.

Example. Consider the partition $\lambda = (3, 2)$ of 5. The canonical Young tableau is

$$T_\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

The row stabilizer is

$$\begin{aligned}\mathcal{R}(T_\lambda) = \{&\varepsilon, (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2), (4\ 5),\\ &(1\ 2)(4\ 5), (2\ 3)(4\ 5), (1\ 3)(4\ 5), (1\ 2\ 3)(4\ 5), (1\ 3\ 2)(4\ 5)\}\end{aligned}$$

Remark. We could proceed one of two ways here, we could either define an equivalence relation on the possible Young tableaux of a given partition λ where two Young tableaux are equivalent if they have the same entries in each row. This gives rise to an equivalence relation on tableaux. We call an equivalence class of tableaux a tabloid. Then we define an action of S_n on tabloids where $\sigma \in S_n$ acts by permuting the entries of the tabloid. Then we take the vector space spanned by the tabloids of shape λ. The irreducible representations of S_n then come from special subspaces of the vector space spanned by tabloids of shape λ called Specht modules. Then it is shown that these subspaces are irreducible. This is the approach taken in [Sag01], [Ste12] and [Pee75]. Our approach will be to define a subspace of the group algebra of S_n that is determined by the shape of the Young diagram λ of n and show that as λ ranges over all the possible Young diagrams n, the subspace of the group algebra is an irreducible representation. This is the construction found in [Mor19], [FH91], and [Eti11]. A new approach to understanding representations of the symmetric group using branching graphs of Young diagrams, can be found in [Lor18, p. 187].

In order to follow the second approach outlined in the remark above, we define two important elements of the group algebra of the symmetric group S_n.

Definition 4.6. Let λ be a partition of n and let T_λ be the canonical Young tableau of λ. Define

$$a_\lambda = \sum_{\sigma \in \mathcal{R}(T_\lambda)} \sigma$$

and

$$b_\lambda = \sum_{\sigma \in \mathcal{C}(T_\lambda)} \operatorname{sgn}(\sigma)\sigma$$

Since a_λ and b_λ are in the group algebra, it makes sense to define their product $a_\lambda \cdot b_\lambda$ in the group algebra. It turns out that this product plays a useful role in determining the irreducible representations the symmetric group.

Definition 4.7 (Young symmetrizer)**.** Given a partition, λ of n and the canonical Young tableau T_λ associated to λ, the *Young symmetrizer* of T_λ is the element of the group algebra of S_n defined by

$$c_\lambda = a_\lambda \cdot b_\lambda$$

Let $\mathbb{C}[S_n]c_\lambda = \{v \cdot c_\lambda \mid v \in \mathbb{C}[S_n]\}$. We define a representation of S_n as follows:

Definition 4.8. The representation of S_n corresponding to T_λ is $(\rho, \mathbb{C}[S_n]c_\lambda)$ where ρ is defined as follows. For any $\sigma \in S_n$ and any $v \in \mathbb{C}[S_n]c_\lambda$

$$\rho_\sigma(v) = \rho \cdot v$$

Where $\rho \cdot v$ denotes multiplication in the group algebra. We define $V_\lambda \doteq \mathbb{C}[S_n]\cdot_{T_\lambda}$

The big result of this section will be that all of the irreducible representations are of the form $(\rho, \mathbb{C}[S_n]c_\lambda)$ as λ ranges over the the possible Young diagram shapes for n.

Example. For a positive integer n the partition $\lambda = (1, \dots, 1)$ consisting of n 1s has the corresponding canonical Young tableau

$$\boxed{1}\boxed{2}\dots\boxed{n}.$$

Since this diagram only has one row, any $\sigma \in S_n$ stabilizes that row, this implies that $\mathcal{R}(T_\lambda) = S_n$. On the other hand, the only way a column can be stabilized by $\sigma \in S_n$ is if the entries in the tableau stay fixed, hence $\mathcal{C}(T_\lambda) = \{\varepsilon\}$. Therefore

$$a_\lambda = \sum_{\sigma \in \mathcal{R}(T_\lambda)} \sigma = \sum_{\sigma \in S_n} \sigma$$

and

$$b_\lambda = \sum_{\sigma \in \mathcal{C}(T_\lambda)} \operatorname{sgn}(\sigma) = \varepsilon.$$

This makes the computation of the Young symmetrizer pretty easy since

$$c_\lambda = a_\lambda b_\lambda = a_\lambda \varepsilon = a_\lambda = \sum_{\sigma \in S_n} \sigma.$$

Since the row stabilizer is a subgroup, and $\mathcal{R}(T_\lambda) = S_n$, this means $\sigma c_\lambda = \sigma a_\lambda = a_\lambda = c_\lambda$ for all $\sigma \in S_n$. Thus for any $v \in \mathbb{C}[S_n]$, $v = \sum_{\sigma \in S_n} k_\sigma \sigma$ where $k_\sigma \in \mathbb{C}$.

$$v \cdot c_\lambda = \left(\sum_{\sigma \in S_n} k_\sigma \sigma \right) c_\lambda = \sum_{\sigma \in S_n} k_\sigma (\sigma c_\lambda) = \left(\sum_{\sigma \in S_n} k_\sigma \right) c_\lambda.$$

Notice that since $k_\sigma \in \mathbb{C}$, $\sum_{\sigma \in S_n} k_\sigma \in \mathbb{C}$ hence any element in V_λ can be written as kc_λ for $k \in \mathbb{C}$. This means that V_λ is one dimensional. Notice that we have $\rho_\sigma(c_\lambda) = \sigma c_\lambda = c_\lambda$. Therefore the representation attached to $T_{(1,\dots,1)}$ is a one dimensional representation of S_n where every element of S_n acts on $\mathbb{C}[S_n] \cdot c_\lambda$ by multiplication by 1 hence it is isomorphic to the trivial representation.

Example. If we take the partition $\lambda = (n)$ of n then the corresponding canonical Young tableau of λ is

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \\ \vdots \\ \begin{array}{|c|} \hline n \\ \hline \end{array}$$

Notice that any $\sigma \in S_n$ stabilizes the only column of T_λ hence $\mathcal{C}(T_\lambda) = S_n$. Only the identity permutation fixes rows in T_λ hence $\mathcal{R}(T_\lambda) = \{\varepsilon\}$. We compute the Young symmetrizer,

$$c_\lambda = a_\lambda b_\lambda = \left(\sum_{\sigma \in \mathcal{R}(T_\lambda)} \sigma \right) \left(\sum_{\sigma \in \mathcal{C}(T_\lambda)} \operatorname{sgn}(\sigma)\sigma \right) = \varepsilon \sum_{\sigma \in \mathcal{C}(T_\lambda)} \operatorname{sgn}(\sigma)\sigma = b_\lambda.$$

For any $\sigma \in \mathbb{C}[S_n]$ we have

$$\sigma c_\lambda = \sigma b_\lambda = \operatorname{sgn}(\sigma) b_\lambda.$$

Therefore for any $v \cdot c_\lambda \in V_\lambda$ we have

$$\sum_{\tau \in S_n} k_\tau \tau \cdot c_\lambda = \sum_{\tau \in S_n} k_\tau \tau c_\lambda = \sum_{\tau \in S_n} k_\tau \operatorname{sgn}(\tau) c_\lambda = \left(\sum_{\tau \in S_n} k_\tau \operatorname{sgn}(\tau) \right) \cdot c_\lambda,$$

thus V_λ is 1 dimensional since $\sum k_\tau \operatorname{sgn}(\tau) \in \mathbb{C}$. Finally we see how $\sigma \in S_n$ acts on c_λ. We have

$$\rho_\sigma(c_\lambda) = \operatorname{sgn}(\sigma) c_\lambda.$$

Therefore the representation attached to (n) is one dimensional and acts by multiplying by the sign of a permutation hence $V_{(n)}$ is isomorphic to the sign representation of S_n.

Example. What representation of S_3 do we get if we look at the partition $\lambda = (2, 1)$? The canonical Young tableau is

$$T_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

We see that the row stabilizer is $\{\varepsilon, (1\ 2)\}$ and the column stabilizer is $\{\varepsilon, (1\ 3)\}$ so we have

$$a_\lambda = \sum_{\sigma \in \mathcal{R}(T_\lambda)} \sigma = \varepsilon - (1\ 2) \qquad \text{and} \qquad b_\lambda = \sum_{\sigma \in \mathcal{C}(T_\lambda)} \mathrm{sgn}(\sigma)\sigma = \varepsilon - (1\ 3),$$

hence $c_\lambda = \varepsilon + (1\ 2) - (1\ 3) - (1\ 3\ 2)$.

$V_{(2,1)}$ is 2 dimensional with basis c_λ and $(1\ 3)c_\lambda$ [Ser77, p.47]. To determine which representation $V_{(2,1)}$ is, we compute its character. We first determine how $(1\ 2)$ acts on the basis vectors c_λ and $(1\ 3)c_\lambda$ to compute its matrix,

$$\rho_{(1\ 2)}(c_\lambda) = (1\ 2) \cdot (\varepsilon + (1\ 2) - (1\ 3) - (1\ 3\ 2)) = (1\ 2) + \varepsilon - (1\ 3\ 2) - (1\ 3) = c_\lambda$$

and

$$\begin{aligned} \rho_{(1\ 2)}((1\ 3)c_\lambda) &= (1\ 2) \cdot ((1\ 3) + (1\ 2\ 3) - \varepsilon - (2\ 3)) \\ &= (1\ 3\ 2) + (2\ 3) - (1\ 2) - (1\ 2\ 3) \\ &= c_\lambda - (1\ 3)c_\lambda \end{aligned}$$

Therefore the matrix of $\rho_{(1\ 2)}$ is

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Hence $\chi((1\ 2)) = 0$

To determine the matrix of $\rho_{(1\ 2\ 3)}$ we determine where it sends basis elements.

$$\rho_{(1\ 2\ 3)}(c_\lambda) = (1\ 2\ 3) \cdot (\varepsilon + (1\ 2) - (1\ 3) - (1\ 3\ 2)) = (1\ 2\ 3) + (1\ 3) - (2\ 3) - \varepsilon = (1\ 3)c_\lambda.$$

and

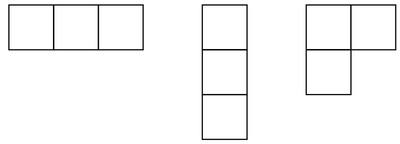
$$\begin{aligned} \rho_{(1\ 2\ 3)}((1\ 3)c_\lambda) &= (1\ 2\ 3) \cdot ((1\ 3) + (1\ 2\ 3) - \varepsilon - (2\ 3)) \\ &= (2\ 3) + (1\ 3\ 2) - (1\ 2\ 3) - (1\ 2) \\ &= -c_\lambda - (1\ 3)c_\lambda. \end{aligned}$$

Hence the matrix of $\rho_{(1\ 2\ 3)}$ is

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

which shows that $\chi(1\ 2\ 3) = -1$. Therefore $V_{(2,1)}$ is isomorphic to the standard representation because it has the same character as the standard representation.

In summary, the three non-isomorphic representations of S_3 are given by V_λ as λ ranges over the 3 possible Young diagram shapes for partitions of S_3:



with the above diagrams corresponding to the trivial, sign and standard representation respectively. This will generalize to S_n. To prove that V_λ consist of all the irreducible representations of S_n as λ ranges over all the possible partitions of n we proceed by a series of lemmas. The two main things we need to prove are that every V_λ is irreducible, and that if λ and μ are two different partitions of n, then V_λ and V_μ are non isomorphic as representations. Since know that the number of non-isomorphic irreducible representations S_n is equal to the number of conjugacy classes of S_n and the conjugacy classes of S_n are in bijection with partitions of n, this will show that the V_λ are all the irreducible representations of S_n.

4.3 Classifying the Irreducible Representations of S_n

First we need to define an ordering for partitions of n for a fixed natural number n.

Definition 4.9. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ are two partitions of n, then we say $\lambda \succ \mu$ if the first nonzero $\lambda_i - \mu_i$ is positive.

Remark. Notice that $\succ$ is a strict total order on the partitions of n. That is, for any partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1 \ldots, \mu_\ell)$ of n, if $\mu \neq \lambda$ then $\mu \succ \lambda$ or $\lambda \succ \mu$. This holds because if $\mu \neq \lambda$, then for some $1 \leq i \leq \min\{k, \ell\}$ we have $\mu_i \neq \lambda_i$ hence $\mu_i - \lambda_i > 0$ or $\lambda_i - \mu_i > 0$.

Example. For the possible partitions of 4 we have

$$(4) \succ (3,1) \succ (2,2) \succ (2,1,1) \succ (1,1,1,1)$$

Definition 4.10. Given a partition λ of n, we define the set

$$\mathcal{R}(T_\lambda)\mathcal{C}(T_\lambda) = \{\sigma \in S_n \mid \sigma = xy \text{ with } x \in \mathcal{R}(T_\lambda)\mathcal{C}(T_\lambda)\}.$$

Remark. If $\sigma \in \mathcal{R}(T_\lambda)\mathcal{C}(T_\lambda)$, then $\sigma = xy$ for a unique $x \in \mathcal{R}(T_\lambda)$ and unique $y \in \mathcal{C}(T_\lambda)$. This follows from the fact that $\mathcal{R}(T_\lambda) \cap \mathcal{C}(T_\lambda) = \{\varepsilon\}$.

Lemma 4.2.

(i) If $x \in \mathcal{R}(T_\lambda)$, then $xa_\lambda = a_\lambda$.

(ii) If $y \in \mathcal{C}(T_\lambda)$, then $yb_\lambda = \operatorname{sgn}(y)b_\lambda$.

Proof. This follows from the definitions of a_λ and b_λ and the fact that $\mathcal{R}(T_\lambda)$ and $\mathcal{C}(T_\lambda)$ are subgroups of S_n. $\square$

Lemma 4.3. If $\sigma \in \mathbb{C}[S_n]$, then

$$a_\lambda \sigma b_\lambda = \begin{cases} 0 & \text{if } \sigma \notin \mathcal{R}(T_\lambda)\mathcal{C}(T_\lambda) \\ \operatorname{sgn}(y) a_\lambda b_\lambda & \text{if } \sigma = xy \text{ with } x \in \mathcal{R}(T_\lambda) \text{ and } y \in \mathcal{C}(T_\lambda) \end{cases}$$

Proof. Suppose $\sigma \in \mathbb{C}[S_n]$ and $\sigma = xy$ with $x \in \mathcal{R}(T_\lambda)$ and $y \in C(T_\lambda)$. We compute

$$a_\lambda \sigma b_\lambda = (a_\lambda x)(y b_\lambda) = a_\lambda \operatorname{sgn}(y) b_\lambda = \operatorname{sgn}(y) a_\lambda b_\lambda.$$

Now suppose that $\sigma \notin \mathcal{R}(T_\lambda)\mathcal{C}(T_\lambda)$. It suffices to show that there is a transposition in $t \in \mathcal{R}(T_\lambda) \cap \mathcal{C}(\sigma \cdot T_\lambda)$. Because if there is such a transposition, then

$$\begin{aligned} a_\lambda \sigma b_\lambda &= (a_\lambda t)\sigma((\sigma^{-1} t \sigma) b_\lambda) \\ &= (a_\lambda t)\sigma(\operatorname{sgn}(\sigma^{-1} t \sigma) b_\lambda) \\ &= a_\lambda \sigma \operatorname{sgn}(t) b_\lambda \\ &= -a_\lambda \sigma b_\lambda. \end{aligned}$$

Thus $a_\lambda \sigma b_\lambda = -a_\lambda \sigma b_\lambda$ which implies that $a_\lambda \sigma b_\lambda = 0$.

We now prove the following claim:

If $\sigma \notin \mathcal{R}(T_\lambda)\mathcal{C}(T_\lambda)$, then there exists a transposition t such that $t \in \mathcal{R}(T_\lambda) \cap \mathcal{C}(\sigma \cdot T_\lambda)$.

We proceed by proving the contrapositive. Suppose that there is no transposition t such that $t \in \mathcal{R}(T_\lambda) \cap \mathcal{C}(\sigma \cdot T_\lambda)$. This means that there is no pair of numbers (i, j) from $1, \ldots, n$ such that (i, j) are in a row of T_λ and (i, j) are both in the same column of $\sigma \cdot T_\lambda$. This means, for every element in the first row of T_λ there is some $y_1 \in \mathcal{C}(\sigma \cdot T_\lambda)$ such that the first row of $y_1 \cdot (\sigma T_\lambda)$ contains all of the elements of the first row of T_λ. Similarly for every row i in T_λ, there exists some y_i such that $y_i \in \mathcal{C}(\sigma \cdot T_\lambda)$ and $y_i \cdot (\sigma T_\lambda)$ contains the same elements as the i'th row of T_λ. Then we can find an $x_i \in \mathcal{R}(T_\lambda)$ such that the i'th row of $x_i \cdot \mathcal{R}(T_\lambda)$ is identical to the i'th row of $y_i \cdot \mathcal{C}(\sigma T_\lambda)$.

Therefore, there are $x \in \mathcal{R}(T_\lambda)$ and $y \in \mathcal{C}(\sigma \cdot T_\lambda)$ such that $x \cdot T_\lambda = y\sigma \cdot (T_\lambda)$, thus $x = y\sigma$. Since $C(\sigma \cdot T_\lambda) = \sigma \mathcal{C}(T_\lambda)\sigma^{-1}$, this means that $y \in \sigma \mathcal{C}(T_\lambda)\sigma^{-1}$ and thus $y = \sigma z \sigma^{-1}$ with $z \in \mathcal{C}(T_\lambda)$. This proves that

$$x = \sigma z \sigma^{-1} \sigma = \sigma z.$$

Thus $\sigma = xz^{-1}$ proving that $\sigma \in \mathcal{R}(T_\lambda)\mathcal{C}(T_\lambda)$ as desired.

□

Lemma 4.4. If $\lambda \succ \mu$, then for $\sigma \in \mathbb{C}[S_n]$ we have $a_\lambda \sigma b_\mu = 0$

Proof. We proceed in a similar way to lemma 4.3. Suppose $\lambda \succ \mu$. If $\lambda_1 > \mu_1$, this means that λ has more columns than μ. The tableau T_λ has the numbers 1 to λ_1 in its first row. We claim that a pair of numbers from this row must occur in a column in the tableau $\sigma\mu$. Indeed the number of rows in μ is strictly less than λ_1 so two of $\{1, 2 \ldots, \lambda_1\}$ must occur in the same column by the pigeonhole principle. This means that there is a transposition t such that $t \in \mathcal{R}(T_\lambda)$ and $t \in \mathcal{C}(\mu)$.

If $\lambda_1 = \mu_1$, then there must be some i such that $\lambda_i > \mu_i$ by the assumption that $\lambda \succ \mu$. If there is no pair of elements from row 1 of T_λ that occur in the same column of $\sigma\mu$, then we can apply $x_1 \in \mathcal{R}T_\lambda$ and $y_1 \in \mathcal{C}(\sigma\mu)$ such that the first row of $x_1 T_\lambda$ is equal to the first row of $y_1(\sigma \cdot \mu)$. We can do the same thing for the first $i-1$ rows, if there are no pairs from each of the rows that occur in the same column. Hence the tableaux $T_\lambda^{'} = x_{i-1} \cdots x_1 T_\lambda$ and $\mu^{'} = y_{i-1} \cdots y_1 \sigma\mu$ have the same entries in the first $i-1$ rows. Now, the λ_i numbers in the i'th row of T_λ' cannot all be placed in different columns because there are fewer than λ_i columns below the i'th row of μ'. Hence by the pigeonhole principle there must one column that has two elements from the i'th row of $T_\lambda^{'}$. Since multiplying T_λ by elements in its row stabilizer and multiplying elements in $\sigma\mu$ by elements in its row stabilizer doesn't change it's column stabilizer, it follows that there must be a transposition $t \in \mathcal{R}(T_\lambda) \cap \mathcal{C}(\sigma\mu)$.

Therefore there is always a transposition t such that $t \in \mathcal{R}(T_\lambda) \cap \mathcal{C}(\sigma \cdot \mu)$. This means that

$$a_\lambda \sigma b_\mu = (a_\lambda t)\sigma(\sigma^{-1} t \sigma b_\mu) = (a_\lambda)\sigma \operatorname{sgn}(t) b_\mu = -a_\lambda \sigma b_\mu,$$

hence $a_\lambda \sigma b_\mu = 0$ as desired. □

Corollary 4.5. If $\lambda \succ \mu$, then $c_\lambda \mathbb{C}[S_n] c_\mu = \{0\}$.

Proof. Suppose $v \in \mathbb{C}[S_n]$, then we have

$$c_\lambda v c_\mu = c_\lambda \left(\sum_{\sigma \in S_n} k_\sigma \sigma \right) c_\mu = \sum_{\sigma \in S_n} k_\sigma c_\lambda \sigma c_\mu = \sum_{\sigma \in S_n} 0 = 0.$$

□

Definition 4.11. Let $L : \mathbb{C}[S_n] \to \mathbb{C}$ be the function defined by

$$L(\sigma) = \begin{cases} 0 & \text{if } \sigma \notin \mathcal{R}(T_\lambda)\mathcal{C}(T_\lambda) \\ \operatorname{sgn}(y) & \text{if } \sigma = xy, \quad x \in \mathcal{R}(T_\lambda), y \in \mathcal{C}(T_\lambda). \end{cases}$$

and extended by linearity for any $v \in \mathbb{C}[S_n]$. That is, for any $v \in \mathbb{C}[S_n]$,

$$L(v) = L\left(\sum_{\sigma \in S_n} k_\sigma \sigma \right) = \sum_{\sigma \in S_n} k_\sigma L(\sigma). \tag{2}$$

Proposition 4.6. For any $v \in \mathbb{C}[S_n]$, $a_\lambda v b_\lambda = L(v) c_\lambda$.

Proof. Suppose $v \in \mathbb{C}[S_n]$, then we have

$$\begin{aligned} a_\lambda v b_\lambda &= a_\lambda \left(\sum_{\sigma \in S_n} k_\sigma \sigma \right) b_\lambda \\ &= \sum_{\sigma \in S_n} k_\sigma a_\lambda \sigma b_\lambda \\ &= \sum_{\sigma \in S_n} k_\sigma L(\sigma) a_\lambda b_\lambda \\ &= \left(\sum_{\sigma \in S_n} k_\sigma L(\sigma) \right) a_\lambda b_\lambda \\ &= L(v) a_\lambda b_\lambda \end{aligned}$$

□

Proposition 4.7. $c_\lambda V_\lambda \subseteq \mathbb{C}c_\lambda$

Proof. Suppose $w \in V_\lambda$ then $w = v \cdot c_\lambda$ for some $v \in \mathbb{C}[S_n]$. We have,

$$c_\lambda v c_\lambda = a_\lambda(b_\lambda v a_\lambda)b_\lambda = L(b_\lambda v a_\lambda)a_\lambda b_\lambda = L(b_\lambda v a_\lambda)c_\lambda.$$

Since $L(b_\lambda v a_\lambda) \in \mathbb{C}$, this shows that $c_\lambda v c_\lambda \in \mathbb{C}c_\lambda$.

□

Proposition 4.8. $c_\lambda^2 = \frac{n!}{\dim(V_\lambda)}c_\lambda$.

Proof. We have

$$c_\lambda^2 = a_\lambda(b_\lambda a_\lambda)b_\lambda = L(b_\lambda a_\lambda)c_\lambda$$

We define a map $f : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ as follows,

$$f(v) = vc_\lambda.$$

for all $v \in \mathbb{C}[S_n]$. It follows that

$$f(f(v)) = f(vc_\lambda) = vc_\lambda^2 = L(b_\lambda a_\lambda)vc_\lambda.$$

Now we calculate the trace of f with respect to two different bases. Consider first, the basis given by the regular representation. For any basis element $\sigma \in S_n$, the coefficient of σ in the product σc_λ is 1 since the coefficient of ε in c_λ is 1. This means the trace of the matrix is a sum of 1's and since there are $n!$ basis elements in $\mathbb{C}[S_n]$, this means $\mathrm{Tr}(f) = n!$. On the other hand we have that f satisfies the relation $f(f(v)) - L(b_\lambda a_\lambda)f(v) = 0$ hence the eigenvalues are either 0 or $L(b_\lambda a_\lambda)$. The number of times the eigenvalue $L(b_\lambda a_\lambda)$ appears is equal to the rank of f which is equal to the dimension of the image of f. The image of f is V_λ since if $x \in \mathbb{C}[S_n]c_\lambda$ then $x = vc_\lambda$ for $v \in \mathbb{C}[S_n]$ and thus $f(v) = vc_\lambda = x$.

$$\mathrm{Tr}(f) = L(b_\lambda a_\lambda)\dim(V_\lambda)$$

Therefore $c_\lambda^2 = \frac{n!}{\dim(V_\lambda)}c_\lambda$. □

Proposition 4.9. $\mathrm{Hom}_{S_n}(V_\lambda, V_\mu)$ is isomorphic to $c_\lambda\mathbb{C}[S_n]c_\mu$ as $\mathbb{C}$-vector spaces.

Proof. See [Eti11, p. 113] □

Theorem 4.10.

- *V_λ is irreducible*
- *If $\lambda \succ \mu$, then V_λ is not isomorphic to V_μ*

Proof. By Proposition 4.9, we know that for any λ and μ we have

$$\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) \cong c_\lambda \mathbb{C}[S_n] c_\mu.$$

To prove (i), notice that by proposition 4.7 $c_\lambda \mathbb{C}[S_n] c_\lambda = c_\lambda V_\lambda \subseteq \mathbb{C} c_\lambda$. Since $\mathbb{C} c_\lambda$ is spanned by c_λ, it is a one-dimensional vector space hence $c_\lambda V_\lambda$ is either $\mathbb{C} c_\lambda$ or $\{0\}$. But $c_\lambda^2 \in c_\lambda V_\lambda$ and we know from proposition 4.8 that $c_\lambda^2 \neq 0$. This implies that $c_\lambda V_\lambda = \mathbb{C} c_\lambda$ which is 1 dimensional, hence $\mathrm{Hom}_{S_n}(V_\lambda, V_\mu)$ is 1-dimensional. By the converse of Schur's lemma, this means that V_λ is irreducible.

To prove (ii), suppose $\lambda \succ \mu$, then $c_\lambda \mathbb{C}[S_n] c_\mu = 0$, hence $\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = 0$, that is, the only morphism of representations of S_n from V_λ to V_μ is the 0 map. Hence V_λ and V_μ are not isomorphic representations. $\square$

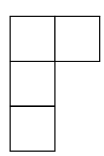
Theorem 4.11. *Every irreducible representation of S_n is isomorphic to V_λ for some partition λ.*

Proof. If λ and μ are two partitions of n such that $\lambda \neq \mu$, then since $\succ$ is a strict total order, either $\lambda \succ \mu$ or $\mu \succ \lambda$ and hence, by Theorem 4.10, V_λ is not isomorphic to V_μ. This means that there are as many pairwise non-isomorphic V_λ as there are partitions of n. The number of non-isomorphic irreducible representations of S_n is equal to the number of conjugacy classes in S_n. The number of conjugacy classes in S_n is equal to the number of partitions of S_n (see [Mor19] for more detail). Therefore every irreducible representation of S_n must be equal to V_λ for some partition λ of n. $\square$

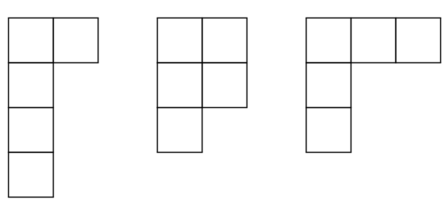
Now that we have proved the main result of this thesis we briefly present another use for Young diagrams. Young diagrams can help use understand how induced representations of irreducible representations decompose into irreducible representations.

Definition 4.12. Let λ be a Young diagram. We define $\mathcal{A}(\lambda)$ to be the set of all Young diagrams that are obtained by adding one box to the Young diagram of λ and obtaining a valid Young diagram. Similarly the set $\mathcal{S}(\lambda)$ is defined to be the set of Young diagrams that are obtained by removing a box from λ and being left with a valid Young diagram.

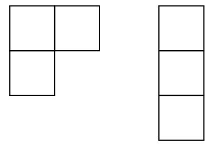
Example. For the Young diagram

$\lambda =$ 

$\mathcal{A}(\lambda)$ consists of the diagrams



and $\mathcal{S}(\lambda)$ consists of the diagrams



Young diagrams give us an easy way of computing induced and restriction representations of the symmetric group.

Theorem 4.12.

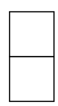
$$\mathrm{Ind}_{S_{n-1}}^{S_n}(V_\lambda) = \bigoplus_{\mu \in \mathcal{A}(\lambda)} V_\mu$$

$$\mathrm{Res}_{S_{n-1}}^{S_n}(V_\lambda) = \bigoplus_{\mu \in \mathcal{S}(\lambda)} V_\mu$$

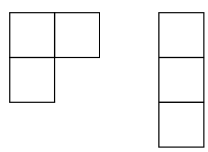
The proof of this theorem can be found in [Sag01].

Using Theorem 4.12 we can compute the induced representations from section 3 in a much simpler way.

Example. Recall that the sign representation of S_2 is represented by the Young diagram



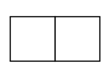
The diagrams that can be obtained by adding a square are



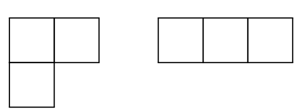
By theorem 4.12 and the fact that $V_{(2,1)}$ is isomorphic to the standard representation of S_3 and $V_{(1,1,1)}$ is isomorphic to the sign representation of S_3,

$$\mathrm{Ind}_{S_3}^{S_2}(\mathrm{sgn}) = V_{(2,1)} \oplus V_{(1,1,1)} = \mathrm{standard} \oplus \mathrm{sgn}\,.$$

Example. The trivial representation of S_2 is represented by the Young diagram



The diagrams that can be obtained by adding a square are



Recalling that the diagrams above correspond to the standard and trivial representation respectively, and making use of Theorem 4.12

$$\mathrm{Ind}_{S_3}^{S_2}(\mathbb{1}) = \mathrm{standard} \oplus \mathbb{1}.$$

Theorem 4.12 agrees with the calculations of the induced representations that we carried out in section 3 but with much less effort. This is a quite useful theorem because it reduces the problem of decomposing induced representations into sums of irreducible representations to a completely mechanical task.

4.4 Conclusion

Young diagrams and Young tableaux give us a powerful way of classifying all of the irreducible representations of the symmetric group up to isomorphism. Furthermore, they help us understand how the induced representation of an irreducible representation of a symmetric group decomposes into irreducible representations. There are other applications of Young diagrams; for example, there is a technique for computing the dimension of V_λ called the Hook Length Formula which can be found in [GN04]. Natural questions that arise from this thesis, are (1), how can we decompose *any* representation into irreducible representations and (2), can we classify irreducible representations of symmetric groups in the case when V is a vector space over a field with positive characteristic? These would be exciting questions to explore in the future.

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