PHIL 473: Philosophy of Logic

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What is Constructive Mathematics?

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Abstract

Undergraduate mathematics students are not traditionally exposed to the principles of constructive mathematics and its applications to provability. By contrasting the use of classical logic in proofs that are valid in classical mathematics but are not considered valid in constructive mathematics this discussion paper aims to motivate doing mathematics constructively. Constructive mathematics restricts the definition of what constitutes a proof and avoids the universal acceptance of the law of excluded middle. These restrictions might seem a nuisance at first glance but they can open a window to insightful proofs and a perspective that enhances instead of opposes classical mathematics.

1 Introduction

All mathematicians, no matter what subfield they are in, are eventually introduced to the concept of proof writing. A proof is a logical argument that asserts that the statement claimed to be true is verifiably true. A proof serves two purposes, first it convinces the person writing the proof of the veracity of a claim, and second, it serves as a way of communicating that result to other mathematicians who will verify that the proof is correct. In practice this works because mathematicians have an agreement as to which principles they are allowed to assume without proof (axioms) and which principles of reasoning they are permitted to use. Constructive mathematics challenges one of the principles of logic that most mathematicians tacitly accept and use, namely the law of excluded middle which says that for every statement P it is either the case that P is true or that the negation of P is true. There are many kinds of mathematics that are considered constructive mathematics but this paper will explore the BHK interpretation of intuitionistic logic and how it has been used in constructive mathematics.

2 Motivation

When mathematicians are introduced to proof writing, they are taught general ways to prove statements. For example a proof of a statement of the form "If P, then Q" is an argument that starts with the assumption of P and then uses deductive reasoning and knowledge of other true mathematical propositions to deduce that Q must also be true. However there are equivalent ways to prove an if-then statement. One is to use the contrapositive because in classical logic, the statement if P then Q is equivalent to if $\neg Q$ then $\neg P$. One of the most important principles that mathematicians use comes from classical logic and it is called the law of excluded middle. This law says that for every statement P, either P or $\neg P$ holds. This is to say that this law cannot be used universally because we should be able to say which of P or $\neg P$ holds. For example, consider an unproven claim like Goldbach's conjecture which states that any positive integer can be written as the sum of two prime numbers. This has been verified up to very large numbers but so far no one has been able to prove or disprove it. In constructive mathematics whenever we have an "or" statement we should be able to determine which of the two conjuncts is true. One can think of constructive mathematics as replacing the notion of truth with the notion of provability where provability means a direct construction.

To illustrate what we mean by construction and provability it helps to consider a classic example of a proof that is not constructively valid.

Theorem 2.1. *There exist irrational numbers x and y such that x^y is rational*

Proof. This proof uses the assumption that the number $\sqrt{2}^{\sqrt{2}}$ is either rational or it is irrational.

Case 1: Suppose that $\sqrt{2}^{\sqrt{2}}$ is rational. If this is the case, since $\sqrt{2}$ is irrational then we can take $x = \sqrt{2}$ and $y = \sqrt{2}$ and we see that x^y is rational.

Case 2: Suppose that $\sqrt{2}^{\sqrt{2}}$ is irrational, then taking $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ we see that

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2$$

is rational. □

This proof which is rather contrived serves as an example of one that is also not very informative. It does not feel like a satisfying proof because we haven't been told which two numbers satisfy the property. A proof that would be more illuminating would be to take the irrational number $\log_2(9)$, the irrational number $\sqrt{2}$ and then use properties of the logarithm and exponents to show that

$$\sqrt{2}^{\log_2(9)} = \sqrt{2}^{\log_2(3^2)} = \sqrt{2}^{2\log_2(3)} = 2^{\log_2(3)} = 3$$

This proof has an advantage over the first proof because it explicitly takes two numbers that are irrational and then proves that they satisfy the statement of (2.1).

There were two objections about the proof of (2.1) that are the motivation for constructive mathematics and intuitionistic logic. The first, has been addressed, namely that an existential claim should explicitly show which objects satisfy the claim and the second has to do with the law of excluded middle. For this, we need to take about the mathematician L.E.J Brouwer. Brouwer was a Dutch mathematician born in 1881 who is credited with creating the philosophy of intuitionism. Brouwer believed that a mathematical object did not exist unless it was explicitly constructed [1]. Without going too far into the philosophy of mathematics, Brouwer believed that time was the only *a priori* notion and that mathematics was a creation of the human mind, and thus to show that an object exists we must construct it. Brouwer also believed that Mathematics came before logic, that is, he did not believe that just because a mathematical proof was valid according to the laws of classical logic that it was a valid mathematical proof. He actually saw logic as an application of mathematics. [1]. Brouwer is also notable for having proved an important theorem in classical mathematics called Brouwer's Fixed Point Theorem whose proof is not constructive. He later went on to view his Fixed Point Theorem as invalid and rejected it [2].

Perhaps because of this belief that mathematics came before logic, Brouwer never formalized intuitionistic logic. It was one of Brouwer's students, Arend Heyting who first described how to interpret logical connectives in intuitionistic logic according to the principles that Brouwer used. This is what we now call the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic.

So what counts as a constructive proof and what kinds of proof techniques are we allowed to use in constructive logic? Are we allowed to prove that $P \to Q$ by proving that $\neg Q \to \neg P$? Are we allowed to use proof by contradiction? To answer these questions, we will look at the BHK interpretation of intuitionistic logic. The definition has been adapted from [1].

Definition 2.1. The BHK interpretation of $\perp$, the logic connectives $\vee, \wedge, \neg, \to$ of first order logic with quantifiers $\forall$, $\exists$ is given as follows :

1. $\perp$: has no proof.
2. $\vee$: A proof of $P \vee Q$ is a proof of either P or a proof of Q.
3. $\wedge$: A proof of $P \wedge Q$ is a proof of P and a proof of Q.
4. $\to$: A proof of $P \to Q$ is a construction that turns a proof of P and a proof of Q.
5. $\neg$: A proof of $\neg P$ is a construction which transforms a proof of P into a proof of $\perp$
6. $\exists$: A proof of $\exists x P(x)$ is a construction of an object a and a proof that $P(a)$ holds.
7. $\forall$: A proof of $\forall x P(x)$ is an algorithm that takes any a in the domain and returns a proof that $P(a)$ holds.

$\perp$ is a symbol used to mean falsity, in this case it has not proof because it is meant to be understood as a contradiction. The BHK interpretation is not a rigorous definition of intuitionistic logic since it does not formally define what a construction is. To formally define intuitionistic logic, we would have to give a syntax and semantics; this can be done using Kripke models (see [3], page 75) but will not be demonstrated here in order to stay on the topic of applications to constructive mathematics. As such, when we demonstrate that a certain logical sentence doesn't hold under the BHK interpretation, we will have to give informal arguments as to why it doesn't instead of specifying a countermodel like we normally would in formal logic. The BHK interpretation is meant to capture the essence of intuitionistic logic and to provide us with a framework for determining what constitutes a constructively valid proof. There are other kinds of constructive mathematics like recursive constructive mathematics, Martin-Löf's intuitionistic type theory, and Errett Bishop's constructive mathematics [1]. We only present intuitionistic constructive mathematics.

3 What's Valid and What's Not?

In intuitionistic logic, $\neg(P \wedge \neg P)$ is always true.

Proof. Suppose we have a proof of $P \wedge \neg P$. According to the BHK interpretation this means we have a proof of P and a proof of $\neg P$. A proof of $\neg P$ is a construction that turns a proof of P into a proof of $\perp$. We have a proof of P so we can apply it to the construction to get a proof of $\perp$. This shows that we can construct of a proof of $\perp$ from a proof of $(P \wedge \neg P)$ therefore $\neg(P \wedge \neg P)$. □

Also $(\neg P \vee \neg Q) \to \neg(P \wedge Q)$ is always true.

Proof. Suppose we have a proof of $(\neg P \vee \neg Q)$. According to the BHK interpretation this means that we have a proof of either $\neg P$ or a proof of $\neg Q$. If we have a proof of $\neg P$ it is a construction that turns a proof of P into a proof of $\perp$. Now if we have a proof of $P \wedge Q$, we have a proof of both P and a proof of Q, and we can use the construction that turns a proof of P to $\perp$ to get a proof of $\perp$. Hence, we can convert a proof of $(P \wedge Q)$ into a proof of $\perp$ meaning that $\neg(P \wedge Q)$ is true. If we have a proof of $\neg Q$, the construction is analogous to that of the case where we had a proof of $\neg P$. □

However, $\neg(P \wedge Q) \to (\neg P \vee \neg Q)$ is not always true. Consider what this would tell us if it were true. It would tell us that we could always decide which of P or Q can be used to construct a proof of $\perp$ given a construction that turns a proof of $P \wedge Q$ into a proof of $\perp$. This is not always the case as will be show later.

4 Examples of Proofs

So which proofs are valid and which are not in constructive mathematics using the BHK interpretation? Well, if we show that P implies a contradiction, *i.e.* if we show how to turn a proof of P into a proof of a contradiction, then by definition of the BHK interpretation we have proved $\neg P$. However, if we prove that $\neg P$ implies a contradiction then all we have shown is that $\neg\neg P$ is true [4]. Intuitively $\neg\neg P$ does not imply P because according to the BHK interpretation all we have done is show that $\neg P$ implies a contradiction, we have not provided a proof of P. The proof that $\sqrt{2}$ is irrational *is* valid because it starts by assuming that $\sqrt{2}$ is rational and concludes by proving a contradiction, hence we know that it is not rational. This would be a valid proof of the irrationality of $\sqrt{2}$ according to a constructivist following the BHK interpretation. Some mathematicians who have not been exposed to constructive mathematics sometimes erroneously get the impression that these proofs are not valid constructively [4].

For the following examples we need the following definitions:

Definition 4.1. A real number a is said to be algebraic if it is the root of a polynomial with rational coefficients. That is, a is algebraic if there exists a polynomial $p(x) = r_n x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0$ with $r_0, r_1, \ldots, r_{n-1}, r_n$ all rational numbers such that

$$p(a) = r_n(a)^n + r_{n-1}(a)^{n-1} + \cdots + r_1(a) + r_0 = 0.$$

A number a is said to be transcendental if it is not algebraic.

We also need the following results which can be found in [5] (Theorem 20.9 and Theorem 20.12.)

Theorem 4.1. *If x and y are algebraic, then $x+y$, $x-y$, xy are algebraic.*

It is known that the numbers e and π are both transcendental and we can now present the following non-constructive proof.

Theorem 4.2 (Non-constructive proof that either $e+\pi$ or $e-\pi$ is irrational)**.** *Either $e+\pi$ or $e-\pi$ is irrational.*

Proof. Suppose both $x = e+\pi$ and $y = e-\pi$ are algebraic. Then $x+y = e+\pi+e-\pi = 2e$ would be transcendental but since a sum of two algebraic numbers is algebraic, this would imply that $2e$ is algebraic. But $2e$ is not algebraic so we have arrived at a contradiction. Similarly, consider $y - x = e - \pi - e - \pi = -2\pi$. Since the difference of two algebraic numbers is also algebraic, it follows that -2π is algebraic, but -2π is not algebraic. Therefore, either $e+\pi$ and $e-\pi$ is transcendental. $\square$

This proof is not constructively acceptable under the BHK interpretation because it does not prove which of $e+\pi$ or $e-\pi$ is transcendental. The proof only shows that $\neg(P \wedge Q)$ but this is not equivalent to $(\neg P \vee \neg Q)$ as we saw earlier.

It is worth noting, that in constructive mathematics we cannot claim that all real numbers are either algebraic or transcendental because this would mean that we had an algorithm for each real number that told us if that number was algebraic or transcendental. Since it is not known whether $e + \pi$ is algebraic or transcendental, we cannot claim that every number is either algebraic or transcendental.

5 So what changes?

So what is at stake here? What do mathematicians lose out by adopting constructive mathematics? Well one thing that gets lost is the axiom of choice. The axiom of choice can be stated as follows:

Definition 5.1 (Choice function)**.** If we have a collection of sets $\mathcal{A}$, a choice function is a function f whose domain is $\mathcal{A}$ and $f(X) \in X$ for every set X in $\mathcal{A}$.

An example to illustrate what a choice function is, is to take the collection of sets

$$\mathcal{A} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$$

And define our choice function as follows:

$$\begin{aligned} f(\{1\}) &= 1 \\ f(\{1, 2\}) &= 2 \\ f(\{1, 2, 3, \}) &= 3 \end{aligned}$$

Through this example we see why f is called a choice function; for $\{1, 2\}$ we could have chosen either $f(\{1, 2\}) = 1$ or we could have chosen $f(\{1, 2\}) = 2$. For a less trivial example consider the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Take $\mathcal{A}$ to be the collection of all subsets of $\mathbb{N}$. Since each subset of $\mathbb{N}$ has a smallest element, our choice function can return the smallest element of each subset.

The axiom of choice might seem like a trivial statement for collections of finite sets and even for some infinite collections of sets like we saw with the subsets of $\mathbb{N}$. But in general, it is not clear how to build a choice function for some collections of sets. As an example, consider the collection of all non-empty subsets of the real numbers. It's hard to think of a way of explicitly describe a choice function for this collection of sets. One cannot choose the smallest real number in each set because a set of real numbers does not necessarily have a smallest element. Despite these difficulties, the axiom of choice states that such a function does exist. As one might expect, constructivists are critical of the axiom of choice, and it can be shown [4] that it implies the law of excluded middle in the sense that if we take the axiom of choice to be true, then for any proposition P we can decide whether P or $\neg P$ holds. Thus the axiom of choice is rejected by constructive mathematicians. However, if an explicit description of a choice function is provided, a constructivist will accept that there is a choice function for that specific collection of sets without implying that it is universally true.

Here is an example of a proof that uses the axiom of choice.

Theorem 5.1 (Surjective functions have right inverses)**.** *For every surjective function $f : A \to B$, there exists a function $g : B \to A$ such that $(f \circ g)(a) = a$ for all $a \in A$. (g is said to be a right inverse of f.)*

Proof. Suppose that $f : A \to B$ is surjective. This means, by definition, for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$. We want to show that there exists a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. For each $b \in B$ consider the set $f^{-1}(\{b\}) = \{a \in A : f(a) = b\}$ which is not empty because f is surjective. If we take the collection of sets $f^{-1}(\{b\})$ for each $b \in B$, the axiom of choice tells us that for each $b \in B$ we can select an $a \in A$ such that $a \in f^{-1}(\{b\})$. This gives us a way of defining a function $g : B \to A$, for each $b \in B$. Simply define $g(b) = a$ where $a \in f^{-1}(\{b\})$. Hence

$$f(g(b)) = f(a) = b$$

for each $b \in B$ which by definition means that $f \circ g = \mathrm{id}_B$. □

In this proof, unless it is explicitly stated, one could miss the application of the axiom of choice. We needed to use it in the proof because there might not be an explicit way of describing how to choose an element from the pre-image $f^{-1}(\{b\})$. It turns out the axiom of choice is *equivalent* to the statement that every surjective function has a right inverse [6]. That is, every surjective function has a right inverse if and only if the axiom of choice holds. Because it is equivalent to the axiom of choice, it is not valid in constructive mathematics.

There are some important statements that are known to be equivalent to the axiom of choice [6] and hence would be lost in constructive mathematics:

- Zorn's lemma.
- Every vector space has a basis.
- Every commutative ring with identity has a maximal ideal.

So giving up the axiom of choice would mean giving up these results from classical mathematics. However some constructive mathematicians do accept the axiom of countable choice [4], which states that countable collections of sets have choice functions.

6 Conclusion

By restricting the definition of what constitutes a proof and avoiding the universal acceptance of the law of excluded middle, mathematicians lose some already proven theorems of mathematics but gain a different perspective. This might seem like an unnecessary challenge, however it might be best seen as an alternative, rather than a replacement of classical mathematics in the same way that non-euclidean geometries do not replace euclidean geometry but give us rich and interesting new theories. Constructive mathematics might not be mainstream but it may deserve more serious attention by mathematicians because much like how non-euclidean geometry provided new insight, constructive mathematics might one day provide support for theories that require truth of a statement to be defined as provability of a statement. Additionally, proofs in constructive mathematics can be more insightful than proofs in classical mathematics since they often provide explicit constructions. Thus the proofs do not only show that a claim is true, but show how to find the mathematical object that makes it true.

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