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#### Introduction to statistics

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# Problems on estimation

## **Objectives**

· Problems on estimation

### Tema 7: Summary (Method of moments)

			Parameter estimates from $E(X) = 1/n \sum_i x_i = ar{x}$
Model	f(x)	E(X)	$E(X^2) = 1/n \sum_i x_i^2$ ,
Bernoulli	$p^x (1-p)^{1-x}$	p	$\hat{p}=ar{x}$
Binomial	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$\hat{p}=rac{ar{x}}{n}$
Shifted geometric	$p(1-p)^{x-1}$	$\frac{1}{p}$	$\hat{p}=rac{1}{ar{x}}$
Negative Binomial	$inom{x+r-1}{x}p^r(1-p)^x$	$r^{rac{1-p}{p}}$	$\hat{p}=rac{r}{ar{x}-r}$
Poisson	$rac{e^{-\lambda}\lambda^x}{x!}$	λ	$\hat{\lambda}=ar{x}$
Exponential	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\hat{\lambda}=rac{1}{ar{x}}$
Normal	$rac{1}{\sqrt{2\pi}\sigma}e^{-rac{(x-\mu)^2}{2\sigma^2}}$	$\mu$	$\hat{\mu}=ar{x}.\hat{\sigma}^2=rac{1}{n}\sum_i x_i^2-ar{x}^2$

# Tema 7: Summary (Maximum likelihood)

• Imagine we make n observations and obtain the values  $(x_1, \dots, x_n)$  is

The likelihood function, the probability of having observed  $(x_1,\dots x_n)$  is  $L(\theta)=\Pi_{i=1..n}f(x_i;\theta)$ 

We can take the log of L,

$$\ln L( heta) = \sum_i \ln(f(x_i; heta))$$

this is called the log-likelihood function

Computing the maximum of  $\ln L(\theta)$  gives us an estimate for  $\theta$  which we call  $\hat{\theta}$ 

#### Problem 1

Consider:

- P(X=0)=1/2
- P(X=1)=a
- $oldsymbol{P}(X=-1)=1/2-a \ oldsymbol{ar{X}}=rac{1}{n}\sum_{i=1}^n X_i ext{ for } a\in(0,1/2)$

a. for 
$$T=rac{ar{X}}{2}+rac{1}{4}$$
 compute  $E(T)$ ,  $V(T)$ 

$$E(T) = rac{E(ar{X})}{2} + rac{1}{4}$$

and

$$E(\bar{X}) = E(X) = \sum_{x=-1,0,1} x P(X = x)$$
  
= -1 \*  $P(X = -1) + 0 * P(X = 0) + 1 * P(X = 1) = 2a - 1/2$ 

then

$$E(T)=a-1/4+1/4=a$$
 and thus  $E(T)$  is an **unbiased** estimator of  $a$ 

$$V(T) = V(\frac{\bar{X}}{2} + \frac{1}{4}) = \frac{1}{4}V(\bar{X}) = \frac{1}{4}\frac{V(X)}{n}$$

so we need to find V(X)

Remember:  $V(X) = E(X^2) - E(X)^2$  we miss  $E(X^2)$ 

$$E(X^2) = \sum_{x=-1,0,1} x^2 P(X=x) = = (-1)^2 * P(X=-1) + 0^2 * P(X=0) + 1^2 * P(X=1) = a + \frac{1}{2} - a = \frac{1}{2}$$

Then

$$V(X) = E(X^2) - E(X)^2 \ = rac{1}{2} - (2a - rac{1}{2})^2 = rac{1}{4} + 2a - 4a^2$$

putting everything together

$$V(T) = rac{1}{4}rac{V(X)}{n} = rac{1/4 + 2a - 4a^2}{4n}$$

since  $V(T)=\sigma_T o 0$ , when n o 0 then T is a **consistent** estimator

#### Problem 2

Consider:

- $E(ar{X})=E(X)=\mu$  then  $ar{X}$  is an **unbiased** estimator of  $E(X)=\mu$
- a. is  $E(ar{X})$  an **unbiased** estimator of  $E(X)^2=\mu^2$

compute:  $E(ar{X}^2)$  (second moment about the origin)

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Remember:  $V(ar{X}) = E(ar{X}^2) - E(ar{X})^2$ 

then 
$$E(ar{X}^2)=E(ar{X})^2+V(ar{X})=E(X)^2+rac{V(X)}{n}=\mu^2+rac{\sigma_X^2}{n}$$

as  $E(ar{X}^2) 
eq \mu^2$  then  $E(ar{X}^2)$  is a **biased** estimator of  $\mu^2$ 

#### Problem 3

Consider:

$$f(x) = \left\{ egin{aligned} (1+ heta)x^ heta, & ext{if } x \in (0,1) \ 0, & otherwise \end{aligned} 
ight.$$

a. compute E(X)

$$egin{aligned} E(X)&=\int_0^1x(1+ heta)x^ heta dx=\int_0^1(1+ heta)x^{1+ heta}dx\ &=rac{(1+ heta)x^{2+ heta}}{2+ heta}\Big|_0^1=rac{1+ heta}{2+ heta} \end{aligned}$$

b. compute  $\hat{\theta}$  using the method of moments

$$E(X) = \mu_1' = rac{1}{n} \sum_i x_i = ar{x}$$
  $rac{1+\hat{ heta}}{2+\hat{ heta}} = ar{x}$ 

solving for  $\hat{ heta}$ 

$$\hat{ heta}=rac{1}{1-ar{x}}-2$$

c. if the result of a random sample is  $x_1=0.92$ ;  $x_2=0.79$ ;  $x_3=0.90$ ;  $x_4=0.65$ ;  $x_5=0.86$ . Compute  $\hat{\theta}$ 

$$ar{x} = rac{0.92 + 0.79 + 0.90 + 0.65 + 0.86}{5} = 0.824$$

then

$$\hat{ heta} = rac{1}{1-ar{x}} - 2 = rac{1}{1-0.824} - 2 = 3.6818$$

#### Problem 4

Consider:

$$f(x) = egin{cases} rac{x}{ heta}e^{rac{-x^2}{2 heta}}, & ext{if } x > 0 \ 0, & otherwise \end{cases}$$

a. Compute  $\hat{ heta}$  by maximum likelihood

If we have a set of observations  $(x_1, \dots x_n)$ , the probability of having observed those numbers is given by the likelihood function:

$$egin{align} L( heta) &= \Pi_{i=1..n} rac{x}{ heta} e^{rac{-x_i^2}{2 heta}} \ &= x_1 \! * \! \ldots \! * \! x_n heta^{-n} e^{\sum_i rac{-x_i^2}{2 heta}} \end{split}$$

the log likelihood is:

$$\ln L( heta) = \sum_i \ln(x_i) - n \ln( heta) - \sum_i rac{x_i^2}{2 heta}$$

deriving with respect to  $\theta$  then and equalling to zero in  $\hat{\theta}$ 

$$\left. \frac{d \ln L(\theta)}{d \theta} \right|_{\hat{\theta}} = -\frac{n}{\hat{\theta}} + \sum_{i} \frac{x_i^2}{2\hat{\theta}^2} = 0$$

solving for  $\hat{\theta}$ 

$$\hat{ heta} = rac{1}{2n} \sum_i x_i^2$$

b. for a random experiment of 4 random sample with values:

$$x_1 = 16.88; x_2 = 10.23; x_3 = 4.59; x_4 = 6.66; x_5 = 13.68$$

compute the estimate of  $\hat{ heta}$ 

since: 
$$\hat{ heta} = rac{1}{2n} \sum_i x_i^2$$

then

$$\hat{ heta} = rac{1}{2*5}(16.88^2 + 10.23^2 + 4.59^2 + 6.66^2 + 13.68^2) = 64.21534$$

#### Problem 5

Consider:

$$f(t) = egin{cases} \lambda e^{-\lambda(t- au)}, & ext{if } t \geq au \ 0, & otherwise \end{cases}$$

a. For random sample:  $T_1,\dots T_n$  and au is known then compute  $\hat{\lambda}$  by maximum likelihood

If we have a set of observations  $(t_1, \dots t_n)$ , the probability of having observed those numbers is given by the likelihood function:

$$L(\lambda) = \Pi_{i=1..n} \lambda e^{-\lambda(t_i- au)}$$

the log likelihood is:

$$\ln L(\lambda) = \sum_i \ln(\lambda) + \sum_i \ln(e^{-\lambda(t_i - au)}) = n \ln(\lambda) - \lambda \sum_i t_i + n \lambda au$$

deriving with respect to  $\lambda$  then and equalling to zero in  $\hat{\lambda}$ 

$$rac{d \ln L(\lambda)}{d \lambda} \Big|_{\hat{\lambda}} = rac{n}{\hat{\lambda}} + n au - \sum_i t_i = 0$$

solving for  $\hat{\lambda}$ 

$$\hat{\lambda} = rac{n}{\sum_i t_i - n au} = rac{1}{ar{t} - au}$$

a. For random sample:  $T_1 \ldots T_n$  and au is known then compute  $\hat{\lambda}$  by them method of moments

For the method of moments we must have

$$E(T) = \bar{t}$$

since we are given  $E(T)= au+rac{1}{\lambda}$ ; and because  $E(Z)=E(T- au)=rac{1}{\lambda}$  for an exponential probability function then

 $\hat{\lambda} = rac{1}{ar{t} - au}$  , the same estimate as maximum likelihood

c. For random sample:  $T_1 \ldots T_n$  and  $\lambda$  is known then compute  $\hat{ au}$  by them method of moments

Again

$$E(T) = \bar{t}$$

but this time it gives us the equation for the estimate of au

$$\hat{ au}+rac{1}{\lambda}=ar{t}$$
 solving for  $\hat{ au}$  then

$$\hat{ au} = ar{t} - rac{1}{\lambda}$$

#### Problem 6

Consider:

$$f(x) = egin{cases} 2 lpha x e^{-lpha x^2}, & ext{if } x \geq 0 \ 0, & otherwise \end{cases}$$

a. Compute  $\hat{lpha}$  by maximum likelihood

Remember problem 10

$$f(x) = egin{cases} rac{x}{ heta}e^{rac{-x^2}{2 heta}}, & ext{if } x > 0 \ 0, & otherwise \end{cases}$$

then  $2lpha=rac{1}{ heta}$  , we can re-parametrize and by maximum likelihood we had found

$$\hat{ heta} = rac{1}{2n} \sum_i x_i^2$$
 then  $\hat{lpha} = rac{n}{\sum_i x_i^2}$ 

#### Problem 7

Consider the exponential density re-parametrized by

$$f(x) = egin{cases} rac{1}{eta}e^{-rac{x}{eta}}, & ext{if } x \geq 0 \ 0, & otherwise \end{cases}$$

then 
$$E(X)=eta$$
 and  $V(X)=eta^2$ 

a. compute  $\hat{eta}$  by maximum likelihood

then

$$L(eta) = \Pi_{i=1..n} rac{1}{eta} e^{-rac{x_i}{eta}}$$

$$\ln L(rac{1}{eta}) = \sum_i \ln(rac{1}{eta}) + \sum_i \ln(e^{-rac{x_i}{eta}}) = -n \ln(eta) - rac{1}{eta} \sum_i x_i$$

deriving with respect to eta then and equalling to zero in  $\hat{eta}$ 

$$rac{d \ln L(eta)}{deta} \Big|_{\hat{eta}} = -rac{n}{\hat{eta}} + rac{1}{\hat{eta}^2} \sum_i x_i = 0$$

solving for  $\hat{eta}$ 

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 $\hat{eta}=ar{x}$  which for the exponential function (see table) is  $\hat{eta}=\hat{\lambda}^{-1}$ 

c. compute the E(B) and V(B), where B is the estimator (random variable) whose values give us the estimate  $\hat{\beta}$ 

Then

$$B=ar{X}$$

and

 $E(B)=E(ar{X})=E(X)=eta$  therefore B is an **unbiased** estimator of eta, or  $\hat{eta}$  is **unbiased** estimate of eta  $V(B)=V(ar{X})=rac{V(X)}{n}$  and then when  $n o\infty$ , V(B) o0 and therefore we say that estimator is **consistent** 

#### Problem 8

Consider

$$f(x) = \left\{ egin{array}{ll} rac{2( heta-x)}{ heta^2}, & ext{if } x \in [0, heta] \ 0, & otherwise \end{array} 
ight.$$

a. For random sample:  $X_1,\dots X_n$  and compute  $\hat{ heta}$  by the method of moments

The method of moments proposes to find the estimate  $\hat{ heta}$  from the equation

 $E(X)=ar{x}$  therefore we need to find E(X)

$$E(X)=\int_0^{ heta}xrac{2( heta-x)}{ heta^2}dx=rac{2}{ heta^2}(rac{ heta x^2}{2}-rac{x^3}{3})\Big|_0^{ heta}=rac{ heta}{3}$$

Therefore, we have

$$E(X) = rac{\hat{ heta}}{3} = ar{x}$$
 and solving for  $\hat{ heta}$ 

then

$$\hat{\theta} = 3\bar{x}$$