Introduction to statistics

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Problems on Confidence intervals

Summary

- $(1-\alpha)$*100% confidence intervals for the mean μ:

when $X \hookrightarrow N(\mu, \sigma^2)$

case 1) and we **know** σ^2 then CI for the estimation of $\hat{\mu}$

CI for μ:

$$(l, u) = (\bar{x} - z_{\alpha/2}\sigma/\sqrt{n}, \bar{x} + z_{\alpha/2}\sigma/\sqrt{n})$$

case 2) and we **don't know** σ^2 then

CI for μ:

$$(l, u) = (\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n}, \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n})$$

- $(1-\alpha)$*100% confidence intervals for the proportion p:

when $K \rightarrow Bernoulli(p)$, where $k=1$ is the event with probability p, and $\bar{K}$ is the average of events in n trials (relative frequency of events) then the CI for the estimation of $\hat{p}$ is

case 3) CI for p:

$$(l, u) = (\bar{k} - z_{\alpha/2}\Big[\frac{\bar{k}(1-\bar{k})}{n}\Big]^{1/2}, \bar{k} + z_{\alpha/2}\Big[\frac{\bar{k}(1-\bar{k})}{n}\Big]^{1/2})$$

- $(1-\alpha)$*100% confidence intervals for the estiamtion of the variance σ^2:

when $X \hookrightarrow N(\mu, \sigma^2)$

case 4) CI for σ:

$$(l, u) = (\frac{s^2(n-1)}{\chi^2_{\alpha/2,n-1}}, \frac{s^2(n-1)}{\chi^2_{1-\alpha/2,n-1}})$$

Remember for the formulas:

- case 1 and 3) $z_{\alpha/2}$ = `qnorm(1-alpha/2)`
- case 2) $t_{\alpha/2,n-1}$ = `qt(1-alpha/2, n-1)`
- case 4) $\chi^2_{\alpha/2,n-1}$ = `qchisq(1-alpha/2, n-1)`

Problem 1

Consider:

- $n = 5$
- $\alpha = 1 - 0.95 = 0.05$
- $CI = (229.7, 233.5)$

a. $P(\mu \in (229.7, 233.5)) = 0.95$?

No. μ is not a random variable it is a parameter of a probability function. Probabilities are defined only for random variables.

Problem 1

b. compute $\bar{x}$ and s

we are given 95% confidence interval

$CI = (l, u) = (229.7, 233.5)$

Let's remember the definition:

The 95% **random** confidence interval

$(L, U) = (\bar{X} - f_{sup}, \bar{X} - f_{inf})$ is such that

$P(\bar{X} - f_{sup} \leq \mu \leq \bar{X} - f_{inf}) = P(f_{inf} \leq \mu - \bar{X} \leq f_{sup}) = 0.95$

Problem 1

if:

- $X \hookrightarrow N(\mu, \sigma^2)$ and
- and we **do not know** the variance σ^2

Then: $\alpha = 0.05$

$(l, u) = (\bar{x} - t_{\alpha/2,n-1} s/\sqrt{n}, \bar{x} + t_{\alpha/2,n-1} s/\sqrt{n})$

$t_{0.025,n-1} * S/\sqrt{n}$

$(l, u) = (\bar{x} - t_{0.025,n-1} s/\sqrt{n}, \bar{x} + t_{0.025,n-1} s/\sqrt{n})$

where

$t_{0.025,n-1} = F^{-1}_{t,n-1}(0.975) =$ `qt(0.975, 4)` $= 2.77$

Problem 1

Therefore we are given

$(\bar{x} - 2.77s/\sqrt{5}, \bar{x} + 2.77s/\sqrt{5}) = (229.7, 233.5)$

and to equations to solve for $\bar{x}$ and 2:

i. $\bar{x} - 1.23s = 229.7$

ii. $\bar{x} + 1.23s = 233.5$

With solutions:

$\bar{x} = (229.7 + 233.5)/2 = 231.6$ and $s = (233.5 - \bar{x})/1.23 = 1.53$

Problem 1

c. compute 99% CI:

we have:

$(l, u) = (\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n}, \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n})$

We leave out a total of $\alpha = 1\%$.

or

$(l, u) = (\bar{x} - t_{0.005,n-1}s/\sqrt{n}, \bar{x} + t_{0.005,n-1}s/\sqrt{n})$

Problem 1

where

$t_{0.005,n-1} = F^{-1}_{t,n-1}(0.995) =$ `qt(0.995, 4)` $= 4.60$

then: $(\bar{x} - 4.60s/\sqrt{5}, \bar{x} + 4.60s/\sqrt{5}) = (228.45, 234.75)$

Problem 2

Consider:

- $n = 1000$
- $x = 17$

Where $X \hookrightarrow Bin(n, p)$

Remember: If we define the average $\bar{K}$ of a Bernoulli trial ($k = 0, 1$):

- $\bar{K} = \sum_{i=1}^{n} K_i = X/n$, $E(\bar{K}) = E(K) = p$
- $S^2 = \bar{K}(1 - \bar{K})$

The $(1 - \alpha)100\%$ CI associated with a set of observations is

$CI = (l, u) = (\bar{k} - z_{\alpha/2}\left[\frac{\bar{k}(1-\bar{k})}{n}\right]^{1/2}, \bar{k} + z_{\alpha/2}\left[\frac{\bar{k}(1-\bar{k})}{n}\right]^{1/2})$

when np and $n(p - 1) > 5$

Problem 2

a. Then we have: $\hat{p} = \bar{k} = 0.017$

$n\hat{p} = 17$ and $n(1 - \hat{p}) = 983 > 5$ and we can use the approximation:

$$Z = \frac{X-\mu}{\sigma} = \frac{X-E(X)}{\sqrt{V(X)}} = \frac{X-np}{\left[np(1-p)\right]^{1/2}} = \frac{\bar{K}-p}{\left[p(1-p)/n\right]^{1/2}} \to N(0,1)$$

Problem 2

b. compute the 99% confidence interval then $\alpha = 0.01$

$$CI = (l,u) = (\bar{k} - z_{0.005}\left[\frac{\bar{k}(1-\bar{k})}{n}\right]^{1/2}, \bar{k} + z_{0.005}\left[\frac{\bar{k}(1-\bar{k})}{n}\right]^{1/2})$$

Since: $z_{0.005} = \Phi^{-1}(0.995) = \texttt{qnorm(0.995)} = 2.575829$

Then

$CI = (0.006474, 0.027526)$

or

$\hat{p} = 0.017 \pm 0.01$

c. the estimate $\hat{p} \leq 0.02752$ we cannot guarantee the conditions of the client with 99% confidence.

Problem 3

Consider:

- $P(Y=0) = a$
- $P(Y=1) = 1-a$
- $na(1-a) \geq 8$

a. Compute n such that for the CI (l,u), $D = u - l$ is maximum.

Y is a Bernoulli variable with $p = 1-a$ and as $n(1-p)p \geq 8$ then

$$CI = (\bar{y} - z_{\alpha/2}\left[\frac{\bar{y}(1-\bar{y})}{n}\right]^{1/2}, \bar{y} + z_{\alpha/2}\left[\frac{\bar{y}(1-\bar{y})}{n}\right]^{1/2})$$

$D = 2z_{\alpha/2}\left[\frac{\bar{y}(1-\bar{y})}{n}\right]^{1/2}$ is maximum when D^2 is maximum.

Problem 3

$\frac{dD^2}{d\bar{y}} = \frac{(2z_{\alpha/2})^2}{n}(1-2\bar{y}) = 0$ then

D is maximum when the estimated proportion $\bar{y} = 1/2$ or $D_{max} = \frac{z_{\alpha/2}}{\sqrt{n}}$

Problem 3

b. Consider $D = 0.02$ and $\alpha = 0.10$ for 90% confidence. Compute minimum n.

$z_{\alpha/2} = z_{0.05} = \Phi^{-1}(0.95) = \texttt{qnorm(0.95)} = 1.644854$

$D_{max} = 0.02 = \frac{1.644854}{\sqrt{n_{min}}}$ then

$n_{min} = (1.644854/0.02)^2 = 6763.862$

or

$n \geq 6764$ (differences with results are because from tables $z_{0.05} \sim 1.65$)

Problem 4

Consider:

- $682, 553, 555, 666, 657, 649, 522, 568, 700, 558$
- $X \hookrightarrow N(\mu, \sigma_X^2)$

  a. Compute 95% CI

- $\bar{x} = 611$
- $s = 65.51$
- $n = 10$
- As we don't know the variance of X then we use s

the observed CI is:

$$(l, u) = (\bar{x} - t_{0.025,n-1}s/\sqrt{n}, \bar{x} + t_{0.025,n-1}s/\sqrt{n})$$

Problem 4

$t_{0.025,n-1} = F_{t,n-1}^{-1}(0.975) =$ `qt(0.975, 9)` $= 2.26$

then: $(l, u) = (\bar{x} - 2.26s/\sqrt{n}, \bar{x} + 2.26s/\sqrt{n}) = (564.135, 657.865)$

Problem 5

Consider:

- $n = 9$
- the 90% CI is $(118.25, 123.55)$
- $X \hookrightarrow N(\mu, \sigma^2)$
- We know σ

the observed CI is: $(l, u) = (\bar{x} - z_{0.05}\sigma/\sqrt{n}, \bar{x} + z_{0.05}\sigma/\sqrt{n})$

Problem 5

where

$z_{0.05,n-1} = \Phi^{-1}(0.95) =$ `qnorm(0.95)` $= 1.644$

then we have two equations

i. $l = \bar{x} - 1.644 * \sigma/\sqrt{9} = 118.25$

ii. $u = \bar{x} + 1.644 * \sigma/\sqrt{9} = 123.55$

  a. y b. Compute $\bar{x}$ and σ_X^2

solving i and ii for $\bar{x}$ and σ_X^2

$\bar{x} = (118.25 + 123.55)/2 = 120.9$ and $\sigma = 3 * (123.55 - \bar{x})/1.644 = 4.83$ or $\sigma^2 = 23.36$

Problem 5

  c. Compute 97% confidence interval, then $\alpha = 0.03$

Remember:

$(l, u) = (\bar{x} - z_{\alpha/2}\sigma/\sqrt{n}, \bar{x} + z_{\alpha/2}\sigma/\sqrt{n})$

as $z_{\alpha/2} = z_{0.015} = \Phi^{-1}(0.985) =$ `qnorm(0.985)` $= 2.170$ then

$(l, u) = (120.9 - 2.170 * 4.83/\sqrt{9}, 120.9 + 2.170 * 4.83/\sqrt{9})$

$= (117.4063, 124.3937)$

Problem 5

d. for $d = 123.55 - 118.25 = 5.3 = (u - l)$ compute n

$d = u - l = 2 * 2.170 * 4.83/\sqrt{n} = 5.3$

then

$n = (\frac{2*2.170*4.83}{5.3})^2 = 15.64 \sim 16$

Problem 6

consider:

- $\bar{x} = 0.5354$
- $s = 0.3479$
- $n = 51$
- $\alpha = 0.05$

a. the 95% CI for the variance

$(l, u) = (\frac{s^2(n-1)}{\chi^2_{\alpha/2,n-1}}, \frac{s^2(n-1)}{\chi^2_{1-\alpha/2,n-1}})$

Problem 6

$l = \frac{s^2(n-1)}{\chi^2_{\alpha/2,n-1}}$

$\chi^2_{\alpha/2,n-1} = \chi^2_{0.025,50} =$ `qchisq(1-0.025, 50)` $= 71.42$

then $l = 0.3479^2 * 50/71.42 = 0.0847$

Problem 6

$u = \frac{s^2(n-1)}{\chi^2_{1-\alpha/2,n-1}}$

$\chi^2_{1-\alpha/2,n-1} = \chi^2_{0.975,50} =$ `qchisq(1-0.975, 50)` $= 32.35$

then $u = 0.3479^2 * 50/32.35 = 0.1870$

$CI = (l, u) = (0.0847, 0.1870)$

Problem 6

consider:

- $X = 17$, number of fisheries with concentrations greater than 0.700 ppm
- $\alpha = 0.01$
- then $\hat{p} = \bar{k} = 1/3$ where K is a Bernoulli trial.

Problem 6

since $n\hat{p} = 17$ and $n(1-\hat{p}) = 34 > 5$ then CI is

$$(l, u) = (\hat{p} - z_{\alpha/2}\left[\frac{\hat{p}(1-\hat{p})}{n}\right]^{1/2}, \hat{p} + z_{\alpha/2}\left[\frac{\hat{p}(1-\hat{p})}{n}\right]^{1/2})$$

since:

$z_{\alpha/2} =$ `qnorm(1-0.005)` $= 2.575$

then

$$(l, u) = (1/3 + 2.575 * \sqrt{\frac{1/3*2/3}{51}}, 1/3 + 2.575 * \sqrt{\frac{1/3*2/3}{51}} = (0.1634, 0.5033)$$

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Problems on hypothesis testing

Summary

Hypothesis testing:

1.From the problem context, identify the parameter of interest: μ, p or σ

2.State the null hypothesis, for instance for μ

- two tailed: $H_0 : \mu = \mu_0$
- or, upper tailed: $H_0 : \mu \leq \mu_0$
- or, lower tailed: $H_0 : \mu \geq \mu_0$

3.Specify an appropriate alternative hypothesis:

- two tailed: $H_1 : \mu \neq \mu_0$
- or, upper tailed: $H_1 : \mu > \mu_0$
- or, lower tailed: $H_1 : \mu < \mu_0$

Note: upper and lower tail refers where you expect μ to be if H_1 is true.

4.Choose a significance level: α (for example $\alpha = 0.05$ when we want 95% confidence).

5.Define the statistic

when $X \hookrightarrow N(\mu, \sigma) = N(\mu, \sigma)$

Hypothesis for μ

case 1) if σ is known then

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

is standard (`pnorm(z)`)

case 2) if σ is not known then

$$T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

is t-distributed with $n-1$ degrees of freedom (`pt(t,n-1)`)

when $X \hookrightarrow N(\mu, \sigma) = N(\mu, \sigma)$ and $n \times p_0$, $n \times (1-p_0) > 5$

Hypothesis for p_0

case 3)

$$Z = \frac{\bar{X} - p_0}{\left[\frac{p_0(1-p_0)}{n}\right]^{1/2}}$$

is standard (`pnorm(z)`).

when $X \hookrightarrow N(\mu, \sigma) = N(\mu, \sigma)$

Hypothesis for σ_0

case 4)

$$W = \frac{(n-1)S^2}{\sigma_0^2}$$

is χ^2-distributed with $n-1$ degrees of freedom (`pchisq(w,n-1)`).

6.Test the hypothesis

- define critical **the region** for the statistic under H_0 with probability of α at the edges
- or, compute confidence intervals at α confidence limit
- or, compute P-value:
  – two tail: P-value$= 2(1 - F(|z|))$,
  – upper tail: P-value$= (1 - F(z))$,
  – lower tail: P-value$= F(z)$

7.Decide

Reject the null hypothesis if:

- the observed statistic falls in the critical region
- or, the $(1-\alpha)100\%$ confidence interval does not contain μ_0 (σ_0, p_0, etc…)
- or, if $P < \alpha$

Otherwise **do not** reject null hypothesis

R code for testing hypothesis:

case 1)

```
library(BSDA)

x <- c(…)

z.test(x, mu = 0, alternative = , sigma.x = , conf.level = )
```

case 2)

```
x <- c(…)

t.test(x, mu = , alternative = , conf.level = )
```

case 3)

```
x <- c(…)

prop.test(x, n =, p = , alternative = , conf.level = , correct = FALSE )
```

case 4)

```
library(EnvStats)

x <- c(…)

VarTest(x, sigma.squared =, alternative = , conf.level = , )
```

Problem 1

- consider the measurements:

204.999, 206.149, 202.150, 207.048, 203.496, 206.343, 203.496, 206.676, 205.831

- $\mu \geq 206.5$

a. At 90% confidence ($\alpha = 0.1$)

- Test for the mean where we do not know σ

Lower tail: $H_0 : \mu \geq 206.5$, $H_1 : \mu < 206.5$

Problem 1

- Statistic: $T = \frac{\bar{X}-\mu_0}{S/\sqrt{9}}$, t-distribution with $n-1$ degrees of freedom
- Observed value: $t = \frac{205.132-206.5}{1.707/\sqrt{9}} = -2.404$

Problem 1

P-value for lower tail:

P-value$= F(T) = P(T < -2.404) =$ `pt(-2.404, 8)`

$= 0.021 < \alpha = 0.10$

Since: $P < \alpha$ we reject the null hypothesis

Problem 1

b. if $\sigma_X^2 = 4$. Test the hypothesis $\mu_0 = 206.5$ at 95% confidencefidence

- Test for the mean where know σ

Two tail: $H_0 : \mu = 206.5$, $H_1 : \mu \neq 206.5$

- Statistic: $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{9}}$, standard distribution
- Observed value: $z = \frac{205.132-206.5}{2/\sqrt{9}} = -2.05$

Problem 1

P-value for two tail:

P-value$= 2(1 - F(|-2.05|)) =$ `2*(1-pnorm(2.05))` $= 0.0403 < \alpha = 0.05$

Since: $P < \alpha$ we reject the null hypothesis

Problem 2

- consider the measurements:

53700, 55500, 53000, 52400, 51000, 62000, 75000, 53800, 56600

- $\mu_0 = 62000$

a. At 95% confidence ($\alpha = 0.5$)

- Test for the mean where we do not know σ

Lower tail: $H_0 : \mu \geq 62000$, $H_1 : \mu < 62000$

Problem 2

- Statistic: $T = \frac{\bar{X}-\mu_0}{s/\sqrt{9}}$, t-distribution with $n - 1 = 8$ degrees of freedom
- Observed value: $t = \frac{57000-62000}{7464.08/\sqrt{9}} = -2.01$

Problem 2

a. critical region

$P(T < t_{0.95}) = 0.05$

$t_{0.95,8} =$ `qt(0.05,8)` $= -1.8595$

The critical region is $T < -1.8595$

since $t = -2.01 < -1.8595$ we then reject the null hypothesis

Problem 2

b. P-value for lower tail:

P-value$= F(T) = P(T < -2.01) =$

`pt(-2.01, 8)` $= 0.03966 < \alpha = 0.05$

Since: $P < \alpha$ we reject the null hypothesis

Problem 2

c. compute the 99% CI for the if $\sigma^2 = 54760000$

When we know the variance then the CI for an n sample of nromal variables is:

$(l, u) = (\bar{x} - z_{\alpha/2}\sigma/\sqrt{n}, \bar{x} + z_{\alpha/2}\sigma/\sqrt{n})$

$\bar{x} = 57000$, $\sigma = 7400$, $n = 9$ and

$z_{\alpha/2} = z_{0.005} =$ `qnorm(0.995)` $= 2.5758$

Putting everything together

$(l, u) = (50646.29, 63353.71)$ (since it contains 62000, we do not reject H_0 with 99% confidence)

Problem 3

consider:

- $\bar{x} = 7750$
- $s = 145$
- $n = 6$
- $\mu_0 = 8000$
- We don't make a claim if $\mu \geq \mu_0$
- We make a claim if $\mu < \mu_0$

a. if $\alpha = 0.1$ should we make a claim?

- Test for the mean where we do not know σ

Lower tail: $H_0 : \mu \geq 8000$, $H_1 : \mu < 8000$

Problem 3

- Statistic: $T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$, t-distribution with $n - 1$ degrees of freedom
- Observed value: $t = \frac{7750 - 8000}{145/\sqrt{6}} = -4.22$

Problem 3

P-value for lower tail:

P-value$= F(T) = P(T < -4.22) =$ `pt(-4.22, 5)`

$= 0.004 < \alpha = 0.01$

Since: $P < \alpha$ we reject the null hypothesis, we make a claim.

Problem 3

b. consider:

- $\sigma_X^2 = 136161$
- $\alpha = 0.05$

Test the same hypothesis.

- Test for the mean where we do not know σ

Lower tail: $H_0 : \mu \geq 8000$, $H_1 : \mu < 8000$

- Statistic: $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$, is a standard variable
- Observed value: $z = \frac{7750-8000}{369/\sqrt{6}} = -1.659$

Problem 3

P-value for lower tail:

P-value$= F(Z) = P(Z < -1.659) =$ `pnorm(-1.659)`

$= 0.004 < \alpha = 0.05$

Since: $P < \alpha$ we reject the null hypothesis

Problem 3

c.

- if $\mu_1 = 7700$ what is type II error or false negative probability?

$\beta = P(accept|H_0 : false)$

We accept H_0 when the observed value z falls in the acceptance region, at $\alpha = 0.05$ that is

$z > z_c =$ `qnorm(0.05)` $= -1.644$

For this critical z_c, the critical $\bar{x}_c$ is

$z_c = \frac{\bar{x}_c-8000}{369/\sqrt{6}} = -1.644$ then $\bar{x}_c = 7752.19$

Problem 3

For accepting the null hypothesis then we need to observe an average that is greater than $\bar{x} = 7752.19$

What is the probability that if $\mu = 7700$, we accept H_0, that is:

$\beta = P(accept|H_0 : false) = P(\bar{X} > 7752.19|\mu = 7700)$

$P(\bar{X} > 7752.19|\mu = 7700) = P(Z > \frac{7752.19-7700}{369/\sqrt{6}})$

$= 1 - \phi(0.35) =$ `1-pnorm(0.35)` $= 0.36316$

Problem 4

- $\mu_0 = 14$
- $\sigma = 4.8$

- $n = 26$
- $\bar{x} = 12.5$
- $s = 2.7$
- Test for the mean where we do not know σ

a.

Lower tail: $H_0 : \mu \geq 14$, $H_1 : \mu < 14$

reject H_0 when H_0 is true is a **false positive** or type I error.

Problem 4

false positive probability:

$\alpha = P(reject|H_0 : true)$

The probability is what we leave out from the rejection zone when H_0 is true.

Problem 4

consider:

- $\alpha = 0.01$ test the hypothesis
- Test for the mean where we do not know σ
- Statistic: $T = \frac{\bar{X}-\mu_0}{s/\sqrt{n}}$, t-distribution with $n-1$ degrees of freedom
- Observed value: $t = \frac{12.5-14}{2.7/\sqrt{26}} = -2.832$

Problem 4

P-value for lower tail:

P-value$= F(T) = P(T < -2.832) =$ `pt(-2.832, 25)`

$= 0.0045 < \alpha = 0.001$

Since: $P < \alpha$ we reject the null hypothesis

Problem 4

c. consider:

- $\sigma = 4.8$
- $\alpha = 0.05$

Test the same hypothesis.

- Test for the mean where we do not know σ

Lower tail: $H_0 : \mu \geq 14$, $H_1 : \mu < 14$

- Statistic: $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$, is a standard variable

- Observed value: $z = \frac{12.5-14}{4.8/\sqrt{26}} = -1.593$

Problem 4

P-value for lower tail:

P-value$= F(Z) = P(Z < -1.593) =$ `pnorm(-1.593)`

$= 0.0555 > \alpha = 0.05$

Since: $P > \alpha$ we **do not** reject the null hypothesis

Problem 5

consider:

- $\bar{x} = 10$
- $s = 1.5$
- $n = 41$

a. What is the CI at 87.886% confidence $\alpha = 1 - 0.87886 = 0.12114$

while we do not **know** σ^2 n is big and then we can use the standard distribution:

$(l, u) = (\bar{x} - z_{\alpha/2}\sigma/\sqrt{n}, \bar{x} + z_{\alpha/2}\sigma/\sqrt{n})$

Problem 5

$(l, u) = (\bar{x} - z_{\alpha/2}\sigma/\sqrt{n}, \bar{x} + z_{\alpha/2}\sigma/\sqrt{n})$

where $z_{\alpha/2} =$ `qnorm(1-0.12114/2)` $= 1.55$

then

$(l, u) = (10 - 1.55 * 1.5/\sqrt{41}, 10 + 1.55 * 1.5/\sqrt{41}) = (9.63, 10.36)$

Problem 5

b. consider:

- $\mu_0 = 10.5$
- $\alpha = 0.05$

Test the hypothesis.

Lower tail: $H_0 : \mu \geq 10.5$, $H_1 : \mu < 10.5$

- Statistic: $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$, is a standard variable
- Observed value: $z = \frac{10-10.5}{1.5/\sqrt{41}} = -2.134$

Problem 5

P-value for lower tail:

P-value$= F(Z) = P(Z < -2.134) =$ `pnorm(-2.134)`

$= 0.016 < \alpha = 0.05$

Since: $P < \alpha$ we reject the null hypothesis

Problem 6

consider:

- measurements: 515, 464, 558, 491
- $\sigma^2 = 10000$
- $\alpha = 0.1$

Lower tail: $H_0 : \sigma^2 \geq 10000$, $H_1 : \sigma^2 < 10000$

- the sample variance is

$s^2 =$ `sd(c(515, 464, 558, 491))^2` $= 1590$

Problem 6

- Statistic: $W = \frac{(n-1)S^2}{\sigma_0^2}$, is a random variable that follows a χ^2 with $n-1$ degrees of freedom.
- Observed value: $w_{obs} = \frac{(4-1)1590^2}{10000} = 0.477$

Problem 6

P-value for lower tail:

P-value$= F(W) = P(W < 0.477)$

$=$ `pchisq(0.477, 3)` $= 0.076 < \alpha = 0.1$

Since: $P < \alpha$ we reject the null hypothesis

Problem 6

consider

- $X \hookrightarrow N(\mu, \sigma^2)$
- $\mu = 500$
- $\sigma = 50$

b. what is n such that $P(|\bar{X} - \mu| < 10) = 0.95$?

$P(-10 < \bar{X} - \mu < 10) = 0.95$

$P(\frac{-10}{\sigma/\sqrt{n}} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \frac{10}{\sigma/\sqrt{n}}) = 0.95$

Problem 6

$\Phi(\frac{10}{\sigma/\sqrt{n}}) - \Phi(\frac{-10}{\sigma/\sqrt{n}}) = 0.95$

$\Phi(\frac{10}{\sigma/\sqrt{n}}) - (1 - \Phi(\frac{10}{\sigma/\sqrt{n}})) = 0.95$

$\Phi(\frac{10}{\sigma/\sqrt{n}}) = (1 + 0.95)/2 = 0.975$

$\frac{10}{\sigma/\sqrt{n}} =$ `qnorm(0.975)` $= 1.959$

$\frac{10}{\sigma/\sqrt{n}} = 1.959$

solving for n then $n_{min} = 96.4$, $n > 97$

Problem 7

consider

- $X \hookrightarrow N(\mu, \sigma^2)$
- $\sigma = 5$
- Upper tail: $H_0 : \mu \leq 80$, $H_1 : \mu > 80$
- $n = 100$
- $\alpha = 0.0505$

compute type II error, false negative:

$\beta = P(accept|H_0 : false)$

We accept H_0 when the observed value z falls in the acceptance region, at $\alpha = 0.0505$

Problem 7

Acceptance region:

$z < z_c =$ `qnorm(1-0.0505)` $= 1.640$

For this critical value z_c we obtain the critical value for $\bar{x}_c$

$z = \frac{\bar{x}_c - 80}{5/\sqrt{100}} = 1.640$ then $\bar{x}_c = (5/10) * 1.64 + 80 = 80.82$

Problem 7

For accepting the null hypothesis H_0 we need to observe an average that is lower than $\bar{x} = 80.82$

What is the probability that if $\mu = 81$, we accept H_0?

that is:

$\beta = P(accept|H_0 : false) = P(\bar{X} < 80.82|\mu = 81)$

$P(\bar{X} < 80.82|\mu = 81) = P(Z < \frac{80.82-80}{5/\sqrt{100}})$

$= \phi(-0.36) =$ `pnorm(-0.36)` $= 0.3594236$

Problem 8

Consider:

- K is a Bernoulli variable $K \rightarrow Bernoully(p)$

- The measurements give

$\bar{k} = \hat{p} = 926/1225 = 0.7559$

Problem 8

a.

upper tail: $H_0 : p \leq p_0 = 0.75$, $H_1 : p > p_0 = 0.75$

- Statistic:

$$Z_{obs} = \frac{\bar{X} - p_0}{\left[\frac{p_0(1-p_0)}{n}\right]^{1/2}}$$

is a standard variable because $np_0, n(1-p_0) > 5$ by the CLT

- Observed value: $z = \frac{0.7559-0.75}{\sqrt{0.75(1-0.75)}/\sqrt{1225}} = 0.47$

Problem 8

P-value for upper tail:

P-value$= 1 - F(Z) = 1 - P(Z < 0.47) = 1 - \Phi(0.47)$

$=$ `1-pnorm(0.47)` $= 0.31$

only with a level of significance of 0.31 we can reject the null hypothesis, and conclude that the method produces the results within the tolerance limits.