Introduction to statistics

author: Alejandro Cáceres date:
autosize: true

Barcelona East School of Engineering
Universitat Politècnica de Catalunya (UPC)

Problems session 4 (Tema 5 part 2)

Resumen

Modelo	outcome	x	f(x)	E(X)	V(X)	R
Uniforme discreto	resultados equiprobables	a,… b	$\frac{1}{n}$	$\frac{b+a}{2}$	$\frac{(b-a+1)^2-1}{12}$	`rep(1/n, n)`
Bernoulli	evento A	0,1	$(1-p)^{1-x}p^x$	p	$p(1-p)$	`c(1-p,p)`
Binomial	# de eventos A en n repeticiones de Bernoulli	0,1,…	$\binom{n}{x}(1-p)^{n-x}p^x$	np	$np(1-p)$	`dbimon(x,n,p)`
Geometrico de eventos	# de eventos B en repeticiones de Bernoulli antes de evento A	0,1,…	$(1-p)^x p$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	`dgeom(x,p)`
Geometrico de ensayos	# de ensayos B en repeticiones de Bernoulli antes de evento A	1,…	$(1-p)^{x-1}p$	$\frac{1}{p}$	$\frac{1}{p^2}$	`dgeom(x-1,p)`
Binomial Negativo de eventos	# de eventos B en repeticiones de Bernoulli hasta r eventos A	0,1,..	$\binom{x+r-1}{x}(1-p)^x p^r$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	`dnbinom(x,r,p)`
Binomial Negativo de ensayos	# de ensayos en repeticiones de Bernoulli hasta r eventos A	r,r+1,..	$\binom{x-1}{r-1}(1-p)^{x-r}p^r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	`dnbinom(x-r,r,p)`

Modelo	outcome	x	f(x)	E(X)	V(X)	R
Hypergeometrico	# de eventos A en una muestra n de población N con K eventos A	$\max(0, n+K-N)$, $\ldots \min(K, n)$	$\frac{1}{\binom{N}{n}}\binom{K}{x}\binom{N-K}{n-x}$	$\frac{nN}{K}$	$\frac{nN}{K}(1-\frac{N}{K})\frac{N-n}{N-1}$	`dhyper(x, K, N-K, n)`
Poisson	# de eventos en un intervalo	0,1, ..	$\frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ	`dpoiss(x, lambda)`
Exponencial	Intervalo entre dos eventos A	$[0, \infty)$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	`dexp(x, lambda)`
Normal	medidas con errores simétricos y con valor mas probable en la media	$(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2	`dnorm(x, mu, sigma)`
Uniforme continuo	resultados equiprobables	(a, b)	$\frac{1}{b-a}$	$\frac{b+a}{2}$	$\frac{(b-a)^2-1}{12}$	`dunif(x, a, b)`

Problem 1

Consider:

- random variables: X particles/minute, Y particles/0.5minutes
- $P(X > 0)$=0.996

a. What is $P(Y < 2)$?

Poisson distribution: $P(X = k) = f(k; \lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$

Problem 1

- We find λ for one minute

$$P(X > 0) = 1 - P(X = 0) = 1 - f(0, \lambda)$$

$$1 - e^{-\lambda} = 0.996$$

then

$$\lambda = -\log(0.004) = 5.52$$

- We find λ in an interval of $0.5m$

$$\lambda_{0.5m} = 5.52 * 0.5 = 2.76$$

- We then compute

$$P(Y < 2) = P(Y \leq 1) = F_{pois}(1; \lambda_{0.5}) = f(0; \lambda_{0.5}) + f(1; \lambda_{0.5})$$

$$= e^{-\lambda_{0.5}} + e^{-\lambda_{0.5}}\lambda_{0.5} = 0.23$$

in R:

$F_{pois}(1; \lambda_{0.5}) =$ `ppois(1, 2.76)`

Problem 1

b. find $q_{0.25}$ such that

$$P(Y \leq q_{0.25}) = F_{pois}(q_{0.25}) = 0.25$$

- From the previous result we know $F_{pois}(1; \lambda_{0.5}) = 0.23$. We are almost there.
- We compute

$F_{pois}(2; \lambda_{0.5}) = \sum_{i=0,1,2} f(i; \lambda_{0.5}) = F_{pois}(1; \lambda_{0.5}) + f(2; \lambda_{0.5})$

$= 0.23 + \frac{e^{-\lambda_{0.5}} \lambda_{0.5}^2}{2} = 0.47$

then $q_{0.25} \in (1, 2)$

in R

from $F_{pois}(q_{0.25}) = 0.25$ we have $q_{0.25} = F_{pois}^{-1}(0.25) =$ `qpois(0.25, 2.76)` $= 2$

Problem 1

c. consider:

- $p = 0.2$ probability of being radioactive particles
- $n = 5$ number of particles selected
- X number of radioactive particles observed

compute: the probability that the majority of particles are not radioactive. That is the minority of particles are radioactive: $P(X \leq 2)$

Problem 1

Binomial distribution:

$P(X = k) = f(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$ or $X \hookrightarrow Bin(n, p)$

$P(X \leq 2) = F_{bin}(2) = f(0; n, p) + f(1; n, p) + f(2; n, p)$

$= \binom{5}{0}(1-p)^5 + \binom{5}{1} p (1-p)^4 + \binom{5}{2} p^2 (1-p)^3 = 0.94$

d. The expected value of radioactive particles is the mean $E(X) = n * 0.2 = 1$. That is, we expect to find 1 radioactive particle when we select 5 particles.

Problem 2

Consider:

- X number of no defective books
- $p = 0.98$ no defective, $q = 1 - p = 0.02$ defective books
- $n = 50$ books are selected

a. Compute $P(X \geq 3)$
$P(X \geq 3) = 1 - P(X < 3)$

Problem 2

Binomial distribution:

$P(X = k) = f(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$ or $X \hookrightarrow Bin(n, p)$

$P(X \geq 3) = 1 - P(X < 3) = 1 - F_{bin}(2; n, p)$

$= 1 - f(0; n, p) - f(1; n, p) - f(2; n, p)$

$= 1 - \binom{50}{0}(1-p)^{50} - \binom{50}{1} p (1-p)^{49} - \binom{50}{2} p^2 (1-p)^{48} \sim 1$

b. Y number of defective books

$P(Y = 3) = f(k; n, q) = \binom{50}{3} q^3 (1-q)^{47}$

$= 0.0606$

- The average (expected) number of defective books in 50 is $E(Y) = np = 50 * 0.02 = 1$
- Think of the 50 books as a “continuous” interval. Then the average nuber of books in the interval is

$\lambda = E(Y) = 1$, but now since we are considering the 50 books as an interval then $Y \hookrightarrow Poiss(\lambda)$

$P(Y=3) = f_{poiss}(3;\lambda) = \texttt{dpois(3,1)} = 0.061$

Considering 50 books as a “continuous” interval is not a bad approximation.

Problem 3

- 6 buses arrive every hour. Then $\lambda = 6$
- number of buses $X \hookrightarrow Pois(\lambda)$

a. compute $P(T > 1/3)$

Exponential density distribution for the time

$f(t;\lambda) = \lambda e^{-\lambda t}$

Problem 3

$P(T > 1/3) = 1 - P(T \leq 1/3) = 1 - F_{exp}(1/3) = 1 - \int_0^{1/3} \lambda e^{-\lambda t} dt$

$= 1 - (-e^{-\lambda t})\Big|_0^{1/3} == e^{-2} = 0.135$

in R

$1 - F_{exp}(1/3) = \texttt{1-pexp(1/3,6)}$

b. compute: $P(T < 1/3|T > 1/6)$

$P(T < 1/3|T > 1/6) = \frac{P(T<1/3 \cap T>1/6)}{P(T>1/6)} = \frac{P(1/6<T<1/3)}{P(T>1/6)}$

$= \frac{F_{exp}(1/3) - F_{exp}(1/6)}{1-F_{exp}(1/6)}$

now; $F_{exp}(x) = 1 - e^{-\lambda x} = 1 - e^{-6x}$

$P(T < 1/3|T > 1/6) = \frac{1-e^{-2}-1+e^{-1}}{1-(1-e^{-1})} = 0.63$

in R

```
(pexp(1/3,6) - pexp(1/6,6))/(1- pexp(1/6,6))
```

Problem 4

consider:

- probability $p = 0.51$ of having a boy, probability $q = 0.49$ of having a girl
- $n = 4$ a family
- X number of boys in a family, Y number of girls in a family

a. compute $P(X = 1) + P(Y = 1)$

- Binomial distribution for boys $X \hookrightarrow Bin(n, p)$
- Binomial distribution for girls $Y \hookrightarrow Bin(n, q)$

Problem 4

- $P(X = 1) = f(k; 4, 0.51) = \binom{4}{1} 0.51^1 (1 - 0.51)^3 = 0.24$
- and $P(Y = 1) = f(k; 4, 0.49) = \binom{4}{1} 0.49^1 (1 - 0.49)^3 = 0.26$
- then $P(X = 1) + P(Y = 1) = 0.5$

Problem 4

b. Compute: $P(X \geq 2)$

$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1)$

$= 1 - F_{binom}(1; n, p)$

$= 1 - f(0; n = 4, p = 0.51) - f(1; n = 4, p = 0.51)$

$= 1 - \binom{4}{0}(1-0.51)^4 - \binom{4}{1}0.51^1(1-0.51)^3 = 0.07023$

Problem 4

c. compute n such that $P(Y \geq 1) = 0.75$

$P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(Y = 0)$
$= 1 - f(0; n, q) = 0.75$

Therefore we have to solve

$1 - f(0; n, q) = 0.75$

where f is the binomial function $f(k; n, q) = \binom{n}{k} q^k (1-q)^{n-k}$

$1 - \binom{n}{0} 0.49^0 * (1-0.49)^n = 0.75$

$0.51^n = 0.25$

Problem 4

- $n = \frac{\log(0.25)}{\log(0.51)} = 2.05$
- or if $P(Y \geq 1) > 0.75$ then $n > 2.05$
- The minimum integer that satisfies the condition is $n = 3$.

Problem 5

- Number of people taking sick days $X \hookrightarrow Pois(\lambda)$

$P(X = k) = f(k; \lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$

- $P(X = 1) = \frac{1}{2} P(X = 0)$

a. compute: $E(X)$

Problem 5

- first we find λ

$P(X = 1) = \frac{1}{2} P(X = 0)$

$f(1; \lambda) = \frac{1}{2} f(0; \lambda)$

$e^{-\lambda}\lambda = \frac{1}{2} e^{-\lambda}$

then $\lambda = \frac{1}{2}$

For a Poisson distribution $E(X) = \lambda = 0.5$

Problem 5

b. Probability that in two consecutive days two people are taking sick days and the following day another two take a sick day.

They are independent events then:

$P(X = 2) * P(X = 2) = f(2; \lambda)^2 = [\frac{e^{-0.5}0.5^2}{2!}]^2 = 0.0057$

Problem 5

c. compute $P(Y \leq 2)$ where Y is the number of people taking sick days in a period of 3 days, and expected value of number of people taking sick days within 3 days is

$\lambda_{3d} = 3\lambda_{1d} = 3/2$

$P(Y \leq 2) = F_{pois}(2) = f(0; \lambda_{3d}) + f(1; \lambda_{3d}) + f(2; \lambda_{3d})$

$= e^{-3/2} + e^{-3/2}3/2 + \frac{e^{-3/2}(3/2)^2}{2} = 0.808$

In R: `ppois(2, 3/2)`

Problem 6

- $f(x) = \lambda^{-\lambda x}$, where $\lambda = 0.01386$

a. Compute the median of $f(x)$

$F_{exp}(q_{0.5}) = 1 - e^{-\lambda q_{0.5}} = 0.5$

then

$q_{0.5} = F_{exp}^{-1}(0.5)$

$= -\frac{\log(0.5)}{\lambda} = 50.01$

```
qexp(0.5, 0.01386)
```

Problem 7

Consider:

- $P(C|A) = 0.8$, 80% of treated patients with A recover
- $P(C|B) = 0.7$, 70% of treated patients with B recover
- $P(A) = 0.4$, 40% are treated with A
- $P(B) = 0.6$, 60% are treated with B

a. what is $P(A|C)$?
$P(A|C) = \frac{P(C|A)P(A)}{P(C|A)P(A)+P(C|B)P(B)} = 0.43$

Problem 7

b. now consider:

- $n = 5$
- $p = P(C)$ probability that a patients recovers

$P(C) = P(C|A)P(A) + P(C|B)P(B) = 0.74$

Compute: $P(X \geq 3)$

$X \hookrightarrow Bin(n, p)$

Problem 7

$P(X = k) = f(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$

$P(X \geq 3) = 1 - P(X < 3) = 1 - P(X \leq 2)$
$= 1 - F_{binom}(2; n, p)$
$= 1 - \binom{5}{0} p^0 (1-p)^5 - \binom{5}{1} p (1-p)^4 - \binom{5}{2} p^2 (1-p)^3 = 0.885$

for $p = 0.74$

Problem 8

consider:

- $p = 0.1$ probability of error

a. Compute $P(X = 15)$ where X is the number of bits received with no error before the first error.

Geometric distribution:

$f(X = k) = (1-p)^k p$, that is $X \hookrightarrow Geom(p)$
$P(X = 15) = (1-p)^{15} p = 0.02$

Problem 8

b. now consider

- $n = 50$
- X number of errors in transmitting of 50 bits

compute: $P(X \leq 3)$

now, $X \hookrightarrow Bin(n, p)$

Then

$= \binom{50}{0}(1-p)^{50} + \binom{50}{1}p(1-p)^{49} + \binom{50}{2}p^2(1-p)^{48}$

$+\binom{50}{3}p^3(1-p)^{47}$

$= 0.25$

Problem 9

- The numnber of constumers that arrive to a cashier every 15min is $X \hookrightarrow Pois(\lambda)$ with $E(X) = 5$, then $\lambda_{15mim} = E(X) = 5$

a. compute: $P(T > 3m)$;

first we compute λ for an interval of a minute m:

$\lambda_{1m} = \frac{1}{15}\lambda_{15m} = \frac{1}{3}$

T follows an exponential model:

$f(t; \lambda_{1m}) = \lambda_{1m}e^{-\lambda_{1m}t}$

$P(T > 3m) = 1 - P(T \leq 3m) = 1 - F_{exp}(3; \lambda_{1m})$

Remember that: $F_{exp}(x) = 1 - e^{-\lambda x}$

$P(T > 3m) = 1 - (1 - e^{-1}) = 0.36$

in R:

```
1 - pexp(3, 1/3)
```

Problem 9

$P(T < 6|T > 3) = \frac{P(3<T<6)}{P(T\geq 3)}$

$= \frac{F_{exp}(6)-F_{exp}(3)}{1-F_{exp}(3)}$

$= \frac{-e^{-6/3}+e^{-3/3}}{e^{-3/3}} = 1 - e^{-1} = 0.63$

in R:

```
(pexp(6, 1/3) - pexp(3, 1/3))/(1-pexp(3, 1/3))
```

Problem 10

Consider:

- $p = 0.6$ for the probability of a component to function.
- $n = 4$
- when $X \geq 2$ the satelite functions.

the number of components that function

$P(X = k) = f(k; n, p) = \binom{n}{k}p^k(1-p)^{n-k}$ or $X \hookrightarrow Bin(n, p)$

Problem 10

a. compute $P(X \geq 2)$

$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1)$

$= 1 - F_{binom}(1; n, p) = 1 - f(0; n, p) - f(1; n, p)$

$= 1 - \binom{4}{0}(1-0.6)^4 - \binom{4}{1}0.6(1-0.6)^3 = 0.82$

Note: in R you can confirm the answer with

```
1-pbinom(1,size=4,prob=0.6)
```

or

```
1-dbinom(0,size=4,prob=0.6)-dbinom(1,size=4,prob=0.6)
```

Problem 10

b. compute $\frac{P(X=k+1)}{P(X=k)}$

remember $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

then $\frac{f(k+1;n,p)}{f(k;n,p)} = \frac{\binom{n}{k+1}}{\binom{n}{k}} \frac{p}{1-p}$

$= \frac{\frac{4!}{(k+1)!(4-k-1)!}}{\frac{4!}{k!(4-k)!}} * 3/2 = \frac{k!(4-k)!}{(k+1)!(4-k-1)!} * 3/2 = \frac{k!(4-k-1)!(4-k)}{k!(k+1)(4-k-1)!} * 3/2$

$= \frac{4-k}{k+1} * 3/2$

Problem 11

- The number of earthquakes in 100 years $X \hookrightarrow Pois(\lambda = 2.1)$

a. What is the probability of an earthquake in a region occurs within the next 25 years if the region has not experienced an earthquake for 10 years.

- compute $P(T \leq 0.25|T > 0.1)$

where T is an exponential variable $f(t;\lambda) = \lambda e^{-\lambda t}$, and $F_{exp}(x) = 1 - e^{-\lambda t}|_0^x = 1 - e^{-\lambda x}$

$P(T \leq 0.25|T > 0.1) = \frac{P(0.1 < T \leq 0.25)}{P(T > 0.1)}$

$= \frac{F_{exp}(0.25) - F_{exp}(0.1)}{1 - F_{exp}(0.1)}$

$= \frac{-e^{-0.25*2.1} + e^{-0.1*2.1}}{e^{-0.1*2.1}} = 1 - e^{-2.1(0.25-0.1)} = 0.2702$

Problem 12

consider:

- $q = 0.4$ probability of a call being answered, $p = 1 - q = 0.6$ probability of a call **not** being answered
- $n = 10$ calls.

number unanswered calls $X \hookrightarrow Bin(n, p)$, then

$P(X = k) = f(k; n, p) = \binom{n}{k}p^k(1-p)^{n-k}$

Problem 12

a. compute $P(X \geq 2)$

$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1) = 1 - F_{binom}(1, n, p)$

$= 1 - f(0; n, p) - f(1; n, p)$

$= 1 - \binom{10}{0}(1-p)^{10} - \binom{10}{1}p(1-p)^9 = 0.998$

in R:

```
1-pbinom(1,size=10,prob=0.6)
```

Problem 13

Consider

- $n = 5$
- Number of components $X \geq 4$ needed for the system to run
- $p = 0.95$

number components that work $X \hookrightarrow Bin(n, p)$, then

$P(X = k) = f(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$

Problem 13

a. what is the probability for the system to run $P(X \geq 4)$

$P(X \geq 4) = 1 - P(X < 4) = 1 - P(X \leq 3) = 1 - F_{binom}(3; n, p)$

$= 1 - f(0; n, p) - f(1; n, p) - f(2; n, p)$

$= 1 - \binom{5}{0}(1-p)^5 - \binom{5}{1} p (1-p)^4 - \binom{5}{2} p^2 (1-p)^3$
$-\binom{5}{3} p^3 (1-p)^2 = 0.9774075$

in R: `1-pbinom(3,size=5,prob=0.95)`

Problem 13

b. consider now:

- $q = P(X \geq 4) = 0.9774075$ the probability that a system works.
- $p = 1 - q = 0.0225925$ the probability that a system does not work.
- Y is the number of systems tested before finding 2 out of order.

Y is a negative binonial variable: $Y \hookrightarrow NB(r = 2, p = 0.0225925)$ for **trials**

Problem 13

$P(Z = k) = f(k; r, p) = \binom{x-1}{r-1} (1-p)^{x-r} p^r$

a. compute $P(Y \geq 4)$

$P(Y \geq 4) = 1 - P(Y < 4) = 1 - P(Y \leq 3)$, $x = r, r+1, \ldots$

$= 1 - F_{NB}(3; r, p) = 1 - f(2; r, p) - f(3; r, p)$

$= 1 - F_{NB}(3; r, p) = 1 - \binom{1}{1}(1-p)^0 p^2 - \binom{2}{1}(1-p)^1 p^1$

$= 1 - 1p^2 - 2(1-p)p^2$

$= 0.9984$

or in R

```
1-dnbinom(2-2, 2, 0.0225925)-dnbinom(3-2, 2, 0.0225925)
```

o

```
1-pnbinom(3-2, 2, 0.0225925)
```

Problem 13

c. compute the expected value of a and b

- $E(X) = np = 4.75$ about 5 components will typically run in a system
- $E(Y) = \frac{2}{p} = 88.52$ about 88 systems need to be tested before finding 2 which do not work.

Problem 14

Consider

- The mean time between two light bolts $\mu_T = 52.8$
- $f(t;\lambda) = e^{-\lambda t}\lambda$, where $E(X) = \frac{1}{\lambda} = \mu_T = 52.8$, $\lambda = 1/52.8$

a. Compute $P(T > 120)$
Remember: $F_{exp}(x) = 1 - e^{-\lambda t}|_0^x = 1 - e^{-\lambda x}$
$P(T > 120) = 1 - P(T \leq 120) = 1 - F_{exp}(x) = -e^{120/52.8} = 0.103$

Problem 14

b. compute $P(T \leq 72 | T > 42)$

$P(T \leq 72 | T > 42) = \frac{P(42 < T \leq 72)}{P(T \geq 42)}$
$= \frac{F_{exp}(72) - F_{exp}(42)}{1 - F_{exp}(42)}$
$= \frac{e^{-42/52.8} - e^{-72/52.8}}{e^{-42/52.8}} = 0.433$

Problem 14

c. compute $P(T \leq M_T) = 0.5$.
$P(T \leq M_T) = F_{exp}(M_T) = 0.5$
$1 - e^{\lambda t} = 0.5$

solving form M_T then
$M_t = -\frac{\log(0.5)}{\lambda} = \log(0.5) * 52.8 = 36.59$

$M_t > \mu_T$

Problem 15

Consider

- $p = 0.3$ probability of a patient having allergy
- $n = 20$ are observed

$P(X = k) = f(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$ or $X \hookrightarrow Bin(n, p)$

a. compute $P(X \geq 3)$

$P(X \geq 3) = 1 - P(X < 3) = 1 - P(X \leq 2) = 1 - F_{binom}(2; n, p)$
$= 1 - f(0; n, p) - f(1; n, p) - f(2; n, p)$
$= 1 - \binom{20}{0}(1-p)^{20} - \binom{20}{1} p (1-p)^{19} - \binom{20}{2} p^2 (1-p)^{18} = 0.964$

in R: `1-pbinom(2,size=20,p=0.3)`

Problem 15

b. now consider

- $K = 5 = 20 * 0.25$ have allergy
- $N = 20$ total number of patients
- $n = 7$ are selected from the 20
- number of patients with allergy $X \hookrightarrow Hyper(N, n, K)$

Problem 15

$f(X; N, n, K) = \frac{\binom{K}{k}}{\binom{N}{n}} \binom{N-K}{n-k}$

Compute $P(x \geq 4)$

$P(x \geq 4) = 1 - P(x < 4) = 1 - P(x \leq 3)$

$= 1 - F_{hyper}(3; N, n, k)$

$= 1 - f(0; N, n, K) - f(1; N, n, K) - f(2; N, n, K) - f(3; N, n, K)$

$= 1 - \frac{\binom{5}{0}}{\binom{20}{7}}\binom{20-5}{7-0} - \frac{\binom{5}{1}}{\binom{20}{7}}\binom{20-5}{7-1} - \frac{\binom{5}{2}}{\binom{20}{7}}\binom{20-5}{7-2} - \frac{\binom{5}{3}}{\binom{20}{7}}\binom{20-5}{7-3}$

$= 0.03069$

```
1-phyper(3,5,15,7)
```

Problem 16

Consider

- X is the lifetime of NNN particle $X \hookrightarrow N(\mu, \sigma^2)$
- $P(X > 42) = 0.9452$
- $P(X > 52) = 0.34458$

Problem 16

a. compute $P(X \leq 48)$
   $P(X \leq 48) = F_{norm}(48; \mu, \sigma^2) = \Phi(\frac{48-\mu}{\sigma})$

Remember: $\Phi(x)$ is the cumulative probability function for the standard distribution that is found in tables.

- we need μ and σ
  i) $P(X > 42) = 0.9452$
  $P(X > 42) = 1 - F_{norm}(42; \mu, \sigma^2) = 1 - \Phi(\frac{42-\mu}{\sigma}) = 0.9452$

then
$\Phi(\frac{42-\mu}{\sigma}) = 1 - 0.9452 = 0.0548$
$\frac{42-\mu}{\sigma} = \Phi^{-1}(0.0548)$

Problem 16

in R

- $\Phi(z)$ is `pnorm(z)`
- $\Phi^{-1}(prob)$ is `qnorm(prob)`
  `qnorm(0.0548)` $= -1.6$

i. $\frac{42-\mu}{\sigma} = -1.6$
   The other equation follows from $P(X > 52) = 0.34458$
   $\frac{52-\mu}{\sigma} = \Phi^{-1}(1 - 0.34458) = \Phi^{-1}(0.65542) =$ `qnorm(0.65542)`

ii. $\frac{52-\mu}{\sigma} = 0.4$

Problem 16

Solving i. and ii. for μ and σ we find $\mu = 50, \sigma = 5$

then

$P(X \leq 48) = \Phi(\frac{48-\mu}{\sigma})$
$= \Phi(\frac{48-50}{5}) =$ `pnorm(-0.4)` $= 0.3445783$

Problem 16

Note on the use of tables:

- Tables do have $\Phi(z)$ only for $z > 0$, or $\Phi^{-1}(p)$, for $p > 0.5$
- We know that $\Phi(z)$ is symetric then $\Phi(-z) = 1 - \Phi(z)$, or $\Phi^{-1}(p) = -\Phi^{-1}(1-p)$

- To compute $\Phi^{-1}(0.0548)$ as $p < 0.5$ we look for $\Phi^{-1}(1 - 0.0548) = 0.9452$ and the z we find we will multiply it by -1.

Problem 16

for $\Phi^{-1}(0.9452)$ we look the cell with the probability 0.9452. We find the closest in 0.9450 that corresponds to 1.64 (row: 1.6, column:0.04), We multiply 1.64 by -1 because $p < 0.5$, then $\Phi^{-1}(0.0548) = -1.6$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
.1	.53983	.54379	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57534
.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
.3	.61791	.62172	.62551	.62930	.63307	.63683	.64058	.64431	.64803	.65173
.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68438	.68793
.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
.7	.75803	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78523
.8	.78814	.79103	.79389	.79673	.79954	.80234	.80510	.80785	.81057	.81327
.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83397	.83646	.83891
1.0	.84134	.84375	.84613	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87285	.87493	.87697	.87900	.88100	.88297
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89616	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91308	.91465	.91621	.91773
1.4	.91924	.92073	.92219	.92364	.92506	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95448
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96637	.96711	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670

Problem 16

for $\Phi^{-1}(0.65542)$ we look the cell with the probability 0.65542. We find the cell that correspons to 0.400 (row: 0.4, column:0.00), since $p > 0.5$ that is the result! $\Phi^{-1}(0.0548) = 0.04$

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z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
.1	.53983	.54379	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57534
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.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83397	.83646	.83891
1.0	.84134	.84375	.84613	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87285	.87493	.87697	.87900	.88100	.88297
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89616	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91308	.91465	.91621	.91773
1.4	.91924	.92073	.92219	.92364	.92506	.92647	.92785	.92922	.93056	.93189

Problem 16

b. compute $P(Y < 50 | Y > 48)$ for $f(y; \lambda) = \lambda e^{-\lambda y}$

What is λ?

We have:

$$P(Y > 48) = 1 - P(Y \leq 48) = 1 - F_{exp}(48) = 1 - (1 - e^{\lambda \cdot 48}) = 0.38122$$

solving for λ;$\lambda = -\frac{\log(0.38122)}{48} = 0.02$

then:

$$P(Y < 50 | Y > 48) = \frac{P(Y < 50 \cap Y > 48}{P(Y > 48)} = \frac{P(48 \leq Y \leq 50}{1 - P(Y \leq 48)} = \frac{F_{exp}(50) - F_{exp}(48)}{1 - F_{exp(48)}}$$

Remember that $F_{exp}(x) = 1 - e^{\lambda x}$

in R:

`(pexp(50, 0.02) - pexp(48, 0.02))/(1-pexp(48, 0.02))` $= 0.03921056$

Problem 17

Consider

- the number of railway disruptions $X \hookrightarrow Pois(\lambda_{1m})$, then $P(X=k)=f(k;\lambda)=\frac{\lambda_{1m}^k}{k!}$
- $P(X \geq 0)=P(X=0)$

Compute $P(Y>1)$, where Y is the number of disruptions in 3 months: We need to compute λ_{3m}

Problem 17

Fiding λ_{3m}:

- We first compute λ_{1m}

$P(X>0)=P(X=0)$

then

$1-P(X=0)=P(X=0)$, and solving for $P(X=0)$

we have

$P(X=0)=0.5$

- now let's compute λ_{1m} from the Poisson function:

$P(X=0)=f(0,\lambda_{1m})=e^{-\lambda_{1m}}=0.5$

and then

$\lambda_{1m}=-\log(0.5)=0.693$

- then change the units to $3m$

$\lambda_{3m}=3*\lambda_{1m}=2.079$

Problem 17

Computing the probability $P(Y>1)$:

$P(Y>1)=1-P(Y\leq 1)=1-f(0,\lambda_{3m})-f(1,\lambda_{3m})$

$=1-e^{-\lambda_{3m}}-e^{-\lambda_{3m}}\lambda_{3m}=0.615$

in R

$P(Y>1)=1-P(Y\leq 1)=1-F_{Poiss}(1;\lambda_{3m})$

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1 - ppois(1,2.079)
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Problem 17

b. Consider now

- The probability of a trimester without disruptions $P(Y=0)=e^{-2.079}=0.125$
- $p=0.125$ is the probability of an event A (for no disruptions in three months) in a Bernoulli trial
- X is the number of trimesters with disruptions (B) observed before an event without disruptions appears.

Then $X \hookrightarrow Geom(p)$; $P(X=k)=f(k)=(1-p)^k p$

Compute $P(X=2)$

$P(X=2)=f(2)=(1-0.125)^2*0.125=0.0957$

Problem 18

Count months in variable Y until finding one that has the event A: a month with at most one accident.

a. compute $P(Y=3)$ where Y

then $Y \hookrightarrow Geom(p)$ and $P(X=k)=f(k;p)=(1-p)^k p$

and p is the probability that A occurs. What is p?

$p=P(A)=P(X\leq 1)$: a month with at most one accident, and X is the number of accidents per month.

Problem 18

Consider:

- $\lambda_{1m} = 3$
- the amount of accidents in a month $X \hookrightarrow Pois(\lambda)$ then $P(X = k) = f_{pois}(k; \lambda) = e^{-\lambda k} \frac{\lambda^k}{k!}$

Here we compute the probability of event A: **a month with at most one accident**.

$P(X \leq 1) = F_{pois}(1)$

Remember that:

$F_{pois}(1; \lambda = 3) =$ `ppois(1, 3)`

- $p = P(X \leq 1) = 0.199$ probability of a month with at most one accident.

Problem 18

a. compute $P(Y = 3)$ where Y is the number of months with more than one accident (B) before a month with at most one accident (A).

then $Y \hookrightarrow Geom(p)$ and $P(X = k) = f(k; p) = (1 - p)^k p$

$P(Y = 3) = (1 - p)^3 p =$ `dgeom(3, 0.199)` $= 0.102$

Problem 18

b. now count the number of days in variable W in a year ($n = 360$) when event A happens: Days with no accident.

They ask to compute the $E(W)$.

- If A occurs with probability p then $W \hookrightarrow Bin(n = 360, p)$

Then the answer is:

$E(W) = np$. But what is p?

$p = P(A) = P(Z = 0)$: the probability of no accidents per day, and Z is the number of accidents per day

Problem 18

Computing p

- the amount of accidents in a day $Z \hookrightarrow Pois(\lambda_d)$ then $P(Z = k) = f_{pois}(k; \lambda_d) = e^{-\lambda_d k} \frac{\lambda_d^k}{k!}$
- $\lambda_{1day} =$?

We re-escale λ; $\lambda_{1d} = \lambda_{1m}/30 = 1/10$

$p = P(A) = P(Z = 0) = e^{-\lambda_{day}} = e^{-1/10} =$ `dpois(0, 1/10)` $= 0.904$

Problem 18

Finally

- $P(Z = 0) = p$ for $W \hookrightarrow Bin(n = 360, p)$ then

$E(W) = np = 360 * 0.904 = 325.74$