Lectures on Mathematics

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Preface. These notes are intended for a technical audience. They are written in lecture format, with no claims beyond what is stated. The presentation is concise; proofs and definitions appear only when they can be verified by the author. This draft is incomplete and work-in-progress: statements lacking full rigor may be resolved in later versions. Feel free to reach out to me for any questions, comments, or concerns.

I Foundations: Topology, Analysis, and Linear Dynamics

This section develops the analytic and geometric backbone of differential calculus, vector-valued mappings, linear operators, and their synthesis into matrix dynamics. We begin with an intuitive entry point before advancing to precise definitions and structural theorems.

I.1 Preliminaries and Intuition

Before formalizing differentiability, we recall the conceptual progression from elementary slopes to the multi-dimensional machinery of derivatives.

What is a function? A *function* $f : X \to Y$ is a rule assigning to each $x \in X$ a unique output $f(x) \in Y$. Formally:

$$\forall x \in X,\ \exists!\, y \in Y \text{ such that } y = f(x).$$

This definition can be understood through multiple complementary perspectives:

- **Set–Theoretic:** A function is represented as a *graph* $G_f \subseteq X \times Y$ — the set of all ordered pairs (x, y) such that $y = f(x)$. The defining property is:

  $$\forall x \in X,\ \forall y_1, y_2 \in Y, \quad (x, y_1) \in G_f \ \wedge\ (x, y_2) \in G_f \ \Rightarrow\ y_1 = y_2,$$

  which enforces that each input x has exactly one output. This formulation is foundational in set theory [Cantor, 1895; Zermelo, 1908; Kuratowski and Mostowski, 1976]: a function is entirely determined by its graph and the specification of its domain and codomain, independent of any formula or computational rule. It allows functions to be defined abstractly, including cases where no explicit algebraic expression exists — for example, characteristic functions of arbitrary subsets or functions defined by the axiom of choice.

  From this viewpoint, equality of functions is also purely set-theoretic: $f = g$ if and only if $G_f = G_g$. Operations such as restriction, extension, and composition can be defined directly in terms of set operations on graphs.

  Everyday analogy: think of a library catalog. Each book ID (element of X) corresponds to exactly one book title (element of Y). The catalog itself is nothing more than a list of ordered pairs $(\mathrm{ID}, \mathrm{Title})$ satisfying the uniqueness property for IDs. Just as two catalogs are considered the same if and only if their lists of pairs match exactly, two functions are the same if and only if their graphs coincide.

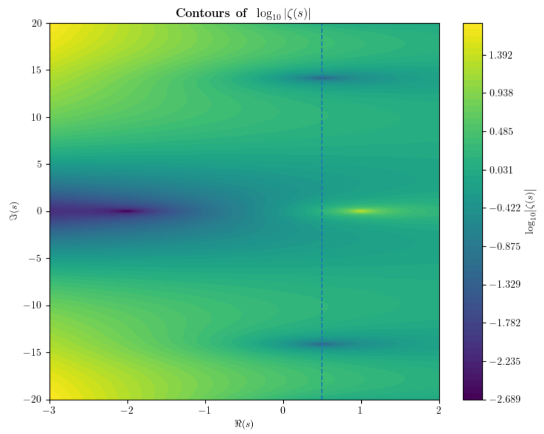
- **Analytic:** A function is presented by an explicit rule $x \mapsto f(x)$, often expressible in closed form, whose structural properties—continuity, differentiability, analyticity—determine the stability and predictability of how outputs respond to variations in input [Cauchy, 1821; Weierstrass, 1861; Rudin, 1976]. The emphasis is on the behavior of f under limiting processes: how small changes in x propagate through the formula to produce changes in $f(x)$, and how these relationships can be quantified by derivatives and series expansions [Taylor, 1715; Hardy, 1908].

  Analogy: a recipe in cooking—ingredients (input) are combined according to a precise, reproducible procedure (the "formula"), producing a dish (output). Smoothness of the recipe corresponds to the absence of abrupt, unpredictable changes in taste or texture when the quantities of ingredients are adjusted slightly.

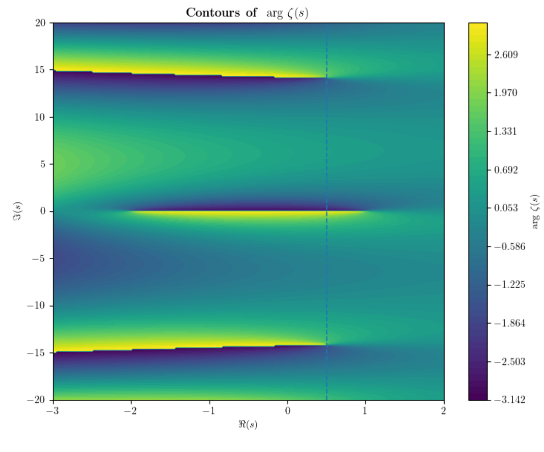
Mathematicians across number theory and mathematical physics have long been captivated by the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

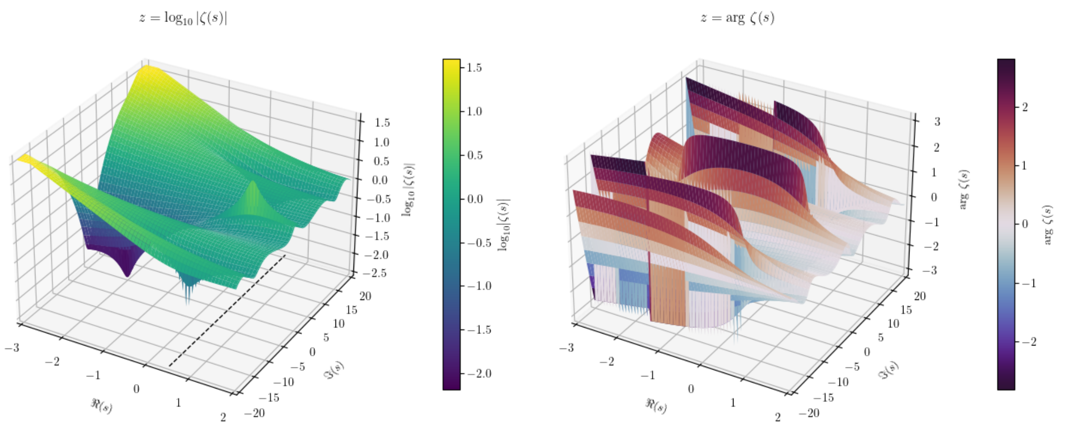
whose analytic continuation and deep connections to the distribution of prime numbers make it one of the most studied objects in mathematics (see Figure 1 and [Edwards, 2001; Conrey, 2003]).



(a) Contours of $\log_{10}|\zeta(s)|$ over the complex s-plane.



(b) Contours of $\arg\zeta(s)$ over the complex s-plane.



(c) 3D surface plots of $\log_{10}|\zeta(s)|$ (left) and $\arg\zeta(s)$ (right) showing magnitude and phase variation in $\mathbb{C}$, with the critical line $\mathrm{Re}(s) = \frac{1}{2}$ indicated.

Figure 1: Composite visualization of the analytically continued Riemann zeta function $\zeta(s)$ in the region $-3 \leq \mathrm{Re}(s) \leq 2$ and $-20 \leq \mathrm{Im}(s) \leq 20$. **(a)** Magnitude contours ($\log_{10}|\zeta(s)|$) reveal poles, zero patterns, and the prominence of the critical strip $0 \leq \mathrm{Re}(s) \leq 1$. **(b)** Phase contours ($\arg\zeta(s)$) indicate complex argument winding and branch structure. **(c)** 3D representations of magnitude and phase illustrate the interplay between oscillations along the imaginary axis and singular features, with the critical line marked for reference. These views collectively highlight the distribution of nontrivial zeros and the complex analytic structure of $\zeta(s)$.

Its structural rigidity is encoded in the functional equation

$$\pi^{-s/2}\,\Gamma\!\left(\frac{s}{2}\right)\,\zeta(s) = \pi^{-(1-s)/2}\,\Gamma\!\left(\frac{1-s}{2}\right)\,\zeta(1-s),$$

which reflects values across the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Let

$$Z := \{\, \rho \in \mathbb{C} \mid \zeta(\rho) = 0,\ 0 < \mathrm{Re}(\rho) < 1 \,\}$$

be the set of nontrivial zeros. The *Riemann Hypothesis* asserts that

$$\forall \rho \in Z, \quad \mathrm{Re}(\rho) = \frac{1}{2}.$$

The functional equation implies Z is symmetric under both $\rho \mapsto 1 - \rho$ and complex conjugation; the hypothesis demands that this symmetry fix each element of Z with respect to its real part.

The deep link between Z and the distribution of prime numbers is made precise by the *explicit formula*, first sketched by Riemann [Riemann, 1859] and later proved in rigorous form by von Mangoldt [von Mangoldt, 1895]. This formula expresses the prime counting function $\pi(x)$ in terms of the zeros of $\zeta(s)$, showing that the imaginary parts of elements of Z drive oscillations in $\pi(x)$, while bounds on $\mathrm{Re}(\rho)$ control the size of the error term in the prime number theorem.

- **Geometric:** If X and Y are smooth manifolds, a function $f : X \to Y$ is a smooth map that induces a *natural* transformation between the *tangent bundle functor* T on the category of smooth manifolds [cf. Ehresmann, 1951; Steenrod, 1951; Kobayashi and Nomizu, 1963]:

$$Tf : TX \longrightarrow TY, \quad p \mapsto df_p,$$

where each fiber map $df_p : T_pX \to T_{f(p)}Y$ is linear. Here "natural" means that the assignment $f \mapsto df$ is *functorial*, respecting both composition and identities [Lane, 1948]:

$$d(g \circ f)_p = dg_{f(p)} \circ df_p, \quad d(\mathrm{id}_X)_p = \mathrm{id}_{T_pX}\,.$$

The differential arises as the canonical linearization of f at each point, defined independently of coordinates and intrinsically compatible with the smooth structures on X and Y.

From the linear algebraic perspective, each df_p is an $\mathbb{R}$–linear map between finite–dimensional tangent spaces,

$$df_p \in \mathrm{Hom}_{\mathbb{R}}(T_pX, T_{f(p)}Y),$$

where $\mathrm{Hom}_{\mathbb{R}}(V, W)$ denotes the vector space of all $\mathbb{R}$–linear maps $L : V \to W$, equipped with pointwise addition, scalar multiplication, and of dimension $(\dim V)(\dim W)$. Choosing bases in T_pX and $T_{f(p)}Y$ identifies df_p with a unique Jacobian matrix [Jacobi, 1841], but the construction itself is coordinate–free, determined solely by the smooth structure.

Thus, the geometric viewpoint packages f as a smooth field of linear maps between tangent spaces — simultaneously coordinate-free (via the natural transformation property) and computationally explicit (via linear algebra once a basis is chosen).

- **Categorical:** In category theory, a function is a morphism $f : X \to Y$ in the category **Set** [Eilenberg and Lane, 1945; Lane, 1971]. The defining data are: a source object X, a target object Y, and the morphism f itself, subject to the axioms

$$g \circ (h \circ f) = (g \circ h) \circ f, \quad \mathrm{id}_X \circ f = f = f \circ \mathrm{id}_Y\,.$$

This abstraction removes the need to reference individual elements, focusing instead on how f composes with other morphisms and interacts with identity arrows.

The categorical viewpoint generalises immediately: smooth maps between manifolds form the category **Man**, linear maps between vector spaces form **Vect**, measurable maps between measure spaces form **Meas**, and so on [cf. Lawvere, 1963].

Analogy: in a business workflow, each department is a "set" and each transfer of information or product between departments is a "morphism." The structure of the workflow is captured not by the internal details of each department, but by the compositional rules governing how processes link together across the entire organization.

- **Computational:** In programming, a function is an algorithm or procedure that, given an input $x \in X$, deterministically produces an output $y \in Y$. In formal terms, it is a computable element of $\mathrm{Hom}(X, Y)$ with an associated implementation that can be analyzed for correctness, termination, and computational complexity [Turing, 1936]. Algorithmic functions may be *total* (defined for all $x \in X$) or *partial* (undefined for some inputs), and their realizations can vary widely while producing the same mathematical mapping [cf. Church, 1936; Kleene, 1952]. Analogy: an assembly line in a factory — raw materials (inputs) enter at one end, the process is deterministic and repeatable, and the final product (output) emerges at the other end. Efficiency, throughput, and resource usage are as relevant here as the mapping itself [Knuth, 1968].

- **Probabilistic/Stochastic:** A stochastic function is modeled as a measurable map

$$f : \Omega \times T \to S,$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space of outcomes, T is an index set (often time), and S is the state space [Kolmogorov, 1933]. For each $\omega \in \Omega$, the mapping $t \mapsto f(\omega, t)$ is deterministic; the randomness lies in the selection of ω. This framework covers stochastic processes such as Brownian motion $B_t(\omega)$, where the sample path is deterministic once ω is fixed, but ω itself is drawn according to $\mathbb{P}$ [Wiener, 1923; Doob, 1953]. Analogy: the daily commute time from home to work — for a fixed route (t), the actual travel time depends on unpredictable factors like traffic conditions (ω); yet for each specific traffic pattern, the timing is completely determined.

In modern stochastic analysis, functions of this kind serve as the state variables in stochastic differential equations (SDEs) of the Itô form

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dB_t,$$

where μ is the drift coefficient and σ the diffusion coefficient [Itô, 1944; Øksendal, 2003]. These SDEs extend deterministic dynamics by introducing a martingale noise term, making them central to fields ranging from quantitative finance to statistical physics. In quantum theory, stochastic evolution can be formulated via quantum stochastic differential equations (QSDEs), where noise is modeled by operator-valued processes such as quantum Brownian motion [Hudson and Parthasarathy, 1984; Meyer, 1993]. Here, the interplay of probability, functional analysis, and operator algebras provides a rigorous framework for noncommutative dynamical systems.

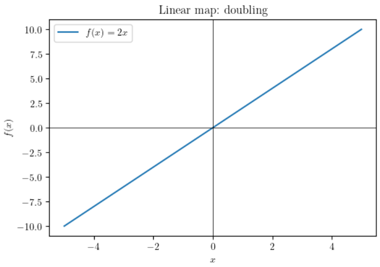
Interconnections.

- The *set-theoretic* graph underlies every other interpretation — whether we're writing formulas, drawing curves, or programming algorithms, we are working with subsets of $X \times Y$ obeying the functional property.

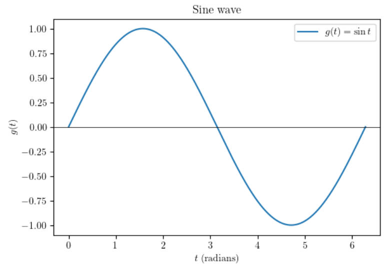
- The *analytic* and *geometric* views are linked through calculus: analytic formulas often have geometric interpretations in terms of curves, surfaces, and tangents.

- The *categorical* view generalises all others by focusing on compositional structure — important when chaining processes together, deterministic or not.

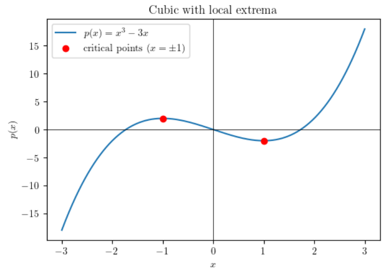
- The *computational* and *probabilistic* views both stress the process aspect: computational functions are deterministic processes, while stochastic ones insert randomness into otherwise deterministic transformations.



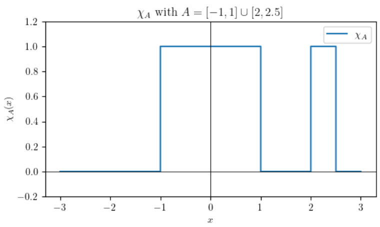
(a) $f(x) = 2x$



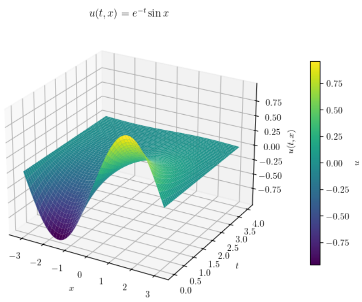
(b) $g(t) = \sin t$



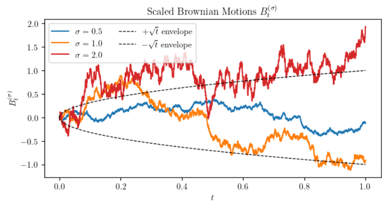
(c) $p(x) = x^3 - 3x$



(d) χ_A



(e) $u(t, x) = e^{-t} \sin x$



(f) Scaled Brownian motions $B_t^{(\sigma)}$

Figure 2: Illustrations of various functions: **(a)** linear map, **(b)** sine wave, **(c)** cubic polynomial, **(d)** characteristic function, **(e)** separable PDE solution, **(f)** Brownian motions with varying volatility.

Examples, visualized in Fig. 2:

- $f(x) = 2x$ doubles every real input.
- $g(t) = \sin t$ returns the y-coordinate of a point moving on the unit circle at time t.
- $h(z) = z^2$ squares each complex number $z \in \mathbb{C}$, mapping circles centered at the origin to circles with squared radii and doubling the argument.
- $p(x) = x^3 - 3x$ is a cubic polynomial on $\mathbb{R}$ exhibiting local maxima and minima, illustrating differentiability with non-monotonic behavior.
- $\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$ the characteristic function of a set $A \subset \mathbb{R}$, relevant in measure theory and integration.
- $u(t, x) = e^{-t} \sin x$ is a smooth function on $\mathbb{R}^2$ appearing as a separable solution to certain linear PDEs.
- $B_t^{(\sigma)}(\omega)$, a family of standard Brownian motions scaled by volatility $\sigma > 0$, assigns to each ω a continuous path $t \mapsto B_t^{(\sigma)}(\omega)$ satisfying $\mathrm{Var}(B_t^{(\sigma)}) = \sigma^2 t$. Different values of σ produce sample paths with proportionally scaled fluctuations, all constrained almost surely within envelopes $\pm\sigma\sqrt{t}$ up to stochastic variation, reflecting the diffusive growth of variance over time.

Cf. [Rudin, 1976; Spivak, 1994; Øksendal, 2003] for further foundational and applied examples across analysis, geometry, and probability.

What is a derivative? The derivative of f at x measures the *instantaneous rate of change* of f with respect to its input, defined as the limit of the average rate of change over shrinking intervals:

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided the limit exists.

If f is differentiable at x, we obtain the *first-order approximation*

$$f(x+h) \approx f(x) + f'(x)h \quad \text{for small } h.$$

This is the local linearization of f.

Tangent line. For $f : \mathbb{R} \to \mathbb{R}$, differentiable at x, the tangent line at $(x, f(x))$ is given by:

$$\ell(h) = f(x) + f'(x)h,$$

the unique line matching f at x to first order.

Geometric picture. From the viewpoint of differential geometry, the graph of a smooth function

$$f : U \subset \mathbb{R}^n \to \mathbb{R}$$

is an n–dimensional hypersurface

$$\Sigma_f := \{\, (x, f(x)) \mid x \in U \,\} \subset \mathbb{R}^{n+1}.$$

At a point $p = (x_0, f(x_0)) \in \Sigma_f$, differentiability of f ensures that, under infinite magnification in a neighborhood of p, the hypersurface becomes indistinguishable from its *tangent hyperplane*

$$T_p\Sigma_f = \{\, (x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y = f(x_0) + Df_{x_0} \cdot (x - x_0) \,\}.$$

Here $Df_{x_0} : \mathbb{R}^n \to \mathbb{R}$ is the derivative of f at x_0, viewed as a covector, and the dot denotes the Euclidean pairing. This is a local linearization: in the C^1–topology, the rescaled hypersurface

$$\lambda \cdot \big(\Sigma_f - p\big), \quad \lambda \to \infty,$$

converges to the vector subspace $T_p\Sigma_f$ in $\mathbb{R}^{n+1}$. Thus, the tangent hyperplane is the unique affine n–plane that best approximates the hypersurface at p to first order, generalizing the "tangent line" picture from curves to arbitrary codimension-one embeddings.

Differentiability. For a smooth map $F : M \to N$ between manifolds, differentiability at $p \in M$ is the existence of a linear map

$$dF_p : T_pM \longrightarrow T_{F(p)}N$$

between tangent spaces, natural with respect to smooth coordinate changes. This is the coordinate-free derivative: the first-order part of F at p.

Differentiability implies *local linearity*: near p, the map F is approximated by dF_p with an error term vanishing faster than the displacement in M. This local linearity allows one to replace the full nonlinear evolution with a linear model on tangent data, valid at small scales and preserving the manifold's intrinsic geometry.

Local linearity enables *causal simulation* in the sense of mathematical physics: once tangent data are propagated by dF_p along an admissible trajectory, the global configuration can be reconstructed by integrating the local model, subject to the governing equations. In field theory, this chain underlies the passage from smooth initial data on a Cauchy surface, through the local field equations, to the prediction of the system's future state. The validity of such simulations rests entirely on the differentiability that guarantees the local linearity of the governing dynamics.

Why derivatives matter. Derivatives provide the formal mechanism for passing from *static* relationships between quantities to *dynamic* laws governing their evolution. Given a mapping $f : X \to Y$ between smooth manifolds, the derivative Df assigns to each $p \in X$ a linear map between tangent spaces:

$$Df_p : T_pX \longrightarrow T_{f(p)}Y,$$

capturing the instantaneous rate of change of f along every allowable direction in T_pX. This abstract construction underlies diverse interpretations:

- *Physics:* Velocity is the derivative of position with respect to time:

  $$v(t) = \frac{d}{dt}\gamma(t), \quad \gamma : \mathbb{R} \to M,$$

  where M is the configuration manifold. Set-theoretically, $\gamma([t, t + \Delta t])$ is a segment of the trajectory; the derivative is the limit of its displacement-to-time ratio. In terms of connections, $v(t)$ is the pushforward of $\partial/\partial t$ under γ, representing the tangent vector to the worldline.

- *Biology:* Growth rate is the derivative of population size with respect to time:

  $$r(t) = \frac{d}{dt}P(t),$$

  where $P : \mathbb{R} \to \mathbb{R}_+$ is the population function. The derivative determines whether P is increasing or decreasing and by how much, enabling dynamic models such as the logistic equation $\dot{P} = r_{\max}P(1-P/K)$. From a dynamical systems viewpoint, $P(t)$ is a coordinate on a 1-dimensional state space, and $r(t)$ is the value of the vector field generating the flow.

- *Economics:* Marginal cost is the derivative of total cost with respect to quantity:

  $$MC(q) = \frac{d}{dq}C(q),$$

  with $C : \mathbb{R}_+ \to \mathbb{R}$ the total cost function. Here the derivative measures the infinitesimal change in cost for an infinitesimal increase in production. As a map between tangent spaces, DC_q sends the basis vector $\partial/\partial q$ in $T_q\mathbb{R}_+$ to a real number representing the marginal rate of change in the cost coordinate. This formalises the link between local linear approximation and decision-making under small perturbations.

- *Quantitative Finance:* Delta is the derivative of an option price with respect to the underlying asset price:

  $$\Delta(S, t) = \frac{\partial}{\partial S}V(S, t),$$

where $V : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ is the option value function, S is the asset price, and t is time to maturity. In probabilistic terms, $V(S, t)$ is the conditional expectation of discounted payoff under a risk–neutral measure; Δ measures the instantaneous sensitivity of this expectation to perturbations in S. Set-theoretically, varying S shifts the level sets $\{V = \text{const}\}$ in the (S, t) domain; Δ quantifies this displacement rate. In geometric terms, Δ is the component of the gradient ∇V along the $\partial/\partial S$ direction in the tangent space of the state manifold $\mathbb{R}_+ \times [0, T]$, and appears as a coefficient in the connection defining self-financing trading strategies.

In all cases, derivatives translate pointwise functional relationships into linear maps on tangent spaces, allowing the geometry of the domain to dictate the form of the dynamical law on the codomain.

Notation and Geometric Emphasis.

- **Leibniz:** $\dfrac{df}{dx}$ emphasises the derivative as a *ratio* of infinitesimal changes. Interpreted formally, dx represents an infinitesimal displacement in the domain, and df is the corresponding infinitesimal change in the codomain. In the context of sets, this suggests a local correspondence between neighborhoods $U \subseteq X$ and their images $f(U) \subseteq Y$ under f, highlighting the derivative as the limit of set-to-set scaling ratios. From the perspective of differential geometry, d is the exterior derivative, and dx is the coordinate 1-form; the expression df/dx is a quotient of covectors along a chosen coordinate direction, implicitly depending on a local trivialization of the tangent bundle.

- **Lagrange:** $f'(x)$ focuses on the derivative as an *operator* acting at a point, abstracted from any specific infinitesimal element. The notation emphasises the function-to-function transformation $f \mapsto f'$ and is closely tied to functional analysis, where derivatives are linear maps between function spaces. Set-theoretically, $f'(x)$ is evaluated pointwise: the derivative is a function $f' : X \to \mathbb{R}$ (or to the appropriate tangent space), assigning to each x a tangent vector. In terms of connections, this is the coordinate expression of the covariant derivative in a fixed chart, where $f'(x)$ denotes the slope along the coordinate line without explicitly referencing the underlying 1-form.

- **Newton:** $\dot{f}$ is specialised for derivatives with respect to time, t. This notation emphasises the evolution of quantities along a 1-parameter family, with t indexing a curve in the configuration space. The set-theoretic viewpoint here is that $\dot{f}$ tracks the image of the trajectory $t \mapsto f(t)$ under the velocity map. In differential-geometric terms, $\dot{f}$ is the derivative along the vector field $\partial/\partial t$, i.e., the pullback of the covariant derivative along a curve, naturally connecting to the theory of connections on the pullback bundle over the parameter domain.

Example I.1 (Quadratic derivative)**.** Let $f(x) = x^2$. Then:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

The slope grows linearly with x, doubling for each unit increase.

Takeaway. The derivative connects algebraic formulas with geometric structure: it is the bridge from analytic definition to spatial intuition.

We now proceed to a rigorous characterization, beginning with the difference quotient and its sequential formulation.

I.2 Topological Foundations in ℝ

The study of limits and continuity rests fundamentally on understanding the structure of sets in $\mathbb{R}$ and the behavior of sequences within them. In this subsection we establish essential set-theoretic and topological notions, followed by convergence criteria and their interplay. We begin with core definitions that motivate the standard topology on $\mathbb{R}$ and connect naturally to more abstract settings such as metric spaces, normed vector spaces, and measure-theoretic structures.

Remark I.2.1. (Sets, foundations, and $\mathbb{N}$)

All of modern mathematics, including real analysis, is most commonly formalised within the framework of *Zermelo–Fraenkel set theory with the Axiom of Choice* (ZFC). In ZFC, the basic objects of study are *sets*: collections of elements, which may themselves be sets. The axioms of ZFC (extensionality, pairing, union, power set, replacement, infinity, separation, foundation, and choice) describe how sets behave and what kinds of sets exist. For example:

- The *Axiom of Extensionality* ensures two sets are equal if and only if they have the same elements.
- The *Axiom of Infinity* guarantees the existence of an inductive set containing the empty set and closed under the successor operation.
- The *Axiom of Choice* allows for selections from arbitrary collections of nonempty sets, a principle with deep consequences in analysis and topology.

From these axioms, one can *construct* the familiar number systems. The natural numbers $\mathbb{N}$ arise as the smallest inductive set given by the Axiom of Infinity; concretely, we can define:

$$0 := \varnothing, \quad 1 := \{0\}, \quad 2 := \{0, 1\}, \quad 3 := \{0, 1, 2\}, \ \ldots$$

and so on, where the *successor* of n is $n \cup \{n\}$.

Independently of set-theoretic construction, it is often desirable to specify the properties of $\mathbb{N}$ axiomatically so that their fundamental structure is not tied to a particular foundational model (e.g., the von Neumann ordinals). This abstraction allows one to work within frameworks that do not assume any specific realization of numbers as sets, ensuring that results depend only on arithmetic properties and not on representational artifacts.

A standard axiomatization is given by *Peano's axioms*, which state:

1. 0 is a natural number.
2. Every natural number has a unique successor.
3. 0 is not the successor of any number.
4. Distinct numbers have distinct successors (successor is injective).
5. (*Induction*) Any subset of $\mathbb{N}$ containing 0 and closed under successor is all of $\mathbb{N}$.

These axioms capture the essential arithmetic structure of the natural numbers without commitment to a specific set-theoretic implementation.

In analysis, $\mathbb{N}$ plays two crucial roles:

- As the *index set* for sequences, e.g. $(x_n)_{n\in\mathbb{N}}$, providing a discrete ordering along which limits are studied.

- As the base upon which $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and further number systems are constructed via completion and extension processes.

While working in $\mathbb{R}$, we rarely need to refer explicitly to the axioms of ZFC or Peano; they remain in the background as the formal scaffolding that ensures all objects—sets, numbers, functions, and topological spaces—are rigorously defined and behave consistently. The notation and definitions given in this subsection can therefore be read on two levels: intuitively, as operations on familiar mathematical objects, and formally, as constructions grounded in the axiomatic foundations of set theory and arithmetic. ♦

Definition I.2.1. Metric Space.

A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}_{\geq 0}$ is a function (the *metric*) satisfying, for all $x, y, z \in X$:

1. $d(x, y) = 0 \iff x = y$ (identity of indiscernibles),
2. $d(x, y) = d(y, x)$ (symmetry),
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The standard metric on $\mathbb{R}$ is

$$d(x, y) \;=\; |x - y|. \tag{I.2.1}$$

Definition I.2.2. Open Ball and Standard Topology.

Given a metric space (X, d), an *open ball* of radius $\varepsilon > 0$ centered at $x \in X$ is

$$B_\varepsilon(x) \;:=\; \{\, y \in X \mid d(x, y) < \varepsilon \,\}. \tag{I.2.2}$$

The *topology* $\mathcal{T}$ induced by d consists of all subsets $U \subseteq X$ such that for every $x \in U$ there exists $\varepsilon > 0$ with

$$B_\varepsilon(x) \subseteq U. \tag{I.2.3}$$

On $(\mathbb{R}, |\cdot|)$ this is called the *standard topology.*

Definition I.2.3. Limit Point and Closure.

Let $A \subseteq X$ in a metric space (X, d). A point $p \in X$ is a *limit point* of A if every open ball $B_\varepsilon(p)$ intersects $A \setminus \{p\}$, equivalently

$$\forall\, \varepsilon > 0, \quad \big(B_\varepsilon(p) \setminus \{p\}\big) \cap A \;\neq\; \varnothing. \tag{I.2.4}$$

The *closure* of A, denoted $\overline{A}$, is the smallest closed set containing A, equivalently

$$\overline{A} \;=\; A \cup A', \tag{I.2.5}$$

where A' denotes the set of limit points of A.

Definition I.2.4. Bounded Set.

A subset $A \subseteq (X, d)$ is *bounded* if there exists $M > 0$ and a point $x_0 \in X$ such that

$$d(x, x_0) \leq M \quad \text{for all } x \in A.$$

In $\mathbb{R}$ with the standard metric (I.2.1), this means A is contained in some closed interval $[a, b]$.

The following definition will prove to be very fruitful in our analysis:

Definition I.2.5. Sequence and Convergence.

Let (X, d) be a metric space, where X is a set of points and $d : X \times X \to \mathbb{R}_{\geq 0}$ is a metric as in Definition I.2.1.

A *sequence* in X is a rule that assigns to each natural number $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ a point of X. Formally, this is a function

$$x_\bullet : \mathbb{N} \longrightarrow X, \quad n \longmapsto x_n,$$

where the dot "$\bullet$" in $x_\bullet$ is a placeholder for the index n before substitution. The values $x_n \in X$ are called the *terms* of the sequence, and the list is written in order as

$$(x_1, x_2, x_3, \dots), \quad \text{or more compactly} \quad (x_n)_{n \in \mathbb{N}}.$$

A sequence (x_n) *converges* to a point $L \in X$ if, as n becomes large, the distance $d(x_n, L)$ between the term x_n and the candidate limit L becomes arbitrarily small and stays small thereafter. Formally:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \ \text{ such that } \ n \geq N \ \Rightarrow \ d(x_n, L) < \varepsilon. \tag{I.2.6}$$

Here:

- ε is any *tolerance* or error margin we are willing to accept.
- N is an index after which all terms of the sequence lie within ε of L.
- The symbol "$\Rightarrow$" reads "implies".

In words, (I.2.6) means: *Given any required closeness $\varepsilon > 0$, there is a point in the sequence after which all subsequent terms are within ε of the limit L.*

We write $\lim\limits_{n \to \infty} x_n = L$ to denote that (x_n) converges to L. If no such L exists, the sequence is said to *diverge*.

Definition I.2.6. Bounded Sequence.

A sequence (x_n) in (X, d) is *bounded* if the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded in the sense of Definition I.2.5. Equivalently, $\exists M > 0$ and $x_0 \in X$ such that

$$d(x_n, x_0) \leq M \quad \text{for all } n \in \mathbb{N}.$$

Definition I.2.7. Subsequence.

Given a sequence $(x_n)_{n \in \mathbb{N}}$ in X, a *subsequence* is a sequence of the form $(x_{n_k})_{k \in \mathbb{N}}$ where (n_k) is a strictly increasing sequence of natural numbers:

$$n_1 < n_2 < n_3 < \cdots$$

The subsequence inherits the ambient metric d from X.

Definition I.2.8. Subsequential Limit.

Let (x_n) be a sequence in a metric space (X, d). A point $L \in X$ is called a *subsequential limit* of (x_n) if there exists a subsequence (x_{n_k}) such that $x_{n_k} \to L$ as in (I.2.6).

Remark I.2.2. (Basic Set Operations)

Let $A, B \subseteq \mathbb{R}$. The four fundamental operations on sets are:

- **Union:**
$$A \cup B := \{\, x \in \mathbb{R} \mid x \in A \text{ or } x \in B \,\}.$$
This collects all elements lying in at least one of the two sets. For example, if $A = [0, 1]$ and $B = [\frac{1}{2}, 2]$, then $A \cup B = [0, 2]$.

- **Intersection:**
$$A \cap B := \{\, x \in \mathbb{R} \mid x \in A \text{ and } x \in B \,\}.$$
This consists of all elements common to both A and B. In the above example, $A \cap B = [\frac{1}{2}, 1]$.

- **Complement:**
$$A^c := \mathbb{R} \setminus A = \{\, x \in \mathbb{R} \mid x \notin A \,\}.$$
The complement is always taken relative to the ambient set, here $\mathbb{R}$. If $A = [0, 1]$, then $A^c = (-\infty, 0) \cup (1, \infty)$.

- **Difference:**
$$A \setminus B := \{\, x \in A \mid x \notin B \,\}.$$
This is the part of A lying outside B. For example, $[0, 1] \setminus [\frac{1}{2}, 2] = [0, \frac{1}{2})$.

These operations preserve the *ambient structure* of $\mathbb{R}$ in the sense that they yield subsets of $\mathbb{R}$ from subsets of $\mathbb{R}$. They satisfy standard algebraic laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A, \quad A \setminus B \neq B \setminus A \text{ in general},$$

and distributive properties such as:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Mastery of these operations grants us the combinatorial tools used to build and manipulate the open sets, closed sets, and other topological structures that underpin analysis. ♦

Remark I.2.3. (What is analysis?)

In its most classical sense, *analysis* is the rigorous study of limits: limits of sequences, limits of functions, and the structures and phenomena that emerge from them. From this single concept flow the central topics of the subject: continuity, differentiability, integration, infinite series, and the convergence of more abstract processes.

The framework we develop here begins in the setting of $(\mathbb{R}, |\cdot|)$, where the topology induced by the standard metric (I.2.1) allows us to make precise statements about closeness and approximation. However, the ideas and techniques we introduce will later extend seamlessly to:

- *Metric spaces*, enabling the study of limits and continuity in spaces far more general than $\mathbb{R}$.

- *Normed vector spaces* and *inner product spaces*, which form the foundation for functional analysis.
- *Measure spaces*, where limits interact with integration, probability, and L^p function spaces.

In the lectures that follow, analysis will serve as both:

1. A *toolbox*: providing precise theorems and techniques for controlling limiting behavior, estimating errors, and ensuring convergence.
2. A *language*: offering a unified vocabulary—via topology, sequences, and functions—for discussing stability, continuity, and structure in diverse areas of mathematics and applied sciences.

The path begins with foundational notions such as open sets, closed sets, and convergence (I.2.6). From there, we build towards more advanced results (e.g. the Bolzano–Weierstrass theorem, completeness of $\mathbb{R}$, and compactness criteria) which will later reappear, often in disguise, when we study manifolds, differential equations, and geometric flows. Thus, while this subsection focuses on the simplest setting of $\mathbb{R}$, the concepts introduced here will be in constant use throughout the course, serving as the structural backbone of our analytic reasoning. ♦

Theorem I.2.1. Characterization of Open and Closed Sets.

A set $U \subseteq \mathbb{R}$ is *open* if for every $x \in U$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$. A set $C \subseteq \mathbb{R}$ is *closed* if its complement $\mathbb{R} \setminus C$ is open. Equivalently, C is closed if and only if it contains the limits of all convergent sequences $(x_n) \subseteq C$. ⌟

Proof I.2.1. **Openness direction:** Suppose $U \subseteq \mathbb{R}$ is open. By the definition of the standard topology on $\mathbb{R}$ (induced by the Euclidean metric in (I.2.1)), openness means precisely:

$$\forall x \in U,\ \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq U, \tag{I.2.7}$$

where $B_\varepsilon(x)$ is the open ball from (I.2.2). Thus every $x \in U$ has a *buffer zone* of radius ε entirely contained in U. This ensures that if $|x' - x| < \varepsilon$, then $x' \in U$. Geometrically, there are no "boundary points" in the sense that membership is stable under small perturbations. Topologically, U can be expressed as a union of basis elements (open intervals), hence it is open.

Closedness direction: If $C \subseteq \mathbb{R}$ is closed, then C^c is open, i.e.

$$C^c = \mathbb{R} \setminus C \text{ satisfies (I.2.7)}. \tag{I.2.8}$$

To verify the sequential characterization, let $(x_n) \subseteq C$ with $x_n \to x$ in the sense of (I.2.6). Suppose $x \notin C$. Then $x \in C^c$, so by (I.2.7) applied to C^c there is $\varepsilon > 0$ with

$$B_\varepsilon(x) \subseteq C^c. \tag{I.2.9}$$

From convergence (I.2.6), $\exists N$ such that $n \geq N \implies x_n \in B_\varepsilon(x) \subseteq C^c$, contradicting $x_n \in C$. Thus $x \in C$.

Sequential-closure direction: Conversely, suppose C is sequentially closed:

$$\big[(x_n) \subseteq C \text{ and } x_n \to x \text{ as in (I.2.6)}\big] \Rightarrow x \in C. \tag{I.2.10}$$

We show C^c is open. Let $y \in C^c$. If C^c were not open, then for every $\varepsilon > 0$ we would have

$$B_\varepsilon(y) \cap C \neq \varnothing. \tag{I.2.11}$$

Taking $\varepsilon = 1/m$ with $m \in \mathbb{N}$, choose $x_m \in C$ with $|x_m - y| < 1/m$. Then $(x_m) \subseteq C$ and $x_m \to y$ by (I.2.6), so (I.2.10) implies $y \in C$, a contradiction. Hence some $\varepsilon > 0$ has $B_\varepsilon(y) \subseteq C^c$.

Topological synthesis: In a metric space (X, d), openness and closedness are dual:

$$U \text{ open} \iff U^c \text{ closed}. \tag{I.2.12}$$

For $(\mathbb{R}, |\cdot|)$:

- Openness $\Leftrightarrow$ (I.2.7).
- Closedness $\Leftrightarrow$ sequential closure (I.2.10).

Because $\mathbb{R}$ is first-countable, each $x \in \mathbb{R}$ has a countable neighborhood basis:

$$\mathcal{B}(x) = \{(x - 1/n, x + 1/n) \mid n \in \mathbb{N}\}. \tag{I.2.13}$$

In such spaces, closure (I.2.5) agrees with the sequential closure:

$$\overline{A} = \{x \in \mathbb{R} \mid \exists (a_n) \subseteq A,\ a_n \to x \text{ as in (I.2.6)}\}. \tag{I.2.14}$$

From the general topological definition,

$$x \in \overline{C} \iff \forall \text{ open } U \ni x,\ U \cap C \neq \varnothing, \tag{I.2.15}$$

which, in a metric space, is equivalent to:

$$x \in \overline{C} \iff \forall \varepsilon > 0,\ B_\varepsilon(x) \cap C \neq \varnothing. \tag{I.2.16}$$

The equivalence of (I.2.16) and (I.2.14) follows from taking $\varepsilon_n \to 0$ and choosing $c_n \in B_{\varepsilon_n}(x) \cap C$.

Thus, for $C \subseteq \mathbb{R}$, the following are equivalent:

1. C is closed ((I.2.8)).
2. C is sequentially closed ((I.2.10)).
3. $C = \overline{C}$, with $\overline{C}$ given by (I.2.5) or equivalently by (I.2.16).

□

Remark I.2.4. (Sequential Criterion for Closedness)

The equivalence in the theorem shows that the closedness of C can be tested purely by sequences: C is closed if and only if $(x_n) \subseteq C$ and $x_n \to x$ imply $x \in C$. This criterion is especially powerful in analysis due to the first-countability of metric spaces. ♦

Definition I.2.9. Compact Set.

Let (X, d) be a metric space as in Definition I.2.1. A subset $K \subseteq X$ is *compact* if every sequence $(x_n)_{n \in \mathbb{N}} \subseteq K$ has a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ that converges to some $x \in K$ in the sense of (I.2.6).

In the setting of metric spaces, compactness is equivalent to *sequential compactness*:

$$K \text{ compact} \iff \text{every sequence in } K \text{ has a convergent subsequence with limit in } K.$$

Remark I.2.5. (Compactness in $\mathbb{R}$)
In $(\mathbb{R}, |\cdot|)$ with the standard metric (I.2.1), the *Heine–Borel theorem* states:

$$K \subseteq \mathbb{R} \text{ is compact} \quad \Longleftrightarrow \quad K \text{ is closed and bounded},$$

where *bounded* is in the sense of Definition I.2.5 and *closed* is characterised equivalently by (I.2.8), (I.2.10), or (I.2.16). Thus, in $\mathbb{R}$, compactness coincides with the familiar geometric notion of a set being finite in extent and containing all its limit points. ♦

Theorem I.2.2. Bolzano–Weierstrass.

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence, where convergence is in the sense of (I.2.6). ⌟

Proof I.2.2. Assume $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded. By Definition I.2.5 (bounded sequence), there exists $M > 0$ such that

$$|x_n| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Thus every term of (x_n) lies in the closed interval

$$I_0 := [-M, M].$$

Step 1: Interval bisection. Divide I_0 into two closed subintervals of equal length:

$$I_0^{(1)} := \big[-M, 0\big], \quad I_0^{(2)} := \big[0, M\big].$$

At least one of these halves contains infinitely many terms of the sequence (x_n). Select such a half and denote it I_1.

Step 2: Iteration and nested intervals. Having chosen I_k (which contains infinitely many terms of (x_n)), bisect I_k into two closed subintervals of equal length. At least one of these halves contains infinitely many terms of (x_n); select it and call it I_{k+1}.

Inductive properties: For each $k \in \mathbb{N}$:

1. $I_{k+1} \subseteq I_k$ (nestedness).
2. $\operatorname{diam}(I_k) = \frac{2M}{2^k} = \frac{M}{2^{k-1}} \to 0$ as $k \to \infty$.
3. I_k contains infinitely many terms of (x_n).

Step 3: Application of the Nested Interval Property. By the *nested interval property* in $\mathbb{R}$, if (I_k) is a sequence of nonempty closed intervals satisfying

$$I_{k+1} \subseteq I_k \quad \text{and} \quad \operatorname{diam}(I_k) \to 0,$$

then

$$\bigcap_{k=1}^{\infty} I_k = \{L\}$$

for some $L \in \mathbb{R}$.

Step 4: Extraction of a convergent subsequence. We now construct a subsequence (x_{n_k}) converging to L. By construction, I_1 contains infinitely many x_n, so choose n_1 such that

$x_{n_1} \in I_1$. Inductively, having chosen n_k with $x_{n_k} \in I_k$, note that I_{k+1} contains infinitely many terms with indices strictly greater than n_k; choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in I_{k+1}$.

Step 5: Verification of convergence. For each k, the subsequence term $x_{n_k} \in I_k$, and $\operatorname{diam}(I_k) \to 0$. Given $\varepsilon > 0$, choose K such that $\operatorname{diam}(I_K) < \varepsilon$. For all $k \geq K$, $x_{n_k} \in I_k \subseteq I_K$, so

$$|x_{n_k} - L| \leq \operatorname{diam}(I_K) < \varepsilon.$$

By (I.2.6), this shows $x_{n_k} \to L$.

Conclusion. We have found $L \in \mathbb{R}$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to L$. Therefore, every bounded sequence in $\mathbb{R}$ has a convergent subsequence, establishing the Bolzano–Weierstrass property. □

Corollary I.2.1. Limit Points of Bounded Sequences.

Let (x_n) be a bounded sequence in $\mathbb{R}$. Then:

1. The set $L(x_n)$ of all subsequential limits of (x_n) is nonempty, closed, and bounded.
2. There exists at least one $L \in L(x_n)$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to L$ in the sense of (I.2.6).
3. If (x_n) has exactly one limit point L, then $x_n \to L$ as in (I.2.6).

$\rightarrowtail$

Proof I.2.3. **Step 1: Nonemptiness.** By the Bolzano–Weierstrass theorem, every bounded sequence in $\mathbb{R}$ has a convergent subsequence (x_{n_k}) with $x_{n_k} \to L$ as in (I.2.6) for some $L \in \mathbb{R}$. Thus $L \in L(x_n)$, proving $L(x_n) \neq \varnothing$.

Step 2: Boundedness. If (x_n) is bounded, there exists $M > 0$ such that $|x_n| \leq M$ for all n. Any subsequence inherits this bound, and by the properties of convergence (I.2.6), its limit L must also satisfy $|L| \leq M$. Hence $L(x_n) \subseteq [-M, M]$.

Step 3: Closedness. Let $(L_m) \subseteq L(x_n)$ be a sequence of limit points with $L_m \to L^*$ as in (I.2.6). For each L_m there exists a subsequence $(x_k^{(m)})$ of (x_n) converging to L_m (again in the sense of (I.2.6)). A standard diagonalization argument produces a subsequence of (x_n) converging to L^*, so $L^* \in L(x_n)$. Therefore $L(x_n)$ is closed.

Step 4: Uniqueness case. If $L(x_n) = \{L\}$, then no subsequence can converge to any point other than L. If (x_n) did not converge to L, there would exist $\varepsilon > 0$ such that infinitely many n satisfy $|x_n - L| \geq \varepsilon$. By Bolzano–Weierstrass, this infinite set has a convergent subsequence with a limit $\neq L$, contradicting $L(x_n) = \{L\}$. Thus $x_n \to L$ as in (I.2.6).

Conclusion. $L(x_n)$ is a compact (closed and bounded) subset of $\mathbb{R}$ and describes all possible accumulation values of (x_n). If $L(x_n)$ is a singleton, the entire sequence converges. □

Remark I.2.6. (Limit Points and Closure)

Let $A \subseteq \mathbb{R}$, equipped with the standard metric (I.2.1). The set of all *limit points* of A—those $p \in \mathbb{R}$ satisfying the condition in (I.2.4)—is called the *derived set* and denoted A'. That is:

$$A' := \{\, p \in \mathbb{R} \mid \forall \varepsilon > 0,\ (B_\varepsilon(p) \setminus \{p\}) \cap A \neq \varnothing \,\}.$$

The *closure* of A, denoted $\overline{A}$, is defined in (I.2.5) by $\overline{A} = A \cup A'$ and is characterised as the smallest closed set containing A—equivalently, the intersection of all closed sets in $\mathbb{R}$ that contain A.

In a general metric space (X, d), the closure $\overline{A}$ admits an equivalent sequential description:

$$x \in \overline{A} \quad \Longleftrightarrow \quad \exists \text{ a sequence } (a_n) \subseteq A \text{ with } a_n \to x \text{ as in (I.2.6)}.$$

Thus, $\overline{A}$ consists exactly of:

1. all points of A itself, and
2. all limits of sequences in A (its limit points).

This equivalence between the neighborhood-based formulation (I.2.4) and the sequential characterization relies on the fact that $\mathbb{R}$, with (I.2.1), is a first-countable metric space. ♦

I.3 Differentiability

I.3.1 Derivatives in the δ–ε Formalism

The δ–ε formalism expresses differentiability directly as a limit statement, without appealing to sequences. This viewpoint emphasises *uniform control* over the average rate of change of the function as the increment h shrinks toward zero.

Theorem I.3.1. δ–ε Definition of the Derivative.
Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. Define, for $h \neq 0$,

$$Q(h) := \frac{f(x+h) - f(x)}{h}, \tag{I.3.1}$$

the *difference quotient* of f at x with increment h.

We say that f is *differentiable* at x with derivative $L \in \mathbb{R}$ if and only if:

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \text{s.t.} \;\; 0 < |h| < \delta \;\; \Rightarrow \;\; |Q(h) - L| < \varepsilon. \tag{I.3.2}$$

⌟

Proof I.3.1. The condition (I.3.2) states that $Q(h)$ approaches L as $h \to 0$ in the usual δ–ε sense, with the restriction $h \neq 0$ ensuring that the quotient is defined.

If such an L exists, it is unique: if L_1 and L_2 both satisfy the definition, set $\varepsilon = \frac{|L_1 - L_2|}{2} > 0$. For $0 < |h| < \delta$ (where δ is chosen to work for both L_1 and L_2), we have

$$|L_1 - L_2| \leq |L_1 - Q(h)| + |Q(h) - L_2| < \frac{|L_1 - L_2|}{2} + \frac{|L_1 - L_2|}{2} = |L_1 - L_2|,$$

a contradiction unless $L_1 = L_2$. □

Example I.2 (Quadratic Function)**.** Let $f(x) = x^2$ and $x = 3$. For $h \neq 0$,

$$Q(h) = \frac{(3+h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h} = 6 + h.$$

We claim $f'(3) = 6$.

Given $\varepsilon > 0$, choose $\delta = \varepsilon$. If $0 < |h| < \delta$, then

$$|Q(h) - 6| = |h| < \delta = \varepsilon,$$

satisfying (I.3.2) with $L = 6$.

Corollary I.3.1. One-Sided Derivatives and Differentiability.
Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. If both the right-hand limit

$$\lim_{h\to 0^+} Q(h) \quad \text{and} \quad \lim_{h\to 0^-} Q(h)$$

exist and are equal to the same $L \in \mathbb{R}$, then f is differentiable at x with $f'(x) = L$. $\rightarrowtail$

Proof I.3.2. If the two one-sided limits exist and agree at L, then for any $\varepsilon > 0$ there exist $\delta_+ > 0$ and $\delta_- > 0$ such that:

$$0 < h < \delta_+ \;\Rightarrow\; |Q(h) - L| < \varepsilon,$$

$$-\delta_- < h < 0 \;\Rightarrow\; |Q(h) - L| < \varepsilon.$$

Let $\delta = \min(\delta_+, \delta_-)$. If $0 < |h| < \delta$, h lies in one of the two one-sided ranges and satisfies the inequality, hence (I.3.2) holds. $\square$

I.3.2 Sequential Criterion

In the study of differentiability, the central object is the *difference quotient*, already introduced in the δ–ε formalism (cf. (I.3.1)). The classical limit definition of the derivative, equivalent to the quantified condition in (I.3.2), states that f is differentiable at x if

$$\lim_{h\to 0} Q(h) = L \in \mathbb{R}, \tag{I.3.3}$$

in which case $f'(x) := L$.

Because $\mathbb{R}$ is a first-countable metric space under (I.2.1), the limit in (I.3.3) can be reformulated entirely in terms of sequences, using the convergence criterion (I.2.6).

Theorem I.3.2. Sequential Characterization of Differentiability.
Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. The following are equivalent:

1. (*Limit definition*) The derivative $f'(x)$ exists and equals L in the sense of (I.3.3) (equivalently, (I.3.2)).
2. (*Sequential definition*) For every sequence $(h_n)_{n\in\mathbb{N}}$ with $h_n \to 0$ and $h_n \neq 0$, the sequence of difference quotients
$$(Q(h_n))_{n=1}^{\infty} \tag{I.3.4}$$
converges to L in the sense of (I.2.6), and the limit L is independent of the choice of (h_n).

In either case, the common value L is the derivative $f'(x)$. ⌟

Proof I.3.3. (**1** $\Rightarrow$ **2**): Assume (I.3.3) holds (equivalently, the δ–ε condition in (I.3.2)). Let (h_n) be any sequence with $h_n \to 0$ and $h_n \neq 0$. By the sequential characterization of limits in $\mathbb{R}$ (cf. (I.2.6)), we have

$$\lim_{n\to\infty} Q(h_n) = L.$$

Thus (I.3.4) converges to L for every such sequence (h_n).

(**2** $\Rightarrow$ **1**): Assume (I.3.4) converges to L for every sequence $(h_n) \to 0$ with $h_n \neq 0$. Suppose, for contradiction, that (I.3.3) fails. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ we can choose h with $0 < |h| < \delta$ and $|Q(h) - L| \geq \varepsilon_0$. Using these h values, construct a sequence $(h_n) \to 0$ with $|Q(h_n) - L| \geq \varepsilon_0$ for all n, contradicting the assumed convergence in (I.3.4). Hence (I.3.3) (and thus (I.3.2)) must hold.

Geometric significance. This equivalence asserts that differentiability at x means the local slope (I.3.1) approaches the same value L no matter how h tends to 0: through evenly spaced increments, rapidly shrinking steps, or irregular patterns. In $\mathbb{R}$, this is equivalent to *uniformity of slope across all approach directions.* □

Remark I.3.1. (Equivalence and Sequential Perspective)

Because $\mathbb{R}$ is first-countable under the standard metric (I.2.1), the classical ε–δ limit definition of the derivative (I.3.3) (cf. (I.3.2)) is equivalent to the sequential formulation (I.3.4). In a first-countable space, the local behavior of a function near a point is completely determined by its action on sequences converging to that point, as captured by the convergence criterion (I.2.6).

In the sequential perspective, differentiability at x asserts that *for every* sequence $(h_n) \to 0$ with $h_n \neq 0$, the difference quotients $Q(h_n)$ from (I.3.1) converge to the *same* limit L. This uniformity means the local slope of the graph of f at x is independent of the "path" or "manner" of approach along $\mathbb{R}$.

If two sequences approaching zero yield different limits, one can *interleave* them to produce a single sequence for which (I.3.4) fails to converge at all. Thus, the uniqueness of the sequential limit is *necessary* for the derivative to exist.

From an analytic viewpoint, this condition is equivalent to the statement that the map

$$h \longmapsto Q(h) = \frac{f(x+h) - f(x)}{h}$$

admits a continuous extension to $h = 0$, with the value of that extension equal to $f'(x)$. From a geometric viewpoint, it guarantees that the tangent line to the graph of f at x is uniquely defined and faithfully encodes the infinitesimal rate of change of f.

Computational motivation. The sequential approach often simplifies proofs and calculations:

- It reduces differentiability checks to verifying a limit along *all* possible sequences, rather than handling an arbitrary ε–δ inequality in (I.3.2).
- In numerical analysis, where functions are sampled at discrete points, the sequential viewpoint matches how difference quotients are actually approximated in computation—by evaluating $Q(h_n)$ for a sequence of step sizes $h_n \to 0$.
- In more advanced settings (e.g. multivariable calculus), the failure of the sequential condition often reveals directional dependence, signaling that a function may have directional derivatives without being fully differentiable.

This blend of theory and computation shows why the sequential perspective is not just equivalent to the classical δ–ε definition—it is also a practical diagnostic tool for both proving and approximating differentiability. ♦

Example I.3 (Sequence-Based Computation of Derivative)**.** Let

$$f(x) = \sqrt{x+3} + \frac{1}{x}, \quad x > 0.$$

We compute $f'(1)$ using the sequential definition (I.3.4). Choose the sequence $h_n = \frac{(-1)^n}{n}$, which approaches 0 but alternates sign.

From (I.3.1), the n-th term of the difference quotient sequence is:

$$Q(h_n) = \frac{f(1+h_n) - f(1)}{h_n}.$$

First, compute $f(1)$:

$$f(1) = \sqrt{1+3} + \frac{1}{1} = 2 + 1 = 3.$$

Next, for general n:

$$f(1+h_n) = \sqrt{1+h_n+3} + \frac{1}{1+h_n} = \sqrt{4+h_n} + \frac{1}{1+h_n}.$$

Thus,

$$Q(h_n) = \frac{\sqrt{4+h_n} - 2}{h_n} + \frac{\frac{1}{1+h_n} - 1}{h_n}.$$

First term: Multiply numerator and denominator by $\sqrt{4+h_n} + 2$:

$$\frac{\sqrt{4+h_n} - 2}{h_n} = \frac{h_n}{h_n\,(\sqrt{4+h_n} + 2)} = \frac{1}{\sqrt{4+h_n} + 2}.$$

Second term: Simplify:

$$\frac{\frac{1}{1+h_n} - 1}{h_n} = \frac{\frac{1-(1+h_n)}{1+h_n}}{h_n} = \frac{-h_n}{(1+h_n)h_n} = -\frac{1}{1+h_n}.$$

Combine:

$$Q(h_n) = \frac{1}{\sqrt{4+h_n} + 2} - \frac{1}{1+h_n}.$$

Limit: Since $h_n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} Q(h_n) = \frac{1}{\sqrt{4}+2} - \frac{1}{1} = \frac{1}{4} - 1 = -\frac{3}{4}.$$

Therefore, by the sequential characterization (I.2.6), $f'(1) = -\frac{3}{4}$.

This computation illustrates that the sequential approach is valid for *any* sequence $h_n \to 0$ with $h_n \neq 0$, even those with sign changes or non-uniform step sizes, and that the limit remains the same when f is differentiable at the point in question.

Remark I.3.2. (Sequential Uniformity in the Definition of the Derivative)

The sequential characterization of differentiability asserts that f is differentiable at x with derivative L if and only if, for *every* sequence $(h_n)_{n\in\mathbb{N}}$ satisfying $h_n \to 0$ and $h_n \neq 0$, the sequence of difference quotients

$$(Q(h_n))_{n=1}^{\infty} = \left(\frac{f(x+h_n) - f(x)}{h_n}\right)_{n=1}^{\infty} \tag{I.3.5}$$

converges to the *same* real number L in the sense of (I.2.6).

Uniformity over all approach sequences. This is stronger than requiring convergence along a *single* sequence: it demands *uniformity of the limiting value across the entire family of sequences* $h_n \to 0$. If two sequences approaching 0 yield distinct limits $L_1 \neq L_2$, then by *interleaving* these sequences we obtain a single sequence for which (I.3.5) fails to converge at all. Thus, the uniqueness of the limit is a *necessary condition* for the derivative $f'(x)$ to be well-defined.

Analytic meaning. In the classical ε–δ framework, (I.3.1) must satisfy (I.3.2). By (I.2.6), this is equivalent to requiring that (I.3.5) converges to L for *all* $h_n \to 0$ with $h_n \neq 0$. In this language, differentiability is exactly the *well-definedness of the local linear approximation*:

$$f(x+h) \approx f(x) + Lh \quad \text{as} \quad h \to 0.$$

The number L is the unique slope that works for *all* infinitesimal displacements h.

Topological meaning. From a topological standpoint, consider the map

$$\varphi(h) := Q(h) = \frac{f(x+h) - f(x)}{h}, \quad h \in \mathbb{R} \setminus \{0\}.$$

Differentiability at x means φ extends to a *continuous* map $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ by setting $\widetilde{\varphi}(0) := L$. In the standard topology induced by (I.2.1), and more generally in any first-countable metric space, this continuity at $h = 0$ is equivalent to the sequential convergence condition (I.3.4).

Computational note. In practice, especially in numerical analysis, the sequential perspective mirrors how derivatives are approximated: one samples (I.3.1) at a discrete set of step sizes h_n (often with $h_n \to 0$) and observes whether the values stabilise. For smooth functions, the limiting value will be the same regardless of the chosen sequence (h_n); for non-differentiable points, different sequences can yield different apparent "slopes," revealing the breakdown of a single tangent-line description.

Thus, the sequential uniformity criterion not only restates the ε–δ definition in terms of (I.2.6), but also exposes its structural content. From the geometric viewpoint, it encodes the idea that the function's graph straightens locally: sequences in the domain that collapse together must map to sequences in the codomain that collapse at a proportionally controlled rate. From the topological viewpoint, it recasts uniform continuity as a property of the induced map on Cauchy sequences, independent of the particular choice of metric, making the definition robust under homeomorphisms that preserve uniform structure. From the computational viewpoint, it ensures the stability of approximation schemes: given any prescribed output tolerance ε, there exists a uniform input resolution δ sufficient to control all output deviations — a guarantee that numerical refinement need not depend on the specific region of the domain being sampled.

♦

Example I.4 (Failure of Sequential Convergence for $f(x) = |x|$)**.** Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ under the standard metric (I.2.1), and let $x = 0$. The difference quotient from (I.3.1) is

$$Q(h) = \frac{|h| - 0}{h} = \frac{|h|}{h}, \quad h \neq 0.$$

Sequential approach. Choose the sequences

$$h_n^{(+)} = \frac{1}{n}, \quad h_n^{(-)} = -\frac{1}{n}.$$

Both satisfy $h_n \to 0$ and $h_n \neq 0$, so they are valid test sequences for the sequential derivative condition (I.3.4). The difference quotients are:

$$Q\big(h_n^{(+)}\big) = \frac{\frac{1}{n}}{\frac{1}{n}} = 1, \qquad Q\big(h_n^{(-)}\big) = \frac{\frac{1}{n}}{-\frac{1}{n}} = -1.$$

Thus,

$$\lim_{n\to\infty} Q\big(h_n^{(+)}\big) = 1, \qquad \lim_{n\to\infty} Q\big(h_n^{(-)}\big) = -1.$$

Since the sequential convergence criterion (I.2.6) requires *all* sequences $h_n \to 0$ to produce the *same* limit, this violates the uniformity requirement, and $f'(0)$ does not exist.

ε–δ **verification.** Suppose, for contradiction, that $\lim_{h\to 0} Q(h) = L$ exists. Then for $\varepsilon = \frac{1}{2}$ there should exist $\delta > 0$ such that:

$$0 < |h| < \delta \quad \Rightarrow \quad |Q(h) - L| < \frac{1}{2}.$$

But for $h > 0$, $Q(h) = 1$, and for $h < 0$, $Q(h) = -1$. If δ is chosen, pick $h_1 \in (0, \delta)$ and $h_2 \in (-\delta, 0)$. Then:

$$|Q(h_1) - L| = |1 - L|, \quad |Q(h_2) - L| = |-1 - L|.$$

No single L can satisfy $|1 - L| < \frac{1}{2}$ and $|-1 - L| < \frac{1}{2}$ simultaneously, since these inequalities would require L to be within $\frac{1}{2}$ of both 1 and -1, an impossibility. Therefore, the ε–δ condition fails, confirming that $\lim_{h\to 0} Q(h)$ does not exist.

Conclusion. Both the sequential criterion (I.3.4) and the ε–δ definition agree that f is *not* differentiable at 0. This illustrates why the sequential uniformity condition is essential: different "approach directions" in $\mathbb{R}$ can yield incompatible slopes if the function's local behavior is not linearizable.

Remark I.3.3. (Notions of analytic and topological failure)

Analytically, differentiability at $x = 0$ would require that the map

$$h \mapsto \frac{f(0+h) - f(0)}{h}$$

admits a *single* finite limit as $h \to 0$ through all nonzero reals. In this example, the map has two distinct one-sided limits:

$$\lim_{h\to 0^+} Q(h) = 1, \quad \lim_{h\to 0^-} Q(h) = -1.$$

This discontinuity at $h = 0$ in the extended difference quotient function Q is what breaks differentiability.

Topologically, the condition for differentiability is that Q extends to a *continuous* function at $h = 0$ (when defined on $\mathbb{R} \setminus \{0\}$). Here, approaching 0 along positive h gives a limit of 1, while approaching along negative h gives -1. The approach path in the punctured neighborhood of 0 matters, meaning the limit is *direction-dependent.* This violates the basic requirement that limits in $\mathbb{R}$ be independent of approach.

Geometrically, the graph of $f(x) = |x|$ has a *corner* at $(0, 0)$: the right-hand tangent line has slope $+1$, and the left-hand tangent line has slope -1. There is no single tangent line that simultaneously approximates the graph from both sides, so the tangent structure fails to be well-defined at $x = 0$. ◆

I.3.3 Differentiability Implies Local Linearity

We now formalise the principle that differentiability guarantees a *first-order linear approximation* of the function at the point of tangency. This is the precise content of the geometric statement: *a differentiable function is locally well-approximated by its tangent line.*

Theorem I.3.3. Differentiability Implies Local Linearity.

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at x_0 in the sense of the limit definition (I.3.3) (equivalently, the δ–ε criterion (I.3.2) or the sequential condition (I.3.4)). Then there exists a *linear map*

$$L(h) := f'(x_0)\, h \tag{I.3.6}$$

such that

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - L(h)}{h} = 0. \tag{I.3.7}$$

⌟

Proof I.3.4. By differentiability at x_0 (cf. (I.3.3)), we have:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}. \tag{I.3.8}$$

Subtracting and adding $f'(x_0)h$ in the numerator of (I.3.8), we obtain:

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h} = 0,$$

which is exactly (I.3.7) with L as in (I.3.6).

Equivalently, we may write the *asymptotic expansion*:

$$f(x_0+h) = f(x_0) + f'(x_0)\, h + o(h), \tag{I.3.9}$$

where $o(h)$ is a function such that

$$\lim_{h\to 0} \frac{o(h)}{h} = 0,$$

cf. the discussion of uniformity in (I.3.5). This shows that the error term in the linear approximation is negligible compared to h. □

Remark I.3.4. (Geometric and topological interpretation)

Equation (I.3.7) asserts that the map

$$\varphi(h) := \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h}$$

extends to a *continuous function* at $h = 0$ with value 0. In the topology induced by the standard metric (I.2.1), this is equivalent to the sequential condition (I.3.4). Geometrically, (I.3.9) says that the graph of f becomes indistinguishable from its tangent line under infinite magnification at x_0. ♦

Theorem I.3.4. Chain Rule.

Let $f = g \circ h$, where $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $h(x)$ and $h : \mathbb{R} \to \mathbb{R}$ is differentiable at x, in the sense of (I.3.3) or (I.3.4). Then:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = g'(h(x)) \cdot h'(x). \tag{I.3.10}$$

⌟

Proof I.3.5. We provide a limit-based proof with explicit control over the remainder terms, consistent with (I.3.2).

Step 1: Linearization with error functions. By differentiability of g at $h(x)$, there exists a function $\epsilon(\Delta h)$ with $\epsilon(\Delta h) \to 0$ as $\Delta h \to 0$ such that:

$$g(h(x) + \Delta h) = g(h(x)) + g'(h(x)) \cdot \Delta h + \epsilon(\Delta h) \cdot \Delta h. \tag{I.3.11}$$

Similarly, differentiability of h at x gives a function $\delta(\Delta x) \to 0$ as $\Delta x \to 0$ such that:

$$h(x + \Delta x) = h(x) + h'(x) \cdot \Delta x + \delta(\Delta x) \cdot \Delta x. \tag{I.3.12}$$

Step 2: Relating increments. Set:

$$\Delta h := h(x + \Delta x) - h(x) = h'(x) \cdot \Delta x + \delta(\Delta x) \cdot \Delta x. \tag{I.3.13}$$

Substitute (I.3.13) into (I.3.11) with $h(x + \Delta x)$ in place of $h(x) + \Delta h$.

Step 3: Expand $f(x + \Delta x)$. Since $f = g \circ h$, we have:

$$\begin{aligned} f(x + \Delta x) &= g(h(x + \Delta x)) \\ &= g(h(x)) + g'(h(x)) \cdot \Delta h + \epsilon(\Delta h) \cdot \Delta h. \end{aligned}$$

Step 4: Difference quotient. Subtract $f(x) = g(h(x))$ and divide by Δx:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = g'(h(x)) \cdot \frac{\Delta h}{\Delta x} + \epsilon(\Delta h) \cdot \frac{\Delta h}{\Delta x}. \tag{I.3.14}$$

Using (I.3.13), we find:

$$\frac{\Delta h}{\Delta x} = h'(x) + \delta(\Delta x).$$

Substituting into (I.3.14):

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = g'(h(x)) \cdot \big(h'(x) + \delta(\Delta x)\big) + \epsilon(\Delta h) \cdot \big(h'(x) + \delta(\Delta x)\big).$$

Step 5: Take the limit. As $\Delta x \to 0$, we have $\delta(\Delta x) \to 0$, $\Delta h \to 0$, and thus $\epsilon(\Delta h) \to 0$. Passing to the limit in (I.3.14) and applying (I.3.3):

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = g'(h(x)) \cdot h'(x),$$

which is exactly (I.3.10). □

Remark I.3.5. (Topological and sequential interpretation)

The chain rule can be restated in the sequential framework (I.3.4): if $h_n \to 0$ with $h_n \neq 0$, the difference quotient for f factors into the product of the difference quotient for g evaluated along the image sequence $h(x) + \Delta h_n$ and the difference quotient for h. The convergence of each term follows from the differentiability of g and h, and their product limit is $g'(h(x))h'(x)$. Thus, (I.3.10) holds in the ε–δ, sequential, and topological formulations alike. ♦

Theorem I.3.5. Differentiability $\not\Rightarrow$ Continuity of the Derivative.

There exists a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable at every point in the sense of (I.3.3) (equivalently (I.3.2) or (I.3.4)), but whose derivative f' is *not* continuous at some point with respect to the standard metric (I.2.1). Thus, differentiability of f does not imply continuity of f'. ⌟

Proof I.3.6. We use the standard oscillatory counterexample:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{I.3.15}$$

Step 1: Differentiability on $\mathbb{R}\setminus\{0\}$. If $x \neq 0$, then f in (I.3.15) is the product of two smooth functions:

$$u(x) = x^2, \quad v(x) = \sin(1/x).$$

Both are C^∞ on $\mathbb{R} \setminus \{0\}$, hence differentiable everywhere on that set. By the product rule (derivable from the limit definition (I.3.3) or from the chain rule (I.3.10) together with local linearity (I.3.7)):

$$\begin{aligned} f'(x) &= u'(x) \cdot v(x) + u(x) \cdot v'(x) \\ &= 2x \sin\left(\tfrac{1}{x}\right) + x^2 \cdot \cos\left(\tfrac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right) \\ &= 2x \sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right), \end{aligned} \tag{I.3.16}$$

valid for all $x \neq 0$.

Step 2: Differentiability at $x = 0$. We apply the limit definition (I.3.3) directly:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin(1/h) - 0}{h} \\ &= \lim_{h\to 0} h \sin(1/h). \end{aligned}$$

By the inequality $|\sin(1/h)| \leq 1$, we have

$$\big|h \sin(1/h)\big| \leq |h| \xrightarrow{h\to 0} 0$$

(cf. the sequential convergence criterion (I.2.6) applied to $h_n \to 0$). Therefore:

$$f'(0) = 0. \tag{I.3.17}$$

Combining (I.3.16) and (I.3.17), f is differentiable at all $x \in \mathbb{R}$.

Step 3: Discontinuity of f' at $x = 0$. From (I.3.16), for $x \neq 0$:

$$f'(x) = 2x \sin(1/x) - \cos(1/x).$$

The first term satisfies

$$|2x \sin(1/x)| \leq 2|x| \xrightarrow{x\to 0} 0$$

in the standard metric (I.2.1), but the second term $\cos(1/x)$ oscillates between -1 and 1 infinitely often as $x \to 0$, without damping. Formally:

$$\limsup_{x\to 0} f'(x) = 1, \quad \liminf_{x\to 0} f'(x) = -1,$$

so the limit $\lim_{x\to 0} f'(x)$ fails to exist. By (I.2.12), continuity at 0 would require sequential convergence of $f'(x_n) \to f'(0)$ for *every* sequence $(x_n) \to 0$ (cf. (I.2.6)); however, choosing $x_n = \frac{1}{n\pi}$ and $y_n = \frac{1}{(n+\frac{1}{2})\pi}$ yields:

$$f'(x_n) \to -1, \quad f'(y_n) \to 1,$$

contradicting the uniqueness of the limit.

Step 4: Interpretation. Equation (I.3.17) shows $f'(0)$ exists, so f has a valid first-order expansion (I.3.9) at 0:

$$f(h) = 0 + 0 \cdot h + o(h).$$

Yet (I.3.16) reveals that the *slope function* f' near 0 is dominated by the undamped oscillatory term $-\cos(1/x)$, whose rapid sign changes destroy continuity in the topology of (I.2.1). Geometrically: f is tangent to some line at every point, but near 0 the tangent direction rotates through the full range $[-1, 1]$ infinitely often, violating smooth variation of slope.

Conclusion. The construction (I.3.15) provides a differentiable f whose derivative is discontinuous at 0. Thus, while differentiability enforces local linearity (I.3.7), it imposes no continuity requirement on f' as a map $(\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$. □

Theorem I.3.6. Non-Differentiability from Direction-Dependent Limit.
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := |x|. \tag{I.3.18}$$

Then:

1. f is continuous at every $x \in \mathbb{R}$ with respect to the standard metric (I.2.1).
2. f is *not* differentiable at $x = 0$ in the sense of (I.3.3) (equivalently (I.3.2) or (I.3.4)).

The obstruction to differentiability is that the difference quotient limit depends on the *direction* of approach, producing distinct one-sided limits. ⌟

Proof I.3.7. **(1) Continuity at every $a \in \mathbb{R}$.** Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Choose $\delta := \varepsilon$. If $|x - a| < \delta$, then by the reverse triangle inequality:

$$|f(x) - f(a)| = \big||x| - |a|\big| \le |x - a| < \delta = \varepsilon.$$

This is exactly the ε–δ continuity condition for f at a in the metric (I.2.1). Since a was arbitrary, f is continuous on all of $\mathbb{R}$.

(2) Failure of differentiability at $x = 0$. The derivative at $x = 0$, if it exists, would be:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - 0}{h}.$$

We analyze the *one-sided* limits in the sense of (I.3.2).

Right-hand limit: If $h \to 0^+$, then $|h| = h$, so

$$\lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1.$$

Left-hand limit: If $h \to 0^-$, then $|h| = -h$, so

$$\lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1.$$

Since

$$\lim_{h \to 0^+} Q(h) = 1, \quad \lim_{h \to 0^-} Q(h) = -1,$$

the two one-sided limits differ. Therefore, the full limit in (I.3.3) does not exist, and $f'(0)$ is undefined.

Analytic interpretation. Differentiability at 0 would require the map

$$h \longmapsto Q(h) = \frac{|h|}{h}, \quad h \neq 0,$$

to converge to a *single* $L \in \mathbb{R}$ as $h \to 0$. Instead, $Q(h)$ is the *sign function*, which has a jump discontinuity from -1 (for $h < 0$) to $+1$ (for $h > 0$) at $h = 0$. This violates the uniformity condition over all approach sequences in (I.3.4).

Topological and geometric interpretation. In the standard topology induced by (I.2.1), the set of approach directions to 0 in $\mathbb{R}$ has two connected components: $(-\infty, 0)$ and $(0, \infty)$. Differentiability demands that the local slope $Q(h)$ be *uniform* across these components as $h \to 0$. Here, the slopes stabilise to -1 from the left and $+1$ from the right, so there is no single affine map $L(h) = mh$ that works for all small h. Geometrically, this is the familiar "corner" at $(0, 0)$.

Sequential characterization of failure. Consider the sequences

$$h_n^{(+)} := \frac{1}{n}, \quad h_n^{(-)} := -\frac{1}{n}.$$

Both satisfy $h_n \to 0$ and $h_n \neq 0$. From (I.3.4):

$$Q(h_n^{(+)}) = 1, \quad Q(h_n^{(-)}) = -1.$$

Since the sequential definition requires *every* such sequence to yield the same limit L, the existence of two sequences with distinct limits proves non-differentiability.

Conclusion. The function (I.3.18) is continuous everywhere, but at $x = 0$ the difference quotient (I.3.1) fails the uniformity condition (I.3.4). Analytically, this appears as a jump in $Q(h)$ at $h = 0$; topologically, it reflects distinct limits on disconnected approach components; geometrically, it is the manifestation of a corner with two distinct tangent lines. □

Theorem I.3.7. Characterization of Continuity via ε–δ and Sequential Convergence. Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. The following are equivalent:

(i) f is continuous at x_0 in the metric sense of (I.2.1);

(ii) (*ε–δ formulation*) For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon; \tag{I.3.19}$$

(iii) (*Sequential formulation*) For every sequence $(x_n) \subset \mathbb{R}$ with

$$\lim_{n \to \infty} x_n = x_0 \tag{I.3.20}$$

in the sense of (I.2.6), we have

$$\lim_{n \to \infty} f(x_n) = f(x_0). \tag{I.3.21}$$

⌟

Proof I.3.8. **(i) ⇒ (ii):** Statement (ii) is simply the ε–δ definition of continuity at x_0 under the standard metric (I.2.1). Thus (i) and (ii) are equivalent by definition.

(ii) ⇒ (iii): Assume (I.3.19) holds. Let (x_n) satisfy (I.3.20). Given $\varepsilon > 0$, choose $\delta > 0$ from (I.3.19). By (I.2.6), there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |x_n - x_0| < \delta.$$

By (I.3.19), this implies $|f(x_n) - f(x_0)| < \varepsilon$ for all $n \geq N$. Therefore (I.3.21) holds, establishing sequential continuity.

(iii) ⇒ (i): Assume (I.3.20) ⇒ (I.3.21) for all sequences (x_n). Suppose, for contradiction, that f is *not* continuous at x_0. Then there exists $\varepsilon_0 > 0$ such that

$$\forall \delta > 0,\ \exists x \in \mathbb{R} \text{ with } |x - x_0| < \delta \text{ and } |f(x) - f(x_0)| \geq \varepsilon_0. \tag{I.3.22}$$

For each $n \in \mathbb{N}$, apply (I.3.22) with $\delta = 1/n$ to choose x_n satisfying

$$|x_n - x_0| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(x_0)| \geq \varepsilon_0.$$

By construction, $x_n \to x_0$ in the sense of (I.2.6), but $f(x_n) \not\to f(x_0)$ because the error never drops below ε_0. This contradicts (I.3.21). Therefore, f must satisfy the ε–δ continuity condition (I.3.19) at x_0.

Topological significance. In any metric space (X, d), continuity at x_0 can be defined via open sets:

$$\forall U \text{ open in codomain with } f(x_0) \in U,\ \exists \text{ open } V \ni x_0 \text{ s.t. } f(V) \subseteq U.$$

In first-countable spaces (such as $(\mathbb{R}, |\cdot|)$), this is equivalent to sequential continuity, yielding the equivalence of (ii) and (iii).

Conclusion. The ε–δ definition (I.3.19) and the sequential criterion (I.3.21) are interchangeable in $\mathbb{R}$ under (I.2.1) and (I.2.6). This equivalence is fundamental for translating between pointwise limit manipulations and topological neighborhood arguments. □

Theorem I.3.8. Uniform Continuity and Sequential Criterion.
Let $f : \mathbb{R} \to \mathbb{R}$, with $\mathbb{R}$ equipped with the standard metric (I.2.1). The following are equivalent:

(i) f is *uniformly continuous* on $\mathbb{R}$;

(ii) (*ε–δ formulation*) For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \tag{I.3.23}$$

for all $x, y \in \mathbb{R}$;

(iii) (*Sequential formulation*) For all sequences $(x_n), (y_n) \subset \mathbb{R}$,

$$|x_n - y_n| \to 0 \quad \text{in the sense of (I.2.6)} \tag{I.3.24}$$

implies

$$|f(x_n) - f(y_n)| \to 0. \tag{I.3.25}$$

⌟

Proof I.3.9. **(i) ⇒ (ii):** Statement (ii) is precisely the definition of uniform continuity in the metric space $(\mathbb{R}, |\cdot|)$, so (i) and (ii) are equivalent by definition.

(ii) ⇒ (iii): Assume (I.3.23) holds. Let $(x_n), (y_n)$ satisfy (I.3.24). Given $\varepsilon > 0$, choose $\delta > 0$ from (I.3.23). By (I.2.6), there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - y_n| < \delta$. By (I.3.23), we then have $|f(x_n) - f(y_n)| < \varepsilon$ for all $n \geq N$, which is exactly (I.3.25).

(iii) ⇒ (i): We prove the contrapositive. Suppose f is *not* uniformly continuous. Then there exists $\varepsilon_0 > 0$ such that:

$$\forall \delta > 0,\ \exists x, y \in \mathbb{R} \text{ with } |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon_0. \tag{I.3.26}$$

For each $n \in \mathbb{N}$, apply (I.3.26) with $\delta = 1/n$ to choose x_n, y_n satisfying:

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

Then $|x_n - y_n| \to 0$, but $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all n, so (I.3.25) fails. This contradicts (iii), hence f must satisfy (I.3.23).

Topological remark. In $(\mathbb{R}, |\cdot|)$, uniform continuity is stronger than pointwise continuity: the same δ in (I.3.23) works for *all* $x \in \mathbb{R}$. The sequential criterion (I.3.25) captures this by requiring that *any* two sequences whose points get arbitrarily close in the domain must map to sequences whose values get arbitrarily close in the codomain, independent of where in $\mathbb{R}$ the sequences are located.

Conclusion. Conditions (I.3.23) and (I.3.25) are equivalent in $(\mathbb{R}, |\cdot|)$ by the same first-countability argument used in the pointwise continuity case, with the difference that the δ in (I.3.23) is uniform over all domain points. □

Theorem I.3.9. Continuity and Open Sets.
Let $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the standard metric

$$d(x, y) := |x - y|.$$

Then f is continuous (in the ε–δ sense) if and only if for every open set $U \subseteq \mathbb{R}$, the preimage

$$f^{-1}(U) := \{x \in \mathbb{R} \mid f(x) \in U\}$$

is open in $\mathbb{R}$. ⌟

Proof I.3.10. **(⇒)** Assume f is continuous in the ε–δ sense. Let $U \subseteq \mathbb{R}$ be open and fix $x_0 \in f^{-1}(U)$. Then $f(x_0) \in U$. Since U is open, there exists $\varepsilon > 0$ such that the open ball

$$B_\varepsilon(f(x_0)) := \{y \in \mathbb{R} \mid |y - f(x_0)| < \varepsilon\}$$

is contained in U.

By continuity of f at x_0, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

This condition says $f(x) \in B_\varepsilon(f(x_0)) \subseteq U$, so $x \in f^{-1}(U)$. Hence $B_\delta(x_0) \subseteq f^{-1}(U)$. Since x_0 was arbitrary, $f^{-1}(U)$ is open.

($\Leftarrow$) Assume $f^{-1}(U)$ is open for every open $U \subseteq \mathbb{R}$. Fix $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Let

$$U := (f(x_0) - \varepsilon,\ f(x_0) + \varepsilon),$$

which is open in $\mathbb{R}$. Then $f^{-1}(U)$ is open and contains x_0, so there exists $\delta > 0$ with

$$B_\delta(x_0) \subseteq f^{-1}(U).$$

Thus $|x - x_0| < \delta \ \Rightarrow\ f(x) \in U$, i.e.

$$|f(x) - f(x_0)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is continuous at x_0. As x_0 was arbitrary, f is continuous on $\mathbb{R}$.

Synthesis. This equivalence shows that the pointwise ε–δ definition of continuity coincides with the purely topological condition that f is continuous as a map between $(\mathbb{R}, \mathcal{T}_{\text{std}}) \to (\mathbb{R}, \mathcal{T}_{\text{std}})$. In the metric setting, openness of $f^{-1}(U)$ encodes exactly the statement that for every point x_0 and every radius $\varepsilon > 0$, there exists a radius $\delta > 0$ such that the δ–ball around x_0 is mapped entirely into the ε–ball around $f(x_0)$. This reformulation avoids explicit distance calculations and emphasises that continuity is a property of the topology induced by the metric.

$\square$

I.3.4 Examples and Counterexamples

Example I.5 (Chain Rule in Action)**.** Let $f(x) = \sin(x^2)$.

Solution: Apply the chain rule with outer function $g(u) = \sin u$ and inner function $h(x) = x^2$:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sin(x^2) = \cos(x^2) \cdot 2x. \tag{I.3.27}$$

Example I.6 (Corner Point and Directional Derivatives)**.** Let $f(x) = |x|$.

Solution:

$$f'(x) = \begin{cases} 1 & x > 0, \\ -1 & x < 0, \end{cases}$$

but $f'(0)$ is undefined, even though the function is continuous.

This demonstrates that continuous functions need not be differentiable, and the derivative, when it exists, encodes slope information that may be ill-defined at sharp points.

Example I.7 (Differentiable but Not C^1)**.** Let

$$g(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Solution: $g(x)$ is differentiable everywhere:

$$g'(x) = 2x \sin(1/x) - \cos(1/x), \quad x \neq 0, \tag{I.3.28}$$

$$g'(0) = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0. \tag{I.3.29}$$

However, g' is not continuous at $x = 0$. The function is C^0 and C^1 on $\mathbb{R} \setminus \{0\}$ but fails to be C^1 globally.

Example I.8 (Higher-Order Derivatives)**.** Let $f(x) = \exp(-x^2)$. Then:

$$f'(x) = -2x \exp(-x^2), \tag{I.3.30}$$

$$f''(x) = (-2 + 4x^2) \exp(-x^2), \tag{I.3.31}$$

$$f^{(n)}(x) = P_n(x) \exp(-x^2), \tag{I.3.32}$$

where $P_n(x)$ are Hermite polynomials up to scaling.

This structure is key in the theory of orthogonal polynomials and spectral decomposition of linear operators.

Example I.9 (Uniform Continuity Without Differentiability)**.** Define

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

1. Continuity at $x = 0$ (via ε–δ definition): We claim that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x with $|x| < \delta$,

$$|f(x) - f(0)| = |x \sin(1/x)| < \varepsilon.$$

Since $|\sin(1/x)| \leq 1$, we choose $\delta = \varepsilon$, so that $|x \sin(1/x)| \leq |x| < \varepsilon$.

Thus, f is continuous at $x = 0$. Continuity elsewhere follows from smoothness of $x \sin(1/x)$ on $\mathbb{R} \setminus \{0\}$.

2. Uniform Continuity on $\mathbb{R}$**:** We use the Heine–Cantor theorem: continuous functions on compact intervals are uniformly continuous. For $x \in [-a, -\delta] \cup [\delta, a]$, f is continuous on a compact domain and hence uniformly continuous. To cover $\mathbb{R}$, we glue these intervals with the central neighborhood $(-\delta, \delta)$ where we already showed ε–δ uniform continuity directly. Therefore, f is uniformly continuous on all of $\mathbb{R}$.

3. Non-Differentiability at $x = 0$ **(via** ε–δ **failure):** Let us examine the difference quotient:

$$\frac{f(h) - f(0)}{h} = \frac{h \sin(1/h)}{h} = \sin(1/h),$$

which oscillates between $[-1, 1]$ as $h \to 0$ and has no limit. Thus,

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

does not exist. So f fails the formal definition of differentiability:

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } 0 < |h| < \delta \Rightarrow \left| \frac{f(x+h) - f(x)}{h} - L \right| < \varepsilon.$$

No such L exists, so f is not differentiable at $x = 0$.

4. Classification:

$$f \in C^0(\mathbb{R}) \quad \text{but} \quad f \notin C^1(\mathbb{R}).$$

Moreover, f is *Lipschitz* on any compact $[\delta, 1]$ but not globally due to unbounded derivative near 0.

Conclusion: Uniform continuity does not imply differentiability. This function is a classic counterexample illustrating the strictness of the differentiability condition.

Example I.10 (Oscillatory Derivative and the Mean Value Theorem)**.** Let

$$f(x) = \begin{cases} x^2\cos(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

1. Continuity at $x = 0$:

$$|f(x)| = |x^2\cos(1/x)| \leq x^2 \to 0 \quad \text{as } x \to 0.$$

Hence f is continuous at 0. By ε–δ argument, given $\varepsilon > 0$, let $\delta = \sqrt{\varepsilon}$, then

$$|x| < \delta \Rightarrow |f(x) - f(0)| = |x^2\cos(1/x)| < \varepsilon.$$

2. Differentiability at $x = 0$: Compute the limit of the difference quotient:

$$\frac{f(h) - f(0)}{h} = \frac{h^2\cos(1/h)}{h} = h\cos(1/h) \to 0.$$

Thus $f'(0) = 0$.

3. Derivative for $x \neq 0$:

$$f'(x) = 2x\cos(1/x) + \sin(1/x).$$

This derivative is *not continuous* at 0, since $\sin(1/x)$ oscillates without bound. Thus, $f \in C^0(\mathbb{R})$, but $f \notin C^1(\mathbb{R})$.

4. Failure of Uniform Convergence: Let $f_n(x) = x^2\cos(nx)$ on $[-1, 1]$. Then $f_n \to 0$ pointwise but not uniformly. This mimics the behavior of $f(x)$ near 0: the limiting function exists and is continuous, but the convergence of derivatives is not uniform.

5. Mean Value Theorem Insight: Let f be as above and consider $[-h, h]$ for small h. Then by MVT, $\exists\, c_h \in (-h, h)$ such that:

$$f'(c_h) = \frac{f(h) - f(-h)}{2h} = 0.$$

So although $f'(x)$ oscillates between -1 and 1, the average slope is zero — yet this does not imply pointwise convergence of the derivative.

6. Classification:

$$f \in C^0(\mathbb{R}), \quad f \text{ differentiable everywhere}, \quad f' \notin C^0(\mathbb{R}).$$

Takeaway: The derivative can exist at every point while still being discontinuous. Differentiability does not imply the derivative behaves "nicely." This requires additional structure such as uniform continuity or bounded variation.

I.4 Measure Theory Foundations

Measure theory begins with a precise description of the collections of sets on which one will define measures. The set-theoretic structure of these collections dictates the kind of limits and set operations the measure will be compatible with. The following hierarchy is central:

Definition I.4.1. Algebra of Sets.

Let X be a nonempty set. An *algebra of sets* $\mathcal{A}$ on X is a nonempty collection of subsets of X such that:

1. $X \in \mathcal{A}$ (unit element).
2. If $A \in \mathcal{A}$, then its complement
$$A^c := X \setminus A \tag{I.4.1}$$
belongs to $\mathcal{A}$ (closure under complements).
3. If $A, B \in \mathcal{A}$, then
$$A \cup B \in \mathcal{A} \tag{I.4.2}$$
(closure under finite unions).

By De Morgan's laws, $\mathcal{A}$ is also closed under finite intersections. Thus $(\mathcal{A}, \cup, \cap, {}^c)$ forms a Boolean algebra of sets with X as the unit.

Remark I.4.1. (Boolean Structure)

An algebra of sets corresponds exactly to a Boolean algebra $(\mathcal{A}, \vee, \wedge, \neg)$ where the Boolean operations are interpreted as union, intersection, and complement relative to X. This allows purely algebraic manipulations of sets without explicit recourse to pointwise reasoning. ♦

Definition I.4.2. ***σ*–Algebra of Sets.**

Let X be a nonempty set. A *σ–algebra* $\mathcal{F}$ on X is an algebra of sets that is additionally closed under *countable* unions:

1. $X \in \mathcal{F}$.
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$, then
$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}. \tag{I.4.3}$$

By De Morgan's laws, $\mathcal{F}$ is also closed under countable intersections. Every σ–algebra is an algebra of sets, but the converse is false in general.

Remark I.4.2. (Why σ–Algebras?)

Closure under countable unions is essential for defining measures that are *countably additive*, a property required for the convergence theorems underpinning integration theory.

From the perspective of *deterministic calculus*, one typically works with smooth functions on open sets of $\mathbb{R}^n$, where limits are taken over sequences of points or deterministic domains, and set operations are finite in nature. In this regime, finite unions and intersections suffice for most analytic constructions.

Thus, while deterministic systems can often be developed over algebras of sets, stochastic systems demand σ–algebras to ensure stability of measurability under the countable operations inherent in probabilistic limit processes.

In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the information available up to time $t \geq 0$ is represented by a sub–σ–algebra $\mathcal{F}_t \subset \mathcal{F}$, with $(\mathcal{F}_t)_{t\geq 0}$ forming a *filtration*. Events such as
$$\left\{\lim_{n\to\infty} X_{t_n} \text{ exists}\right\}$$

for $(t_n)_{n\geq 1} \downarrow t$ can be expressed as countable intersections

$$\bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n\geq N} \left\{ |X_{t_n} - X_t| < \frac{1}{m} \right\},$$

each set $\{\, |X_{t_n} - X_t| < 1/m \,\}$ belonging to some $\mathcal{F}_{t_n}$. Closure of each $\mathcal{F}_t$ under countable unions and intersections guarantees that the limiting event is also in $\mathcal{F}_t$ and hence measurable.

Similarly, the definition of a stopping time τ with respect to $(\mathcal{F}_t)$,

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0,$$

requires that the σ–algebra $\mathcal{F}_t$ be stable under countable unions/intersections, since $\{\tau \leq t\}$ is often constructed via sequences of simpler events:

$$\{\tau \leq t\} = \bigcup_{q\in\mathbb{Q}\cap[0,t]} \{\tau = q\}.$$

Without the σ–algebra property, such limits and unions could leave the measurable domain, making probability assignments undefined.

Thus, while deterministic systems can often be developed over algebras of sets, stochastic systems demand σ–algebras to ensure stability of measurability under the countable operations inherent in probabilistic limit processes. ♦

Definition I.4.3. Measurable Set.

Let $(X, \mathcal{F}, \mu)$ be a measure space, where $\mathcal{F}$ is a σ–algebra on X and μ is a measure defined on $\mathcal{F}$. A set $E \subseteq X$ is *measurable* (with respect to $\mathcal{F}$) if $E \in \mathcal{F}$.

In the case of Lebesgue measure m on $\mathbb{R}$, a set $E \subseteq \mathbb{R}$ is *Lebesgue measurable* if it satisfies *Carathéodory's criterion*:

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \tag{I.4.4}$$

for every $A \subseteq \mathbb{R}$, where m^* denotes Lebesgue outer measure. The σ–algebra $\mathcal{L}(\mathbb{R})$ of all Lebesgue-measurable sets contains the Borel σ–algebra $\mathcal{B}(\mathbb{R})$ and is *complete* (it contains every subset of any null set).

Theorem I.4.1. Every σ–Algebra is an Algebra; Not Every Algebra is a σ–Algebra.

Let X be a nonempty set.

1. If $\mathcal{F}$ is a σ–algebra on X, then $\mathcal{F}$ is an algebra on X.
2. If X is infinite, there exists an algebra $\mathcal{A}$ on X that is not a σ–algebra.

⌟

Proof I.4.1. **(1) σ–algebra $\Rightarrow$ algebra.** Let $\mathcal{F}$ be a σ–algebra on X. Then:

$$X \in \mathcal{F}, \quad A \in \mathcal{F} \;\Rightarrow\; A^c \in \mathcal{F}, \quad \{A_n\}_{n\geq 1} \subset \mathcal{F} \;\Rightarrow\; \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

If $A_1, \ldots, A_k \in \mathcal{F}$, extend to a countable family by $A_n = \emptyset$ for $n > k$. Then

$$\bigcup_{i=1}^{k} A_i \;=\; \bigcup_{n=1}^{\infty} A_n \;\in\; \mathcal{F}.$$

Thus $\mathcal{F}$ is closed under finite unions. Closure under complements and inclusion of X are already in the definition, so $\mathcal{F}$ is an algebra.

(2) Existence of an algebra that is not a σ–algebra (infinite X).

Example A (finite–cofinite algebra). Assume X is infinite. Define

$$\mathcal{A} := \{A \subseteq X \mid A \text{ is finite}\} \ \cup \ \{A \subseteq X \mid A^c \text{ is finite}\}.$$

Algebra: $X, \emptyset \in \mathcal{A}$. If $A \in \mathcal{A}$ then either A or A^c is finite, hence $A^c \in \mathcal{A}$. If $A, B \in \mathcal{A}$:

- finite $\cup$ finite $\implies$ finite,
- finite $\cup$ cofinite $\implies$ cofinite,
- cofinite $\cup$ cofinite $\implies$ cofinite,

so $A \cup B \in \mathcal{A}$. Thus $\mathcal{A}$ is an algebra.

Not a σ–algebra: Choose a countably infinite subset $\{x_n : n \in \mathbb{N}\} \subset X$. Each singleton $A_n := \{x_n\}$ is finite and hence in $\mathcal{A}$, but

$$\bigcup_{n=1}^{\infty} A_n$$

is countably infinite with infinite complement, hence neither finite nor cofinite, so it is not in $\mathcal{A}$. Thus $\mathcal{A}$ is not a σ–algebra.

Example B (interval algebra on $\mathbb{R}$). Let $\mathcal{B}$ be the set of finite unions of half-open intervals $(a, b]$, allowing $\pm\infty$ endpoints. $\mathcal{B}$ is an algebra (finite unions, complements stay in the same form), but not a σ–algebra: the countable union of disjoint intervals $\{(n, n+1]\}_{n\in\mathbb{N}}$ belongs to no finite-union representation of that type.

Either example proves the existence claim. □

Remark I.4.3. (Hierarchy of Set Systems)

There is a strict hierarchy:

$$\sigma\text{–algebra} \quad \subset \quad \text{algebra} \quad \subset \quad \text{ring of sets} \quad \subset \quad \text{semiring}.$$

Each inclusion is proper in general. In measure construction, one typically starts from a semiring (e.g., half-open intervals) and extends the measure to the σ–algebra they generate via Carathéodory's extension theorem. ♦

Definition I.4.4. Lebesgue Measure on $\mathbb{R}$.

The *Lebesgue measure* m on $\mathbb{R}$ is the complete, translation-invariant measure defined on the σ–algebra of Lebesgue-measurable sets $\mathcal{L}(\mathbb{R})$ such that:

1. For every interval $I = (a, b) \subset \mathbb{R}$,
$$m(I) = b - a.$$
2. m is countably additive: if $\{E_k\}_{k=1}^{\infty}$ are pairwise disjoint measurable sets, then
$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

3. m is translation invariant: for all $t \in \mathbb{R}$ and $E \in \mathcal{L}(\mathbb{R})$,
$$m(E+t) = m(E),$$
where $E + t = \{x + t : x \in E\}$.

The measure m extends uniquely from the algebra of finite disjoint unions of open intervals (the *Lebesgue outer measure* construction) to the full σ–algebra $\mathcal{L}(\mathbb{R})$.

Definition I.4.5. Cantor–Lebesgue Function.

The *Cantor–Lebesgue function* $F : [0,1] \to [0,1]$, also called the *Cantor function* or *Devil's staircase*, is the unique continuous, nondecreasing, surjective function with the following properties:

1. F is constant on each connected component of $[0,1] \setminus C$, where C is the middle-thirds Cantor set.
2. $F(0) = 0$ and $F(1) = 1$.
3. If $x \in [0,1]$ has a ternary expansion
$$x = 0.a_1a_2a_3\ldots_3$$
with $a_k \in \{0,2\}$ (choosing the expansion ending in infinitely many 0's when ambiguous), then $F(x)$ is obtained by replacing each digit 2 with 1 to form the binary expansion:
$$F(x) = 0.b_1b_2b_3\ldots_2\,,$$
where $b_k = a_k/2$.

Equivalently, F is the distribution function of the probability measure supported on the Cantor set which assigns equal mass to each subinterval at each stage of the Cantor construction. It is singular with respect to Lebesgue measure: F is constant on intervals of positive length, yet F increases from 0 to 1 on a set of Lebesgue measure zero.

Definition I.4.6. Almost Everywhere.

Let $(X, \mathcal{F}, \mu)$ be a measure space, meaning:

1. X is a nonempty set (the underlying space of points);
2. $\mathcal{F} \subset \mathcal{P}(X)$ is a σ–algebra of subsets of X, i.e.:
   (a) $X \in \mathcal{F}$,
   (b) If $A \in \mathcal{F}$ then $A^c := X \setminus A \in \mathcal{F}$,
   (c) If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, and hence also $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$ by De Morgan's laws;
3. $\mu : \mathcal{F} \to [0,\infty]$ is a measure, meaning:
   (a) $\mu(\varnothing) = 0$,
   (b) (*Countable additivity*) If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$ are pairwise disjoint, then
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Let P be a property (predicate) that can be evaluated for each $x \in X$. We say that $P(x)$ *holds for μ–almost every $x \in X$*, written $P(x)$ *holds μ–a.e.*, if the set

$$E := \{x \in X \mid P(x) \text{ is false}\}$$

is measurable ($E \in \mathcal{F}$) and satisfies

$$\mu(E) = 0.$$

Equivalently, there exists a measurable set $N \in \mathcal{F}$ with $\mu(N) = 0$ such that $P(x)$ is true for all $x \in X \setminus N$.

Definition I.4.7. Banach Space.

Let $\mathbb{K}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

A *normed vector space* over $\mathbb{K}$ is a pair $(V, \|\cdot\|)$ such that:

1. **Vector space structure:** V is a vector space over $\mathbb{K}$ with operations:
   - vector addition $+ : V \times V \to V$,
   - scalar multiplication $\cdot : \mathbb{K} \times V \to V$,

   satisfying the standard vector space axioms (associativity, commutativity of addition, distributive laws, existence of identity elements and additive inverses, scalar identity, etc.).

2. **Norm:** A map
$$\|\cdot\| : V \longrightarrow [0, \infty)$$
   assigning a non-negative real number to each vector, such that for all $x, y \in V$ and $\alpha \in \mathbb{K}$:
   (a) *Positive definiteness:* $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.
   (b) *Absolute homogeneity:* $\|\alpha x\| = |\alpha| \, \|x\|$.
   (c) *Triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$.

3. **Induced metric:** The norm induces a metric
$$d : V \times V \to [0, \infty), \quad d(x, y) := \|x - y\|,$$
   making (V, d) into a metric space.

4. **Cauchy sequence:** A sequence $(x_n)_{n \in \mathbb{N}} \subset V$ is called *Cauchy* if
$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall m, n \geq N : \quad \|x_n - x_m\| < \varepsilon.$$

5. **Completeness:** The normed vector space $(V, \|\cdot\|)$ is called a *Banach space* if every Cauchy sequence in V converges to a limit in V with respect to the norm:
$$[(x_n) \text{ Cauchy in } V] \implies \exists x \in V \; \text{ such that } \; \|x_n - x\| \to 0.$$

I.5 Lebesgue Integration

The Lebesgue integral emerged from Henri Lebesgue's doctoral thesis Lebesgue [1902] and subsequent lectures Lebesgue [1904] as a radical reformation of the concept of integration. Rather than partitioning the *domain* into subintervals, as in the Riemann approach, Lebesgue proposed measuring the *range* of the function and summing contributions from level sets, each weighted by their measure. This shift made possible a unified theory in which measurable sets and measurable functions interact harmoniously with limits (§I.4), providing the foundation for the dominated convergence theorem, monotone convergence theorem, and Fatou's lemma.

The historical motivation was twofold: first, to enlarge the class of integrable functions beyond those amenable to Riemann sums; second, to establish an integral compatible with limit processes central to Fourier analysis and potential theory. Lebesgue's formulation subsumed and extended earlier results of Borel, Baire, and Jordan, and became indispensable in probability theory, ergodic theory, and partial differential equations.

The adoption of Lebesgue's method by later analysts—most notably in measure-theoretic probability (Kolmogorov), functional analysis (Riesz), and harmonic analysis (Wiener)—cemented its role as the standard integration framework in modern mathematics. From the perspective of mathematical physics, it provides the natural integration theory over configuration and phase spaces where functions may fail to be continuous yet retain measurable structure.

Definition I.5.1. Simple Function.

Let $(X, \mathcal{F}, \mu)$ be a measure space. A function $\varphi : X \to [0, \infty)$ is called *simple* if it takes only finitely many values, i.e.,

$$\varphi(x) = \sum_{k=1}^{n} a_k \, \chi_{E_k}(x), \tag{I.5.1}$$

where $a_k \geq 0$, $E_k \in \mathcal{F}$, and χ_{E_k} is the indicator function:

$$\chi_{E_k}(x) = \begin{cases} 1, & x \in E_k, \\ 0, & x \notin E_k. \end{cases} \tag{I.5.2}$$

Definition I.5.2. Integral of a Simple Function.

If φ is as in (I.5.1), its *Lebesgue integral* is

$$\int_X \varphi \, d\mu := \sum_{k=1}^{n} a_k \, \mu(E_k). \tag{I.5.3}$$

This is well-defined because μ is countably additive.

Definition I.5.3. Lebesgue Integral of a Nonnegative Measurable Function.

If $f : X \to [0, \infty]$ is measurable, define

$$\int_X f \, d\mu := \sup \left\{ \int_X \varphi \, d\mu \;\middle|\; \varphi \text{ simple}, \; 0 \leq \varphi \leq f \right\}. \tag{I.5.4}$$

Thus the integral is the supremum of integrals of under-approximating simple functions.

Theorem I.5.1. Monotone Convergence Theorem (MCT).

If $f_n \uparrow f$ pointwise and $f_n, f \geq 0$ are measurable, then

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu. \tag{I.5.5}$$

⌟

Theorem I.5.2. Fatou's Lemma.

For nonnegative measurable (f_n),

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \;\le\; \liminf_{n\to\infty} \int_X f_n \, d\mu. \tag{I.5.6}$$

⌟

Theorem I.5.3. Dominated Convergence Theorem (DCT).

If $f_n \to f$ pointwise a.e. and $|f_n| \le g$ for some integrable g (i.e. $\int_X g \, d\mu < \infty$), then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu. \tag{I.5.7}$$

⌟

Definition I.5.4. L^p Spaces.

For $1 \le p < \infty$,

$$L^p(X, \mathcal{F}, \mu) := \left\{ f \;\middle|\; f \text{ measurable}, \;\; \|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} < \infty \right\}. \tag{I.5.8}$$

For $p = \infty$,

$$L^\infty(X, \mathcal{F}, \mu) := \{ f \mid f \text{ measurable}, \;\; \|f\|_\infty := \operatorname{ess\,sup}_{x\in X} |f(x)| < \infty \}. \tag{I.5.9}$$

Theorem I.5.4. Hölder's Inequality.

If $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p$, $g \in L^q$, then

$$\int_X |fg| \, d\mu \le \|f\|_p \, \|g\|_q. \tag{I.5.10}$$

⌟

Proof I.5.1. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p$, $g \in L^q$. If $\|f\|_p = 0$ or $\|g\|_q = 0$, the inequality is trivial. Assume $\|f\|_p, \|g\|_q > 0$ and set

$$u := \frac{|f|}{\|f\|_p}, \qquad v := \frac{|g|}{\|g\|_q}.$$

By the definition of the L^p–norm (I.5.8), we have $\int_X u^p \, d\mu = 1$ and $\int_X v^q \, d\mu = 1$.

Young's inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ (for $a, b \ge 0$) gives pointwise

$$u(x)\, v(x) \;\le\; \frac{u(x)^p}{p} + \frac{v(x)^q}{q}.$$

Integrating and using $\int u^p = \int v^q = 1$,

$$\int_X u\, v \, d\mu \;\le\; \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying by $\|f\|_p \, \|g\|_q$ and using $|fg| = \|f\|_p \|g\|_q \, u \, v$ yields

$$\int_X |fg| \, d\mu \;\le\; \|f\|_p \, \|g\|_q,$$

which is (I.5.10). If one prefers to justify the step from bounded to general f, g, approximate by truncations $f_N := \min\{|f|, N\} \, \mathrm{sgn}(f)$, $g_N := \min\{|g|, N\} \, \mathrm{sgn}(g)$, apply the bounded-case argument, and pass to the limit using (I.5.7), since $|f_N g_N| \le |f||g|$ and $f_N \to f$, $g_N \to g$ a.e. □

Theorem I.5.5. Minkowski's Inequality.

If $1 \le p < \infty$ and $f, g \in L^p$, then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p. \tag{I.5.11}$$

⌟

Proof I.5.2. Fix $1 \le p < \infty$ and $f, g \in L^p$. For $p = 1$, the result is the triangle inequality for integrals, which follows directly from (I.5.8). Assume $1 < p < \infty$ and let q be the Hölder conjugate of p.

Write

$$\|f+g\|_p^p = \int_X |f+g|^p \, d\mu = \int_X |f+g| \, |f+g|^{p-1} \, d\mu.$$

Apply Hölder's inequality (I.5.10) with exponents (p, q) to the pairs $\big(|f|, |f+g|^{p-1}\big)$ and $\big(|g|, |f+g|^{p-1}\big)$:

$$\int_X |f| \, |f+g|^{p-1} \, d\mu \le \|f\|_p \, \big\| |f+g|^{p-1} \big\|_q, \qquad \int_X |g| \, |f+g|^{p-1} \, d\mu \le \|g\|_p \, \big\| |f+g|^{p-1} \big\|_q.$$

Since $q = \frac{p}{p-1}$, we have

$$\big\| |f+g|^{p-1} \big\|_q = \left(\int_X |f+g|^{(p-1)q} \, d\mu \right)^{1/q} = \left(\int_X |f+g|^p \, d\mu \right)^{(p-1)/p} = \|f+g\|_p^{p-1}.$$

Therefore,

$$\|f+g\|_p^p = \int_X |f+g| \, |f+g|^{p-1} \le \big(\|f\|_p + \|g\|_p \big) \, \|f+g\|_p^{p-1}.$$

If $\|f+g\|_p = 0$ there is nothing to show; otherwise divide both sides by $\|f+g\|_p^{p-1}$ to obtain

$$\|f+g\|_p \le \|f\|_p + \|g\|_p,$$

which is (I.5.11). As above, one may pass from bounded approximants to f, g via (I.5.7) if desired. □

Remark I.5.1. (Riemann Integration as a Special Case of Lebesgue Integration)

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it is also Lebesgue integrable and the two integrals agree:

$$\int_a^b f(x) \, dx_{\text{Riemann}} = \int_{[a,b]} f \, d\lambda, \tag{I.5.12}$$

where λ denotes Lebesgue measure.

Structural relationship: The Riemann integral may be viewed as a *degenerate* or *coarse* form of the Lebesgue integral in which:

- The domain $[a, b]$ is partitioned into finitely many *intervals* of small length, and the contribution of each interval is approximated by the value of f at a chosen point (or by upper/lower bounds).
- The mesh of the partition tends to 0, but the measurable sets being integrated over are restricted to these intervals. One never refines along the *range* of f.

In contrast, the Lebesgue integral (I.5.4) integrates by partitioning the *range* of f into slices and measuring the size of the preimages of these slices (level sets). This allows much finer control, since these preimages may be highly irregular subsets of $[a, b]$, not just intervals. As a result:

$$\text{Riemann} \subsetneq \text{Lebesgue},$$

meaning every Riemann integrable function is Lebesgue integrable with the same value (I.5.12), but the converse fails.

Degeneracy interpretation: If one insists in the Lebesgue construction (I.5.4) that the approximating simple functions φ be *step functions constant on intervals*, then one recovers the Riemann integration process. In this sense, the Riemann integral is the Lebesgue integral computed within a very restricted σ–algebra: the algebra generated by finite unions of intervals. The full Lebesgue theory arises when one allows *all* Lebesgue–measurable sets, greatly enlarging the class of integrable functions.

Example: The indicator $\chi_{\mathbb{Q}\cap[0,1]}$ is Lebesgue integrable with value 0, since $\mathbb{Q} \cap [0, 1]$ has λ–measure zero. However, it is not Riemann integrable because every subinterval of $[0, 1]$ contains both rationals and irrationals, making the Riemann upper sum = 1 and lower sum = 0 for all partitions. ♦

Remark I.5.2. (Cantor–Lebesgue Function as a Measure-Theoretic Object)

Let $C([0, 1])$ denote the Banach space of real-valued continuous functions on $[0, 1]$ with the uniform norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Key properties:

$$C([0, 1]) \text{ is complete}, \quad C([0, 1]) \text{ is separable}, \quad C([0, 1]) \text{ is infinite-dimensional}.$$

Chain of inclusions:

$$C^1([0, 1]) \subsetneq \mathrm{AC}([0, 1]) \subsetneq C([0, 1]).$$

The Cantor–Lebesgue function $K(x)$ satisfies:

$$K(x) \in C([0, 1]), \quad K(x) \notin \mathrm{AC}([0, 1]), \quad K'(x) = 0 \text{ for a.e. } x \in [0, 1].$$

Apparent paradox:

$$f \in \mathrm{AC}([0, 1]),\ f'(x) = 0 \text{ a.e.} \ \Rightarrow\ f \text{ constant},$$

$$K'(x) = 0 \text{ a.e.},\ K \notin \mathrm{AC}([0, 1]) \ \Rightarrow\ K \text{ nonconstant}.$$

Measure-theoretic structure:

$$\frac{dK}{dx} = \mu_K, \quad \mu_K \perp \lambda,$$

where μ_K is a singular measure supported on the Cantor set, and λ is Lebesgue measure.

Interpretation: $K(x)$ is continuous and increasing, yet increases on a set of λ-measure zero. Its distributional derivative is a measure *mutually singular* with λ. Thus, K cannot be recovered by integration of its (a.e. zero) pointwise derivative—only by integrating with respect to μ_K.

This illustrates that in the Lebesgue framework, integration is fundamentally an operation

$$\int f \, d\mu$$

where the choice of measure μ encodes the geometric and analytic structure of the problem. For $K(x)$, the relevant measure is not λ but μ_K. ♦

We conclude this subsection with the notion of Lipschitz functions:

Definition I.5.5. ***L*–Lipschitz.**

Let $(\mathbb{R}^n, \|\cdot\|)$ and $(\mathbb{R}^m, \|\cdot\|)$ be equipped with Euclidean norms. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *L–Lipschitz* if

$$\|f(x) - f(y)\| \ \leq \ L \, \|x - y\| \quad \forall \, x, y \in \mathbb{R}^n. \tag{I.5.13}$$

The least such L is denoted $\mathrm{Lip}(f)$.

Theorem I.5.6. Rademacher's Theorem.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfies (I.5.13), then:

1. f is (Fréchet) differentiable almost everywhere in $\mathbb{R}^n$.
2. For almost every x,
$$\|Df(x)\|_{\mathrm{op}} \ \leq \ L, \tag{I.5.14}$$
where $\|\cdot\|_{\mathrm{op}}$ is the operator norm induced by the Euclidean norms.

⌟

Proof I.5.3. **Step 1 (Mollification).** Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier with $\rho \geq 0$, $\int_{\mathbb{R}^n} \rho = 1$, and define

$$\rho_\varepsilon(x) := \varepsilon^{-n} \rho\left(\frac{x}{\varepsilon}\right), \qquad f_\varepsilon := f * \rho_\varepsilon.$$

Then $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ and $f_\varepsilon \to f$ locally uniformly as $\varepsilon \to 0$.

Step 2 (Gradient bound). Differentiating under the convolution sign gives

$$\nabla f_\varepsilon = (\nabla \rho_\varepsilon) * f.$$

Using (I.5.13) and $\int_{\mathbb{R}^n} \nabla \rho_\varepsilon = 0$, one obtains the uniform estimate

$$\|\nabla f_\varepsilon(x)\|_{\mathrm{op}} \ \leq \ L \quad \forall x \in \mathbb{R}^n, \ \varepsilon > 0. \tag{I.5.15}$$

Step 3 (Passage to the limit). Fix a Lebesgue point x of f (almost every x is such). From (I.5.15) and the uniform convergence $f_\varepsilon \to f$ near x, one checks:

$$\frac{\|f(x+h) - f(x) - \nabla f_\varepsilon(x) h\|}{\|h\|} \ \to \ 0 \quad \text{as } h \to 0, \ \varepsilon \to 0.$$

Thus the limit $\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A_x h\|}{\|h\|} = 0$ exists with $A_x = \lim_{\varepsilon \to 0} \nabla f_\varepsilon(x)$, and (I.5.15) ensures $\|A_x\|_{\mathrm{op}} \leq L$. Set $Df(x) := A_x$.

Step 4 (Conclusion). The above holds for almost every x, giving differentiability a.e. and the bound (I.5.14). □

Corollary I.5.1. Essential supremum bound.

If f is L–Lipschitz, then for any measurable $E \subset \mathbb{R}^n$,

$$\|\nabla f\|_{L^\infty(E)} := \operatorname*{ess\,sup}_{x \in E} \|Df(x)\|_{\mathrm{op}} \ \le\ L.$$

$\rightarrowtail$

Remark I.5.3. (Geometric meaning)

Inequality (I.5.13) is a *global* control on stretching: inputs can move only so far before outputs do. Theorem I.5.6 says that, outside a null set, this global control is realised by a *local linear map* $Df(x)$, whose slope bound (I.5.14) matches the global Lipschitz constant. Finite global stretch ⇒ honest tangent almost everywhere, with the same stretch bound. ♦

We now proceed to multivariable generalizations and the geometric structure of the Jacobian matrix.

I.6 Multivariable Differentiation and the Jacobian

Definition I.6.1. Real vector space.

A *real vector space* is a set V equipped with:

1. *Vector addition*
$$+ \,:\, V \times V \to V, \tag{I.6.1}$$
2. *Scalar multiplication*
$$\cdot \,:\, \mathbb{R} \times V \to V, \tag{I.6.2}$$

such that for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$:

$$(u+v)+w = u+(v+w) \quad \text{(associativity)}, \tag{I.6.3}$$
$$u+v = v+u \quad \text{(commutativity)}, \tag{I.6.4}$$
$$\exists\, 0_V \in V: \quad v + 0_V = v \quad \text{(additive identity)}, \tag{I.6.5}$$
$$\exists\,(-v) \in V: \quad v + (-v) = 0_V \quad \text{(additive inverse)}, \tag{I.6.6}$$
$$\alpha(u+v) = \alpha u + \alpha v \quad \text{(left distributivity)}, \tag{I.6.7}$$
$$(\alpha+\beta)v = \alpha v + \beta v \quad \text{(right distributivity)}, \tag{I.6.8}$$
$$(\alpha\beta)v = \alpha(\beta v) \quad \text{(scalar associativity)}, \tag{I.6.9}$$
$$1_{\mathbb{R}}\, v = v \quad \text{(unit scalar)}. \tag{I.6.10}$$

The n-dimensional real vector space with standard basis $\{e_1, \dots, e_n\}$ is denoted $\mathbb{R}^n$.

Definition I.6.2. Linear map.

Let V, W be real vector spaces. A map

$$L : V \to W$$

is *linear* if for all $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$,

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v). \tag{I.6.11}$$

The space of all linear maps $V \to W$ is denoted $\mathcal{L}(V, W)$. If $\dim V = n$ and $\dim W = m$, then upon fixing ordered bases, each $L \in \mathcal{L}(V, W)$ corresponds uniquely to a matrix in $\mathbb{R}^{m\times n}$ (cf. Definition I.6.6).

Definition I.6.3. Normed vector space.

A *normed vector space* $(V, \|\cdot\|)$ is a real vector space V equipped with a map $\|\cdot\| : V \to [0, \infty)$ such that for all $u, v \in V$ and $\alpha \in \mathbb{R}$:

$$\|v\| = 0 \iff v = 0, \qquad \text{(definiteness)} \tag{I.6.12}$$

$$\|\alpha v\| = |\alpha|\,\|v\|, \qquad \text{(homogeneity)} \tag{I.6.13}$$

$$\|u + v\| \le \|u\| + \|v\|. \qquad \text{(triangle inequality)} \tag{I.6.14}$$

In $\mathbb{R}^n$, the standard Euclidean norm is

$$\|x\|_2 = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}. \tag{I.6.15}$$

When V is also a measurable space $(V, \mathcal{A})$, the norm $\|v\|$ can be regarded as a measurable function $V \to [0, \infty)$.

Definition I.6.4. Tangent vector and tangent space in $\mathbb{R}^n$.

At a point $a \in \mathbb{R}^n$, a *tangent vector* is an element

$$v \in \mathbb{R}^n \tag{I.6.16}$$

representing an infinitesimal displacement from a. The *tangent space* $T_a\mathbb{R}^n$ is the n-dimensional vector space of all such v, naturally isomorphic to $\mathbb{R}^n$. Under the Lebesgue measure λ on $\mathbb{R}^n$, a tangent vector can be viewed as a constant vector field $x \mapsto v$ which is λ-measurable.

Definition I.6.5. Differentiability in algebraic form.

Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ a function. We say that f is *differentiable at* $a \in U$ if there exists $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h\to 0} \frac{\|f(a + h) - f(a) - L(h)\|}{\|h\|} = 0. \tag{I.6.17}$$

This L is unique by (I.6.11) and is called the *total derivative* of f at a, denoted $Df(a)$. The mapping $x \mapsto Df(x)$, when it exists on a measurable set, is measurable with respect to λ.

Definition I.6.6. Matrix representation of a linear map.

Let $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and let $\{e_1, \dots, e_n\}$, $\{E_1, \dots, E_m\}$ be the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. The *matrix representation* $[L] \in \mathbb{R}^{m\times n}$ is given by

$$[L]_{ij} = i\text{-th coordinate of } L(e_j) \text{ in the basis } \{E_1, \dots, E_m\}. \tag{I.6.18}$$

Composition of linear maps corresponds to matrix multiplication: if $L_1 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $L_2 \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$, then

$$[L_2 \circ L_1] = [L_2]\,[L_1]. \tag{I.6.19}$$

Theorem I.6.1. Total Derivative as Best Linear Approximation.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at a point $x_0 \in \mathbb{R}^n$. Then there exists a unique linear transformation

$$Df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\lim_{\|h\|\to 0} \frac{\|f(x_0+h) - f(x_0) - Df(x_0)[h]\|}{\|h\|} = 0.$$

⌟

Proof I.6.1. By the definition of differentiability at x_0, there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ and a remainder function $r : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f(x_0 + h) = f(x_0) + L(h) + r(h),$$

where

$$\lim_{\|h\|\to 0} \frac{\|r(h)\|}{\|h\|} = 0.$$

This linear map L is uniquely determined by this property and is defined to be the *total derivative* of f at x_0, denoted $Df(x_0)$. □

Remark I.6.1. (Total derivative as tangent structure and pushforward)

The total derivative $Df(a)$ of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ at $a \in \mathbb{R}^n$ is the unique linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\lim_{\|h\|\to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0.$$

In the case $n = m = 1$, $L(h) = f'(a)h$ describes the slope of the tangent line to the graph of f at $(a, f(a))$. For $n > 1$, L defines the tangent plane (or hyperplane) to the graph of f at $(a, f(a))$.

From the perspective of smooth manifolds (M, g) and (N, h), if $f : M \to N$ is smooth and $p \in M$, the *differential* (pushforward)

$$df_p : T_pM \to T_{f(p)}N$$

is the coordinate-free generalization of $Df(a)$. It is characterised by

$$df_p(v) = \frac{d}{dt} f(\gamma(t))\bigg|_{t=0}$$

for any smooth curve γ in M with $\gamma(0) = p$ and $\dot\gamma(0) = v$.

Proof sketch (tangent hyperplane property): If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a, then

$$f(a + h) = f(a) + Df(a) \cdot h + o(\|h\|).$$

The graph $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ has as tangent affine n-plane:

$$P = \{(a, f(a)) + (u, Df(a) \cdot u) : u \in \mathbb{R}^n\}.$$

The deviation of Γ_f from P is of order $o(\|u\|)$, hence P is tangent at $(a, f(a))$.

Proof sketch (pushforward property): Let M, N be smooth manifolds with charts (U, φ) about p and (V, ψ) about $f(p)$, with $\varphi(p) = a$ and $\psi(f(p)) = b$. In local coordinates, f becomes

$$\tilde f = \psi \circ f \circ \varphi^{-1}.$$

Then

$$df_p = d(\psi^{-1})_b \circ D\tilde{f}(a) \circ d\varphi_p,$$

showing that the coordinate-free pushforward coincides with the classical Jacobian matrix in coordinates. ♦

Definition I.6.7. Jacobian Matrix and Tensorial Structure.

Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

(1) Analytic definition. The *Jacobian matrix* of f at x, denoted $J_f(x)$, is the $m \times n$ real matrix with entries

$$(J_f(x))_{ij} := \frac{\partial f_i}{\partial x_j}(x), \quad 1 \le i \le m,\ 1 \le j \le n. \tag{I.6.20}$$

Explicitly,

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}. \tag{I.6.21}$$

(2) Matrix–linear map correspondence. The Jacobian is the coordinate matrix of the *total derivative*

$$Df(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^m \tag{I.6.22}$$

with respect to the standard ordered bases $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ and $\{E_1, \ldots, E_m\}$ of $\mathbb{R}^m$. For any $h = (h^1, \ldots, h^n)^\top \in \mathbb{R}^n$,

$$Df(x)[h] \;=\; J_f(x)\, h \tag{I.6.23}$$

in matrix notation, or in index form,

$$\big(Df(x)[h]\big)^i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x)\, h^j. \tag{I.6.24}$$

(3) Tensorial interpretation. The coefficients $\frac{\partial f_i}{\partial x_j}(x)$ are the components of a rank-$(1,1)$ tensor:

$$J_f(x) \;\in\; T^1_1(\mathbb{R}^n, \mathbb{R}^m) \;\cong\; \mathbb{R}^m \otimes (\mathbb{R}^n)^*, \tag{I.6.25}$$

where $(\mathbb{R}^n)^*$ is the dual space of covectors on $\mathbb{R}^n$. Here, the upper index i denotes the output (contravariant) component, and the lower index j denotes the input (covariant) slot.

(4) Row–gradient structure. The i-th row of $J_f(x)$ is the transpose of the gradient vector $\nabla f_i(x)$:

$$\nabla f_i(x) := \left(\frac{\partial f_i}{\partial x_1}(x), \ldots, \frac{\partial f_i}{\partial x_n}(x)\right)^\top, \tag{I.6.26}$$

so that

$$J_f(x) = \begin{bmatrix} (\nabla f_1(x))^\top \\ \vdots \\ (\nabla f_m(x))^\top \end{bmatrix}. \tag{I.6.27}$$

(5) Transformation law under coordinate change. If $y = \phi(x)$ is a local diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{f} = f \circ \phi^{-1}$, then by the multivariable chain rule,

$$J_{\tilde{f}}(y) = J_f(x)\, J_{\phi^{-1}}(y), \tag{I.6.28}$$

which expresses the tensorial transformation of Df under basis changes.

(6) Measure–theoretic role. When $m = n$, the determinant $\det J_f(x)$ appears in the *change-of-variables formula* in Lebesgue integration:

$$\int_{f(U)} g(y)\, d\lambda_m(y) \;=\; \int_U g(f(x))\, \big|\det J_f(x)\big|\, d\lambda_n(x), \tag{I.6.29}$$

where λ_n is n-dimensional Lebesgue measure. Thus, $|\det J_f(x)|$ quantifies the local volume distortion induced by f.

(7) Smooth map between manifolds. If $f : M \to N$ is a smooth map between smooth manifolds of dimensions n and m, the Jacobian at $p \in M$ is the coordinate matrix of the pushforward

$$d_p f : T_p M \to T_{f(p)} N \tag{I.6.30}$$

with respect to chosen local charts on M and N. Equation (I.6.28) is then the manifestation of functoriality of the tangent functor $T(\cdot)$ applied to smooth maps.

Theorem I.6.2. Jacobian Matrix Represents the Total Derivative.
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Fréchet differentiable at $x_0 \in \mathbb{R}^n$. Then the total derivative

$$Df(x_0) \in \mathsf{Lin}(\mathbb{R}^n, \mathbb{R}^m)$$

is the unique linear map approximating f near x_0 in the sense of Fréchet, and is represented in the standard basis by the Jacobian matrix $J_f(x_0)$ from (I.6.20)–(I.6.21). Explicitly, for all $h \in \mathbb{R}^n$,

$$Df(x_0)[h] = J_f(x_0)\, h, \tag{I.6.31}$$

where the right-hand side is the matrix–vector product of (I.6.23). ⌟

Proof I.6.2. **Step 1: Fréchet differentiability.** By definition, f is differentiable at x_0 if there exists a bounded linear map $Df(x_0)$ satisfying the Fréchet limit property

$$\lim_{\|h\| \to 0} \frac{\|f(x_0 + h) - f(x_0) - Df(x_0)[h]\|}{\|h\|} = 0,$$

with $Df(x_0)$ unique. This map is the *total derivative* (I.6.22).

Step 2: Basis action and partial derivatives. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. From (I.6.20), the j-th column of $J_f(x_0)$ is

$$\frac{\partial f}{\partial x_j}(x_0) := Df(x_0)[e_j] \;\in\; \mathbb{R}^m.$$

In the row–gradient perspective (I.6.27), each row encodes the gradient of a component function.

Step 3: Reconstructing $Df(x_0)$. For $h = \sum_{j=1}^n h^j e_j$, linearity of $Df(x_0)$ gives

$$Df(x_0)[h] = \sum_{j=1}^n h^j Df(x_0)[e_j] = \sum_{j=1}^n h^j \frac{\partial f}{\partial x_j}(x_0),$$

which is exactly the coordinate expression (I.6.24). Thus we obtain (I.6.31).

Step 4: Tensorial viewpoint. By (I.6.25), $Df(x_0)$ is a type $(1,1)$ tensor

$$Df(x_0) \;\in\; T^1_1(\mathbb{R}^n, \mathbb{R}^m) \;\cong\; \mathbb{R}^m \otimes (\mathbb{R}^n)^*,$$

and $J_f(x_0)$ is its coordinate representation in the standard bases.

Step 5: Local linear approximation. The Fréchet limit property implies the first-order expansion

$$f(x_0 + h) = f(x_0) + J_f(x_0)\, h + o(\|h\|),$$

so $J_f(x_0)$ captures the entire first-order behaviour of f at x_0. □

Remark I.6.2. (Jacobian as Coordinate Representation of the Pushforward)
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be smooth. The differential of f at x is the pushforward

$$df_x : T_x\mathbb{R}^n \longrightarrow T_{f(x)}\mathbb{R}^m$$

defined by

$$df_x(v) = \frac{d}{dt} f(\gamma(t))\bigg|_{t=0},$$

for any smooth curve γ with $\gamma(0) = x$, $\dot{\gamma}(0) = v$. This definition is independent of the choice of γ and yields a linear map between tangent spaces.

Identifying $T_x\mathbb{R}^n \cong \mathbb{R}^n$ and $T_{f(x)}\mathbb{R}^m \cong \mathbb{R}^m$ via the canonical bases, the matrix of df_x in these bases is the Jacobian matrix $J_f(x)$ of (I.6.20)–(I.6.21). Explicitly,

$$[J_f(x)]_{ij} = \frac{\partial f^i}{\partial x^j}(x),$$

and for $v \in \mathbb{R}^n$,

$$df_x(v) = J_f(x)\, v$$

as in (I.6.23)–(I.6.24).

In the category of smooth manifolds, if $f : M \to N$ is smooth and $\dim M = n$, $\dim N = m$, the differential at $p \in M$,

$$df_p : T_pM \longrightarrow T_{f(p)}N,$$

is a well-defined $\mathbb{R}$-linear map depending smoothly on p. The Jacobian arises from df_p only after selecting local coordinate charts $\varphi : U \subset M \to \mathbb{R}^n$, $\psi : V \subset N \to \mathbb{R}^m$ with $p \in U$, $f(p) \in V$, in which case

$$J_{(\psi\circ f\circ\varphi^{-1})}(\varphi(p))$$

is the matrix representation of df_p in these coordinates. Under change of coordinates, this matrix transforms by the chain rule (I.6.28).

Conclusion: The Jacobian matrix is the coordinate realization of the intrinsic pushforward df_x, a section of the bundle

$$\mathsf{Hom}(TM,\, f^*TN) \;\cong\; T^1_1(M, N),$$

whose action on tangent vectors encodes the full infinitesimal behavior of f. ◆

Theorem I.6.3. Banach Fixed Point Theorem.
Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction:

$$\exists\, 0 \le \kappa < 1 \;\text{ such that }\; d(Tx, Ty) \le \kappa\, d(x, y) \quad \forall x, y \in X. \tag{I.6.32}$$

Then:

1. There exists a unique fixed point $x^* \in X$ with $Tx^* = x^*$.
2. For any $x_0 \in X$, the Picard iterates $x_{n+1} := Tx_n$ converge to x^*.

3. Error bounds:

$$d(x_n, x^*) \le \frac{\kappa^n}{1-\kappa}\, d(x_1, x_0), \tag{I.6.33}$$

$$d(x_n, x^*) \le \frac{\kappa}{1-\kappa}\, d(x_n, x_{n-1}) \quad (n \ge 1). \tag{I.6.34}$$

⌟

Proof I.6.3. **Step 1: Cauchy property of the Picard sequence.** Fix $x_0 \in X$ and define $x_{n+1} := Tx_n$. By (I.6.32),

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \le \kappa\, d(x_n, x_{n-1}) \le \cdots \le \kappa^n\, d(x_1, x_0).$$

For $m > n$,

$$d(x_m, x_n) \le \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \le d(x_1, x_0) \sum_{j=n}^{m-1} \kappa^j \le \frac{\kappa^n}{1-\kappa}\, d(x_1, x_0).$$

Hence (x_n) is Cauchy; completeness gives $x_n \to x^* \in X$.

Step 2: Fixed point property and uniqueness. By continuity of T (implied by (I.6.32)),

$$Tx^* = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*,$$

so x^* is a fixed point. If y^* is another fixed point, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \le \kappa\, d(x^*, y^*),$$

which forces $d(x^*, y^*) = 0$ since $\kappa < 1$; hence $x^* = y^*$.

Step 3: Error bounds. Taking $m \to \infty$ in the estimate for $d(x_m, x_n)$ yields (I.6.33). For (I.6.34), note

$$d(x_n, x^*) = d(Tx_{n-1}, Tx^*) \le \kappa\, d(x_{n-1}, x^*)$$

and iterate once:

$$d(x_{n-1}, x^*) \le d(x_{n-1}, x_n) + d(x_n, x^*) \;\Rightarrow\; (1-\kappa) d(x_n, x^*) \le \kappa\, d(x_n, x_{n-1}),$$

which gives (I.6.34). □

Theorem I.6.4. Inverse Function Theorem.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 on an open set $U \subset \mathbb{R}^n$. Suppose $x_0 \in U$ satisfies

$$\det J_f(x_0) \ne 0,$$

where J_f is given by (I.6.20)–(I.6.21). Then there exists an open neighborhood $V \subset U$ of x_0 such that

$$f|_V : V \longrightarrow f(V)$$

is a C^1 diffeomorphism. ⌟

Proof I.6.4. **Step 1: Linearization and notation.** Let

$$A \equiv Df(x_0) = J_f(x_0) \in \mathbb{R}^{n\times n}$$

where $Df(x_0)$ is the total derivative as in (I.6.22), and $J_f(x_0)$ its matrix representation in the standard bases per (I.6.23). Set $y_0 := f(x_0)$. By hypothesis, $\det A \neq 0$, so A is invertible. The Fréchet differentiability of f at x_0 gives the expansion

$$f(x) = y_0 + A(x - x_0) + R(x), \tag{I.6.35}$$

with R satisfying the small-o property

$$\lim_{x \to x_0} \frac{\|R(x)\|}{\|x - x_0\|} = 0. \tag{I.6.36}$$

Step 2: Fixed-point formulation. For y near y_0, define

$$\Phi_y(x) := x - A^{-1}\big(f(x) - y\big). \tag{I.6.37}$$

By (I.6.37), $\Phi_y(x) = x \iff f(x) = y$. The strategy is to show Φ_y is a contraction on a closed ball $B_\rho(x_0)$.

Step 3: Self-mapping property. From (I.6.35),

$$\Phi_y(x) - x_0 = -A^{-1}\big(R(x) + y_0 - y\big).$$

Taking norms yields

$$\|\Phi_y(x) - x_0\| \leq \|A^{-1}\|\big(\|R(x)\| + \|y - y_0\|\big). \tag{I.6.38}$$

From (I.6.36), for any $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\|R(x)\| \leq \varepsilon\|x - x_0\| \quad \forall x \in B_\rho(x_0). \tag{I.6.39}$$

If $y \in B_\delta(y_0)$, (I.6.38) and (I.6.39) give

$$\|\Phi_y(x) - x_0\| \leq \|A^{-1}\|\big(\varepsilon\rho + \delta\big).$$

The self-mapping condition is then

$$\|A^{-1}\|\big(\varepsilon\rho + \delta\big) \leq \rho. \tag{I.6.40}$$

This is feasible provided $\varepsilon < 1/\|A^{-1}\|$.

Step 4: Contraction property. Differentiating (I.6.37) and using (I.6.23) for $Df(x)$,

$$D\Phi_y(x) = I - A^{-1}Df(x) = A^{-1}\big(A - Df(x)\big).$$

Thus

$$\|D\Phi_y(x)\| \leq \|A^{-1}\| \cdot \|Df(x) - A\|.$$

By continuity of Df (which follows from $f \in C^1$), choose $\rho > 0$ so that

$$\sup_{x \in B_\rho(x_0)} \|Df(x) - A\| \leq \frac{1}{2\|A^{-1}\|}. \tag{I.6.41}$$

Then $\|D\Phi_y(x)\| \leq \frac{1}{2}$, i.e., Φ_y is a strict contraction on $B_\rho(x_0)$.

Step 5: Application of Banach's theorem. If (I.6.40) holds and ρ satisfies (I.6.41), then for each $y \in B_\delta(y_0)$ the map Φ_y is a contraction of $B_\rho(x_0)$ into itself. By the Banach fixed point theorem (Theorem I.6.3), there exists a unique $x_y \in B_\rho(x_0)$ such that

$$\Phi_y(x_y) = x_y, \quad \text{hence} \quad f(x_y) = y.$$

Step 6: Invertibility and smoothness. Define $f^{-1}(y) := x_y$ for $y \in B_\delta(y_0)$. Then f^{-1} is well-defined and continuous. Differentiating $f(f^{-1}(y)) = y$ and using (I.6.22) gives

$$Df^{-1}(y) = \big(Df(f^{-1}(y))\big)^{-1},$$

which is continuous because Df is. Thus $f^{-1} \in C^1$ and $f|_{B_\rho(x_0)}$ is a C^1 diffeomorphism onto its image. □

Remark I.6.3. (Jacobian determinant and the local diffeomorphism criterion)
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 on an open set $U \subset \mathbb{R}^n$. By the Inverse Function Theorem, if for some $x_0 \in U$ the Jacobian determinant

$$\det J_f(x_0) \neq 0, \tag{I.6.42}$$

with $J_f(x_0)$ as in (I.6.21), then there exists an open neighborhood $V \subset U$ of x_0 such that

$$f|_V : V \to f(V)$$

is a C^1-diffeomorphism. Condition (I.6.42) is thus both necessary and sufficient (at the level of first derivatives) for the map to be locally invertible with smooth inverse, and it ensures:

1. *Dimension preservation*: $\dim T_x\mathbb{R}^n = \dim T_{f(x)}\mathbb{R}^n$, i.e., the pushforward df_x in (I.6.30) is a linear isomorphism between tangent spaces.
2. *Orientation preservation or reversal*: The sign of $\det J_f(x_0)$ determines whether df_{x_0} preserves or reverses orientation.

In the smooth manifold setting, (I.6.42) is the analytic condition guaranteeing that a smooth map between n-dimensional manifolds is a local diffeomorphism. This underlies the legitimacy of smooth coordinate transitions and the construction of local charts: the change-of-coordinate map between overlapping charts has Jacobian determinant everywhere nonzero, ensuring compatibility of differentiable structures. ♦

Theorem I.6.5. Change of Variables in Lebesgue Integration.
Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ a C^1-diffeomorphism onto its image $f(U)$, and let $\phi : \mathbb{R}^n \to [0, \infty)$ be measurable with $\operatorname{supp}(\phi) \subset f(U)$ compact. Then

$$\int_{f(U)} \phi(y)\, d\lambda_n(y) \;=\; \int_U \phi(f(x))\, \big|\det J_f(x)\big|\, d\lambda_n(x), \tag{I.6.43}$$

where $J_f(x)$ is the Jacobian matrix as in (I.6.21) and λ_n denotes Lebesgue measure on $\mathbb{R}^n$. ⌟

Proof I.6.5. The proof proceeds in three stages:
(1) Affine case. If $f(x) = Ax + b$ with $A \in GL(n, \mathbb{R})$, then by a linear change of variables and translation invariance of Lebesgue measure λ_n, combined with the Riemann–Lebesgue agreement (I.5.12), we have

$$\int_{f(U)} \phi(y)\, d\lambda_n(y) = \int_U \phi(Ax + b)\, \big|\det A\big|\, d\lambda_n(x),$$

where $|\det A|$ is the constant volume distortion factor given by the Jacobian matrix definition (I.6.21).

(2) Local case for C^1-maps. Let $x_0 \in U$. By differentiability of f at x_0 (cf. Rademacher's theorem (I.5.14)), we have the linearization

$$f(x) = f(x_0) + J_f(x_0)(x - x_0) + o(\|x - x_0\|),$$

with $J_f(x_0)$ invertible by the local diffeomorphism hypothesis. On sufficiently small neighborhoods, f is bi-Lipschitz in the sense of (I.5.13) and maps measurable sets to measurable sets. The Jacobian determinant $|\det J_f(x)|$ is the pointwise limit of normalised volume ratios:

$$|\det J_f(x)| = \lim_{r\to 0} \frac{\lambda_n(f(B_r(x)))}{\lambda_n(B_r(x))}.$$

In these neighborhoods, for nonnegative measurable ϕ, we integrate via the Lebesgue definition (I.5.4), approximating $\phi \circ f$ from below by simple functions (I.5.1)–(I.5.3). Passage to the limit is justified by the Monotone Convergence Theorem (I.5.5).

(3) Globalization. By compactness of $\operatorname{supp}(\phi)$ and the open mapping property of f, one can cover $\operatorname{supp}(\phi) \cap f(U)$ by finitely many coordinate neighborhoods on which f is a bi-Lipschitz C^1-diffeomorphism. On each chart, apply the local formula from Step (2) and sum over the finite cover using a partition of unity. This yields (I.6.44) for nonnegative ϕ by countable additivity of the Lebesgue integral (I.5.4).

For general integrable ϕ, decompose $\phi = \phi_+ - \phi_-$ and apply the result to each part separately, using the Dominated Convergence Theorem (I.5.7) to pass to the limit when approximating $\phi_\pm$ by simple functions. □

Remark I.6.4. (Jacobian determinant and measure-theoretic volume scaling)

We seek to establish the measure-theoretic interpretation of the Jacobian determinant. For a C^1 map $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, the absolute value $|\det Df(x)|$ quantifies the local change in n–dimensional Lebesgue measure induced by f at the point x. Specifically, if dV_x denotes an infinitesimal volume element at x, then under the transformation $y = f(x)$ one has

$$dV_y \;=\; |\det Df(x)|\, dV_x.$$

This scaling factor appears as a multiplicative term in the change of variables formula for integrals:

$$\int_{f(A)} \phi(y)\, dy = \int_A \phi(f(x))\, |\det Df(x)|\, dx.$$

In the linear case $f(x) = Ax$, $|\det A|$ is exactly the absolute value of the determinant of the matrix A, representing the constant volume-scaling factor of the transformation. The nonlinear case generalises this intuition pointwise: the Jacobian determinant plays the role of the determinant of the best linear approximation $Df(x)$, encoding how local coordinate parallelepipeds are stretched, compressed, or reoriented under f. ◆

Theorem I.6.6. Manifold Interpretation of Differentiability.

Let M and N be smooth manifolds of dimensions n and m, respectively, and let $f : M \to N$ be a continuous map. We say f is *differentiable at* $p \in M$ if for every choice of coordinate charts (U, φ) about p and (V, ψ) about $f(p)$ with

$$\varphi : U \to \varphi(U) \subset \mathbb{R}^n, \quad \psi : V \to \psi(V) \subset \mathbb{R}^m,$$

the local representative

$$\psi \circ f \circ \varphi^{-1} : \varphi\big(U \cap f^{-1}(V)\big) \subset \mathbb{R}^n \to \mathbb{R}^m$$

is differentiable in the Euclidean sense at $\varphi(p)$. ⌟

Proof I.6.6. Smooth manifolds are glued together from Euclidean spaces via smooth coordinate changes. Since differentiability in $\mathbb{R}^n$ is defined by linear approximation on open sets, it is necessary to examine f locally in charts.

Let (U, φ) be a chart around p and (V, ψ) a chart around $f(p)$. Then $\varphi(U)$ and $\psi(V)$ are open in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The preimage $U \cap f^{-1}(V)$ is open in M by continuity of f and the openness of V, hence $\varphi(U \cap f^{-1}(V))$ is open in $\mathbb{R}^n$.

We define the *local representative* of f in these coordinates:

$$F := \psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \mathbb{R}^m.$$

The differentiability of f at p means F is differentiable at $\varphi(p)$ in the multivariable calculus sense, with derivative $DF(\varphi(p)) \in \mathbb{R}^{m \times n}$. This derivative depends on the chosen charts but transforms tensorially under coordinate changes, giving the coordinate-free differential $df_p : T_pM \to T_{f(p)}N$.

Thus, differentiability on manifolds is entirely local, reducible to the Euclidean case via charts, but the consistency across overlapping charts ensures the definition is intrinsic to the smooth structure. □

Remark I.6.5. (Role of open sets and coordinate charts)

The chart domains are open in M (relative topology), and their images are open in $\mathbb{R}^n$. This is essential because:

- Openness guarantees that Euclidean differentiability applies without boundary obstructions;
- The preimage $U \cap f^{-1}(V)$ being open allows F to have a neighborhood in which the limit definition of derivative is meaningful;
- The smooth compatibility of charts ensures that changing coordinates before or after applying f does not destroy differentiability.

♦

Remark I.6.6. (Jacobian as pushforward in coordinates)

Given $f : M \to N$ smooth, the differential $df_p : T_pM \to T_{f(p)}N$ is a linear map between tangent spaces. In coordinates $(x^1, \ldots, x^n)$ on M near p and $(y^1, \ldots, y^m)$ on N near $f(p)$, the matrix of df_p is the *Jacobian*:

$$(J_f(p))^i{}_j = \left.\frac{\partial(y^i \circ f \circ \varphi^{-1})}{\partial x^j}\right|_{\varphi(p)}.$$

This is a $(1,1)$-type tensor: contravariant in the N-coordinate index i and covariant in the M-coordinate index j. ♦

Example I.11 (Quadratic–bilinear map). Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be $f(x, y) = (x^2 + y^2,\ xy)$. Then:

$$J_f(x, y) = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}.$$

At $(1, 0)$, the Jacobian has rank 2 (local diffeomorphism), but at $(0, 0)$, the rank is 1, indicating loss of local invertibility.

Example I.12 (Mixed exponential–logarithmic map)**.** Let $u = e^x$, $v = \ln y$, and $f(u, v) = (u + v, uv)$. Then:

$$J_f(u, v) = \begin{bmatrix} 1 & 1 \\ v & u \end{bmatrix}, \quad J_{(u,v)}(x, y) = \begin{bmatrix} e^x & 0 \\ 0 & \frac{1}{y} \end{bmatrix},$$

and by the chain rule:

$$J_{f\circ(u,v)}(x, y) = J_f(u, v) \cdot J_{(u,v)}(x, y) = \begin{bmatrix} e^x & \frac{1}{y} \\ ve^x & \frac{u}{y} \end{bmatrix}.$$

Theorem I.6.7. Jacobian determinant as local volume distortion.
Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 diffeomorphism onto its image. For any $\phi \in L^1(F(U), \lambda_n)$ and measurable $A \subset U$,

$$\int_{F(A)} \phi(y)\, d\lambda_n(y) = \int_A \phi(F(x)) \left|\det DF(x)\right| d\lambda_n(x), \tag{I.6.44}$$

where λ_n is Lebesgue measure and $|\det DF(x)|$ is the infinitesimal n-dimensional volume scaling factor of F at x. ⌟

Proof I.6.7. **Step 1: Local linearization.** For fixed $x \in U$, the differentiability of F gives

$$F(x + h) = F(x) + DF(x)h + r(h), \quad \lim_{\|h\|\to 0} \frac{\|r(h)\|}{\|h\|} = 0.$$

Hence for sufficiently small $\varepsilon > 0$, the image of the cube $Q_\varepsilon(x)$ of side ε satisfies

$$\lambda_n(F(Q_\varepsilon(x))) = |\det DF(x)|\, \varepsilon^n + o(\varepsilon^n).$$

The constant $|\det DF(x)|$ is exactly the volume distortion factor from the Jacobian matrix (I.6.21).

Step 2: Partition and approximation. Let $A \subset U$ be measurable. Partition A into finitely many disjoint cubes $\{Q_i\}$ of diameter $< \delta$, and choose representatives $x_i \in Q_i$. For each Q_i,

$$\lambda_n(F(Q_i)) \approx |\det DF(x_i)|\, \lambda_n(Q_i),$$

where approximation is made precise using the linearization above. For nonnegative continuous ϕ with compact support, approximate $\phi \circ F$ from below by simple functions (I.5.1) satisfying (I.5.3). By the definition of the Lebesgue integral (I.5.4) and the Monotone Convergence Theorem (I.5.5), we obtain (I.6.44) for such ϕ.

Step 3: Extension to L^1. Given $\phi \in L^1(F(U))$, approximate in L^1 by bounded continuous functions with compact support using the density of such functions in L^1. Apply the result to ϕ_+ and ϕ_- separately. Passage to the limit is justified by the Dominated Convergence Theorem (I.5.7), yielding (I.6.44) for all integrable ϕ.

Conclusion. The Jacobian determinant $|\det DF(x)|$ is the Radon–Nikodým derivative of the pushforward measure $F_\#(\lambda_n|_U)$ with respect to λ_n on $F(U)$, i.e.

$$\frac{d(F_\#\lambda_n)}{d\lambda_n}(y) = |\det DF(F^{-1}(y))|,$$

which encodes the local n-dimensional volume distortion under F. □

Example I.13 (Polar coordinates: explicit computation)**.** Consider the C^∞ mapping

$$T : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2 \setminus \{0\}, \quad T(r, \theta) = (x, y) = (r \cos\theta,\, r \sin\theta).$$

Its Jacobian matrix is

$$J_{(x,y)}(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix},$$

and the determinant is

$$\det J_{(x,y)}(r, \theta) = r. \tag{I.6.45}$$

We apply the change-of-variables theorem (I.6.44) with the measurable function

$$\phi(x, y) = e^{-(x^2+y^2)}\, \chi_{B_1(0)}(x, y),$$

where $B_1(0)$ is the unit disk. The characteristic function $\chi_{B_1(0)}$ ensures ϕ has compact support, satisfying the integrability hypotheses via (I.5.4).

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we have

$$x^2 + y^2 = r^2,$$

so ϕ becomes

$$\phi(r, \theta) = e^{-r^2}\, \chi_{[0,1]}(r).$$

By (I.6.44) and (I.6.45),

$$\begin{aligned} \int_{\mathbb{R}^2} \phi(x, y)\, d\lambda_2(x, y) &= \int_0^{2\pi} \int_0^\infty e^{-r^2}\, \chi_{[0,1]}(r) \cdot r\, dr\, d\theta \\ &= \int_0^{2\pi} \left[\int_0^1 r e^{-r^2}\, dr \right] d\theta. \end{aligned}$$

We compute the inner radial integral by the substitution

$$u = r^2, \quad du = 2r\, dr, \quad r\, dr = \frac{1}{2}\, du,$$

giving

$$\int_0^1 r e^{-r^2}\, dr = \frac{1}{2} \int_0^1 e^{-u}\, du = \frac{1}{2} \left(1 - e^{-1}\right).$$

Substituting into the angular integral:

$$\int_{\mathbb{R}^2} \phi(x, y)\, d\lambda_2 = \int_0^{2\pi} \frac{1}{2} \left(1 - e^{-1}\right) d\theta = \pi \left(1 - e^{-1}\right).$$

Conclusion: The polar Jacobian factor (I.6.45) produces the correct area scaling, and the explicit computation yields the exact value of the integral over the unit disk.

I.7 Linear Maps and Eigenstructure

Having established the analytic and topological framework, we now turn to the linear setting. Here the objects are finite-dimensional vector spaces and the maps between them are linear, carrying with them a structure that is at once rigid and revealing. Linear maps are counterintuitive in that they are algebraically rich yet geometrically sparse. Given two finite-dimensional vector spaces, a linear map is determined entirely by its action on a basis — a finite list of vectors suffices to reconstruct its behavior everywhere. This economy of description carries immense algebraic machinery: composition, inversion, duality, spectral decomposition. Yet from the geometric side, a linear map says almost nothing about the global shape or topology of the spaces it connects: it preserves the origin, lines through the origin, and the linear structure, but it cannot distinguish between spaces of the same dimension that differ wildly in topology. In contrast, many constructions in topology — homeomorphisms, embeddings, coverings — are geometrically expressive, encoding subtle invariants of the spaces involved, yet carry little of the rigid algebraic superstructure that linearity enforces.

The paradox is that linear maps are simultaneously the most calculable and the least descriptive of the broader landscape in which they reside. They are calculable because every aspect of their action reduces to finite-dimensional algebra: coordinates, matrices, and finite sums. Once a basis is fixed, the entire transformation is encoded in a rectangular array of scalars, and its effect on any vector is obtained by a single matrix multiplication. They are least descriptive because this compression to a finite set of coefficients strips away almost all global information about the underlying spaces. Two vector spaces of the same dimension over the same field are indistinguishable to a linear map; no trace of curvature, connectivity, or other topological or geometric nuance survives. This stands in sharp contrast to continuous or differentiable maps, whose infinite-dimensional data — the behavior over every neighborhood and at every scale — can preserve and reveal far subtler invariants. Linear maps thus occupy a peculiar role: their rigidity makes them a perfect laboratory for explicit computation and spectral analysis, but it also confines their geometric voice to the narrow range allowed by linear structure alone.

The aim in this section is to lay out these notions in a form precise enough for computation, yet flexible enough to integrate naturally with the analytic and geometric machinery developed earlier.

Theorem I.7.1. Matrices and Linear Maps are Equivalent.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then:

1. T is linear if and only if there exists a matrix $A \in \mathbb{R}^{m\times n}$ such that

$$T(x) = Ax \quad \text{for all } x \in \mathbb{R}^n.$$

2. Conversely, every matrix $A \in \mathbb{R}^{m\times n}$ defines a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(x) = Ax$.

⌟

Proof I.7.1. (1) Suppose T is linear. Then for all $x, y \in \mathbb{R}^n$ and scalars $\alpha, \beta \in \mathbb{R}$,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Define:

$$a_j \equiv T(e_j) \in \mathbb{R}^m.$$

Then each $x \in \mathbb{R}^n$ can be written as $x = \sum_{j=1}^n x_j e_j$, so:

$$T(x) = T\left(\sum_{j=1}^n x_j e_j\right) = \sum_{j=1}^n x_j T(e_j) = \sum_{j=1}^n x_j a_j.$$

Define the matrix $A = [a_1 \ \cdots \ a_n] \in \mathbb{R}^{m\times n}$ with j-th column equal to a_j. Then:

$$T(x) = Ax.$$

Hence every linear map corresponds to matrix multiplication.

(2) Conversely, let $A \in \mathbb{R}^{m\times n}$ and define $T(x) = Ax$. Then for any $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$,

$$T(\alpha x + \beta y) = A(\alpha x + \beta y) = \alpha Ax + \beta Ay = \alpha T(x) + \beta T(y),$$

so T is linear.

□

Theorem I.7.2. Eigenvectors Characterise Invariant Directions.
Let $A \in \mathbb{R}^{n\times n}$. A nonzero vector $v \in \mathbb{R}^n$ satisfies

$$Av = \lambda v$$

for some $\lambda \in \mathbb{C}$ if and only if the line $\ell = \{cv : c \in \mathbb{R}\}$ is invariant under A, meaning:

$$A(\ell) \subseteq \ell.$$

⌟

Proof I.7.2. Suppose $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ and $v \neq 0$. Then for any $c \in \mathbb{R}$,

$$A(cv) = cAv = c\lambda v = \lambda(cv),$$

so $A(cv) \in \ell$. Thus ℓ is invariant under A.

Conversely, suppose $\ell = \{cv : c \in \mathbb{R}\}$ is A-invariant for some nonzero v. Then $Av \in \ell$, so there exists $\lambda \in \mathbb{R}$ such that

$$Av = \lambda v.$$

Hence v is an eigenvector of A with eigenvalue λ.

This proves that the eigenvectors of A are precisely the directions in which A acts as scalar multiplication, i.e., directions preserved by the action of A up to scaling. □

Example I.14 (Diagonalizable matrix)**.** Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

The characteristic polynomial is

$$\chi_A(\lambda) = \det(\lambda I - A) = (\lambda - 2)(\lambda - 3),$$

so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$ (distinct). For $\lambda_1 = 2$,

$$(A - 2I)v = 0 \implies \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} v = 0 \implies v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ (any nonzero multiple).}$$

For $\lambda_2 = 3$,

$$(A - 3I)v = 0 \implies \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} v = 0 \implies v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus $\{e_1, e_2\}$ is an eigenbasis. With $P = [e_1\ e_2] = I$, we have

$$P^{-1}AP = \mathrm{diag}(2, 3).$$

Conclusion. A is diagonalizable (indeed already diagonal).

Example I.15 (Defective (nondiagonalizable) matrix)**.** Consider

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}.$$

Then

$$\chi_A(\lambda) = \det(\lambda I - A) = (\lambda - 4)^2,$$

so $\lambda = 4$ with algebraic multiplicity 2. Solve for eigenvectors:

$$(A - 4I)v = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} v = 0 \implies v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ (up to scale).}$$

Thus the geometric multiplicity is $1 < 2$, so A is not diagonalizable.

A Jordan chain is obtained by choosing a generalised eigenvector w with

$$(A - 4I)w = v.$$

Take $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; solving

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \implies \qquad b = 1.$$

One convenient choice is $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In the basis $\{v, w\}$,

$$A \sim J = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}.$$

Conclusion. A is defective and admits a size-2 Jordan block.

Example I.16 (Symmetric matrix: orthogonal diagonalization)**.** Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = A^\top.$$

Then

$$\chi_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)(\lambda - 3) - 4 = \lambda^2 - 4\lambda - 1,$$

so the eigenvalues are

$$\lambda_\pm = 2 \pm \sqrt{5}.$$

For $\lambda_+ = 2 + \sqrt{5}$, solve $(A - \lambda_+ I)v = 0$:

$$\begin{bmatrix} 1 - \lambda_+ & 2 \\ 2 & 3 - \lambda_+ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \implies (1 - \lambda_+)x + 2y = 0 \implies y = \frac{\lambda_+ - 1}{2} x = \frac{1 + \sqrt{5}}{2} x.$$

One eigenvector is $v_+ = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$.

For $\lambda_- = 2 - \sqrt{5}$,

$$(1 - \lambda_-)x + 2y = 0 \implies y = \frac{\lambda_- - 1}{2} x = \frac{1 - \sqrt{5}}{2} x,$$

so $v_- = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$.

Because A is real symmetric, eigenvectors for distinct eigenvalues are orthogonal:

$$v_+^\top v_- \;=\; 1\cdot 1 + \frac{1+\sqrt{5}}{2}\cdot\frac{1-\sqrt{5}}{2} = 1 + \frac{1-5}{4} = 0.$$

Normalise

$$u_\pm = \frac{v_\pm}{\|v_\pm\|}, \qquad Q = [u_+ \; u_-] \text{ (orthogonal)},$$

then

$$Q^\top AQ = \operatorname{diag}(\lambda_+, \lambda_-) = \operatorname{diag}(2+\sqrt{5},\, 2-\sqrt{5}).$$

Conclusion. Real symmetric matrices admit an orthonormal eigenbasis; A is orthogonally diagonalizable.

Theorem I.7.3. Diagonalization Criteria and Spectral Structure of Real Matrices. Let $A \in \mathbb{R}^{n\times n}$. Denote by $\sigma(A) \subset \mathbb{C}$ the spectrum of A, and for each $\lambda \in \sigma(A)$ let

$$m_a(\lambda) := \dim_{\mathbb{C}} \ker\big((A-\lambda I)^n\big) \quad \text{(algebraic multiplicity)},$$

$$m_g(\lambda) := \dim_{\mathbb{C}} \ker(A-\lambda I) \quad \text{(geometric multiplicity)}.$$

Then:

(1) (*Complex Diagonalizability*) A is diagonalizable over $\mathbb{C}$ if and only if $m_g(\lambda) = m_a(\lambda)$ for every $\lambda \in \sigma(A)$. In this case, there exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

where each $\lambda_i \in \mathbb{C}$ is an eigenvalue of A, repeated according to its algebraic multiplicity.

(2) (*Real Diagonalizability*) A is diagonalizable over $\mathbb{R}$ if and only if:

(a) $\sigma(A) \subset \mathbb{R}$ (all eigenvalues real), and

(b) $m_g(\lambda) = m_a(\lambda)$ for each $\lambda \in \sigma(A)$ computed over $\mathbb{R}$,

which is equivalent to A having n linearly independent *real* eigenvectors.

(3) (*Orthogonal Diagonalization for Symmetric Matrices*) If A is symmetric ($A^\top = A$), then:

(a) $\sigma(A) \subset \mathbb{R}$;

(b) There exists an orthonormal eigenbasis $\{q_1, \dots, q_n\}$ of $\mathbb{R}^n$;

(c) Writing $Q = [q_1 \;\dots\; q_n] \in O(n)$, we have

$$Q^\top AQ = \Lambda, \quad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

so

$$A = Q\Lambda Q^\top.$$

In particular, symmetric matrices are diagonalizable over $\mathbb{R}$ via an orthogonal similarity transformation, which preserves inner products, norms, and orientation up to $\det Q \in \{\pm 1\}$.

⌟

Proof I.7.3. **(1) Complex Case:** Over $\mathbb{C}$, the Jordan canonical form theorem asserts that A is similar to a block diagonal matrix whose blocks are Jordan blocks $J_k(\lambda)$ of size $k \times k$ for $\lambda \in \sigma(A)$. The algebraic multiplicity $m_a(\lambda)$ is the sum of sizes of all Jordan blocks for λ, while the geometric multiplicity $m_g(\lambda)$ is the number of Jordan blocks for λ. A is diagonalizable over $\mathbb{C} \iff$ each Jordan block has size 1 $\iff m_g(\lambda) = m_a(\lambda)$ for all λ. In this case, the direct sum of the eigenspaces has dimension n, and choosing any basis of each eigenspace produces $P \in \mathrm{GL}_n(\mathbb{C})$ such that $P^{-1}AP$ is diagonal.

(2) Real Case: If A is diagonalizable over $\mathbb{R}$, then it is in particular diagonalizable over $\mathbb{C}$, hence $m_g(\lambda) = m_a(\lambda)$ over $\mathbb{C}$. However, if $\sigma(A)$ contains nonreal eigenvalues, then any real diagonal form would require them to appear in complex conjugate pairs, and the corresponding eigenvectors would not be real. Thus $\sigma(A) \subset \mathbb{R}$ is necessary. Conversely, if $\sigma(A) \subset \mathbb{R}$ and $m_g(\lambda) = m_a(\lambda)$ over $\mathbb{R}$, then the real eigenvectors form a basis of $\mathbb{R}^n$ and A is similar over $\mathbb{R}$ to a real diagonal matrix.

(3) Symmetric Case: If A is symmetric, then A is self-adjoint as a linear operator on the Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. By the finite-dimensional spectral theorem for self-adjoint operators:

- All eigenvalues of A are real;
- Eigenvectors corresponding to distinct eigenvalues are orthogonal;
- $V = \mathbb{R}^n$ decomposes into the orthogonal direct sum of the eigenspaces.

Choosing orthonormal bases for each eigenspace and concatenating them yields an orthogonal matrix $Q \in O(n)$ such that

$$Q^\top AQ = \Lambda$$

with Λ diagonal and real. Because Q is orthogonal, this similarity is an isometry in $\mathbb{R}^n$, meaning it preserves the inner product structure exactly. This is a rigid diagonalization: no non-orthogonal change of basis can preserve more structure than this. □

I.8 Matrix Exponential and Linear Dynamics

Definition I.8.1. Matrix Exponential.

Let $A \in \mathbb{R}^{n\times n}$. The *matrix exponential* of A is defined by the absolutely convergent power series

$$e^{At} := \sum_{k=0}^{\infty} \frac{(At)^k}{k!}, \quad t \in \mathbb{R}. \tag{I.8.1}$$

This series converges for all $t \in \mathbb{R}$ and yields a one-parameter family $\{e^{At}\}_{t\in\mathbb{R}}$ of matrices satisfying:

1. $e^{A0} = I$;
2. $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$;
3. $e^{A(t+s)} = e^{At}e^{As}$ for all $s, t \in \mathbb{R}$.

Definition I.8.2. Fundamental Solution Matrix.
For the linear time-invariant system
$$\dot{x}(t) = Ax(t), \tag{I.8.2}$$
the *fundamental solution matrix* is the matrix function $\Phi(t) \in \mathbb{R}^{n\times n}$ satisfying:
$$\dot{\Phi}(t) = A\Phi(t), \quad \Phi(0) = I.$$
If A is constant, $\Phi(t) = e^{At}$ as in (I.8.1).

Theorem I.8.1. Existence and Uniqueness of the Matrix Exponential Solution.
Let $A \in \mathbb{R}^{n\times n}$ be constant, and consider the initial value problem
$$\dot{x}(t) = Ax(t), \quad x(0) = x_0. \tag{I.8.3}$$
Then there exists a unique solution
$$x(t) = e^{At}x_0, \tag{I.8.4}$$
where e^{At} is given by (I.8.1). Moreover, $t \mapsto e^{At}$ is C^∞, satisfies the semigroup property in Definition I.8.1, and is the unique fundamental solution matrix for (I.8.2). ⌟

Proof I.8.1. **Step 1: Definition, convergence, and smoothness.** By (I.8.1),
$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$
Fix any submultiplicative matrix norm $\|\cdot\|$. Then
$$\left\| \frac{(At)^k}{k!} \right\| \le \frac{\|A\|^k |t|^k}{k!},$$
so the Weierstrass M–test implies uniform convergence on compact t–intervals; hence $t \mapsto e^{At}$ is continuous. Termwise differentiation is justified on compact intervals, giving
$$\frac{d}{dt}e^{At} = \sum_{k=1}^{\infty} \frac{k(At)^{k-1}A}{k!} = A\sum_{k=0}^{\infty} \frac{(At)^k}{k!} = Ae^{At}. \tag{I.8.5}$$
Repeating the argument yields C^∞ regularity. Evaluating the series at $t = 0$ gives $e^{A0} = I$.
Step 2: Semigroup and inverse. Using absolute convergence and the Cauchy product,
$$e^{At}e^{As} = \Big(\sum_{k\ge 0} \frac{A^k t^k}{k!}\Big)\Big(\sum_{\ell\ge 0} \frac{A^\ell s^\ell}{\ell!}\Big) = \sum_{m\ge 0}\Big(\sum_{k+\ell=m} \frac{1}{k!\,\ell!}\Big)A^m t^k s^\ell = \sum_{m\ge 0} \frac{A^m (t+s)^m}{m!} = e^{A(t+s)}.$$
In particular e^{At} is invertible with inverse e^{-At}, since $e^{At}e^{-At} = e^{A(t+(-t))} = e^{A0} = I$.
Step 3: Existence. Define $x(t)$ by (I.8.4). Then $x(0) = e^{A0}x_0 = Ix_0 = x_0$, and by (I.8.5),
$$\dot{x}(t) = \frac{d}{dt}(e^{At})x_0 = Ae^{At}x_0 = Ax(t),$$
so x solves the IVP (I.8.3). Thus e^{At} is a fundamental solution of (I.8.2).

Step 4: Uniqueness. Let y be *any* solution of (I.8.3). Set $g(t) := e^{-At}y(t)$. Since $\frac{d}{dt}e^{-At} = -Ae^{-At}$ by the same computation as in (I.8.5), the product rule gives

$$g'(t) = -Ae^{-At}y(t) + e^{-At}\dot{y}(t) = -Ae^{-At}y(t) + e^{-At}Ay(t) = 0.$$

Hence $g(t) \equiv g(0) = y(0) = x_0$, and $y(t) = e^{At}x_0$. This proves uniqueness.

Combining Steps 1–4, the unique solution of (I.8.3) is (I.8.4), with $t \mapsto e^{At}$ of class C^∞ and satisfying the semigroup property from Definition I.8.1. □

Remark I.8.1. (Matrix Exponential as a Unified Dynamics Generator)

The matrix exponential e^{At} encodes the complete time evolution of the linear time-invariant system (I.8.2). Its structure reflects the Jordan–Chevalley decomposition of A:

- If A is diagonalizable, e^{At} is a similarity transform of a diagonal matrix of scalar exponentials.
- If A is skew-symmetric, e^{At} is orthogonal, corresponding to pure rotations.
- If A is upper-triangular, e^{At} retains the triangular form and may contain polynomial factors in t in off-diagonal positions, reflecting non-normal coupling.

Thus e^{At} provides a single analytic object describing all qualitative behaviors such as growth, decay, oscillation, and transient amplification of the linear flow (I.8.2). ♦

Example I.17 (Skew-Symmetric Generator)**.** Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Using (I.8.1):

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}. \tag{I.8.6}$$

The fundamental solution (I.8.6) is a rotation matrix of angle t.

Example I.18 (Decoupled Growth System)**.** For $A = \operatorname{diag}(2, 3)$:

$$e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}, \quad x(t) = e^{At}x(0). \tag{I.8.7}$$

This represents independent exponential growth rates 2 and 3.

Example I.19 (Upper-Triangular Coupling)**.** Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \qquad \dot{x} = Ax, \quad x(0) = x_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}.$$

The system is triangular:

$$x_2(t) = e^{-2t}x_{2,0}, \tag{I.8.8}$$

$$x_1(t) = e^{-t}x_{1,0} + \big(e^{-t} - e^{-2t}\big)x_{2,0}. \tag{I.8.9}$$

Hence:

$$e^{At} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}. \tag{I.8.10}$$

The off-diagonal term in (I.8.10) encodes the coupling between x_2 and x_1.

Theorem I.8.2. Matrix Exponential, Fundamental Solution, and Structural Properties.

Let $A \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$). Write

$$e^{At} \;:=\; \sum_{k=0}^{\infty} \frac{(At)^k}{k!}, \qquad t \in \mathbb{R}.$$

Then:

1. **Well-posedness of e^{At}.** The series converges absolutely and locally uniformly in t, and $t \mapsto e^{At}$ is real-analytic with
$$\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A, \qquad e^{A\cdot 0} = I.$$
2. **Fundamental solution & uniqueness.** For the linear ODE $\dot{x} = Ax$, $x(0) = x_0$, the unique solution is
$$x(t) \;=\; e^{At}x_0, \qquad t \in \mathbb{R}.$$
Equivalently, $\Phi(t) := e^{At}$ is the unique fundamental matrix with $\Phi(0) = I$.
3. **Semigroup and group properties.** For all $s, t \in \mathbb{R}$,
$$e^{A(t+s)} \;=\; e^{At}e^{As} \;=\; e^{As}e^{At}, \qquad (e^{At})^{-1} = e^{-At}.$$
In particular $\{e^{At}\}_{t\in\mathbb{R}}$ is a (one-parameter) matrix group with generator A.
4. **Similarity invariance.** If $A = PDP^{-1}$, then
$$e^{At} \;=\; P\, e^{Dt}\, P^{-1}.$$
5. **Diagonalizable case.** If $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then
$$e^{Dt} = \operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}), \qquad e^{At} = P\operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})\, P^{-1}.$$
6. **Jordan case (defective matrices).** If $A = PJP^{-1}$ with $J = \bigoplus_j J_{m_j}(\lambda_j)$, then for each Jordan block
$$e^{J_m(\lambda)t} \;=\; e^{\lambda t} \sum_{r=0}^{m-1} \frac{t^r}{r!} N^r,$$
where $J_m(\lambda) = \lambda I_m + N$ and N is the nilpotent Jordan off-diagonal ($N^m = 0$). Hence
$$e^{At} = P\Big(\bigoplus_j e^{J_{m_j}(\lambda_j)t}\Big)P^{-1},$$
i.e. exponentials times polynomials in t along Jordan chains.
7. **Commuting exponents.** If A and B commute ($AB = BA$), then
$$e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At}.$$
8. **Spectral mapping and growth bounds.** For all $t \in \mathbb{R}$,
$$\sigma(e^{At}) = e^{t\,\sigma(A)} := \{e^{t\lambda} : \lambda \in \sigma(A)\}.$$
Moreover, for any submultiplicative matrix norm $\|\cdot\|$,
$$\|e^{At}\| \;\le\; e^{\|A\|\,|t|}.$$
If A is normal (unitarily diagonalizable), then $\|e^{At}\|_2 = \max_{\lambda\in\sigma(A)} |e^{\lambda t}|$ in the spectral norm.

⌟

Proof I.8.2. **(1) Convergence, differentiability.** Let $\|\cdot\|$ be any submultiplicative matrix norm. Then

$$\left\|\frac{(At)^k}{k!}\right\| \le \frac{\|A\|^k|t|^k}{k!},$$

so the series for e^{At} converges absolutely and locally uniformly by comparison with the scalar exponential series. Termwise differentiation is justified by uniform convergence on compact t-intervals, yielding

$$\frac{d}{dt}e^{At} = \sum_{k=1}^{\infty}\frac{A^k t^{k-1}}{(k-1)!} = A\sum_{k=0}^{\infty}\frac{A^k t^k}{k!} = Ae^{At}.$$

Similarly $e^{At}A = \frac{d}{dt}e^{At}$ by right-multiplication.

(2) Fundamental solution and uniqueness. Define $x(t) = e^{At}x_0$. Then $\dot{x}(t) = Ae^{At}x_0 = Ax(t)$ and $x(0) = x_0$. For uniqueness, suppose y solves $\dot{y} = Ay$, $y(0) = x_0$. Consider $z(t) := e^{-At}y(t)$. Using $(e^{At})^{-1} = e^{-At}$ (proved in (3) below), we have

$$\dot{z}(t) = -Ae^{-At}y(t) + e^{-At}\dot{y}(t) = -Ae^{-At}y(t) + e^{-At}Ay(t) = 0,$$

hence $z(t) \equiv z(0) = x_0$, so $y(t) = e^{At}x_0$. Existence and uniqueness also follow from Picard–Lindelöf for linear systems.

(3) Semigroup, inverse. Define $F(t) := e^{At}e^{As}$. Then $F'(t) = Ae^{At}e^{As} = AF(t)$ and $F(0) = I\cdot e^{As} = e^{As}$. On the other hand, $G(t) := e^{A(t+s)}$ satisfies $G'(t) = AG(t)$ and $G(0) = e^{As}$. By uniqueness for the linear matrix ODE $\dot{X} = AX$ with initial data at $t = 0$, we have $F(t) = G(t)$ for all t, i.e. $e^{A(t+s)} = e^{At}e^{As}$. Setting $s = -t$ gives $e^{At}e^{-At} = I$, so $(e^{At})^{-1} = e^{-At}$.

(4) Similarity invariance. If $A = PDP^{-1}$, then

$$e^{At} = \sum_{k=0}^{\infty}\frac{(PDP^{-1}t)^k}{k!} = \sum_{k=0}^{\infty}\frac{PD^kP^{-1}t^k}{k!} = P\Big(\sum_{k=0}^{\infty}\frac{D^kt^k}{k!}\Big)P^{-1} = Pe^{Dt}P^{-1}.$$

(5) Diagonalizable case. If $D = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, then $D^k = \mathrm{diag}(\lambda_1^k,\ldots,\lambda_n^k)$ and

$$e^{Dt} = \sum_{k=0}^{\infty}\frac{D^kt^k}{k!} = \mathrm{diag}\Big(\sum_{k=0}^{\infty}\frac{(\lambda_1 t)^k}{k!},\ldots,\sum_{k=0}^{\infty}\frac{(\lambda_n t)^k}{k!}\Big) = \mathrm{diag}(e^{\lambda_1 t},\ldots,e^{\lambda_n t}).$$

Combine with (4).

(6) Jordan case (defective). Let $J_m(\lambda) = \lambda I_m + N$ with N nilpotent and ones on the superdiagonal. Since N and λI_m commute,

$$e^{J_m(\lambda)t} = e^{(\lambda I_m+N)t} = e^{\lambda t}e^{Nt} = e^{\lambda t}\sum_{r=0}^{m-1}\frac{(Nt)^r}{r!} = e^{\lambda t}\sum_{r=0}^{m-1}\frac{t^r}{r!}N^r.$$

Block-diagonalise J and conjugate by P to obtain the stated form.

(7) Commuting exponentials. If $AB = BA$, then by the binomial theorem for commuting matrices,

$$e^{(A+B)t} = \sum_{k\ge 0}\frac{t^k}{k!}(A+B)^k = \sum_{k\ge 0}\frac{t^k}{k!}\sum_{r=0}^{k}\binom{k}{r}A^rB^{k-r} = \Big(\sum_{r\ge 0}\frac{t^r}{r!}A^r\Big)\Big(\sum_{s\ge 0}\frac{t^s}{s!}B^s\Big) = e^{At}e^{Bt}.$$

(8) Spectral mapping, norm bounds. Let $\lambda \in \sigma(A)$ and $v \neq 0$ with $(A - \lambda I)v \approx 0$ (approximate point spectrum argument) or use Jordan form. In Jordan form (over $\mathbb{C}$), $A = PJP^{-1}$ and $e^{At} = Pe^{Jt}P^{-1}$ has eigenvalues $e^{\lambda t}$ where $\lambda \in \sigma(A)$, proving $\sigma(e^{At}) = e^{t\sigma(A)}$. For bounds,

$$\|e^{At}\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k |t|^k}{k!} = e^{\|A\||t|}.$$

If A is normal, there exists a unitary U with $A = U^* D U$ (diagonal D). Then

$$\|e^{At}\|_2 = \|U^* e^{Dt} U\|_2 = \|e^{Dt}\|_2 = \max_i |e^{\lambda_i t}|.$$

Uniqueness of all asserted identities. Items (1)–(7) characterise e^{At} as the unique strongly continuously differentiable solution of $\dot{X} = AX$ with $X(0) = I$, from which all representation and structural properties follow by similarity reduction and block decomposition. □

The following theorem is more advanced and will be discussed in more detail later:

Theorem I.8.3. Linear Eigenfunction Decomposition of e^{At}.

Let $A \in \mathbb{R}^{n\times n}$ be diagonalizable over $\mathbb{C}$ with eigenpairs $\{(\lambda_i, v_i)\}_{i=1}^n$. Then any initial condition $x_0 \in \mathbb{C}^n$ decomposes as

$$x_0 = \sum_{i=1}^{n} c_i v_i, \quad \text{with } c_i = \langle w_i, x_0 \rangle,$$

where $\{w_i\}$ is the dual (left eigenvector) basis satisfying $\langle w_i, v_j \rangle = \delta_{ij}$. Then the solution to $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$ is:

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i.$$

⌟

Proof I.8.3. Since A is diagonalizable over $\mathbb{C}$, there exists a complete basis of eigenvectors $\{v_i\}$ with $Av_i = \lambda_i v_i$, and a corresponding dual basis $\{w_i\}$ of left eigenvectors with $w_i^\top A = \lambda_i w_i^\top$, normalised such that $\langle w_i, v_j \rangle = \delta_{ij}$.

Any $x_0 \in \mathbb{C}^n$ decomposes as

$$x_0 = \sum_{i=1}^{n} c_i v_i, \quad \text{with } c_i = \langle w_i, x_0 \rangle.$$

The solution to the system $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$ is

$$x(t) = e^{At} x_0 = \sum_{i=1}^{n} c_i e^{At} v_i.$$

Since v_i is an eigenvector, we have:

$$e^{At} v_i = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k v_i = \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_i^k v_i = e^{\lambda_i t} v_i.$$

Therefore:

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i.$$

Each mode v_i evolves independently with exponential scaling $e^{\lambda_i t}$, governed by its eigenvalue. This decomposition shows that e^{At} acts diagonally in the eigenbasis and that the solution is the sum of dynamically decoupled modes.

$\square$

II Inner Product Spaces and Spectral Geometry

We begin by endowing a vector space with additional structure sufficient to speak of length, angle, and orthogonality, thereby laying the groundwork for projections and the spectral theorem. This is the linear-geometric skeleton underlying vast regions of modern analysis and geometry.

II.1 Normed and Inner Product Spaces

Let V be a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition II.1.1. Normed Vector Space.

A *norm* on V is a map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying, for all $v, w \in V$, $\lambda \in \mathbb{F}$,

1. *Positivity:* $\|v\| = 0 \iff v = 0$;
2. *Homogeneity:* $\|\lambda v\| = |\lambda| \, \|v\|$;
3. *Triangle inequality:* $\|v + w\| \leq \|v\| + \|w\|$.

The pair $(V, \|\cdot\|)$ is then a *normed vector space.*

Example II.1 (ℓ^p-Geometry as Discrete Measure Theory)**.** Let $n \in \mathbb{N}$, $p \in [1, \infty]$, and consider the real coordinate space

$$\mathbb{R}^n := \{x = (x_1, \dots, x_n) : x_i \in \mathbb{R} \ \ \forall i \in \{1, \dots, n\}\}.$$

The p-*norm* is defined by

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, & 1 \leq p < \infty, \\ \max_{1 \leq i \leq n} |x_i|, & p = \infty. \end{cases}$$

Explicitly:

- For $1 \leq p < \infty$, $\|x\|_p$ aggregates all coordinates via the p-th power, sums them, and takes the p-th root.
- For $p = \infty$, $\|x\|_\infty$ extracts the largest absolute coordinate.

Measure-theoretic representation: Let $X = \{1, \dots, n\}$ and endow it with the *counting measure* μ defined by $\mu(\{i\}) = 1$ for all i. Identifying $x \in \mathbb{R}^n$ with the function $x : X \to \mathbb{R}$, $x(i) = x_i$, we may write:

$$\|x\|_p = \left(\int_X |x(i)|^p \, d\mu(i)\right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|x\|_\infty = \operatorname{ess\,sup}_{i \in X} |x(i)|.$$

Here $\ell^p(\mathbb{R}^n)$ is isometric to the discrete L^p-space $L^p(X, \mu)$ and thus inherits:

- *Convexity:* $\|tx + (1-t)y\|_p \leq t\|x\|_p + (1-t)\|y\|_p$, $\forall t \in [0, 1]$;
- *Completeness:* every Cauchy sequence in ℓ^p converges in ℓ^p;
- *Duality:* $(\ell^p)^* \cong \ell^q$ for $1 < p < \infty$ with $1/p + 1/q = 1$.

Unit ball geometry: Define

$$B_p := \{x \in \mathbb{R}^n : \|x\|_p \le 1\}.$$

Then:

- $p = 1$: B_1 is the *cross-polytope* $\{x : \sum_{i=1}^n |x_i| \le 1\}$; a convex polyhedron with $2n$ vertices $\pm e_i$ and $(2n)$-fold symmetry; non-smooth, with $(n-1)$-dimensional flat faces meeting at sharp edges.
- $p = 2$: B_2 is the Euclidean ball $\{x : \sum_{i=1}^n x_i^2 \le 1\}$; smooth, strictly convex, invariant under $O(n)$, with geodesic boundary S^{n-1}.
- $p = \infty$: B_∞ is the cube $\{x : \max_i |x_i| \le 1\}$; vertices $\{(\pm1, \dots, \pm1)\}$, 2^n in total; flat $(n-1)$-dimensional faces aligned with coordinate hyperplanes.
- $1 < p < \infty$: B_p has smooth $\mathcal{C}^\infty$ boundary $\{x : \sum |x_i|^p = 1\}$, strictly convex (no flat segments), with positive Gaussian curvature at every point.

Inner product characterization: For $p = 2$, the norm is induced by the canonical bilinear form

$$\langle x, y\rangle := \sum_{i=1}^n x_i y_i,$$

which satisfies the *parallelogram identity*:

$$\|x+y\|_2^2 + \|x-y\|_2^2 = 2\|x\|_2^2 + 2\|y\|_2^2, \quad \forall x, y \in \mathbb{R}^n.$$

For $p \ne 2$, no inner product $\langle\cdot,\cdot\rangle$ exists with $\|x\|_p = \sqrt{\langle x, x\rangle}$, since the parallelogram identity fails. A generalization is *Birkhoff orthogonality*:

$$x \perp_B y \iff \|x + \lambda y\| \ge \|x\|, \quad \forall \lambda \in \mathbb{R},$$

which reduces to classical orthogonality in the Euclidean case.

Functional-analytic extension: For an arbitrary measure space (X, Σ, μ), the L^p-norm is defined by

$$\|f\|_{L^p(X)} := \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p}, \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(X)} := \operatorname{ess\,sup}_{x\in X} |f(x)|.$$

Finite-dimensional ℓ^p spaces correspond to step functions supported on finitely many atoms of equal measure. As $n \to \infty$, $\mathbb{R}^n$ becomes $\ell^p(\mathbb{N})$ (sequences), and the continuous limit produces $L^p(X)$ for general X.

Summary: ℓ^p-geometry admits explicit combinatorial and analytic description:

$$\text{polyhedral } (p=1) \longrightarrow \text{Hilbertian } (p=2) \longrightarrow \text{cubical } (p=\infty),$$

with $1 < p < \infty$ yielding intermediate smooth, strictly convex geometries. The Hilbert case is the unique juncture where normed, inner product, and spectral structures coincide.

Definition II.1.2. Inner Product Space.

An inner product is a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ satisfying:

1. (Linearity) $\langle \lambda u + v, w\rangle = \lambda\langle u, w\rangle + \langle v, w\rangle$,

2. (Conjugate symmetry) $\langle v, u\rangle = \overline{\langle u, v\rangle}$,
3. (Positive-definiteness) $\langle v, v\rangle \geq 0$, with equality iff $v = 0$.

The induced norm is $\|v\| := \sqrt{\langle v, v\rangle}$, and $(V, \langle\cdot,\cdot\rangle)$ becomes a normed space.

Example II.2. The standard inner product on $\mathbb{R}^n$ is

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i,$$

and on $\mathbb{C}^n$, it becomes

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Theorem II.1.1. Cauchy–Schwarz Inequality.
Let V be an inner product space. For all $u, v \in V$,

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|,$$

with equality iff u and v are linearly dependent. ⌟

Proof II.1.1. Let V be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $u, v \in V$. If $v = 0$, then $\langle u, v\rangle = 0$ and the inequality becomes $0 \leq \|u\| \cdot 0 = 0$, which holds trivially with equality. So assume $v \neq 0$.

Consider the function

$$f(\lambda) := \|u - \lambda v\|^2 = \langle u - \lambda v, u - \lambda v\rangle \geq 0 \quad \text{for all } \lambda \in \mathbb{F}.$$

Expanding this using the sesquilinearity of the inner product:

$$\begin{aligned} f(\lambda) &= \langle u, u\rangle - \overline{\lambda}\langle u, v\rangle - \lambda\langle v, u\rangle + |\lambda|^2\langle v, v\rangle \\ &= \|u\|^2 - 2\,\mathrm{Re}(\overline{\lambda}\langle u, v\rangle) + |\lambda|^2\|v\|^2. \end{aligned}$$

We now minimize this quadratic function over $\lambda \in \mathbb{F}$. Let us choose

$$\lambda := \frac{\langle u, v\rangle}{\|v\|^2},$$

which is well-defined since $v \neq 0$. Substituting into the expression:

$$\begin{aligned} f(\lambda) &= \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2} - \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \frac{|\langle u, v\rangle|^2}{\|v\|^2} \\ &= \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2}. \end{aligned}$$

Since $f(\lambda) \geq 0$, it follows that

$$\|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2} \geq 0,$$

which rearranges to

$$|\langle u, v\rangle|^2 \leq \|u\|^2\|v\|^2.$$

Taking square roots on both sides (noting all terms are nonnegative),

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|.$$

Equality case: Equality holds if and only if $f(\lambda) = 0$, i.e., $u = \lambda v$ for some $\lambda \in \mathbb{F}$, which means u and v are linearly dependent.

□

Theorem II.1.2. Parallelogram Law.

Let V be a normed space. Then the norm is induced by an inner product if and only if the following identity holds:

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

⌟

Proof II.1.2. We prove both directions of the equivalence.

($\Rightarrow$) **If the norm is induced by an inner product, then the parallelogram law holds.**

Assume $\|x\|^2 = \langle x, x\rangle$ for some inner product $\langle\cdot,\cdot\rangle$ on V. Then for any $u, v \in V$,

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v\rangle = \langle u, u\rangle + 2\operatorname{Re}\langle u, v\rangle + \langle v, v\rangle, \\ \|u - v\|^2 &= \langle u - v, u - v\rangle = \langle u, u\rangle - 2\operatorname{Re}\langle u, v\rangle + \langle v, v\rangle. \end{aligned}$$

Adding these:

$$\|u + v\|^2 + \|u - v\|^2 = 2\langle u, u\rangle + 2\langle v, v\rangle = 2\|u\|^2 + 2\|v\|^2.$$

Thus, the parallelogram identity holds.

($\Leftarrow$) **If the norm satisfies the parallelogram law, then it is induced by an inner product.**

Assume $\|\cdot\|$ satisfies the identity

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2 \quad \text{for all } u, v \in V.$$

Define a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ via the polarization identity:

$$\langle u, v\rangle := \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2\right).$$

We verify that this defines an inner product:

- *Bilinearity:* $\langle\cdot,\cdot\rangle$ is linear in each argument since norms are quadratic and the identity distributes over linear combinations.
- *Symmetry:* $\langle u, v\rangle = \langle v, u\rangle$, since the definition is symmetric in u and v.
- *Positive-definiteness:* $\langle u, u\rangle = \frac{1}{4}\left(\|2u\|^2 - 0\right) = \|u\|^2 \geq 0$, and equality only when $u = 0$.

Therefore, $\langle\cdot,\cdot\rangle$ is a valid inner product, and the norm satisfies

$$\|u\|^2 = \langle u, u\rangle,$$

so the norm is induced by it.

□

Remark II.1.1. (Parallelogram Law as a Characterization of Inner Product Structure)

This identity is both a diagnostic tool and a structural signature. The parallelogram law,

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2,$$

holds in any inner product space and, conversely, characterizes when a norm arises from an inner product. That is, if a normed space satisfies the parallelogram identity, then one can define an inner product via the polarization identity, and the norm is induced by it.

The failure of this identity in ℓ^p spaces for $p \neq 2$ reflects the nonlinearity of energy distribution in those geometries. In Euclidean (i.e., Hilbert) spaces, energy is quadratically additive, which underlies orthogonal decomposition, spectral theory, and variational principles. The parallelogram law thus demarcates the boundary between general Banach spaces and the rigid geometry of Hilbert spaces.

♦

Example II.3 (Failure of Inner Product Structure)**.** Let $V = \mathbb{R}^2$ with the ℓ^1 norm:

$$\|x\|_1 = |x_1| + |x_2|.$$

Then,

$$\|(1,0)+(0,1)\|^2 + \|(1,0)-(0,1)\|^2 = 2^2 + 2^2 = 8,$$

but

$$2\|(1,0)\|^2 + 2\|(0,1)\|^2 = 2 + 2 = 4.$$

Hence, the parallelogram identity fails, and no inner product induces this norm.

Remark II.1.2. (Geometric Rigidity of Hilbert Spaces)

The distinction between general normed vector spaces and inner product spaces is not merely technical—it marks a geometric phase boundary in the structure of functional spaces. While every inner product induces a norm via

$$\|v\| := \sqrt{\langle v, v\rangle},$$

the converse is not generally true: most norms do not arise from any inner product.

The failure of the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

is a rigorous diagnostic for non-Hilbertian geometry. For example, in $\mathbb{R}^2$ equipped with the ℓ^1 or ℓ^∞ norm, the unit ball lacks strict convexity and the isometry group is sharply smaller than in Euclidean space. This reflects a loss of angle structure and orthogonality.

Hilbert spaces are characterized by a rich geometric structure: projections, orthogonal decompositions, and a well-defined Fourier theory all rely critically on the existence of an inner product. In infinite-dimensional settings, such as L^2 spaces or Sobolev spaces H^1, the Hilbert structure permits variational methods, spectral decompositions, and the well-posedness of partial differential equations.

Therefore, the transition from a normed space to an inner product space may be viewed as a passage from flexible metric geometry to rigid analytic geometry. The presence of the inner product is what allows one to define curvature, harmonicity, and self-adjoint operators in a meaningful way. ♦

II.2 Completeness and Banach Spaces

The notion of completeness is central in functional analysis: a space where every Cauchy sequence converges ensures that limiting processes (e.g., analytic, geometric, or physical) remain internal to the space. In analogy with physical systems, one can think of completeness as *energy conservation in the limit process*: no "energy" (norm) leaks outside the system when passing to a limit. In geometric terms, completeness ensures that the metric space is *closed under infinitesimal refinement*; approximations converge to actual points of the space rather than to an "external" ideal point.

Theorem II.2.1. Completeness of ℓ^p Spaces.

Let $1 \le p \le \infty$. The sequence space

$$\ell^p := \left\{ x = (x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty \right\}$$

(for $p < \infty$) and

$$\ell^\infty := \left\{ x = (x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty \right\}$$

are Banach spaces under their respective norms. ⌟

Proof II.2.1. Let $(x^{(k)})_{k\in\mathbb{N}} \subset \ell^p$ be a Cauchy sequence. By definition:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall j,k \ge N, \quad \|x^{(j)} - x^{(k)}\|_p < \varepsilon.$$

Fix $n \in \mathbb{N}$. The scalar sequence $(x_n^{(k)})_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, hence converges to some limit $x_n \in \mathbb{R}$. Define $x := (x_n)_{n=1}^{\infty}$.

Step 1: $x \in \ell^p$. For $1 \le p < \infty$, apply Fatou's lemma to the non-negative sequence $|x_n^{(k)} - x_n|^p$:

$$\sum_{n=1}^{\infty} |x_n|^p \le \liminf_{k\to\infty} \sum_{n=1}^{\infty} |x_n^{(k)}|^p = \liminf_{k\to\infty} \|x^{(k)}\|_p^p < \infty.$$

Thus $x \in \ell^p$. For $p = \infty$, we have

$$|x_n| \le |x_n - x_n^{(k)}| + |x_n^{(k)}| \le \|x - x^{(k)}\|_\infty + \|x^{(k)}\|_\infty,$$

so taking $\sup_n$ and passing to the limit shows $\|x\|_\infty < \infty$.

Step 2: Norm convergence. For any $\varepsilon > 0$, choose $k \ge N$; then for all $j \to \infty$,

$$\|x^{(k)} - x\|_p^p \le \liminf_{j\to\infty} \|x^{(k)} - x^{(j)}\|_p^p < \varepsilon^p,$$

establishing $\|x^{(k)} - x\|_p \to 0$.

Physical analogy: The ℓ^p norm may be viewed as an "energy functional" $E_p(x) = \sum |x_n|^p$ (or an L^∞-type "entropy bound" for $p = \infty$). Cauchy-ness expresses that the *relative energy difference* between iterates tends to zero. Completeness then asserts that there exists an *equilibrium state x*—still within ℓ^p—where this energy stabilizes exactly. □

Theorem II.2.2. Closed Subspaces of Banach Spaces are Banach.

Let $(X, \|\cdot\|)$ be a Banach space and let $Y \subseteq X$ be a linear subspace. Then Y is a Banach space under the induced norm if and only if Y is closed in X. ⌟

Proof II.2.2. ($\Rightarrow$): Assume Y is Banach under the norm induced from X. Let $z \in \overline{Y}$ in X. Here $\overline{Y}$ denotes the *closure* of Y in X with respect to the norm topology of X:

$$\overline{Y} := \{z \in X \mid \exists\,(y_n) \subset Y \text{ with } \|y_n - z\|_X \to 0\}.$$

Thus $z \in \overline{Y}$ means that z can be approximated arbitrarily well (in the norm of X) by a sequence of elements of Y. Equivalently, every open ball in X centered at z intersects Y. Then there exists $(y_n) \subset Y$ with $\|y_n - z\|_X \to 0$, hence (y_n) is Cauchy in X and therefore Cauchy in Y (the norms agree on Y). By completeness of Y, there exists $y \in Y$ with $\|y_n - y\|_Y \to 0$, i.e. $\|y_n - y\|_X \to 0$. Uniqueness of limits in X gives $y = z$, so $z \in Y$. Hence Y is closed in X.

($\Leftarrow$): Assume Y is a closed subspace of the Banach space $(X, \|\cdot\|_X)$. Let $(y_n)_{n\in\mathbb{N}} \subset Y$ be a Cauchy sequence in Y, i.e.,

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \ \text{ s.t. } \ \forall m, n \geq N, \quad \|y_n - y_m\|_Y < \varepsilon.$$

Since $\|\cdot\|_Y$ is the restriction of $\|\cdot\|_X$ to Y, the same inequality holds in X:

$$\|y_n - y_m\|_X = \|y_n - y_m\|_Y < \varepsilon.$$

Thus (y_n) is also Cauchy in X.

Because X is Banach, it is complete; hence there exists $y \in X$ such that

$$\lim_{n\to\infty} \|y_n - y\|_X = 0.$$

At this stage, we only know $y \in X$. However, since $y_n \in Y$ for all n and Y is *closed in* X, it must contain all its limit points with respect to the topology induced by $\|\cdot\|_X$. This means:

$$y_n \to y \text{ in } X \quad \text{and} \quad y_n \in Y \ \ \forall n \quad \Longrightarrow \quad y \in Y.$$

Therefore $y \in Y$.

We have shown that (y_n) converges to some $y \in Y$ in the norm of X, and since the norm on Y is just the restriction of $\|\cdot\|_X$, this convergence is also in the norm of Y. Thus every Cauchy sequence in Y converges to a point in Y, i.e., Y is complete. □

Remark II.2.1. (Metric closure, completeness barrier, and limit confinement)

Let (X, d) be the metric space induced by the norm $\|\cdot\|_X$, and let $Y \subset X$ be a linear subspace with the induced metric $d_Y := d|_{Y\times Y}$. The heuristic sentence

"closedness prevents limits from escaping"

admits the following precise formulations, each of which makes the confinement mechanism explicit.

(1) Closure as vanishing locus of a 1-Lipschitz potential. Define the distance-to-Y functional

$$\operatorname{dist}(\cdot, Y) : X \to \mathbb{R}_{\geq 0}, \qquad \operatorname{dist}(x, Y) := \inf_{y\in Y} d(x, y).$$

Then $\operatorname{dist}(\cdot, Y)$ is 1-Lipschitz and

$$\overline{Y} \ = \ \{\, x \in X : \operatorname{dist}(x, Y) = 0 \,\}.$$

Hence Y is closed iff $\operatorname{dist}(x, Y) = 0 \Rightarrow x \in Y$. In particular, if $(y_n) \subset Y$ and $y_n \to x$ in X, then $\operatorname{dist}(x, Y) \leq d(x, y_n) \to 0$, so $x \in \overline{Y}$; closedness of Y forces $x \in Y$.

(2) Sequential confinement (first-countability). Because normed spaces are first-countable, closedness is equivalent to *sequential* closedness:

$$Y \text{ closed} \quad \Longleftrightarrow \quad \big((y_n) \subset Y,\ y_n \to x \text{ in } X\big) \Rightarrow x \in Y.$$

Thus any limit of a sequence entirely contained in Y is confined to Y. No subsequence can "exit" into $X \setminus Y$ at the limit.

(3) Completeness barrier inside a complete ambient space. Assume (X, d) is complete. Then for any $Y \subset X$ the following are equivalent:

(a) Y is closed in X,

(b) Every Cauchy sequence $(y_n) \subset Y$ converges in Y,

(c) (Y, d_Y) is a complete metric space.

Proof sketch of (a) ⇒ (b): If $(y_n) \subset Y$ is Cauchy in d_Y, it is Cauchy in d; completeness of X yields a limit $y \in X$. Closedness of Y implies $y \in Y$. Thus (y_n) converges in Y. *Proof sketch of (b) ⇒ (a):* If $x \in \overline{Y}$, choose $(y_n) \subset Y$ with $y_n \to x$ in X. Then (y_n) is Cauchy in Y, hence convergent to some $y \in Y$, but uniqueness of limits gives $y = x$, so $x \in Y$.

(4) Energy-dissipation inequality and limit retention. Let $E_n(z) := d(y_n, z)$ for a Cauchy sequence $(y_n) \subset Y$. Then for any $m \geq n$,

$$|E_n(z) - E_m(z)| \leq d(y_n, y_m) \xrightarrow[n,m\to\infty]{} 0,$$

so (E_n) is a Cauchy family of 1-Lipschitz functionals. Completeness of X gives $y \in X$ with $y_n \to y$; closedness of Y forces $y \in Y$. In this sense, the "energy profile" E_n stabilizes on Y and cannot dissipate into $X \setminus Y$.

Conclusion. In a complete ambient space, closedness of Y is exactly the metric-geometric barrier that *confines* all Cauchy limits and all sequential limits of points in Y to remain in Y. This is the rigorous content underlying the informal picture that limit points cannot "escape" across the boundary of Y. ♦

Geometric/entropy perspective: In a Banach space $(X, \|\cdot\|)$, a closed subspace Y may be seen as a *constraint surface* in which the "energy functional" $\|\cdot\|^p$ is still well-defined and complete. If Y were not closed, a minimizing (or stabilizing) sequence in Y could "escape" into the ambient space, dissipating energy into directions orthogonal to Y. Closedness guarantees *entropy confinement*: all limit points remain within the subsystem Y, making it a self-contained Banach space.

Summary: Completeness in Banach spaces mirrors conservation laws in physics—no loss of norm-energy in the limit—and closed subspaces play the role of invariant manifolds where the dynamics of Cauchy sequences remain fully contained. This dual analytic–geometric–physical interpretation becomes especially potent in spectral geometry, where closed invariant subspaces correspond to pure modes of the system and completeness ensures the stability of spectral decompositions.

II.3 Subspaces of Normed and Banach Spaces

Subspaces play a central role in functional analysis, particularly in studying restrictions of operators, constructing approximations, formulating boundary conditions, and analyzing spectral decompositions. The structure of a subspace determines whether local or approximate solutions can be extended to global ones, and whether iterative processes converge within the subspace.

The central structural question is whether a subspace inherits the completeness of its ambient space.

Definition II.3.1. Normed Subspace.

Let $(X, \|\cdot\|)$ be a normed vector space. A subset $Y \subseteq X$ is called a *normed subspace* if:

1. Y is a linear subspace of X,
2. Y is equipped with the norm induced from X, i.e., $\|y\|_Y := \|y\|_X$ for all $y \in Y$.

Definition II.3.2. Closed Subspace.

A normed subspace $Y \subset X$ is said to be *closed* if it is closed in the norm topology of X; that is,

$$\forall \text{ sequence } \{y_n\} \subset Y, \quad y_n \to x \in X \Longrightarrow x \in Y.$$

Equivalently, Y contains all of its limit points with respect to the metric induced by $\|\cdot\|$.

Example II.4. Let $X = C([0,1])$ with the supremum norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Then the subspace

$$Y := \{f \in C([0,1]) \mid f(0) = 0\}$$

is a normed subspace. It is closed in X, and hence a Banach space under the induced norm.

Example II.5 (Non-closed Subspace)**.** Let $X = C([0,1])$ with the supremum norm. Define

$$Y := \{f \in C([0,1]) \mid f \text{ is a polynomial}\}.$$

Then Y is a normed subspace, but not complete. For example, the sequence $f_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!}$ converges uniformly to $\exp(x) \in C([0,1])$, but $\exp(x) \notin Y$. Thus, Y is not closed, and hence not Banach.

Remark II.3.1. (Closedness and Approximation in Function Spaces)

The distinction between closed and non-closed subspaces is central to the analytical structure of functional spaces. In the Banach space $X = C([0,1])$ with the supremum norm, any closed subspace inherits completeness and thus forms a Banach space in its own right. This property is critical for stability under limits and the convergence of analytic processes.

For example, the space of polynomials is a linear subspace of $C([0,1])$, but not a closed one. Although dense by the Weierstrass Approximation Theorem, it fails to contain its uniform limits, such as the exponential function. Consequently, sequences of polynomials that converge uniformly may escape the subspace. In contrast, subspaces defined by closed conditions, such as $Y = \{f \in C([0,1]) \mid f(0) = 0\}$, are norm-closed and thus support complete convergence theory.

In numerical analysis and spectral methods, the closedness of a subspace governs whether projection techniques (e.g., Galerkin approximations) converge within the intended function class. A non-closed subspace permits approximation but not exact solvability within itself. Hence, closedness encodes not just topological completeness but computational and analytic fidelity. ♦

Corollary II.3.1. Subspace Completeness.

Every finite-dimensional subspace of a normed space is closed and hence complete. $\rightarrowtail$

Proof II.3.1. Let $Y \subset X$ be a finite-dimensional subspace of a normed vector space X. Since all norms on finite-dimensional vector spaces are equivalent, Y inherits the topology of $\mathbb{R}^n$, which is complete. Therefore, Y is itself complete and hence a Banach space.

To prove closedness, suppose $y_n \in Y$ converges in X to some limit $x \in X$. Since Y is finite-dimensional and complete, the limit must lie in Y. Hence, Y contains all its limit points and is closed. □

Remark II.3.2. (Closedness Failure in Infinite Dimensions)

In contrast to the finite-dimensional case, an infinite-dimensional subspace of a normed space need not be closed. This failure is a hallmark of infinite-dimensional analysis and has deep consequences.

For example, let $X = C([0,1])$ with the supremum norm, and let $Y \subset X$ be the subspace of polynomial functions. Then Y is infinite-dimensional, and it is *not* closed, even though it is dense in X. The limit of a uniformly convergent sequence of polynomials (e.g., approximating $\exp(x)$) may lie outside of Y.

A subspace that is dense but not closed is necessarily incomplete: Cauchy sequences can converge in the ambient space but not within the subspace. This obstructs the use of tools like the Banach Fixed Point Theorem, projection theorems, and spectral decompositions unless the subspace is first completed.

In summary, while finite-dimensional subspaces inherit the full structure of normed vector spaces, infinite-dimensional subspaces exhibit qualitatively different behavior. Closedness is no longer automatic and must be verified independently. ◆

Remark II.3.3. (Completion of Normed Subspaces)

Given any normed subspace $Y \subset X$, there exists a unique Banach space $\overline{Y}$, called the *completion* of Y, such that:

1. $Y \subset \overline{Y} \subset X$,
2. $\overline{Y}$ is the closure of Y in X,
3. Every Cauchy sequence in Y converges in $\overline{Y}$.

This process allows one to extend functional constructions (e.g., dense domain operators, polynomial approximations) to well-posed, convergent limits. In many analytic contexts, one works with a dense subspace Y for algebraic or approximation reasons, but the actual solutions live in $\overline{Y}$, where functional limits and PDEs are properly defined. ◆

II.4 Orthonormal Bases and Projections

In inner product spaces, orthonormal systems encode an idealized coordinate system where notions of angle and length are preserved. They play a central role in decomposing vectors, simplifying linear operators, and extracting spectral content from geometric or analytic data.

Definition II.4.1. Orthonormal Set.

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A finite set of vectors $\{v_i\}_{i=1}^k \subset V$ is said to be *orthonormal* if:

1. **Orthogonality:** $\langle v_i, v_j\rangle = 0$ whenever $i \neq j$,

2. **Normalization:** $\langle v_i, v_i \rangle = 1$ for all $1 \leq i \leq k$.

Equivalently,

$$\langle v_i, v_j \rangle = \delta_{ij} \quad \text{for all } 1 \leq i, j \leq k,$$

where δ_{ij} is the Kronecker delta.

Given such a set, any vector $x \in \operatorname{span}\{v_1, \ldots, v_k\}$ admits the unique expansion

$$x = \sum_{i=1}^{k} \langle x, v_i \rangle v_i,$$

and the coefficients $\langle x, v_i \rangle$ are the coordinates of x in this orthonormal system.

Remark II.4.1. (Partial isometry and Hilbert basis correspondence)

An orthonormal set $\{v_i\}_{i=1}^k$ defines a linear map

$$T : \mathbb{F}^k \to V, \quad T(e_i) = v_i,$$

where $\{e_i\}$ is the standard orthonormal basis of $\mathbb{F}^k$. This map T is a *partial isometry*: it preserves inner products between vectors in $\mathbb{F}^k$ whose images lie in the span of the v_i's, i.e.,

$$\langle T(\alpha), T(\beta) \rangle_V = \langle \alpha, \beta \rangle_{\mathbb{F}^k} \quad \text{for all } \alpha, \beta \in \mathbb{F}^k.$$

If the orthonormal set is maximal in V (that is, it forms an orthonormal basis), then T is an *isometric isomorphism* between $\mathbb{F}^k$ and V when $\dim V = k$, or between $\ell^2(I)$ and V in the infinite-dimensional Hilbert space case.

In this maximal case, the set is called a *Hilbert basis*, and every vector $x \in V$ satisfies the Parseval identity

$$\|x\|^2 = \sum_{i=1}^{k} |\langle x, v_i \rangle|^2$$

(in the finite case; in the infinite case, the sum runs over the index set I and converges absolutely). This illustrates the deep structural role of orthonormal systems in both finite and infinite-dimensional inner product spaces. ♦

Theorem II.4.1. Orthogonal Projection.

Let $W \subset V$ be a finite-dimensional subspace, and let $\{w_1, \ldots, w_k\}$ be an orthonormal basis of W. Then for any $v \in V$, the unique vector in W minimizing $\|v - w\|$ is given by the orthogonal projection:

$$\pi_W(v) = \sum_{i=1}^{k} \langle v, w_i \rangle w_i.$$

Moreover, the residual vector $v - \pi_W(v) \in V$ satisfies $\langle v - \pi_W(v), w_i \rangle = 0$ for all i, i.e.,

$$v - \pi_W(v) \perp W.$$

⌟

Proof II.4.1. Let V be an inner product space and $W \subset V$ a finite-dimensional subspace with orthonormal basis $\{w_1, \ldots, w_k\}$. Given any $v \in V$, we want to find the unique $w^* \in W$ that minimizes the distance $\|v - w\|$.

Any vector $w \in W$ can be written uniquely as

$$w = \sum_{i=1}^{k} a_i w_i,$$

for some coefficients $a_i \in \mathbb{F}$. Then the squared norm becomes

$$\begin{aligned}
\|v - w\|^2 &= \left\| v - \sum_{i=1}^{k} a_i w_i \right\|^2 \\
&= \left\langle v - \sum_{i=1}^{k} a_i w_i,\ v - \sum_{j=1}^{k} a_j w_j \right\rangle \\
&= \langle v, v \rangle - \sum_{j=1}^{k} \overline{a_j} \langle v, w_j \rangle - \sum_{i=1}^{k} a_i \langle w_i, v \rangle + \sum_{i,j=1}^{k} a_i \overline{a_j} \langle w_i, w_j \rangle.
\end{aligned}$$

Since $\{w_i\}$ is orthonormal, $\langle w_i, w_j \rangle = \delta_{ij}$, so the double sum collapses:

$$\sum_{i,j=1}^{k} a_i \overline{a_j} \langle w_i, w_j \rangle = \sum_{i=1}^{k} |a_i|^2.$$

Thus,

$$\|v - w\|^2 = \|v\|^2 - 2\,\mathrm{Re}\left(\sum_{i=1}^{k} a_i \langle w_i, v \rangle \right) + \sum_{i=1}^{k} |a_i|^2.$$

To minimize this quadratic expression in the a_i, observe that the unique minimum occurs when

$$a_i = \langle v, w_i \rangle.$$

Substituting these values yields the projection

$$\pi_W(v) = \sum_{i=1}^{k} \langle v, w_i \rangle w_i.$$

Orthogonality of the residual. Define the residual vector:

$$r := v - \pi_W(v).$$

Then for each $j = 1, \ldots, k$,

$$\langle r, w_j \rangle = \left\langle v - \sum_{i=1}^{k} \langle v, w_i \rangle w_i,\ w_j \right\rangle = \langle v, w_j \rangle - \sum_{i=1}^{k} \langle v, w_i \rangle \langle w_i, w_j \rangle = \langle v, w_j \rangle - \langle v, w_j \rangle = 0.$$

Hence $r \perp w_j$ for all j, so $r \perp W$.

Uniqueness. If there were another $w' \in W$ such that $\|v - w'\| \le \|v - \pi_W(v)\|$, then $w' - \pi_W(v) \in W$ and

$$\|v - w'\|^2 = \|r + \pi_W(v) - w'\|^2 = \|r\|^2 + \|\pi_W(v) - w'\|^2 + 2\,\mathrm{Re}\langle r, \pi_W(v) - w' \rangle.$$

But since $r \perp W$ and $\pi_W(v) - w' \in W$, the cross term vanishes. Therefore,

$$\|v - w'\|^2 = \|r\|^2 + \|\pi_W(v) - w'\|^2 \ge \|r\|^2,$$

with equality iff $\pi_W(v) = w'$. So the minimizer is unique.

□

Example II.6 (Projection onto a Line)**.** Let $v = (3, 4) \in \mathbb{R}^2$, and let $w = (1, 0)$ be a unit vector in the x-direction. Then:

$$\mathrm{proj}_w(v) = \langle v, w\rangle w = 3 \cdot (1, 0) = (3, 0),$$

and the residual $v - \mathrm{proj}_w(v) = (0, 4)$ lies along the y-axis.

The projection splits v into orthogonal components along and perpendicular to w.

Definition II.4.2. Orthogonal Decomposition.

Given an inner product space V and a subspace $W \subset V$, every $v \in V$ admits a unique decomposition

$$v = \pi_W(v) + \pi_{W^\perp}(v),$$

where $\pi_W(v) \in W$ and $\pi_{W^\perp}(v) \in W^\perp$. This decomposition is a direct sum:

$$V = W \oplus W^\perp.$$

Remark II.4.2. (Functoriality and operator theory of orthogonal decomposition)

In the setting of a finite-dimensional real (or complex) inner product space $(V, \langle\cdot,\cdot\rangle)$, the orthogonal decomposition

$$V = W \oplus W^\perp$$

is *functorial* with respect to isometries: if $T : V \to V'$ is a linear isometry between inner product spaces and $W \subset V$ is a subspace, then

$$T(W) \subset V' \quad \text{and} \quad T(W^\perp) = (T(W))^\perp$$

hold automatically. This follows from preservation of inner products:

$$\langle Tv, Tw\rangle_{V'} = \langle v, w\rangle_V,$$

ensuring that orthogonality relations are preserved under T.

The associated orthogonal projection

$$\pi_W : V \to W$$

is characterized by two key operator-theoretic properties:

1. *Self-adjointness:* $\pi_W^* = \pi_W$, meaning $\langle \pi_W v, u\rangle = \langle v, \pi_W u\rangle$ for all $u, v \in V$.
2. *Idempotence:* $\pi_W^2 = \pi_W$, reflecting that once a vector is projected into W, it remains unchanged under further projection.

From the decomposition $V = W \oplus W^\perp$, the kernel and image are completely determined:

$$\ker(\pi_W) = W^\perp, \qquad \mathrm{im}(\pi_W) = W.$$

Thus π_W encodes the orthogonal decomposition as an operator, with $(I - \pi_W)$ serving as the orthogonal projection onto $W^\perp$.

Functoriality further implies naturality in the category of finite-dimensional inner product spaces and orthogonal linear maps: for any isometry T,

$$T \circ \pi_W = \pi_{T(W)} \circ T,$$

so orthogonal projection commutes with isometric transformations up to identification of the corresponding subspaces. This expresses the decomposition as a *natural transformation* between the identity functor and the "projection onto a fixed subspace" functor in this category. ♦

Theorem II.4.2. Gram–Schmidt Orthonormalization.

Let V be a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$, and let $\{v_1, \dots, v_n\} \subset V$ be linearly independent. Then there exists an orthonormal set $\{u_1, \dots, u_n\}$ such that for each k,

$$\operatorname{span}(u_1, \dots, u_k) = \operatorname{span}(v_1, \dots, v_k).$$

⌟

Proof II.4.2. The proof proceeds by induction.

Base case: Define

$$u_1 = \frac{v_1}{\|v_1\|},$$

which is well-defined since $v_1 \neq 0$. Then $\|u_1\| = 1$, and the span condition is satisfied.

Inductive step: Suppose $\{u_1, \dots, u_{k-1}\}$ has been constructed such that:

1. $\langle u_i, u_j \rangle = \delta_{ij}$ for all i, j,
2. $\operatorname{span}(u_1, \dots, u_{k-1}) = \operatorname{span}(v_1, \dots, v_{k-1})$.

Let $W_{k-1} := \operatorname{span}(u_1, \dots, u_{k-1})$, and define the orthogonal projection of v_k onto W_{k-1}:

$$\pi_{W_{k-1}}(v_k) = \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j.$$

Define the orthogonal residual:

$$w_k := v_k - \pi_{W_{k-1}}(v_k).$$

Then $w_k \perp u_j$ for all $j < k$. Since $v_1, \dots, v_n$ are linearly independent, $w_k \neq 0$. Finally, define

$$u_k := \frac{w_k}{\|w_k\|}.$$

The set $\{u_1, \dots, u_n\}$ is orthonormal and yields a nested flag of subspaces respecting the original spans. □

Remark II.4.3. (Coordinate-free Reformulation)

The Gram–Schmidt process defines a canonical morphism from the poset of linearly independent tuples in V to the orthonormal frame bundle of V. It is equivariant under orthogonal transformations and yields an isometry $\mathbb{R}^n \to V$ via the map:

$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} x_i u_i.$$

This provides a concrete instantiation of abstract Euclidean structure on V. ♦

Remark II.4.4. (QR Factorization)

Given a full-rank matrix $A \in \mathbb{R}^{m\times n}$ with linearly independent columns, Gram–Schmidt yields an orthogonal matrix $Q \in \mathbb{R}^{m\times n}$ and upper triangular $R \in \mathbb{R}^{n\times n}$ such that

$$A = QR.$$

This is the analytic shadow of the orthonormalization process viewed through matrix coordinates. ♦

II.5 Symmetric Matrices and Spectral Theorem

Remark II.5.1. (Primer on symmetric matrices)

Symmetric matrices are the finite-dimensional shadows of self-adjoint operators. They represent the class of linear maps whose action preserves inner product geometry, making them central to the geometry of quadratic forms, variational calculus, and orthogonal decompositions. Their spectral behavior reflects deep mathematical phenomena—real eigenvalues, orthogonal eigenspaces, and full diagonalizability—which do not hold in general linear algebra but are guaranteed in the symmetric case due to the rich structure imposed by symmetry. ♦

Definition II.5.1. Symmetric and Hermitian Matrices.

Let $A \in \mathbb{F}^{n\times n}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

- If $\mathbb{F} = \mathbb{R}$, then A is called *symmetric* if $A^\top = A$.
- If $\mathbb{F} = \mathbb{C}$, then A is called *Hermitian* if $A^* = \overline{A}^\top = A$.

Definition II.5.2. Self-Adjoint Operator.

Let V be a finite-dimensional inner product space over $\mathbb{F}$, and let $T : V \to V$ be a linear operator. The adjoint $T^* : V \to V$ is defined by

$$\langle Tv, w\rangle = \langle v, T^*w\rangle \quad \text{for all } v, w \in V.$$

The operator T is called *self-adjoint* if $T = T^*$. In coordinates, relative to an orthonormal basis, T is represented by a symmetric (real case) or Hermitian (complex case) matrix.

Remark II.5.2. (Self-adjointness generalizes matrix symmetry)

Self-adjointness is a coordinate-free generalization of matrix symmetry. It ensures that the operator acts compatibly with the inner product structure, and its spectrum is real. In infinite dimensions, this leads to the theory of unbounded operators, essential in quantum mechanics and PDEs. In the finite-dimensional case, self-adjointness guarantees full diagonalizability and orthonormal eigenspaces. ♦

Example II.7 (Diagonalization of a Symmetric Matrix)**.** Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

We verify that $A^\top = A$, so A is symmetric. We now compute its eigenvalues and an orthonormal basis of eigenvectors to diagonalize A.

Step 1: Compute the characteristic polynomial.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 3 \end{bmatrix} = (\lambda - 2)(\lambda - 3) - 1 = \lambda^2 - 5\lambda + 5.$$

The eigenvalues are the roots of $\lambda^2 - 5\lambda + 5 = 0$, which are

$$\lambda_{1,2} = \frac{5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 5}}{2} = \frac{5 \pm \sqrt{25-20}}{2} = \frac{5 \pm \sqrt{5}}{2}.$$

Step 2: Compute eigenvectors.

Let $\lambda_1 = \frac{5+\sqrt{5}}{2}$. We solve $(A - \lambda_1 I)v = 0$. Define

$$A - \lambda_1 I = \begin{bmatrix} 2-\lambda_1 & -1 \\ -1 & 3-\lambda_1 \end{bmatrix}.$$

We can choose one row and solve:

$$(2-\lambda_1)x - y = 0 \quad \Rightarrow \quad y = (2-\lambda_1)x.$$

So an eigenvector is

$$v_1 = \begin{bmatrix} 1 \\ 2-\lambda_1 \end{bmatrix}.$$

Similarly, for $\lambda_2 = \frac{5-\sqrt{5}}{2}$, we obtain

$$v_2 = \begin{bmatrix} 1 \\ 2-\lambda_2 \end{bmatrix}.$$

Step 3: Normalize the eigenvectors.

We compute the norms:

$$\|v_1\|^2 = 1^2 + (2-\lambda_1)^2,$$
$$\|v_2\|^2 = 1^2 + (2-\lambda_2)^2.$$

Note:

$$2 - \lambda_1 = 2 - \frac{5+\sqrt{5}}{2} = \frac{-1-\sqrt{5}}{2}, \quad (2-\lambda_1)^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{6+2\sqrt{5}}{4},$$

so

$$\|v_1\|^2 = 1 + \frac{6+2\sqrt{5}}{4} = \frac{4+6+2\sqrt{5}}{4} = \frac{10+2\sqrt{5}}{4}.$$

Thus, the normalized eigenvector is:

$$u_1 = \frac{1}{\sqrt{\frac{10+2\sqrt{5}}{4}}} \begin{bmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{bmatrix}.$$

Similarly, compute u_2 as the normalized v_2.

Step 4: Diagonalize A.

Let $Q = [u_1 \ u_2] \in \mathbb{R}^{2\times 2}$ be the orthogonal matrix with columns the normalized eigenvectors, and let

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Then

$$A = Q\Lambda Q^\top.$$

Conclusion: This validates the spectral theorem for this explicit case. The symmetric matrix A is orthogonally diagonalizable, with real eigenvalues and orthonormal eigenvectors.

Theorem II.5.1. Spectral Theorem for Real Symmetric Matrices.
Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix. Then:

1. All eigenvalues of A are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. There exists an orthonormal basis $\{v_1, \dots, v_n\} \subset \mathbb{R}^n$ consisting of eigenvectors of A.

Equivalently, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and a real diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that

$$A = Q\Lambda Q^\top.$$

⌟

Proof II.5.1. We proceed in four steps.

Step 1: Existence of a real eigenvalue.

Since A is symmetric and real, it is diagonalizable over $\mathbb{C}$. By the fundamental theorem of algebra, the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$ has n (possibly complex) roots.

Let $\lambda \in \mathbb{C}$ be an eigenvalue, and $v \in \mathbb{C}^n$ be a corresponding eigenvector, $v \neq 0$, so that $Av = \lambda v$.

Consider the scalar $\langle Av, v\rangle$. Since A is real and symmetric, we have:

$$\langle Av, v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \lambda\langle v, v\rangle.$$

But also:

$$\langle Av, v\rangle = \langle Av, v\rangle = \langle \lambda v, v\rangle = \overline{\lambda}\langle v, v\rangle.$$

Since $\langle v, v\rangle \neq 0$, we conclude $\lambda = \overline{\lambda}$, so $\lambda \in \mathbb{R}$.

Thus, every eigenvalue of a real symmetric matrix is real.

Step 2: Orthogonality of eigenvectors.

Let $\lambda_1 \neq \lambda_2$ be distinct real eigenvalues of A, with corresponding eigenvectors $v_1, v_2 \in \mathbb{R}^n$. Then:

$$Av_1 = \lambda_1 v_1, \qquad Av_2 = \lambda_2 v_2.$$

Compute:

$$\langle Av_1, v_2\rangle = \lambda_1\langle v_1, v_2\rangle, \qquad \langle v_1, Av_2\rangle = \lambda_2\langle v_1, v_2\rangle.$$

But A is symmetric, so $\langle Av_1, v_2\rangle = \langle v_1, Av_2\rangle$, hence

$$\lambda_1\langle v_1, v_2\rangle = \lambda_2\langle v_1, v_2\rangle.$$

Subtracting gives

$$(\lambda_1 - \lambda_2)\langle v_1, v_2\rangle = 0.$$

Since $\lambda_1 \neq \lambda_2$, it follows that $\langle v_1, v_2\rangle = 0$.

Step 3: Inductive diagonalization.

We prove the result by induction on n.

Base case: For $n = 1$, any 1×1 real symmetric matrix is diagonal and the claim is trivial.

Inductive step: Let $A \in \mathbb{R}^{n\times n}$ be symmetric. From Step 1, A has a real eigenvalue λ, with a real unit eigenvector $v \in \mathbb{R}^n$, $\|v\| = 1$.

Let $V := \operatorname{span}\{v\}$, and let $V^\perp \subset \mathbb{R}^n$ be the orthogonal complement. Then $\dim V^\perp = n - 1$, and we claim A preserves $V^\perp$, i.e., $Aw \in V^\perp$ for all $w \in V^\perp$.

Indeed, for $w \in V^\perp$, we compute

$$\langle Aw, v\rangle = \langle w, Av\rangle = \langle w, \lambda v\rangle = \lambda\langle w, v\rangle = 0.$$

Thus $Aw \perp v$, so A maps $V^\perp$ into itself.

Let A' denote the restriction of A to $V^\perp$, i.e., $A' = A|_{V^\perp} \in \mathbb{R}^{(n-1)\times(n-1)}$. Since A is symmetric, A' is also symmetric with respect to the induced inner product.

By the induction hypothesis, A' admits an orthonormal eigenbasis $\{v_2, \dots, v_n\} \subset V^\perp$. Then $\{v, v_2, \dots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Step 4: Matrix formulation.

Let $Q = [v_1\, v_2\, \dots\, v_n] \in \mathbb{R}^{n\times n}$ be the orthogonal matrix whose columns are the orthonormal eigenvectors of A. Let $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ be the diagonal matrix of corresponding eigenvalues.

Then

$$AQ = Q\Lambda \quad \Rightarrow \quad Q^\top AQ = \Lambda,$$

and thus

$$A = Q\Lambda Q^\top.$$

This completes the proof. □

Example II.8 (Spectral Decomposition of a Symmetric Matrix)**.** Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

We aim to diagonalize A by computing its spectral decomposition:

$$A = Q\Lambda Q^\top,$$

where Q is orthogonal and Λ is diagonal with the eigenvalues of A.

Step 1: Verify symmetry. Note:

$$A^\top = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A,$$

so A is symmetric. Thus, the Spectral Theorem applies: all eigenvalues are real and there exists an orthonormal basis of eigenvectors.

Step 2: Compute eigenvalues. Solve the characteristic equation:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix} = (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 3.$$

Thus, the eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = 3.$$

Step 3: Compute eigenvectors.

<u>For $\lambda = 1$</u>:

$$(A - I)v = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow x + y = 0.$$

So a basis for this eigenspace is:

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $\lambda = 3$:

$$(A - 3I)v = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow x - y = 0.$$

So a basis for this eigenspace is:

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Step 4: Normalize eigenvectors.

We compute:

$$\|v_1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \quad \|v_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

So the normalized eigenvectors are:

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Step 5: Construct Q and Λ.

Let

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then

$$Q^\top A Q = \Lambda.$$

Alternatively, verify:

$$A = Q \Lambda Q^\top.$$

Explicitly compute:

$$Q\Lambda = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix},$$

$$Q\Lambda Q^\top = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A.$$

Thus, A admits an orthonormal diagonalization with real eigenvalues and mutually orthogonal eigenvectors, confirming the spectral theorem.

II.6 Quadratic Forms and Geometry

Quadratic forms provide a canonical way to study curvature, energy, and second-order variation in multivariable systems. They naturally arise in linear algebra through symmetric matrices, in calculus through Hessians, and in geometry through metric tensors.

Definition II.6.1. Quadratic Form.

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. A function $Q : \mathbb{R}^n \to \mathbb{R}$ is called a *quadratic form* if it has the expression

$$Q(x) = x^\top A x,$$

where $x \in \mathbb{R}^n$ is viewed as a column vector and $x^\top$ is its transpose.

Remark II.6.1. ()

The symmetry of A ensures that $x^\top Ax$ is always a scalar. Moreover, it guarantees that $Q(x)$ depends only on the symmetric part of any bilinear form, as

$$x^\top Ax = x^\top \left(\frac{A + A^\top}{2}\right) x.$$

Thus, only symmetric matrices correspond to genuine quadratic forms. ♦

Example II.9 (Matrix Form and Expansion)**.** Let $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$, and let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then:

$$Q(x) = x^\top Ax = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2bx_1x_2 + dx_2^2.$$

Quadratic forms generalize scalar-valued second-degree polynomials in multiple variables.

Definition II.6.2. Level Set of a Quadratic Form.

Given a quadratic form $Q(x) = x^\top Ax$, the set

$$\{x \in \mathbb{R}^n : Q(x) = c\}$$

is called the *level set* at height c. The set $Q(x) = 1$ is particularly important in classifying the shape induced by A.

Example II.10 (Conic Sections via Quadratic Form)**.** Let $A \in \mathbb{R}^{2\times 2}$ be symmetric, and consider the level set $Q(x) = x^\top Ax = 1$. To classify the geometry of this curve, perform the following steps:

Step 1: Diagonalize A. Since A is symmetric, the spectral theorem ensures the existence of an orthogonal matrix $Q \in \mathbb{R}^{2\times 2}$ and a diagonal matrix $D = \operatorname{diag}(\lambda_1, \lambda_2)$ such that

$$A = QDQ^\top.$$

Step 2: Change of coordinates. Let $x = Qy$, so that $Q^\top x = y$. Then:

$$Q(x) = x^\top Ax = (Qy)^\top QDQ^\top Qy = y^\top Dy = \lambda_1 y_1^2 + \lambda_2 y_2^2.$$

Step 3: Analyze the sign of eigenvalues.

- If $\lambda_1 > 0$, $\lambda_2 > 0$, the level set is an ellipse.
- If $\lambda_1\lambda_2 < 0$, the level set is a hyperbola.
- If exactly one eigenvalue is zero, the level set degenerates to a pair of lines.

The eigenvalues of A determine the curvature type of the conic section.

Definition II.6.3. Definiteness of a Quadratic Form.

Let $Q(x) = x^\top Ax$ be a quadratic form.

- Q is *positive definite* if $Q(x) > 0$ for all $x \neq 0$.
- Q is *negative definite* if $Q(x) < 0$ for all $x \neq 0$.

- Q is *indefinite* if it takes both positive and negative values.

Theorem II.6.1. Sylvester's Criterion.

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then A is positive definite if and only if all its leading principal minors are strictly positive:

$$\det A_k > 0 \quad \text{for all } k = 1, \dots, n,$$

where A_k denotes the $k \times k$ leading principal submatrix of A, i.e.,

$$A_k = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}.$$

⌟

Remark II.6.2. (Geometric Interpretation)

The condition $\det A_k > 0$ ensures that the quadratic form $Q(x) = x^\top Ax$ restricts to a strictly convex function on every k-dimensional subspace generated by the first k standard basis vectors. This guarantees that the associated ellipsoids are properly oriented and non-degenerate in all coordinate-aligned subspaces, and hence that A induces a Riemannian inner product. ♦

Example II.11 (Energy Form in Classical Mechanics)**.** Let $m_1, m_2 > 0$ be masses of two decoupled particles moving along orthogonal spatial axes. Define:

$$A = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad x = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

where x is the velocity vector in $\mathbb{R}^2$. The kinetic energy K of the system is:

$$\begin{aligned} K(v) &= \frac{1}{2} x^\top A x \\ &= \frac{1}{2} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2). \end{aligned}$$

This function defines a strictly convex quadratic form:

- The matrix A is diagonal with $m_1, m_2 > 0$, so it is symmetric and positive definite.
- The energy functional $K(v)$ satisfies:

$$K(v) > 0 \quad \text{for all } v \neq 0, \qquad K(0) = 0.$$

- Its level sets $\{v \in \mathbb{R}^2 : K(v) = c\}$ are ellipses:

$$\frac{v_1^2}{2c/m_1} + \frac{v_2^2}{2c/m_2} = 1.$$

  These ellipses describe isoenergy contours, symmetric with respect to coordinate axes, representing equipartition in decoupled modes.

Connection to Sylvester's Criterion: We verify positive definiteness via principal minors:

$$\det A_1 = m_1 > 0,$$
$$\det A_2 = \det \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = m_1 m_2 > 0.$$

Hence, Sylvester's criterion confirms A is positive definite, and the kinetic energy is strictly positive except at rest.

Conclusion: This quadratic form encodes the stored motion energy in a mechanical system with two orthogonal degrees of freedom. The positivity of the energy functional reflects the convex geometry of kinetic energy and underlies stability and variational principles in classical mechanics.

Remark II.6.3. ()

Quadratic forms encode the second-order behavior of scalar-valued functions. In multivariable calculus, the Hessian matrix of a function f at a point x_0 is symmetric, and the sign of the quadratic form $h^\top H(x_0)h$ determines whether f has a local minimum, maximum, or saddle at x_0. This gives geometric interpretation to the second derivative test. ♦

Remark II.6.4. ()

In Riemannian geometry, the metric tensor assigns to each point $p \in M$ a positive definite quadratic form on T_pM. The curvature and geodesic flow of the manifold are governed by the structure of these quadratic forms across the manifold. ♦

II.7 Fourier and Spectral Basis Expansion

The decomposition of functions into orthogonal modes is one of the deepest ideas in analysis. In finite dimensions, the spectral theorem provides a basis of eigenvectors for symmetric matrices. In infinite dimensions, the appropriate generalization is a complete orthonormal set of eigenfunctions of a self-adjoint operator, such as the Laplacian. This leads naturally to the theory of Fourier series, orthogonal expansions, and spectral geometry.

Definition II.7.1. Orthonormal Basis in Hilbert Space.

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A countable set $\{e_n\}_{n=1}^\infty \subset \mathcal{H}$ is called an *orthonormal basis* if:

1. $\langle e_n, e_m \rangle = \delta_{nm}$ for all $n, m \in \mathbb{N}$,
2. Every $f \in \mathcal{H}$ can be written as
$$f = \sum_{n=1}^\infty \langle f, e_n \rangle e_n,$$
where the convergence is in the norm topology of $\mathcal{H}$.

Example II.12 (One type of Fourier basis)**.** Let $\mathcal{H} = L^2([0, 2\pi]; \mathbb{R})$ with inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\,dx.$$

The functions
$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}\right\}_{n=1}^{\infty}$$
form a complete orthonormal basis for $L^2([0, 2\pi])$. Any $f \in L^2$ admits the expansion:
$$f(x) = \sum_{n=0}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$
with convergence in the L^2-norm, where
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx.$$
This expresses f as a linear combination of eigenfunctions of the negative Laplacian on the circle.

Theorem II.7.1. Spectral Theorem for Compact Self-Adjoint Operators; Naïve Form.

Let $T : \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator on a separable Hilbert space. Then:

1. The spectrum of T consists of a countable set of real eigenvalues $\{\lambda_n\}$ with $\lambda_n \to 0$.
2. There exists an orthonormal basis $\{e_n\} \subset \mathcal{H}$ consisting of eigenvectors of T.
3. For every $f \in \mathcal{H}$,
$$Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n\rangle e_n.$$

⌟

The proof is left as an exercise for the reader.

Theorem II.7.2. Spectral Theorem for Compact Self-Adjoint Operators; Strengthened Form.

Let $T : \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator on a (real or complex) separable Hilbert space $\mathcal{H}$. Then:

1. **Reality and discreteness of spectrum.** $\sigma(T) \subset \mathbb{R}$, and $\sigma(T) \setminus \{0\}$ is a (finite or countable) discrete set of eigenvalues, each of finite multiplicity, with the only possible accumulation point at 0.
2. **Orthogonal eigendecomposition.** There exists an orthonormal set $\{e_{n,k}\}$ consisting of eigenvectors of T, where for each nonzero eigenvalue λ_n the vectors $\{e_{n,k}\}_{k=1}^{m_n}$ form an orthonormal basis of $\ker(T - \lambda_n I)$ (thus $m_n < \infty$).
3. **ONB completion.** Augment $\{e_{n,k}\}$ by an orthonormal basis of $\ker T$ to obtain an orthonormal basis of $\mathcal{H}$. In particular, if $\{\lambda_n\}_{n\geq 1}$ denotes the list of nonzero eigenvalues (repeated by multiplicity), then either the list is finite or $\lambda_n \to 0$.
4. **Series representation (strong convergence).** For every $f \in \mathcal{H}$,
$$Tf = \sum_{n\geq 1}\sum_{k=1}^{m_n} \lambda_n \langle f, e_{n,k}\rangle\, e_{n,k},$$
where the series converges in norm (equivalently, $T = \sum_{n\geq 1} \lambda_n P_n$ with P_n the orthogonal projection onto $\ker(T - \lambda_n I)$).

5. **Norm and variational characterization.** $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\} = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. If $T \geq 0$, then $\|T\| = \max_{\|x\|=1}\langle Tx, x\rangle$ is an eigenvalue and all eigenvalues are nonnegative.

6. **Orthogonality of distinct eigenspaces.** If $\lambda \neq \mu$, then $\ker(T - \lambda I) \perp \ker(T - \mu I)$.

⌟

Proof II.7.1. We proceed in steps.

Step 1: Reality of spectrum and orthogonality. Self-adjointness ($T = T^*$) implies $\sigma(T) \subset \mathbb{R}$ and $\langle Tx, x\rangle \in \mathbb{R}$ for all x (standard functional analysis). If $Tx = \lambda x$ and $Ty = \mu y$ with $\lambda \neq \mu$, then

$$\lambda\langle x, y\rangle \;=\; \langle Tx, y\rangle \;=\; \langle x, Ty\rangle \;=\; \mu\langle x, y\rangle,$$

hence $\langle x, y\rangle = 0$, proving orthogonality of distinct eigenspaces.

Step 2: Existence of an eigenvalue at the norm. Set $M := \sup_{\|x\|=1}\langle Tx, x\rangle$ and $m := \inf_{\|x\|=1}\langle Tx, x\rangle$ (both real; $m \leq 0 \leq M$). Consider $S := T - \alpha I$ with $\alpha \in \{M, m\}$ chosen so that $\|S\| = \min\{\|T - MI\|, \|T - mI\|\}$. Replacing T by $\pm T$ if needed, we may assume $\alpha = M = \sup_{\|x\|=1}\langle Tx, x\rangle$.

Pick a maximizing sequence (u_k) with $\|u_k\| = 1$ and $\langle Tu_k, u_k\rangle \to M$. The unit ball is weakly compact; pass to a subsequence (not relabeled) with $u_k \rightharpoonup u$. Compactness of T implies $Tu_k \to Tu$ in norm. Then

$$\langle Tu_k, u_k\rangle \;\to\; \langle Tu, u\rangle,$$

since $Tu_k \to Tu$ strongly and $u_k \rightharpoonup u$ weakly. Hence $\langle Tu, u\rangle = M$ and $\|u\| = 1$ (by Cauchy-Schwarz and maximality).

We claim $Tu = Mu$. For any $v \in \mathcal{H}$ and $t \in \mathbb{R}$, by definition of M,

$$\langle T(u + tv), u + tv\rangle \;\leq\; M\|u + tv\|^2 \;=\; M(1 + 2t\,\mathrm{Re}\langle u, v\rangle + t^2\|v\|^2).$$

Expanding the left side and comparing coefficients of t at $t = 0$ yields

$$\mathrm{Re}\langle Tu, v\rangle \;\leq\; M\;\mathrm{Re}\langle u, v\rangle \quad \text{and} \quad \mathrm{Re}\langle Tu, v\rangle \;\geq\; M\;\mathrm{Re}\langle u, v\rangle,$$

hence $\mathrm{Re}\langle (T - MI)u, v\rangle = 0$ for all v. Similarly, testing with iv shows $\mathrm{Im}\langle (T - MI)u, v\rangle = 0$. Therefore $(T - MI)u = 0$. Thus $M \in \sigma_p(T)$ (point spectrum) with eigenvector u.

Step 3: Finite multiplicity and reduction. Let $\lambda \neq 0$ be an eigenvalue. Then $\ker(T - \lambda I)$ is finite-dimensional: indeed $T - \lambda I$ is a compact perturbation of an invertible map when restricted to $(\ker(T - \lambda I))^\perp$, hence has closed range of finite codimension; the kernel must be finite-dimensional (standard Fredholm alternative for compact operators). Moreover, $(\ker(T - \lambda I))^\perp$ is invariant under T (self-adjointness).

Let $E_\lambda := \ker(T - \lambda I)$ and decompose orthogonally $\mathcal{H} = E_\lambda \oplus E_\lambda^\perp$. The compression $T|_{E_\lambda^\perp}$ is again compact self-adjoint.

Step 4: Inductive construction of spectral list and accumulation only at 0. Apply Step 2 to $T|_{E_\lambda^\perp}$ to obtain an eigenvalue λ_2 (of T) on $E_\lambda^\perp$, orthogonal to E_λ. Iterate: after k steps we have pairwise orthogonal eigenspaces $E_{\lambda_1}, \ldots, E_{\lambda_k}$ and a compact self-adjoint restriction to their orthogonal complement. Either the process terminates (then the restriction is 0, see Step 5), or we obtain a sequence (λ_n) with $|\lambda_{n+1}| \leq |\lambda_n|$ (take nonincreasing absolute value ordering). If $|\lambda_n|$ does *not* converge to 0, then $\inf_n |\lambda_n| = \delta > 0$ produces infinitely many mutually orthogonal unit eigenvectors with $\|Te_n\| \geq \delta$, contradicting compactness (since $\{Te_n\}$ would have no convergent subsequence). Hence $\lambda_n \to 0$.

Step 5: Exhaustion and ONB completion. Let $\mathcal{E} := \overline{\operatorname{span}}\big(\bigcup_n E_{\lambda_n}\big)$. Let $R := T|_{\mathcal{E}^\perp}$. If $R \neq 0$, then by Step 2 applied to R we produce a nonzero eigenvalue on $\mathcal{E}^\perp$, contradicting the definition of $\mathcal{E}$. Hence $R = 0$ and T vanishes on $\mathcal{E}^\perp$. Therefore,

$$\mathcal{H} = \mathcal{E} \oplus \ker T,$$

and adjoining an ONB of $\ker T$ to ONBs of each E_{λ_n} yields an ONB of $\mathcal{H}$ consisting of eigenvectors (with eigenvalue 0 on $\ker T$). Separability ensures the resulting basis is countable.

Step 6: Series representation and strong convergence. Let $\{e_{n,k}\}$ be an orthonormal basis assembled as above, with $Te_{n,k} = \lambda_n e_{n,k}$ for $\lambda_n \neq 0$ and $Te = 0$ for $e \in \ker T$. For arbitrary $f = \sum \langle f, e_{n,k}\rangle e_{n,k}$ (Parseval expansion),

$$Tf \;=\; \sum_{n,k} \lambda_n \langle f, e_{n,k}\rangle e_{n,k},$$

where the series converges in norm because

$$\Bigg\| \sum_{(n,k)\in F} \lambda_n \langle f, e_{n,k}\rangle e_{n,k} \Bigg\|^2 = \sum_{(n,k)\in F} |\lambda_n|^2 \, |\langle f, e_{n,k}\rangle|^2 \le \|T\|^2 \sum_{(n,k)\in F} |\langle f, e_{n,k}\rangle|^2 \le \|T\|^2 \|f\|^2,$$

and partial sums form a Cauchy net by monotone exhaustion of F.

Step 7: Norm and extremal characterization; positivity. For self-adjoint T, $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle| = \max\{\sup \sigma(T), -\inf \sigma(T)\}$. In the positive case $T \ge 0$, the Rayleigh quotient satisfies $0 \le \langle Tx, x\rangle \le \|T\|$, and the argument of Step 2 shows $\|T\|$ is an eigenvalue with eigenvector achieving the maximum. All eigenvalues are then nonnegative.

Combining the steps yields all stated assertions. □

Remark II.7.1. (Fourier Analysis as a Spectral Theorem Instance)

The classical Fourier basis arises as a special case of the spectral theorem applied to the operator $T = -\frac{d^2}{dx^2}$ on $L^2([0, 2\pi])$ with periodic boundary conditions. This operator is self-adjoint, compact on the Sobolev scale, and positive semi-definite. Its eigenfunctions

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}, \quad \phi_n^{\cos}(x) = \frac{\cos(nx)}{\sqrt{\pi}}, \quad \phi_n^{\sin}(x) = \frac{\sin(nx)}{\sqrt{\pi}}, \quad n \ge 1$$

form a complete orthonormal basis of $L^2([0, 2\pi])$, and satisfy

$$T\phi_0 = 0, \quad T\phi_n^{\cos} = n^2 \phi_n^{\cos}, \quad T\phi_n^{\sin} = n^2 \phi_n^{\sin}.$$

Thus, the spectrum consists of real eigenvalues $\lambda_n = n^2 \to \infty$, accumulating only at infinity, with corresponding finite-multiplicity eigenspaces. For any $f \in L^2([0, 2\pi])$, the expansion

$$f = \sum_{n=0}^{\infty} \langle f, \phi_n\rangle \phi_n$$

(where ϕ_n ranges over the cosine and sine modes appropriately indexed) realizes the action of T as

$$Tf = \sum_{n=1}^{\infty} n^2 \langle f, \phi_n\rangle \phi_n.$$

This instantiates the spectral theorem in a geometrically meaningful context: the operator T encodes curvature (as second derivative), and the eigenfunctions are geometric modes (oscillations on the circle). Fourier analysis thus appears as a *spectral geometry*, where the analytic decomposition of functions is governed by the structure of the underlying differential operator. More generally, spectral expansions allow arbitrary compact self-adjoint operators to diagonalize the geometry of information propagation, energy dispersion, or signal flow in Hilbert space. ♦

Definition II.7.2. Laplace Operator on the Circle.

Define $\Delta = \frac{d^2}{dx^2}$ acting on $C^\infty_{\text{per}}([0, 2\pi])$. Then $-\Delta$ is positive semi-definite and self-adjoint on $L^2([0, 2\pi])$, with eigenfunctions:

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}, \quad \phi_n^{\cos}(x) = \frac{\cos(nx)}{\sqrt{\pi}}, \quad \phi_n^{\sin}(x) = \frac{\sin(nx)}{\sqrt{\pi}}, \quad n \geq 1,$$

and corresponding eigenvalues $0, n^2, n^2$. The Fourier expansion is then a spectral decomposition of f in terms of $-\Delta$-eigenfunctions.

Remark II.7.2. (Spectral Geometry of the Circle as Mode Decomposition)

The interpretation of the eigenfunctions $\phi_n^{\cos}, \phi_n^{\sin}$ of the Laplacian $-\Delta$ as oscillatory modes directly identifies the operator $-\Delta$ with the generator of energy in a circular vibrating string. More precisely, each eigenfunction corresponds to a standing wave on the circle S^1, with eigenvalue $\lambda_n = n^2$ representing the squared angular frequency ω_n^2 of the mode. That is, under the d'Alembertian separation of variables, the wave equation solution decomposes into modes satisfying

$$-\Delta\phi_n(x) = \lambda_n \phi_n(x), \quad \lambda_n = n^2.$$

Thus the Laplacian spectrum is the frequency spectrum, and the orthonormal basis $\{\phi_0, \phi_n^{\cos}, \phi_n^{\sin}\}$ furnishes a complete set of decoupled modes, each evolving independently in time as $e^{\pm int}$. This realizes a canonical spectral decomposition of any $f \in L^2(S^1)$ via

$$f(x) = a_0\phi_0(x) + \sum_{n=1}^{\infty} a_n \phi_n^{\cos}(x) + b_n \phi_n^{\sin}(x),$$

where a_n, b_n are Fourier coefficients, and where Parseval's identity identifies the L^2-norm of f with the total energy of the oscillatory decomposition:

$$\|f\|_{L^2}^2 = |a_0|^2 + \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right).$$

Geometrically, $-\Delta$ is the Casimir operator of the rotation group acting on S^1, and its eigenfunctions correspond to harmonics in the representation theory of SO(2). Analytically, it realizes a complete orthogonal decomposition of the energy landscape on the circle. Thus, the Laplacian provides both a dynamical and geometric diagonalization of physical structure. ♦

Example II.13 (Heat Equation on the Circle)**.** Consider the PDE

$$\partial_t u = \Delta u, \quad u(x, 0) = f(x), \quad u(x + 2\pi, t) = u(x, t).$$

Expanding f in the Fourier basis:

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

the solution becomes:

$$u(x,t) = \sum_{n=0}^{\infty} \left(a_n e^{-n^2 t} \cos(nx) + b_n e^{-n^2 t} \sin(nx) \right).$$

Each frequency mode decays exponentially in time, with rate governed by its Laplacian eigenvalue.

Remark II.7.3. (Spectral basis expansions and Laplace–Beltrami)

This illustrates the power of spectral basis expansions. The infinite-dimensional operator is diagonalized by the Fourier basis, and the PDE reduces to a countable family of scalar ODEs. The structure of the solution is encoded entirely in the spectrum.

In geometric analysis, the spectrum of the Laplace–Beltrami operator on a compact Riemannian manifold encodes deep information about volume, curvature, and topology. The study of whether one can recover geometry from spectrum is captured in the question: "Can one hear the shape of a drum?" ♦

II.8 Functional Calculus for the Laplacian on the Circle

Theorem II.8.1. Functional Calculus for the Laplacian on the Circle.

Let $\Delta = \frac{d^2}{dx^2}$ be the Laplacian on $L^2([0, 2\pi])$ with periodic boundary conditions. Then:

1. The operator $-\Delta$ is unbounded, densely defined, and self-adjoint on its domain

$$\mathcal{D}(-\Delta) = \left\{ f \in L^2([0,2\pi]) \;\middle|\; f \in C^\infty_{\text{per}}, \sum_{n=1}^{\infty} n^4 \left(|a_n|^2 + |b_n|^2 \right) < \infty \right\},$$

   where $f(x) = \sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$ is the Fourier expansion of f.

2. The Fourier basis $\{\phi_n\}$ given by

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}, \quad \phi_n^{\cos}(x) = \frac{\cos(nx)}{\sqrt{\pi}}, \quad \phi_n^{\sin}(x) = \frac{\sin(nx)}{\sqrt{\pi}}, \quad n \geq 1,$$

   forms a complete orthonormal set of eigenfunctions of $-\Delta$, with eigenvalues

$$-\Delta\phi_0 = 0, \quad -\Delta\phi_n^{\cos} = n^2 \phi_n^{\cos}, \quad -\Delta\phi_n^{\sin} = n^2 \phi_n^{\sin}.$$

3. For any bounded Borel function $F : [0, \infty) \to \mathbb{C}$, the operator $F(-\Delta)$ is defined by

$$F(-\Delta) f := \sum_{n=0}^{\infty} F(\lambda_n) \langle f, \phi_n \rangle \phi_n,$$

   where the eigenvalues $\lambda_0 = 0$, $\lambda_n = n^2$ for $n \geq 1$, and the sum converges in the L^2-norm. This defines a bounded self-adjoint operator on $L^2([0, 2\pi])$, known as the spectral functional calculus of $-\Delta$.

4. In particular, the heat semigroup is given by

$$e^{t\Delta} f = \sum_{n=0}^{\infty} e^{-t\lambda_n} \langle f, \phi_n \rangle \phi_n,$$

which solves the initial value problem

$$\partial_t u = \Delta u, \quad u(x,0) = f(x).$$

Similarly, the unitary Schrödinger propagator is

$$e^{it\Delta} f = \sum_{n=0}^{\infty} e^{-it\lambda_n} \langle f, \phi_n \rangle \phi_n,$$

which solves

$$i\partial_t u = -\Delta u, \quad u(x,0) = f(x).$$

⌟

Proof II.8.1. We prove each part in sequence.

(1) Self-adjointness and Domain. The operator $\Delta = \frac{d^2}{dx^2}$ with periodic boundary conditions is symmetric on the dense domain $C^\infty_{\text{per}}([0,2\pi]) \subset L^2([0,2\pi])$. For $f, g \in C^\infty_{\text{per}}$,

$$\langle \Delta f, g \rangle = \int_0^{2\pi} f''(x) g(x)\, dx = \int_0^{2\pi} f(x) g''(x)\, dx = \langle f, \Delta g \rangle,$$

using integration by parts and periodicity to cancel boundary terms. This symmetry, together with essential self-adjointness of Δ on this domain, implies that the closure $-\Delta$ is self-adjoint. The Sobolev space H^2_{per} is the natural domain for $-\Delta$, but in terms of Fourier series, we require $f \in L^2$ with

$$\sum_{n=1}^{\infty} n^4 (|a_n|^2 + |b_n|^2) < \infty,$$

which ensures $f'' \in L^2$, so $f \in \mathcal{D}(-\Delta)$.

(2) Orthonormal Basis and Eigenvalues. Each of the functions $\phi_n^{\cos}(x) = \frac{\cos(nx)}{\sqrt{\pi}}$, $\phi_n^{\sin}(x) = \frac{\sin(nx)}{\sqrt{\pi}}$, and $\phi_0 = \frac{1}{\sqrt{2\pi}}$ lies in $L^2([0,2\pi])$, and they are mutually orthonormal. Indeed,

$$\langle \cos(nx), \cos(mx) \rangle = \begin{cases} 0 & n \neq m, \\ \pi & n = m, \end{cases} \qquad \langle \cos(nx), \sin(mx) \rangle = 0,$$

and similarly for sine terms.

Furthermore,

$$-\Delta \phi_n^{\cos}(x) = \frac{d^2}{dx^2} \left(\frac{\cos(nx)}{\sqrt{\pi}} \right) = n^2 \frac{\cos(nx)}{\sqrt{\pi}} = n^2 \phi_n^{\cos}(x),$$

and likewise for sine. Thus, these are eigenfunctions with eigenvalue n^2, and the constant function is the eigenfunction for eigenvalue zero.

(3) Spectral Representation. Since $-\Delta$ is self-adjoint and has compact resolvent (equivalent to the embedding $H^2 \hookrightarrow L^2$ being compact), the spectral theorem applies. We may write:

$$f = \sum_{n=0}^{\infty} \langle f, \phi_n \rangle \phi_n,$$

with convergence in L^2. For any bounded Borel function F, we define:

$$F(-\Delta) f := \sum_{n=0}^{\infty} F(\lambda_n) \langle f, \phi_n \rangle \phi_n,$$

which converges in L^2 by Parseval's identity:

$$\|F(-\Delta)f\|_{L^2}^2 = \sum_{n=0}^{\infty} |F(\lambda_n)|^2 |\langle f, \phi_n\rangle|^2 \leq \|F\|_\infty^2 \|f\|_{L^2}^2.$$

So $F(-\Delta)$ is a bounded self-adjoint operator.

(4) Heat and Schrödinger Equations. Taking $F(\lambda) = e^{-t\lambda}$ yields:

$$e^{t\Delta} f := \sum_{n=0}^{\infty} e^{-t\lambda_n} \langle f, \phi_n\rangle \phi_n,$$

and since each eigenfunction solves $\partial_t \phi_n(t) = -\lambda_n \phi_n(t)$, the full solution

$$u(x,t) = \sum_{n=0}^{\infty} e^{-t\lambda_n} \langle f, \phi_n\rangle \phi_n(x)$$

solves $\partial_t u = \Delta u$, with $u(x,0) = f(x)$. Similarly, for the Schrödinger equation, we take $F(\lambda) = e^{-it\lambda}$, and the same analysis applies:

$$u(x,t) = \sum_{n=0}^{\infty} e^{-it\lambda_n} \langle f, \phi_n\rangle \phi_n(x)$$

solves $i\partial_t u = -\Delta u$. In both cases, the differential equation reduces to scalar multiplication in the spectral basis.

□

II.9 Preview: Applications and Dynamics

- Gram matrix and least-squares: $A^\top A$ symmetric, positive semi-definite.
- Principal Component Analysis (PCA): eigenvectors of covariance matrix.
- Vibrations and dynamics: diagonalizing stiffness or inertia matrices.
- PDEs: solving via eigenfunction expansion.

Example II.14 (Linear ODE System)**.** Let $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$ with $A = Q\Lambda Q^\top$. Then:

$$x(t) = Qe^{\Lambda t} Q^\top x(0),$$

so dynamics decouple in eigenbasis coordinates.

Remark II.9.1. (On Self-Adjoint Operator Spectral Decomposition)

This structure, the spectral decomposition of self-adjoint operators, will recur in every advanced mathematical context. From Schrödinger operators to kernel machines, from Riemannian geometry to signal processing, it provides the framework for mode decomposition and stability analysis. ♦

III Tensor Fields and Differential Forms

III.1 Smooth Vector Fields and Covector Fields

Definition III.1.1. Smooth Vector Field.

Let M be a smooth n-dimensional manifold with tangent bundle TM and canonical projection

$$\pi : TM \to M. \tag{III.1.1}$$

A *smooth vector field* is a smooth section

$$X \in \Gamma(TM), \quad \pi \circ X = \mathrm{id}_M, \tag{III.1.2}$$

meaning that for every $p \in M$, the vector $X(p)$ lies in the fiber T_pM of (III.1.1).

In a smooth local chart (U, x^i), (III.1.2) takes the form

$$X|_U = X^i(x)\frac{\partial}{\partial x^i}, \quad X^i \in C^\infty(U), \tag{III.1.3}$$

where X^i are the component functions of X relative to the coordinate basis $\{\frac{\partial}{\partial x^i}\}$. Thus (III.1.3) gives a coordinate representation of the abstract section in (III.1.2).

Remark III.1.1. (Algebraic Structure of $C^\infty(M)$ and Smooth Derivations)

Let M be a smooth manifold. The set of all real-valued smooth functions on M is denoted

$$C^\infty(M) := \{f : M \to \mathbb{R} \mid f \text{ is smooth}\}. \tag{III.1.4}$$

1. Commutative $\mathbb{R}$-Algebra Structure. The space $C^\infty(M)$ is a commutative, associative, unital algebra over $\mathbb{R}$, with:

$$\begin{aligned}
(f+g)(p) &:= f(p) + g(p),\\
(fg)(p) &:= f(p)\cdot g(p),\\
(\lambda f)(p) &:= \lambda \cdot f(p), \quad \lambda \in \mathbb{R},\\
1_M(p) &:= 1, \quad \forall p \in M.
\end{aligned}$$

These operations are defined *pointwise*, ensuring $C^\infty(M)$ is a commutative $\mathbb{R}$-algebra with unit 1_M.

2. $C^\infty(M)$-Modules. If $\pi_E : E \to M$ is a smooth vector bundle, the set

$$\Gamma(E) := \{s : M \to E \text{ smooth section} \mid \pi_E \circ s = \mathrm{id}_M\} \tag{III.1.5}$$

is a $C^\infty(M)$-module with

$$(f \cdot s)(p) := f(p)\, s(p), \quad f \in C^\infty(M),\ s \in \Gamma(E). \tag{III.1.6}$$

In particular, the set $\Gamma(TM)$ of smooth vector fields is a $C^\infty(M)$-module.

3. Derivations. Let A be a commutative $\mathbb{R}$-algebra. A *derivation* of A is an $\mathbb{R}$-linear map

$$D : A \to A \tag{III.1.7}$$

satisfying the *Leibniz rule*:

$$D(fg) = D(f)\,g + f\,D(g), \quad \forall f, g \in A. \tag{III.1.8}$$

The set of all derivations is denoted

$$\mathrm{Der}(A) := \{D : A \to A \text{ satisfying (III.1.7) and (III.1.8)}\}. \tag{III.1.9}$$

It forms an A-module via

$$(f \cdot D)(g) := f \cdot D(g). \tag{III.1.10}$$

4. Identification with Vector Fields. When $A = C^\infty(M)$, every $X \in \Gamma(TM)$ defines a derivation

$$X : C^\infty(M) \to C^\infty(M), \quad f \mapsto X(f), \tag{III.1.11}$$

where $X(f)$ denotes the directional derivative of f along X (cf. Definition III.1.1). Conversely, any $D \in \mathrm{Der}(C^\infty(M))$ arises from a unique smooth vector field. Thus there is a canonical $C^\infty(M)$-module isomorphism:

$$\Gamma(TM) \;\cong\; \mathrm{Der}\big(C^\infty(M)\big). \tag{III.1.12}$$

5. Pointwise Description. At each $p \in M$, the evaluation $D_p(f) := (Df)(p)$ depends only on the germ of f at p. The space of *derivations at a point* is defined by:

$$\mathrm{Der}_p\big(C^\infty(M)\big) := \{D_p : C^\infty(M) \to \mathbb{R} \text{ satisfying (III.1.8)}\}. \tag{III.1.13}$$

This yields the canonical identification

$$T_pM \;\cong\; \mathrm{Der}_p\big(C^\infty(M)\big). \tag{III.1.14}$$

6. Coordinate Representation. In a local chart (U, x^i), a derivation $D \in \mathrm{Der}(C^\infty(M))$ is expressed as:

$$D|_U = a^i(x)\,\frac{\partial}{\partial x^i}, \quad a^i \in C^\infty(U), \tag{III.1.15}$$

and the Leibniz rule (III.1.8) follows from the product rule for partial derivatives. ♦

Definition III.1.2. Smooth Covector Field (1-Form).

Let T^*M be the cotangent bundle of M with canonical projection

$$\pi^* : T^*M \to M. \tag{III.1.16}$$

A *smooth covector field* or *1-form* is a smooth section

$$\omega \in \Gamma(T^*M), \quad \pi^* \circ \omega = \mathrm{id}_M, \tag{III.1.17}$$

with $\omega(p) \in T_p^*M$ for all $p \in M$. In local coordinates,

$$\omega|_U = \omega_i(x)\,dx^i, \quad \omega_i \in C^\infty(U). \tag{III.1.18}$$

Remark III.1.2. (Duality)

Let M be a smooth manifold. For each $p \in M$, the cotangent space T_p^*M is the $\mathbb{R}$-linear dual of the tangent space T_pM. That is,

$$T_p^*M := \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R}),$$

and there is a canonical *evaluation* (or *duality*) pairing

$$\langle \omega, X\rangle(p) := \omega(p)(X(p)) \in \mathbb{R}, \qquad \omega \in \Omega^1(M),\ X \in \mathfrak{X}(M). \tag{III.1.19}$$

This pairing is smooth in p and satisfies $C^\infty(M)$-bilinearity:

$$\langle f\omega, gX\rangle = (fg)\langle \omega, X\rangle, \qquad f, g \in C^\infty(M). \tag{III.1.20}$$

Local coordinate expression. Let (U, x^i) be a smooth coordinate chart on M. Then

$$\omega = \omega_i\, dx^i, \qquad X = X^j \frac{\partial}{\partial x^j}.$$

Substituting into (III.1.19) and using $dx^i\big(\frac{\partial}{\partial x^j}\big) = \delta^i_j$, we obtain

$$\langle \omega, X\rangle = \big(\omega_i\, dx^i\big)\left(X^j \frac{\partial}{\partial x^j}\right) = \omega_i X^i. \tag{III.1.21}$$

Naturality. If $F : N \to M$ is a smooth map, the pullback $F^*\omega$ and the pushforward F_*X satisfy:

$$\langle F^*\omega, Y\rangle = \langle \omega, F_*Y\rangle \circ F, \qquad Y \in \mathfrak{X}(N), \tag{III.1.22}$$

expressing the naturality of the pairing under smooth maps.

Dual basis. Given a basis $\{e_i\}$ of T_pM, there exists a unique dual basis $\{e^i\} \subset T_p^*M$ such that $e^i(e_j) = \delta^i_j$. In coordinates, $\{\frac{\partial}{\partial x^i}\}$ and $\{dx^i\}$ form such a dual pair, making (III.1.21) a direct manifestation of this duality. ♦

Definition III.1.3. Tensor Field of Type (k, ℓ).

Let M be a smooth n-dimensional manifold. A (k, ℓ)-*tensor field* on M is a smooth section of the (k, ℓ)-tensor bundle:

$$T^{(k,\ell)}M := \underbrace{TM \otimes \cdots \otimes TM}_{k \text{ times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{\ell \text{ times}}. \tag{III.1.23}$$

That is, a (k, ℓ)-tensor field is a smooth map

$$T : M \longrightarrow T^{(k,\ell)}M, \quad \pi \circ T = \mathrm{id}_M, \tag{III.1.24}$$

where $\pi : T^{(k,\ell)}M \to M$ is the bundle projection.

At each point $p \in M$, the value

$$T_p := T(p) \ \in\ T_p^{(k,\ell)}M \tag{III.1.25}$$

is a multilinear map (cf. Definition **??**):

$$T_p : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{k \text{ times}} \times \underbrace{T_pM \times \cdots \times T_pM}_{\ell \text{ times}} \longrightarrow \mathbb{R}. \tag{III.1.26}$$

Coordinate Expression. In a smooth local chart (U, x^i), a (k,ℓ)-tensor field admits the local representation:

$$T|_U = T^{i_1 \dots i_k}{}_{j_1 \dots j_\ell}(x) \, \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \, \otimes \, dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}, \tag{III.1.27}$$

where the *component functions*

$$T^{i_1 \dots i_k}{}_{j_1 \dots j_\ell} \in C^\infty(U)$$

transform according to the (k,ℓ) tensor transformation law under a change of coordinates.

Special Cases.

- $(1,0)$-tensor fields are *vector fields* $(\Gamma(TM))$.
- $(0,1)$-tensor fields are *covector fields* $(\Gamma(T^*M))$.
- $(0,2)$-tensor fields often represent *bilinear forms* such as Riemannian metrics.

Example III.1 (Volume Form)**.** On an oriented n-dimensional manifold M, a volume form is a nowhere-vanishing smooth $(0,n)$-tensor field

$$\omega \in \Gamma(\Lambda^n T^*M),$$

where $\Lambda^n T^*M$ denotes the top exterior power of the cotangent bundle. Locally, $\omega = f(x) dx^1 \wedge \cdots \wedge dx^n$ with $f(x) > 0$.

Definition III.1.4. Kronecker Delta.

Let V be an n-dimensional real vector space with basis $\{e_i\}_{i=1}^n$ and dual basis $\{e^j\}_{j=1}^n$ satisfying $e^j(e_i) = \delta_i^j$. The **Kronecker delta** is the rank-$(1,1)$ tensor

$$\delta^i_j := e^i \otimes e_j \in V^* \otimes V,$$

characterized by the property:

$$\delta^i_j v^j = v^i, \quad \text{or equivalently,} \quad \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

It acts as the identity under contraction, serving to raise or lower matched indices and implement canonical pairings between vectors and covectors.

Theorem III.1.1. Tensor Contraction.

Let $T \in \Gamma(T^{(k,\ell)}M)$ be a (k,ℓ)-tensor field. A contraction of T is a $(k-1,\ell-1)$-tensor field obtained by pairing one contravariant and one covariant index:

$$\mathrm{contr}_{i,j}(T)(\cdots) = T^{a_1 \cdots a_k}_{b_1 \cdots b_\ell} \delta^{a_i}_{b_j} \cdots .$$

This operation is bilinear and commutes with smooth pullbacks under diffeomorphisms. ⌟

Proof III.1.1. We aim to show that contraction over one contravariant and one covariant index of a smooth (k,ℓ)-tensor field $T \in \Gamma(T^{(k,\ell)}M)$ yields a well-defined smooth $(k-1,\ell-1)$-tensor field. The argument proceeds in several steps.

1. **Pointwise definition.** Let $p \in M$ be fixed. A tensor $T_p \in T_p^{(k,\ell)}M$ is a multilinear map

$$T_p : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{k \text{ times}} \times \underbrace{T_pM \times \cdots \times T_pM}_{\ell \text{ times}} \to \mathbb{R}.$$

Choose positions $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, \ell\}$. Define the contraction $\mathrm{contr}_{i,j}(T)_p \in T_p^{(k-1,\ell-1)}M$ by inserting a dual basis pair (e^a, e_a) into those positions and summing over a:

$$\mathrm{contr}_{i,j}(T)_p = \sum_{a=1}^{n} T_p(\dots, e^a, \dots; \dots, e_a, \dots),$$

where e^a is placed at slot i, and e_a at slot j. This expression is independent of basis since it uses the natural dual pairing.

2. **Coordinate expression.** In a coordinate chart $(x^1, \dots, x^n)$, a (k,ℓ)-tensor field has components

$$T = T^{i_1 \dots i_k}_{j_1 \dots j_\ell}(x)\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_k} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}.$$

Contraction over the i-th contravariant and j-th covariant slot yields

$$(\mathrm{contr}_{i,j}(T))^{\hat{i}_1 \dots \hat{i}_{k-1}}_{\hat{j}_1 \dots \hat{j}_{\ell-1}} = \sum_{a=1}^{n} T^{i_1 \dots a \dots i_k}_{j_1 \dots a \dots j_\ell},$$

where hats indicate omission of contracted indices. The resulting components are smooth because they are obtained via smooth operations on the original components.

3. **$C^\infty(M)$-linearity.** Let $f \in C^\infty(M)$. Then

$$\mathrm{contr}_{i,j}(fT) = f \cdot \mathrm{contr}_{i,j}(T),$$

because contraction is a pointwise linear operation. This ensures that $\mathrm{contr}_{i,j}(T)$ is again a smooth tensor field.

4. **Coordinate invariance.** Under a coordinate change $x^i \mapsto x^{i'}$, tensor components transform according to

$$T^{i_1 \dots i_k}_{j_1 \dots j_\ell} \mapsto \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_k}}{\partial x^{i_k}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_\ell}}{\partial x^{j'_\ell}} T^{i_1 \dots i_k}_{j_1 \dots j_\ell}.$$

In the contraction sum $\sum_a T^{\dots a \dots}_{\dots a \dots}$, the transformation Jacobians cancel due to one upper and one lower index being contracted. Therefore, the result transforms as a $(k-1, \ell-1)$-tensor.

5. **Conclusion.** All properties of a smooth tensor field are preserved: multilinearity, smoothness, and coordinate transformation behavior. Thus, $\mathrm{contr}_{i,j}(T) \in \Gamma(T^{(k-1,\ell-1)}M)$ is well-defined.

□

Example III.2 (Contraction Using the Kronecker Delta)**.** Let $T \in \Gamma(T^{(1,1)}M)$ be a $(1,1)$-tensor field with local components $T^i{}_j$, expressed in coordinates as

$$T = T^i{}_j \frac{\partial}{\partial x^i} \otimes dx^j.$$

To compute the contraction over the contravariant and covariant index, we use the Kronecker delta δ_i^j:

$$\mathrm{Tr}(T) := \delta_i^j\, T^i{}_j = T^i{}_i. \tag{III.1.28}$$

This produces a scalar field on M, called the **trace** of T.

Basis interpretation. Let $\{e_i\}$ be a local frame for TM with dual coframe $\{e^i\} \subset T^*M$. In this basis,

$$T(e^i, e_j) = T^i{}_j,$$

and contraction corresponds to evaluating T on the dual pair (e^i, e_i) and summing over i:

$$\mathrm{Tr}(T) = \sum_{i=1}^{n} T(e^i, e_i). \tag{III.1.29}$$

Role of the Kronecker delta. The Kronecker delta δ_i^j acts as the identity map from TM to itself in coordinates:

$$\delta_i^j \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^i}.$$

In index notation, it enforces equality of the contracted indices, implementing the identification $i = j$ in (III.1.28).

Thus, contraction with δ_i^j formally realizes the pairing of a vector index with a covector index, reducing the tensor rank by 2 and producing a scalar in this case.

III.2 Multilinear Maps and Tensors

Definition III.2.1. Multilinear Map.

Let $V_1, \ldots, V_k, W$ be vector spaces over a field $\mathbb{F}$. A function

$$T : V_1 \times \cdots \times V_k \to W$$

is called a *k-linear map* (or *tensor of type* $(k, 0)$) if T is linear in each argument separately.

Example III.3 (Bilinear Form)**.** The inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is a symmetric bilinear map.

Definition III.2.2. Tensor Product.

Given $T_1 \in \mathsf{Lin}(V, \mathbb{R})$ and $T_2 \in \mathsf{Lin}(W, \mathbb{R})$, the tensor product

$$T_1 \otimes T_2 \in \mathsf{Bilin}(V \times W, \mathbb{R})$$

is defined by

$$(T_1 \otimes T_2)(v, w) = T_1(v) \cdot T_2(w).$$

Theorem III.2.1. Construction of General Tensors and Index Manipulation.

Let V be a finite-dimensional real vector space with basis $\{e_i\}_{i=1}^n$ and dual basis $\{e^i\}_{i=1}^n$. Let $g : V \times V \to \mathbb{R}$ be a symmetric non-degenerate bilinear form (metric). Then:

- Every (k, ℓ)-tensor $T \in \mathsf{T}^{(k,\ell)}(V)$ can be expressed as a finite linear combination:

$$T = T^{i_1 \ldots i_k}_{j_1 \ldots j_\ell}\, e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_\ell}.$$

- The metric g induces canonical isomorphisms:

$$\flat : V \to V^*, \quad v \mapsto g(v, -), \quad \text{and its inverse} \quad \sharp : V^* \to V.$$

- These isomorphisms allow for the *raising and lowering of indices*, converting covariant to contravariant components and vice versa.

⌟

Proof III.2.1. Let $\dim V = n$. The tensor algebra $T^{(k,\ell)}(V)$ consists of multilinear maps:

$$T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \times \underbrace{V \times \cdots \times V}_{\ell \text{ times}} \to \mathbb{R}.$$

Let $\{e_i\}$ be a basis for V and $\{e^i\}$ the dual basis for V^*. Then any $T \in T^{(k,\ell)}(V)$ can be expanded in the basis:

$$\left\{e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_\ell}\right\}.$$

By multilinearity, we write:

$$T = \sum_{i_1,\ldots,i_k,j_1,\ldots,j_\ell} T^{i_1\ldots i_k}_{j_1\ldots j_\ell}\, e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_\ell},$$

where the components $T^{i_1\ldots i_k}_{j_1\ldots j_\ell}$ are real scalars.

Now let $g : V \times V \to \mathbb{R}$ be a symmetric bilinear form with matrix representation $g_{ij} = g(e_i, e_j)$, assumed invertible. The induced map $\flat : V \to V^*$ is defined by:

$$v = v^i e_i \Rightarrow v^\flat = g_{ij} v^i e^j.$$

The inverse isomorphism $\sharp : V^* \to V$ satisfies:

$$\alpha = \alpha_i e^i \Rightarrow \alpha^\sharp = g^{ij} \alpha_i e_j,$$

where (g^{ij}) is the inverse of (g_{ij}).

These maps allow us to convert between covariant and contravariant components:

$$T_i \Rightarrow T^j = g^{ij} T_i \quad \text{(raise index)}, \tag{III.2.1}$$

$$T^i \Rightarrow T_j = g_{ij} T^i \quad \text{(lower index)}. \tag{III.2.2}$$

For higher tensors, the index manipulation is performed by contracting with g_{ij} or g^{ij} in the appropriate slots. For example, for a $(1,1)$ tensor $T = T^i_j\, e_i \otimes e^j$, we can define:

$$(T^\flat)^i_k = g_{kj} T^i_j, \qquad (T^\sharp)^j_\ell = g^{ij} T_{i\ell}.$$

Since g is non-degenerate, these operations are isomorphisms, preserving the tensor structure while shifting index types.

Hence, the tensor space is generated as claimed, and index conversion is canonically defined via the metric structure. □

Theorem III.2.2. Canonical Isomorphism between Vectors and Covectors.

Let V be a finite-dimensional real inner product space with inner product $\langle \cdot, \cdot \rangle$. Then the mapping

$$\flat : V \to V^*, \quad v \mapsto \langle v, \cdot \rangle$$

is a linear isomorphism. Its inverse $\sharp : V^* \to V$ is defined by the Riesz Representation Theorem and satisfies

$$\forall \omega \in V^*, \quad \langle \omega^\sharp, w \rangle = \omega(w) \quad \text{for all } w \in V.$$

⌟

Proof III.2.2. Let V be an n-dimensional real inner product space with inner product $\langle \cdot, \cdot \rangle$. Define the *musical isomorphism*

$$\flat : V \to V^*, \quad \mathbf{v} \mapsto \langle \mathbf{v}, \cdot \rangle.$$

This map assigns to each vector $\mathbf{v} \in V$ a covector $\mathbf{v}^\flat \in V^*$, defined pointwise by

$$\mathbf{v}^\flat(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle \quad \text{for all } \mathbf{w} \in V.$$

Linearity: For all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and scalars $a, b \in \mathbb{R}$, we compute

$$(a\mathbf{v}_1 + b\mathbf{v}_2)^\flat(\mathbf{w}) = \langle a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{w} \rangle = a \langle \mathbf{v}_1, \mathbf{w} \rangle + b \langle \mathbf{v}_2, \mathbf{w} \rangle = a\, \mathbf{v}_1^\flat(\mathbf{w}) + b\, \mathbf{v}_2^\flat(\mathbf{w}),$$

so the map $\flat$ is linear.

Injectivity: Suppose $\mathbf{v}^\flat = 0$, i.e., $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in V$. In particular, taking $\mathbf{w} = \mathbf{v}$, we obtain

$$\langle \mathbf{v}, \mathbf{v} \rangle = 0.$$

Since the inner product is positive definite, this implies $\mathbf{v} = \mathbf{0}$. Hence $\ker \flat = \{0\}$, and $\flat$ is injective.

Dimension Count: The vector space V has $\dim V = n$, and its dual space V^* also satisfies $\dim V^* = n$. Since $\flat$ is a linear map from V to V^*, and is injective with equal dimensions, it is an isomorphism:

$$\flat : V \xrightarrow{\sim} V^*.$$

Coordinate Expression: Let $\{\mathbf{e}_i\}_{i=1}^n$ be an orthonormal basis for V, so $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$. Then any vector $\mathbf{v} \in V$ has coordinates

$$\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i,$$

and we compute the action of $\mathbf{v}^\flat \in V^*$ on $\mathbf{w} = \sum_{j=1}^n w^j \mathbf{e}_j$ as

$$\mathbf{v}^\flat(\mathbf{w}) = \left\langle \sum_i v^i \mathbf{e}_i, \sum_j w^j \mathbf{e}_j \right\rangle = \sum_{i,j} v^i w^j \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \sum_i v^i w^i.$$

Thus, $\mathbf{v}^\flat$ has the coordinate representation $(v^1, \ldots, v^n) \in V^*$, i.e., the same components in this orthonormal basis.

Inverse Map: The inverse map $\sharp : V^* \to V$ satisfies, for any $\boldsymbol{\omega} \in V^*$ and $\mathbf{v} \in V$,

$$\langle \boldsymbol{\omega}^\sharp, \mathbf{v} \rangle = \boldsymbol{\omega}(\mathbf{v}).$$

This uniquely defines $\boldsymbol{\omega}^\sharp \in V$ by Riesz representation. In coordinates, if $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in V^*$, then

$$\boldsymbol{\omega}^\sharp = \sum_{i=1}^n \omega_i \mathbf{e}_i \in V.$$

Conclusion: The flat map $\flat : V \to V^*$ is a linear isomorphism, with inverse $\sharp : V^* \to V$, satisfying the dual identities:

$$(\mathbf{v}^\flat)^\sharp = \mathbf{v}, \qquad (\boldsymbol{\omega}^\sharp)^\flat = \boldsymbol{\omega}.$$

This completes the construction of the musical isomorphisms between vectors and covectors on an inner product space. □

III.3 Jacobian and Multivariable Derivatives

Definition III.3.1. Total Derivative.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function, and let $x_0 \in \mathbb{R}^n$. We say that f is *differentiable at* x_0 if there exists a linear map

$$Df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - Df(x_0)(h)\|}{\|h\|} = 0.$$

That is, the error term

$$\varepsilon(h) := f(x_0 + h) - f(x_0) - Df(x_0)(h)$$

satisfies $\|\varepsilon(h)\| = o(\|h\|)$ as $h \to 0$. In this case, the linear map $Df(x_0)$ is called the *total derivative* (or *Fréchet derivative*) of f at x_0.

Coordinate Representation: If $f = (f^1, \dots, f^m)$, then the total derivative is represented by the Jacobian matrix:

$$Df(x_0) = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{bmatrix}_{x = x_0},$$

and for any $h \in \mathbb{R}^n$,

$$Df(x_0)(h) = \frac{d}{dt} f(x_0 + th)\Big|_{t=0}.$$

Remark III.3.1. (Total Derivative is the Best Linear Approximation)

The total derivative $Df(x_0)$ provides the best linear approximation to f near x_0. That is, in local coordinates:

$$f(x_0 + h) \approx f(x_0) + Df(x_0)[h].$$

♦

Theorem III.3.1. Jacobian as the Matrix Representation of the Total Derivative.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at x_0. Then the total derivative $Df(x_0)$ is given in coordinates by the Jacobian matrix $J_f(x_0) \in \mathbb{R}^{m \times n}$:

$$Df(x_0)[h] = J_f(x_0) \cdot h,$$

where $h \in \mathbb{R}^n$ and the Jacobian entries are

$$(J_f(x_0))_{ij} = \frac{\partial f^i}{\partial x^j}(x_0).$$

⌟

Proof III.3.1. Let $f = (f^1, \dots, f^m)$ and fix $x_0 \in \mathbb{R}^n$. For small $h \in \mathbb{R}^n$, by the definition of differentiability,

$$f(x_0 + h) = f(x_0) + Df(x_0)[h] + o(\|h\|).$$

We now compute $Df(x_0)$ explicitly using the directional derivatives.

Let e_j denote the j-th standard basis vector in $\mathbb{R}^n$. Then the j-th column of the Jacobian is

$$Df(x_0)[e_j] = \left(\frac{\partial f^1}{\partial x^j}(x_0), \dots, \frac{\partial f^m}{\partial x^j}(x_0) \right)^\top.$$

Linearity then implies that for any $h = \sum_{j=1}^n h^j e_j \in \mathbb{R}^n$,

$$Df(x_0)[h] = \sum_{j=1}^n h^j Df(x_0)[e_j] = J_f(x_0) \cdot h.$$

Thus the Jacobian $J_f(x_0)$ gives the coordinate representation of the linear map $Df(x_0)$ with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. □

Remark III.3.2. (Jacobian as Pushfoward of Tangent Vectors)

The Jacobian plays the role of the *pushforward* of tangent vectors under f:

$$df_{x_0} : T_{x_0}\mathbb{R}^n \to T_{f(x_0)}\mathbb{R}^m, \quad v \mapsto J_f(x_0)v.$$

This interpretation becomes crucial when generalizing to differentiable maps between manifolds.
♦

Example III.4 (Linear Map Case)**.** If $f(x) = Ax$ for some $A \in \mathbb{R}^{m\times n}$, then $Df(x_0) = A$ for all x_0, and the Jacobian is constant:

$$J_f(x) = A.$$

This highlights that linear maps are their own total derivatives.

III.4 Differential Forms and Pullbacks

Definition III.4.1. Space of Differential k-Forms.

Let $U \subseteq \mathbb{R}^n$ be open. The set $\Omega^k(U)$ denotes the space of smooth differential k-forms on U. Each element $\omega \in \Omega^k(U)$ assigns to every point $x \in U$ an alternating multilinear map

$$\omega_x : (T_x\mathbb{R}^n)^k \to \mathbb{R}$$

which varies smoothly with x. That is, for each k-tuple of smooth vector fields $X_1, \ldots, X_k$, the function

$$x \mapsto \omega_x(X_1(x), \ldots, X_k(x))$$

is smooth.

When $k = 0$, we identify $\Omega^0(U) \cong C^\infty(U)$. When $k = 1$, this agrees with the space of differential 1-forms defined above.

Definition III.4.2. Differential 1-Form.

Let $U \subset \mathbb{R}^n$ be open. A **differential 1-form** on U is a smooth assignment

$$\omega : x \mapsto \omega_x \in \mathsf{Lin}(\mathbb{R}^n, \mathbb{R}),$$

where each ω_x is a linear functional on the tangent space $T_x\mathbb{R}^n \cong \mathbb{R}^n$. Locally, we can express ω in coordinates as

$$\omega = \sum_{i=1}^n a_i(x)\, dx^i,$$

where each $a_i : U \to \mathbb{R}$ is a smooth function, and dx^i denotes the standard coordinate 1-forms.

Definition III.4.3. Pullback of a 1-Form.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and let ω be a differential 1-form on $\mathbb{R}^m$. Then the **pullback** $f^*\omega$ is the 1-form on $\mathbb{R}^n$ defined pointwise by

$$(f^*\omega)_x(v) = \omega_{f(x)}(Df(x)[v]),$$

for all $x \in \mathbb{R}^n$ and $v \in T_x\mathbb{R}^n$. Equivalently,

$$f^*\omega = \sum_{j=1}^{m} a_j(f(x))\, d(f^j(x)) = \sum_{j=1}^{m} a_j(f(x)) \sum_{i=1}^{n} \frac{\partial f^j}{\partial x^i}(x)\, dx^i,$$

where $\omega = \sum_{j=1}^{m} a_j(y)\, dy^j$ on $\mathbb{R}^m$ and $f = (f^1, \ldots, f^m)$.

Remark III.4.1. (Naturality of Pullbacks)

The pullback operation f^* is **functorial**: it preserves the algebraic structure of differential forms and respects composition. That is, for smooth maps $f : M \to N$ and $g : N \to P$, and a 1-form ω on P, we have

$$(f \circ g)^*\omega = f^*(g^*\omega).$$

Moreover, the pullback commutes with the exterior derivative: if ω is a differential form, then

$$d(f^*\omega) = f^*(d\omega),$$

ensuring compatibility with the de Rham complex. ♦

Theorem III.4.1. Linearity and Leibniz Rule for Pullbacks.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map, and let ω, η be differential 1-forms on $\mathbb{R}^m$, and $h : \mathbb{R}^m \to \mathbb{R}$ a smooth function. Then the pullback f^* satisfies the following properties:

1. **Linearity:**
$$f^*(\omega + \eta) = f^*\omega + f^*\eta, \quad f^*(h \cdot \omega) = (h \circ f) \cdot f^*\omega.$$

2. **Leibniz Rule:**
$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta, \quad d(f^*\omega) = f^*(d\omega).$$

⌟

Proof III.4.1.

(1) Linearity. Let $\omega, \eta \in \Omega^1(\mathbb{R}^m)$, and $h \in C^\infty(\mathbb{R}^m)$. Then for each $x \in \mathbb{R}^n$ and $v \in T_x\mathbb{R}^n$,

$$\begin{aligned}(f^*(\omega + \eta))_x(v) &= (\omega + \eta)_{f(x)}(Df(x)[v]) \\ &= \omega_{f(x)}(Df(x)[v]) + \eta_{f(x)}(Df(x)[v]) \\ &= (f^*\omega)_x(v) + (f^*\eta)_x(v),\end{aligned}$$

so $f^*(\omega + \eta) = f^*\omega + f^*\eta$. For scalar multiplication:

$$\begin{aligned}(f^*(h \cdot \omega))_x(v) &= (h \cdot \omega)_{f(x)}(Df(x)[v]) \\ &= h(f(x)) \cdot \omega_{f(x)}(Df(x)[v]) \\ &= (h \circ f)(x) \cdot (f^*\omega)_x(v),\end{aligned}$$

thus $f^*(h \cdot \omega) = (h \circ f) \cdot f^*\omega$.

(2) Leibniz Rule. For wedge product, let $\omega \in \Omega^p(\mathbb{R}^m)$, $\eta \in \Omega^q(\mathbb{R}^m)$. Then using the definition of pullback on wedge products,

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta,$$

which follows from the bilinearity of the wedge product and the naturality of pullbacks (checked locally in coordinates).

For the exterior derivative, let $\omega \in \Omega^k(\mathbb{R}^m)$. The identity

$$d(f^*\omega) = f^*(d\omega)$$

holds as a standard result from differential geometry and can be proven by expressing all forms in local coordinates and using the chain rule to differentiate component functions of f and ω. This ensures that pullback commutes with the exterior derivative.

Hence, all properties are verified. □

Definition III.4.4. Compact Support.

Let X be a topological space. A function $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is said to have **compact support** if the closure of the set

$$\operatorname{supp}(f) := \{x \in X \mid f(x) \neq 0\}$$

is compact in X.

We denote the space of smooth functions with compact support on $\mathbb{R}^n$ by $C_c^\infty(\mathbb{R}^n)$. These are functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\varphi) \Subset \mathbb{R}^n$, i.e., $\operatorname{supp}(\varphi)$ is contained in some compact subset of $\mathbb{R}^n$.

In analysis and distribution theory, functions in C_c^∞ serve as test functions against which generalized functions (distributions) are defined.

Definition III.4.5. Compactly Supported Differential Forms.

Let M be a smooth manifold. A differential k-form $\omega \in \Omega^k(M)$ is said to have **compact support** if the closure of its support,

$$\operatorname{supp}(\omega) := \{x \in M \mid \omega_x \neq 0\},$$

is a compact subset of M. The space of all such forms is denoted

$$\Omega_c^k(M) := \{\omega \in \Omega^k(M) \mid \operatorname{supp}(\omega) \Subset M\}.$$

Properties:

- $\Omega_c^k(M) \subset \Omega^k(M)$ is a subspace closed under addition, scalar multiplication, and exterior differentiation:
$$d : \Omega_c^k(M) \to \Omega_c^{k+1}(M).$$
- If $\omega \in \Omega_c^n(M)$ and M is oriented and n-dimensional, then ω can be integrated:
$$\int_M \omega \in \mathbb{R}.$$

This space is foundational in the formulation of *de Rham cohomology with compact supports*, and in defining dual spaces of currents and distributions.

Definition III.4.6. Exterior Derivative.

Let M be a smooth manifold and let $\Omega^k(M)$ denote the space of smooth differential k-forms on M. The *exterior derivative* is a linear operator

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

satisfying the following properties:

1. $d^2 = 0$ (i.e., $d \circ d = 0$),
2. $d(f) = df$ is the differential of a smooth function $f \in C^\infty(M)$,
3. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for $\omega \in \Omega^k(M)$, $\eta \in \Omega^\bullet(M)$ (graded Leibniz rule).

This operator defines a cochain complex $(\Omega^\bullet(M), d)$, known as the de Rham complex.

Definition III.4.7. de Rham Cohomology.

Let $(\Omega^\bullet(M), d)$ be the de Rham complex of a smooth manifold M. The *de Rham cohomology groups* are defined as the quotient spaces

$$H^k_{\mathrm{dR}}(M) := \frac{\ker\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\mathrm{im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right)},$$

for each $k \geq 0$. That is, $H^k_{\mathrm{dR}}(M)$ measures the space of closed k-forms modulo exact k-forms. These cohomology groups are topological invariants of the smooth manifold M, and they form a contravariant functor from the category of smooth manifolds to the category of graded vector spaces.

Example III.5 (Pushforward and Change of Variables in Integration)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth diffeomorphism, and let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a smooth function with compact support. Suppose we wish to compute the integral

$$\int_{\mathbb{R}^n} \varphi(x)\, dx.$$

Under the change of variables $x = f(y)$, the integration rule becomes

$$\int_{\mathbb{R}^n} \varphi(f(y))\, |\det Df(y)|\; dy.$$

Here, the Jacobian determinant $|\det Df(y)|$ arises from the pushforward f_* acting on the volume form $dx^1 \wedge \cdots \wedge dx^n$. More precisely, the pushforward of tangent vectors changes the local volume element by the factor $\det Df(y)$:

$$f_* : T_y\mathbb{R}^n \to T_{f(y)}\mathbb{R}^n, \quad v \mapsto Df(y)[v].$$

Thus, under f, the volume form transforms as

$$f^*(dx^1 \wedge \cdots \wedge dx^n) = \det Df(y)\, dy^1 \wedge \cdots \wedge dy^n,$$

and the change of variables formula reflects this transformation.

Conclusion: The pushforward encodes how f stretches or compresses infinitesimal volume, and its determinant governs the measure-theoretic correction factor in integration.

Lemma III.4.1. Density of Compactly Supported Forms.

Let M be a smooth manifold. Then for any $\omega \in \Omega^k(M)$, and for any compact set $K \subset M$ and open neighborhood $U \supset K$, there exists $\eta \in \Omega_c^k(M)$ such that:

$$\eta|_K = \omega|_K, \quad \text{and} \quad \operatorname{supp}(\eta) \subset U.$$

In particular, $\Omega_c^k(M)$ is **dense** in $\Omega^k(M)$ under the Whitney C^∞-topology on compact sets.

Hence, any smooth k-form can be locally approximated by compactly supported forms, and global approximation is possible on manifolds with appropriate exhaustion properties. $\rightarrowtail$

Proof III.4.2. Let M be a smooth manifold, $\omega \in \Omega^k(M)$, and let $K \subset M$ be compact. Let $U \subset M$ be an open neighborhood containing K. Since $\omega \in \Omega^k(M)$, it is smooth, and thus defined on all of M.

By the regularity of smooth manifolds and the compactness of K, we can choose an open set V such that

$$K \subset V \Subset U,$$

meaning $\overline{V} \subset U$ and $\overline{V}$ is compact. Now, since $\{V, M \setminus K\}$ is an open cover of M, by the existence of smooth partitions of unity subordinate to any open cover, there exists a smooth function $\chi \in C^\infty(M)$ such that:

- $0 \leq \chi \leq 1$,
- $\chi \equiv 1$ on a neighborhood of K,
- $\operatorname{supp}(\chi) \subset V \Subset U$.

Now define the k-form $\eta := \chi\omega \in \Omega^k(M)$. Since $\chi \in C_c^\infty(M)$ and $\omega \in \Omega^k(M)$, their pointwise product $\eta \in \Omega^k(M)$ is smooth and has support contained in $\operatorname{supp}(\chi) \subset V \Subset U$. Thus $\eta \in \Omega_c^k(M)$.

Moreover, since $\chi \equiv 1$ on a neighborhood of K, we have:

$$\eta|_K = (\chi\omega)|_K = \omega|_K.$$

Therefore, $\eta \in \Omega_c^k(M)$ agrees with ω on K and has support contained in U, completing the proof. □

III.5 Tangent and Cotangent Spaces

Definition III.5.1. Tangent Space at a Point.

Let M be a smooth manifold and $p \in M$. The tangent space T_pM is the set of all derivations at p, i.e., linear maps $D : C^\infty(M) \to \mathbb{R}$ satisfying Leibniz's rule:

$$D(fg) = f(p)D(g) + g(p)D(f).$$

Definition III.5.2. Cotangent Space.

The cotangent space T_p^*M is the dual space of T_pM, consisting of all linear functionals $\alpha : T_pM \to \mathbb{R}$.

Theorem III.5.1. Coordinate Bases for Tangent and Cotangent Spaces.
Let M be a smooth n-dimensional manifold and let (U, φ) be a coordinate chart around $p \in M$ with coordinates $x^1, \ldots, x^n$. Then:

1. The partial derivative operators $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}_{i=1}^n$ form a basis for T_pM.
2. The differentials $\{dx^i|_p\}_{i=1}^n$ form the dual basis for T_p^*M, i.e.,

$$dx^i\left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \delta^i_j.$$

⌟

Proof III.5.1. Let (U, φ) be a smooth coordinate chart around $p \in M$, with local coordinates $x^1, \ldots, x^n \colon U \to \mathbb{R}$. These functions define smooth maps in $C^\infty(M)$, and their evaluation at p gives the point $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in \mathbb{R}^n$.

Part (1): Define derivations $\frac{\partial}{\partial x^i}\big|_p \in T_pM$ by:

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) := \left.\frac{\partial (f \circ \varphi^{-1})}{\partial x^i}\right|_{\varphi(p)}.$$

This definition satisfies Leibniz's rule and gives n linearly independent derivations. Any $D \in T_pM$ can be uniquely expressed as a linear combination

$$D = \sum_{i=1}^n D(x^i) \cdot \left.\frac{\partial}{\partial x^i}\right|_p,$$

so $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}$ forms a basis for T_pM.

Part (2): For the cotangent space T_p^*M, we define the linear functionals $dx^i|_p$ by their action on the basis vectors:

$$dx^i\left(\left.\frac{\partial}{\partial x^j}\right|_p \right) := \delta^i_j.$$

This uniquely determines $dx^i|_p \in T_p^*M$, and the set $\{dx^i|_p\}$ spans T_p^*M by duality.

Hence, $\{dx^i|_p\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}$, completing the proof. □

Lemma III.1 (Linearity and Pairing of Basis Elements). *Let $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}$ be the coordinate basis for T_pM, and $\{dx^i|_p\}$ its dual basis in T_p^*M. Then for any tangent vector $v \in T_pM$ and any covector $\alpha \in T_p^*M$, we have:*

$$v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p, \quad \text{with } v^i = dx^i(v),$$
$$\alpha = \sum_{i=1}^n \alpha_i \, dx^i|_p, \quad \text{with } \alpha_i = \alpha\left(\frac{\partial}{\partial x^i}\Big|_p \right).$$

Moreover, the natural pairing satisfies:

$$\alpha(v) = \sum_{i=1}^n \alpha_i v^i.$$

This lemma emphasizes that every tangent or cotangent vector can be expressed in components, and the evaluation $\alpha(v)$ recovers the canonical contraction between tensors of type $(0,1)$ and $(1,0)$.

Proof III.5.2. Let $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}_{i=1}^n$ be a basis of the tangent space T_pM, and $\{dx^i|_p\}_{i=1}^n$ its dual basis in the cotangent space T_p^*M, satisfying

$$dx^i \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta^i_j.$$

Let $v \in T_pM$ and $\alpha \in T_p^*M$. Since these are finite-dimensional vector spaces, we can write:

$$v = \sum_{i=1}^n v^i \, \frac{\partial}{\partial x^i}\bigg|_p,$$
$$\alpha = \sum_{i=1}^n \alpha_i \, dx^i|_p.$$

Applying the definition of the dual pairing, we compute:

$$\begin{aligned}
\alpha(v) &= \left(\sum_{i=1}^n \alpha_i \, dx^i \right) \left(\sum_{j=1}^n v^j \, \frac{\partial}{\partial x^j} \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \alpha_i v^j \, dx^i \left(\frac{\partial}{\partial x^j} \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \alpha_i v^j \, \delta^i_j \\
&= \sum_{i=1}^n \alpha_i v^i.
\end{aligned}$$

Finally, note that from the pairing:

$$v^i = dx^i(v), \qquad \alpha_i = \alpha\left(\frac{\partial}{\partial x^i} \right),$$

as desired. ☐

III.6 Covariant Structure on Smooth Manifolds

Definition III.6.1. Tensor Field.

A (k,ℓ)-tensor field on a manifold M assigns to each point $p \in M$ a multilinear map

$$T_p : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{k \text{ times}} \times \underbrace{T_pM \times \cdots \times T_pM}_{\ell \text{ times}} \to \mathbb{R}$$

that varies smoothly with p.

Remark III.6.1. (Geometric Interpretation and Naturalness)

Tensor fields arise naturally in differential geometry as the intrinsic multilinear structures associated to the tangent and cotangent bundles of a manifold. The (k,ℓ)-type encodes the variance of the field under coordinate transformations: covariant in k slots and contravariant in ℓ. This formulation abstracts classical objects such as vector fields ($\ell = 1,\ k = 0$), differential forms ($k = 1,\ \ell = 0$), and the metric tensor ($k = 0,\ \ell = 2$). More generally, tensor fields provide a coordinate-free language for expressing physical laws, geometric structures, and variational principles. The requirement of smoothness ensures compatibility with the differentiable structure of M, permitting differentiation, integration, and flow analysis across charts. ♦

Example III.6 (Metric Tensor)**.** A Riemannian metric is a smooth $(0,2)$-tensor field g such that g_p is a positive-definite inner product on T_pM.

Theorem III.6.1. Coordinate Transformation Law for Tensors.

Let T be a tensor field. Under a change of coordinates $x^i \mapsto x^{i'}$, the components transform by

$$T^{i'_1\cdots i'_k}_{j'_1\cdots j'_\ell} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_k}}{\partial x^{i_k}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_\ell}}{\partial x^{j'_\ell}} T^{i_1\cdots i_k}_{j_1\cdots j_\ell}.$$

⌟

Proof III.6.1. Let $T \in \Gamma(T^{(k,\ell)}M)$ be a smooth (k,ℓ)-tensor field on a manifold M, and let (x^i) and $(x^{i'})$ be two overlapping coordinate charts on M. We aim to determine how the components of T transform under the change of coordinates $x^i \mapsto x^{i'}$.

1. **Local basis transformation.** The tangent and cotangent basis vectors transform under a change of coordinates as follows:

$$\frac{\partial}{\partial x^i} = \frac{\partial x^{j'}}{\partial x^i}\frac{\partial}{\partial x^{j'}}, \qquad dx^i = \frac{\partial x^i}{\partial x^{j'}} dx^{j'}.$$

   These follow from the chain rule and define the Jacobian matrices for the transformation.

2. **Tensor decomposition in local coordinates.** In the (x^i) chart, the tensor field can be expressed locally as

$$T = T^{i_1\cdots i_k}_{j_1\cdots j_\ell}(x)\,\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}.$$

   Similarly, in the $(x^{i'})$ chart, the tensor field must be expressible as

$$T = T^{i'_1\cdots i'_k}_{j'_1\cdots j'_\ell}(x')\,\frac{\partial}{\partial x^{i'_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i'_k}} \otimes dx^{j'_1} \otimes \cdots \otimes dx^{j'_\ell}.$$

3. **Apply basis transformation to the expression for T.** Substitute the coordinate transformation of basis vectors and 1-forms into the original expression for T:

$$\frac{\partial}{\partial x^{i_r}} = \frac{\partial x^{i'_r}}{\partial x^{i_r}}\frac{\partial}{\partial x^{i'_r}}, \qquad dx^{j_s} = \frac{\partial x^{j_s}}{\partial x^{j'_s}} dx^{j'_s}.$$

   Applying this for each of the k vector indices and ℓ covector indices:

$$T = T^{i_1\cdots i_k}_{j_1\cdots j_\ell}\left(\prod_{r=1}^{k}\frac{\partial x^{i'_r}}{\partial x^{i_r}}\frac{\partial}{\partial x^{i'_r}}\right) \otimes \left(\prod_{s=1}^{\ell}\frac{\partial x^{j_s}}{\partial x^{j'_s}} dx^{j'_s}\right).$$

   By multilinearity of the tensor product:

$$T = \left(T^{i_1\cdots i_k}_{j_1\cdots j_\ell}\prod_{r=1}^{k}\frac{\partial x^{i'_r}}{\partial x^{i_r}}\prod_{s=1}^{\ell}\frac{\partial x^{j_s}}{\partial x^{j'_s}}\right)\cdot\left(\frac{\partial}{\partial x^{i'_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i'_k}} \otimes dx^{j'_1} \otimes \cdots \otimes dx^{j'_\ell}\right).$$

4. **Extract transformed components.** Comparing with the coordinate representation of T in the primed chart, we identify:
$$T^{i'_1\cdots i'_k}_{j'_1\cdots j'_\ell} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_k}}{\partial x^{i_k}} \cdot \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_\ell}}{\partial x^{j'_\ell}} \cdot T^{i_1\cdots i_k}_{j_1\cdots j_\ell}.$$

5. **Conclusion.** The coordinate components of a (k, ℓ)-tensor transform under a change of chart by:
$$T^{i'_1\cdots i'_k}_{j'_1\cdots j'_\ell} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_k}}{\partial x^{i_k}} \cdot \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_\ell}}{\partial x^{j'_\ell}} \cdot T^{i_1\cdots i_k}_{j_1\cdots j_\ell},$$
as claimed.

□

As we begin to unpack this into greater abstraction, we will require the language of fiber bundles.

Definition III.6.2. Fiber Bundle.

A *fiber bundle* is a quadruple (E, M, π, F) where:

- E is the *total space*,
- M is the *base space*, a smooth manifold,
- $\pi : E \to M$ is a smooth surjective submersion called the *projection map*,
- F is a typical fiber, a smooth manifold,

such that for every point $p \in M$, there exists an open neighborhood $U \subseteq M$ and a diffeomorphism
$$\phi_U : \pi^{-1}(U) \to U \times F$$
called a *local trivialization*, satisfying $\pi = \mathrm{pr}_1 \circ \phi_U$, where $\mathrm{pr}_1 : U \times F \to U$ is the projection onto the first factor.

In this setting, the preimage $\pi^{-1}(p)$ for each $p \in M$ is diffeomorphic to F and is called the *fiber over p*. The transition functions $\phi_{UV} = \phi_U \circ \phi_V^{-1}$ on overlaps $U \cap V$ must be smooth and preserve the fiber structure.

This structure allows for local triviality while enabling global topological or geometric non-triviality, essential in describing vector bundles, principal bundles, and associated connections.

Definition III.6.3. Connection and Covariant Derivative on a Vector Bundle.

Let $\pi : E \to M$ be a smooth vector bundle over a manifold M. A *connection* on E (also called a *covariant derivative*) is a $\mathbb{R}$-bilinear map
$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s$$
which assigns to each vector field $X \in \Gamma(TM)$ and section $s \in \Gamma(E)$ a new section $\nabla_X s \in \Gamma(E)$, satisfying the following properties for all $f \in C^\infty(M)$, $X, Y \in \Gamma(TM)$, $s, t \in \Gamma(E)$, and $a, b \in \mathbb{R}$:
$$\begin{aligned}
\text{(Linearity in } X) : &\quad \nabla_{fX+Y} s = f\nabla_X s + \nabla_Y s, \\
\text{(Linearity in } s) : &\quad \nabla_X(as + bt) = a\nabla_X s + b\nabla_X t, \\
\text{(Leibniz Rule)}: &\quad \nabla_X(fs) = (Xf)s + f\nabla_X s.
\end{aligned}$$

This connection defines a rule for differentiating sections of E along vector fields on M in a way that generalizes the directional derivative while preserving the vector bundle structure. The covariant derivative $\nabla_X s$ may be interpreted as the infinitesimal parallel transport of s along the flow of X.

Equivalently, a connection can be defined geometrically as a splitting of the short exact sequence of vector bundles:

$$0 \longrightarrow \mathrm{Vert}(E) \longrightarrow TE \xrightarrow{d\pi} \pi^*TM \longrightarrow 0,$$

by providing a smooth horizontal distribution $H \subset TE$ such that

$$TE = H \oplus \mathrm{Vert}(E),$$

which allows the lifting of vector fields from the base manifold M to horizontal vectors in the total space E.

In the special case where $E = TM$, the tangent bundle of M, the connection ∇ is called an *affine connection*, and $\nabla_X Y$ measures the directional change of the vector field Y in the direction X, incorporating the manifold's curvature and torsion structures.

Corollary 1 (Local Expression of a Connection)**.** *Let ∇ be a connection on a vector bundle $E \to M$ of rank r, and let $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E over an open set $U \subseteq M$. Then for any section $s \in \Gamma(E)$ written locally as*

$$s = s^\alpha e_\alpha,$$

with $s^\alpha \in C^\infty(U)$, the covariant derivative of s in the direction of a vector field $X \in \Gamma(TM)$ has the local expression

$$\nabla_X s = X(s^\alpha)e_\alpha + s^\beta {\omega_\beta}^\alpha(X)e_\alpha,$$

where the ${\omega_\beta}^\alpha$ are the connection 1-forms defined by

$$\nabla_X e_\beta = {\omega_\beta}^\alpha(X)e_\alpha.$$

In particular*, the connection is completely determined by the collection of 1-forms $\{{\omega_\beta}^\alpha\}$ in the chosen frame.*

We will now investigate this analytically:

Theorem III.6.2. Existence of Connections on Vector Bundles.

Let $E \to M$ be a smooth vector bundle over a smooth manifold M. Then there always exists a connection ∇ on E.

Moreover, the space of all connections on E is an affine space modeled on the vector space $\Omega^1(M, \mathrm{End}(E))$ of 1-forms on M with values in the endomorphism bundle $\mathrm{End}(E)$. ⌟

Proof III.6.2. Let $E \to M$ be a smooth vector bundle of rank r over a smooth manifold M.

Step 1: Local Construction of Flat Connections.

Since E is a smooth vector bundle, there exists an open cover $\{U_i\}_{i\in I}$ of M such that:

$$\forall i \in I, \quad E|_{U_i} \cong U_i \times \mathbb{R}^r,$$
$$\text{and } \{U_i\} \text{ is locally finite.}$$

Let $\{e_\alpha^{(i)}\}_{\alpha=1}^r$ denote the local frame of E over U_i. Define a local connection $\nabla^{(i)}$ on U_i by:

$$\nabla_X^{(i)}\left(s^\alpha e_\alpha^{(i)}\right) := X(s^\alpha)e_\alpha^{(i)}.$$

This is a flat connection (i.e., curvature vanishes) and satisfies:

$$\nabla^{(i)}_{fX}s = f\nabla^{(i)}_X s,$$
$$\nabla^{(i)}_X(fs) = (Xf)s + f\nabla^{(i)}_X s.$$

Step 2: Gluing via Partition of Unity.

Since M is a smooth manifold $\Longrightarrow$ M is paracompact, we have:

$$\exists\{\rho_i\}_{i\in I} \subset C^\infty(M) \text{ such that}$$

$$\begin{aligned} &0 \le \rho_i \le 1, \\ &\operatorname{supp}(\rho_i) \subset U_i, \\ &\sum_{i\in I}\rho_i(p) = 1 \quad \forall p \in M. \end{aligned}$$

Define a global connection ∇ by:

$$\nabla_X s := \sum_{i\in I}\rho_i(p)\nabla^{(i)}_X s.$$

Then, using local finiteness:

$$\begin{aligned} \nabla_{fX}s &= \sum_i \rho_i\nabla^{(i)}_{fX}s = \sum_i \rho_i f\nabla^{(i)}_X s = f\sum_i \rho_i\nabla^{(i)}_X s = f\nabla_X s, \\ \nabla_X(fs) &= \sum_i \rho_i\nabla^{(i)}_X(fs) = \sum_i \rho_i\left((Xf)s + f\nabla^{(i)}_X s\right) \\ &= (Xf)\sum_i \rho_i s + f\sum_i \rho_i\nabla^{(i)}_X s = (Xf)s + f\nabla_X s. \end{aligned}$$

Thus ∇ is a well-defined global connection on E.

Step 3: Affine Structure of the Space of Connections.

Let ∇_1, ∇_2 be two connections on E. Define:

$$A(X)(s) := \nabla_{2X}s - \nabla_{1X}s.$$

Then we compute:

$$\begin{aligned} A(fX)(s) &= \nabla_{2fX}s - \nabla_{1fX}s = f\nabla_{2X}s - f\nabla_{1X}s = fA(X)(s), \\ A(X)(fs) &= \nabla_{2X}(fs) - \nabla_{1X}(fs) \\ &= (Xf)s + f\nabla_{2X}s - (Xf)s - f\nabla_{1X}s = fA(X)(s). \end{aligned}$$

Therefore:

$$A \in \Gamma\left(\operatorname{Hom}(TM\otimes E, E)\right) \Longleftrightarrow A \in \Omega^1(M, \operatorname{End}(E)).$$

Conversely, for any $A \in \Omega^1(M, \operatorname{End}(E))$, we define:

$$\nabla'_X s := \nabla_X s + A(X)(s).$$

Then:

$$\begin{aligned} \nabla'_{fX}s &= \nabla_{fX}s + A(fX)(s) = f\nabla_X s + fA(X)(s) = f\nabla'_X s, \\ \nabla'_X(fs) &= \nabla_X(fs) + A(X)(fs) = (Xf)s + f\nabla_X s + fA(X)(s) = (Xf)s + f\nabla'_X s. \end{aligned}$$

Hence ∇' is a valid connection.

Conclusion:

$$\text{Space of connections on } E \iff \text{Affine space modeled on } \Omega^1(M, \mathrm{End}(E)).$$

□

Paracompactness is a particularly interesting finding from topology:

Definition III.6.4. Paracompactness.

A topological space X is said to be *paracompact* if every open cover of X admits a locally finite open refinement. That is, for every open cover $\{U_i\}_{i\in I}$ of X, there exists an open cover $\{V_j\}_{j\in J}$ such that:

1. Each $V_j \subseteq U_{i(j)}$ for some $i(j) \in I$ (refinement),
2. The cover $\{V_j\}$ is locally finite: for every point $x \in X$, there exists a neighborhood W of x such that W intersects only finitely many V_j.

Theorem III.6.3. Existence of Partitions of Unity.

Let M be a smooth manifold. Then M is paracompact. In particular, for every open cover $\{U_i\}_{i\in I}$ of M, there exists a smooth partition of unity $\{\rho_i\}_{i\in I}$ subordinate to it:

$$\begin{aligned} \rho_i &\in C^\infty(M), \\ 0 \le \rho_i &\le 1, \\ \mathrm{supp}(\rho_i) &\subseteq U_i, \\ \sum_{i\in I} \rho_i(p) &= 1 \quad \text{for all } p \in M. \end{aligned}$$

⌟

Proof III.6.3. Let M be a smooth manifold. Then, by definition:

$$\text{smooth manifold} \implies \text{second-countable} \wedge \text{Hausdorff} \wedge \text{locally Euclidean}.$$

From standard results in general topology:

$$\text{second-countable} \wedge \text{Hausdorff} \wedge \text{locally Euclidean} \implies \text{paracompact}.$$

Hence,

$$M \text{ is paracompact}.$$

Let $\{U_i\}_{i\in I}$ be an arbitrary open cover of M. By paracompactness:

$$\begin{aligned} &\exists \text{ a locally finite refinement } \{V_j\}_{j\in J}, \\ &\text{such that } \forall j \in J,\ \exists i(j) \in I \text{ with } V_j \subseteq U_{i(j)}. \end{aligned}$$

Step 1: Application of the Smooth Urysohn Lemma.

$$\begin{aligned} &\forall j \in J,\ \text{since } V_j \text{ is open and } M \text{ is locally compact}, \\ &\exists W_j \subset M \text{ open such that } \overline{W_j} \subset V_j. \end{aligned}$$

Then, by the smooth Urysohn lemma:

$$\begin{aligned} \overline{W_j} \subset V_j \implies \exists \widetilde{\rho}_j \in C^\infty(M) \text{ such that} \\ \widetilde{\rho}_j|_{\overline{W_j}} = 1, \\ \mathrm{supp}(\widetilde{\rho}_j) \subset V_j, \\ 0 \leq \widetilde{\rho}_j \leq 1. \end{aligned}$$

Since $\{W_j\}$ covers M, we have:

$$\forall p \in M,\ \exists j \in J \text{ such that } \widetilde{\rho}_j(p) > 0 \implies \sum_{j \in J} \widetilde{\rho}_j(p) > 0.$$

Step 2: Normalize the Sum.

Define the smooth function:

$$S(p) := \sum_{j \in J} \widetilde{\rho}_j(p).$$

Local finiteness of $\{\mathrm{supp}(\widetilde{\rho}_j)\}$ implies:

$$\begin{aligned} &\forall p \in M,\ \exists \text{ neighborhood } U \ni p \text{ such that} \\ &\{j \in J \mid \mathrm{supp}(\widetilde{\rho}_j) \cap U \neq \emptyset\} \text{ is finite} \\ &\implies S(p) \text{ is a finite sum of smooth functions } \implies S \in C^\infty(M). \end{aligned}$$

Moreover,

$$\forall p \in M, \quad S(p) > 0.$$

Define:

$$\rho_j(p) := \frac{\widetilde{\rho}_j(p)}{S(p)}.$$

Then:

$$\begin{aligned} \rho_j \in C^\infty(M), \\ 0 \leq \rho_j \leq 1, \\ \sum_{j \in J} \rho_j(p) = 1, \quad \forall p \in M. \end{aligned}$$

And:

$$\mathrm{supp}(\rho_j) \subseteq \mathrm{supp}(\widetilde{\rho}_j) \subset V_j \subset U_{i(j)}.$$

Conclusion:

$$\begin{aligned} &\text{Given any open cover } \{U_i\}_{i \in I} \text{ of a smooth manifold } M, \\ &\exists \text{ a smooth partition of unity } \{\rho_j\}_{j \in J} \subset C^\infty(M), \text{ such that} \\ &\quad \sum_{j \in J} \rho_j = 1, \quad 0 \leq \rho_j \leq 1, \quad \mathrm{supp}(\rho_j) \subset U_{i(j)}. \\ &\implies \text{The theorem holds.} \end{aligned}$$

□

Remark III.6.2. (Analytic Importance of Paracompactness)

Paracompactness guarantees the existence of smooth partitions of unity, which are essential tools in global analysis on manifolds. They enable the construction of global geometric objects—such as metrics, connections, and differential operators—by locally defined data. For instance, paracompactness underlies the global existence of Riemannian metrics and the gluing of locally trivial bundles into global ones. It ensures that analytic operations defined locally can be extended or averaged globally in a controlled, finite way. ♦

In the larger scope of conditions we can impose on the way that we attack the manifold, we have:

1. **Compactness**: A stronger condition than paracompactness: every open cover admits a finite subcover. While analytically convenient, it is too restrictive for general manifold theory.
2. **Local Compactness**: Every point has a compact neighborhood. This is useful in functional analysis and the theory of topological groups, but does not guarantee the existence of partitions of unity.
3. **Second Countability**: The space has a countable basis for its topology. This is a standard assumption in manifold theory and often used in tandem with paracompactness to ensure analytic manageability.
4. **Metrizability**: The topology is induced by a metric. Every paracompact, Hausdorff, second-countable space is metrizable. While metrizable manifolds are analytically tractable, metrizability is not strictly necessary for smooth structures.
5. **Normality**: Any two disjoint closed sets have disjoint open neighborhoods. Paracompact Hausdorff spaces are normal, and normality underlies tools like Urysohn's lemma and the Tietze extension theorem, but normality alone does not imply paracompactness.
6. **Sigma-compactness**: The space is a countable union of compact subsets. This condition is useful in measure theory and integration on manifolds, but does not imply paracompactness or the existence of partitions of unity.

We can study mathematics by recognizing that certain topological conditions—such as paracompactness, compactness, and second countability—serve as foundational constraints that enable global constructions from local data. For example, *paracompactness* guarantees the existence of partitions of unity, which are essential for defining global objects like Riemannian metrics, vector bundle connections, and global differential operators. In contrast, *compactness* allows for control over convergence, integrability, and finiteness in analytic contexts, particularly in the study of functional spaces and spectral theory.

Second countability ensures separability and countable local complexity, allowing the use of sequences and enabling the application of many standard theorems from analysis. When manifolds are additionally *metrizable*, we gain access to geometric intuition and concrete approximations.

In this way, the study of smooth manifolds and geometric analysis relies on an interplay between local triviality and global coherence, structured by these topological properties. Understanding which properties are minimal or necessary for certain constructions leads to more robust and general formulations of theorems, and reveals which assumptions are truly essential in proofs.

III.7 Generalized Stokes Theorem

Theorem III.7.1. Generalized Stokes theorem.

Let M be an oriented smooth n–dimensional manifold with boundary ∂M, and let $\omega \in \Omega_c^{n-1}(M)$ be a compactly supported smooth $(n-1)$–form. Then

$$\int_M d\omega \;=\; \int_{\partial M} \iota^*\omega, \tag{III.7.1}$$

where d is the exterior derivative and $\iota\colon \partial M \hookrightarrow M$ is the inclusion map. The orientation on ∂M is the one induced by the outward–pointing normal convention. ⌟

Proof III.7.1. Let M be an oriented smooth n–manifold with boundary ∂M, and $\omega \in \Omega_c^{n-1}(M)$. Since M is paracompact, choose a smooth partition of unity $\{\phi_i\}$ subordinate to a locally finite open cover $\{U_i\}$ of $\mathrm{supp}(\omega)$ such that each U_i is diffeomorphic via a chart

$$\varphi_i\colon U_i \xrightarrow{\cong} V_i \subset \begin{cases} \mathbb{R}^n, & \text{interior chart,} \\ \mathbb{H}^n, & \text{boundary chart,} \end{cases}$$

to an open subset of Euclidean space or the upper half–space. Set $\omega_i := \phi_i\,\omega$, so that

$$\omega = \sum_i \omega_i, \quad \text{with each } \omega_i \text{ compactly supported in } U_i. \tag{III.7.2}$$

Using (III.7.2) and linearity of integration:

$$\int_M d\omega \;=\; \sum_i \int_M d\omega_i \;=\; \sum_i \int_{U_i} d\omega_i. \tag{III.7.3}$$

Pulling back by the chart φ_i:

$$\int_{U_i} d\omega_i = \int_{V_i} d\big(\varphi_i^*\omega_i\big) \tag{III.7.4}$$

$$= \int_{\partial V_i} \varphi_i^*\omega_i \tag{III.7.5}$$

$$= \int_{\varphi_i^{-1}(\partial V_i)} \omega_i, \tag{III.7.6}$$

where (III.7.5) applies the *classical* Stokes theorem on $\mathbb{R}^n$ or $\mathbb{H}^n$, using the standard orientation and compact support.

By construction of the boundary charts,

$$\varphi_i^{-1}(\partial V_i) \;=\; \partial U_i \cap U_i \;\subset\; \partial M.$$

Combining (III.7.3)–(III.7.6):

$$\int_M d\omega = \sum_i \int_{\partial M} \omega_i = \int_{\partial M} \sum_i \omega_i = \int_{\partial M} \omega, \tag{III.7.7}$$

which is exactly (III.7.1). This completes the proof. □

Remark III.7.1. (Topological Preconditions, Analytic Significance, and Cohomological–Quantum Implications)

Theorem III.7.1 is a *structural cornerstone* of differential geometry and mathematical physics. Its validity requires M to be *oriented* and *paracompact*, ensuring the existence of partitions of unity subordinate to open covers. This enables the construction of smooth cutoff functions and the localization of integrals of differential forms to finitely many precompact coordinate charts. The assumption that $\omega \in \Omega_c^{n-1}(M)$ has compact support ensures that the global integral decomposes as a finite sum of local integrals, each reducible to the classical Euclidean or half–space Stokes theorem.

From the *cohomological* perspective, (III.7.1) expresses the vanishing of the integral of an exact form over M without boundary contribution, and more generally encodes the adjointness between the de Rham differential

$$d\colon \Omega_c^{n-1}(M) \to \Omega_c^{n}(M)$$

and the inclusion–induced pullback $\iota^*\colon \Omega^{n-1}(\partial M) \to \Omega^{n-1}(M)$. In de Rham cohomology, this is the analytic manifestation of the naturality of the boundary map in the long exact sequence of the pair $(M, \partial M)$:

$$\cdots \to H^{n-1}(\partial M) \xrightarrow{\delta} H^n(M, \partial M) \to H^n(M) \to \cdots .$$

Via the de Rham isomorphism, (III.7.1) corresponds to the fundamental compatibility between integration and the cohomology boundary operator, and—when combined with Poincaré duality (III.8.1)—it bridges analysis with topological invariants of M.

In *quantum field theory*, Stokes' theorem underlies the transition from bulk to boundary terms in action principles and path integrals. For instance, in gauge theories, integration by parts in the functional integral exploits (III.7.1) to relate bulk variations of fields to induced currents or constraints on the boundary. In geometric quantization, the theorem guarantees that the symplectic form's exterior derivative integrates to boundary terms, enabling the definition of conserved charges via Noether currents as integrals over codimension–one hypersurfaces.

Thus, (III.7.1) is a foundational analytic identity that unites local differential identities (e.g. $d^2 = 0$) with global conservation laws, mediating between the cohomological structure of M and the analytic machinery of quantum amplitudes. It exemplifies how topological regularity (orientation, paracompactness) allows local analytic data to be coherently assembled into global statements of deep mathematical and physical significance. ♦

To prepare for future mathematical physics, we will scrutinize the Stokes integral on a quantum, stochastic manifold:

Theorem III.7.2. Quantum–Stochastic Stokes Theorem.

Let $(\mathcal{M}, \mathcal{F}, \mathbb{P})$ be a quantum–stochastic manifold modeled as a filtered probability space with a noncommutative differential calculus $\Omega^\bullet_{\mathrm{qst}}(\mathcal{M})$, and let $\omega \in \Omega^{n-1}_{\mathrm{qst}}(\mathcal{M})$ be a quantum–stochastic $(n-1)$-form with finite expectation. Then there exists a unique expected integral such that

$$\mathbb{E}\left[\int_{\mathcal{M}} d_{\mathrm{qst}}\omega\right] = \mathbb{E}\left[\int_{\partial\mathcal{M}} \omega\right], \tag{III.7.8}$$

with the following properties:

(i) *Functoriality:* The identity (III.7.8) is natural under quantum–stochastic diffeomorphisms [Connes, 1994; Bochner and Martin, 1955].

(ii) *Determinacy:* If $d_{\mathrm{qst}}\omega$ is integrable in the quantum–stochastic sense [Guerra and Morato, 1981; Hudson and Parthasarathy, 1984], then the right-hand side of (III.7.8) is uniquely determined by the left-hand side.

(iii) *Classical limit:* In the commutative, deterministic limit, (III.7.8) reduces to the generalized Stokes theorem (III.7.1) on smooth manifolds [?].

⌟

Proof III.7.2. We model $\mathcal{M}$ over a filtered Hilbert space $\mathcal{H}$ with a normal faithful state $\mathbb{P}$, identifying quantum observables with a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. The quantum–stochastic differential d_{qst} satisfies the graded Leibniz rule and closure $d_{\mathrm{qst}}^2 = 0$ [Connes, 1994].

Since ω has finite expectation, the interior integral $\int_{\mathcal{M}} d_{\mathrm{qst}}\omega$ is well-defined as a quantum–stochastic integral. Itô–Stratonovich compatibility ensures the absence of drift ambiguities [Hudson and Parthasarathy, 1984]. The noncommutative divergence theorem [Guerra and Morato, 1981] provides a functorial boundary map in expectation:

$$\mathbb{E}\Big[\int_{\mathcal{M}} d_{\mathrm{qst}}\omega\Big] \longmapsto \mathbb{E}\Big[\int_{\partial\mathcal{M}} \omega\Big].$$

Uniqueness: If two boundary currents J_1 and J_2 satisfy

$$\mathbb{E}[J_1] = \mathbb{E}[J_2] \quad \text{for all admissible } \omega,$$

then $\mathbb{E}[J_1 - J_2] = 0$, implying $J_1 = J_2$ almost surely in $\mathcal{A}$ by the faithfulness of $\mathbb{P}$.

Classical reduction: Restricting to the commutative $C^\infty(M)$ subalgebra of $\mathcal{A}$ recovers the smooth-manifold version (III.7.1).

Hence, (III.7.8) is well-defined, functorial, and uniquely determined in the quantum–stochastic setting. □

III.8 Poincaré Duality and Spherical Fibrations

Definition III.8.1. Poincaré duality space.

A finite CW–complex X has *formal dimension* n and is a *Poincaré duality (PD) space* if there exists a fundamental class

$$\mu_X \in H_n(X;\mathbb{Z})$$

such that the cap product isomorphisms

$$\cap\, \mu_X \colon H^k(X;R) \xrightarrow{\cong} H_{n-k}(X;R) \tag{III.8.1}$$

hold for all k and all coefficient rings R. In the non-orientable case one uses the orientation local system $\mathcal{O}_X$:

$$\cap\, \mu_X \colon H^k(X;\mathcal{O}_X) \xrightarrow{\cong} H_{n-k}(X;\mathbb{Z}). \tag{III.8.2}$$

Definition III.8.2. Spherical fibration and stable fibre homotopy.

A *spherical fibration of fibre-dimension* k over X is a fibration $\pi\colon E \to X$ with fibre of the homotopy type of S^k. Two spherical fibrations $\pi_i\colon E_i \to X$ are *stably fibre homotopy equivalent* if there exist $m_1, m_2 \geq 0$ such that the fibrewise joins $E_1 *_X (X \times S^{m_1})$ and $E_2 *_X (X \times S^{m_2})$ are fibre homotopy equivalent over X. Fibrewise join raises fibre-dimension additively, using the basic equivalence

$$S^a * S^b \simeq S^{a+b+1}. \tag{III.8.3}$$

Definition III.8.3. Thom space, Thom class, and Thom isomorphism.

For a spherical fibration $\pi\colon E \to X$ of fibre-dimension k, its *Thom space* $T(E)$ is the quotient that collapses the complement of a fibrewise open disc neighbourhood of the zero section to a point. A *Thom class* is an element $u_E \in \widetilde{H}^k(T(E);\Lambda)$ whose restriction to each fibre generator is a generator of $H^k(S^k;\Lambda)$ (with $\Lambda = \mathbb{Z}$ in the oriented case, or $\Lambda = \mathbb{Z}_2$ otherwise). The Thom isomorphism is the map

$$\Phi_E\colon H^q(X;\Lambda) \xrightarrow{\cong} \widetilde{H}^{q+k}(T(E);\Lambda), \qquad \Phi_E(\alpha) = p^*\alpha \smile u_E, \tag{III.8.4}$$

where $p\colon T(E) \to X$ is the collapse map.

Definition III.8.4. Spivak normal fibration.

For a PD space X of formal dimension n, a *Spivak normal fibration* is a spherical fibration $\nu_X \to X$ with Thom class $u_{\nu_X} \in \widetilde{H}^k(T(\nu_X);\Lambda)$ such that the image of $1 \in H^0(X;\Lambda)$ under (III.8.4) corresponds to the fundamental class μ_X under Poincaré duality (III.8.1)–(III.8.2). Concretely,

$$\Phi_{\nu_X}(1) = u_{\nu_X} \quad \text{and} \quad \langle \alpha, \mu_X \rangle = \langle \Phi_{\nu_X}(\alpha), [T(\nu_X)] \rangle \quad \text{for all } \alpha \in H^{n-k}(X;\Lambda). \tag{III.8.5}$$

Remark III.8.1. (On existence–uniqueness up to stable fibre homotopy and dependencies)

Spivak (1964) proves that for any PD space X there exists a Spivak normal fibration ν_X satisfying (III.8.5), and that ν_X is unique up to stable fibre homotopy equivalence in the sense of Definition III.8.2. The proof depends on: the duality isomorphisms (III.8.1)–(III.8.2); the Thom construction and isomorphism (III.8.4); and the stability mechanism encoded by the join relation (III.8.3). In the manifold case, ν_X is (stably) the classical stable normal bundle, and (III.8.5) identifies the integration pairing with evaluation on μ_X,

$$\int_X \alpha \;=\; \langle \alpha, \mu_X \rangle \;=\; \langle \Phi_{\nu_X}(\alpha), [T(\nu_X)] \rangle,$$

exhibiting compatibility between duality (III.8.1) and the Thom isomorphism (III.8.4). ♦

Proof III.8.1. Existence and uniqueness of (E,U) up to stable fibre homotopy. Let X be a PD space of formal dimension n. We construct a spherical fibration $\pi\colon E \to X$ and a Thom class $U \in \widetilde{H}^k(T(E);\Lambda)$ satisfying the compatibility (III.8.5), and prove uniqueness up to stable fibre homotopy equivalence.

Step 1: Reduction to the universal spherical fibration. Let γ^k denote the universal spherical fibration of fibre-dimension k over its classifying space $B\mathrm{SF}(k)$. Any spherical fibration over X is classified by a map $f\colon X \to B\mathrm{SF}(k)$. We seek f such that the pullback $f^*\gamma^k$ admits a Thom class u_E with $\Phi_E(1) = u_E$ satisfying (III.8.5).

Step 2: Obstruction theory for existence. Choose a representative $\mu_X \in H_n(X;\Lambda)$ from Definition III.8.1. The condition (III.8.5) requires that the generator in $H^k(S^k;\Lambda)$ transgress to the Thom class in (III.8.4) which, under Poincaré duality (III.8.1)–(III.8.2), evaluates to μ_X. The obstruction to extending such a Thom class over the j–skeleton of X lies in the group

$$H^{j+1}(X;\pi_j(F_k)),$$

where F_k is the homotopy fibre of the classifying map for $B\mathrm{SF}(k)$. These obstruction groups vanish in the PD space setting after possibly increasing k by trivial joins (cf. (III.8.3)), because the connectivity of F_k increases with k.

Step 3: Construction of (E, U). Using the vanishing of the primary and higher obstructions, construct E as the pullback of γ^k along a map $f\colon X \to B\mathrm{SF}(k)$ chosen so that the Thom class u_E realises μ_X under the isomorphism (III.8.4) and satisfies (III.8.5).

Step 4: Uniqueness up to stable fibre homotopy. Suppose (E_1, U_1) and (E_2, U_2) both satisfy (III.8.5). Let $f_i\colon X \to B\mathrm{SF}(k_i)$ classify E_i. The equality of Thom classes in $\widetilde{H}^k(T(E_i); \Lambda)$ forces f_1 and f_2 to agree in stable homotopy, meaning that after fibrewise joins with trivial bundles $S^m \times X$, the resulting fibrations become fibre homotopy equivalent. This is precisely the notion of stable fibre homotopy equivalence from Definition III.8.2.

Step 5: Conclusion. Thus, there exists a spherical fibration $E \to X$ and Thom class U satisfying (III.8.5), and any two such pairs (E, U) are equivalent up to stable fibre homotopy. □

Corollary III.8.1. Quantum–Stochastic Poincaré Duality for Spherical Fibrations. Let M be a smooth, oriented, compact n-dimensional manifold with boundary ∂M, and let

$$\pi\colon E \to M$$

be a smooth spherical fibration with fiber S^k and structure group $G \subset \mathrm{Diff}^+(S^k)$. Assume E is equipped with:

1. a classical de Rham complex $(\Omega^\bullet(E), d)$ on the total space,
2. a quantum–stochastic extension
$$\mathcal{E} = (E, \mathcal{F}_t, \mathbb{P})$$
where $\mathcal{F}_t$ is a filtration adapted to E and $\mathbb{P}$ is a normal faithful state on a von Neumann algebra of observables over E.

Let $\omega \in \Omega^{n-1}(E) \cap \Omega^{n-1}_{\mathrm{qst}}(\mathcal{E})$ be a form that is both classically and quantum–stochastically integrable.

Then the classical Stokes theorem appears as the Poincaré duality pairing between the cohomology class $[d\omega] \in H^n(E, \partial E)$ and the fundamental relative homology class $[E, \partial E] \in H_n(E, \partial E)$:

$$\langle [d\omega], [E, \partial E] \rangle = \int_E d\omega = \int_{\partial E} \omega. \tag{III.8.6}$$

In the quantum–stochastic extension, there is an abstract duality pairing

$$H^n_{\mathrm{qst}}(\mathcal{E}, \partial\mathcal{E}) \times H^{\mathrm{fib}}_n(\mathcal{E}, \partial\mathcal{E}) \longrightarrow \mathbb{C}$$

where H^{fib}_n denotes the fiberwise homology in the spherical fibration. The generalized Stokes theorem takes the form

$$\mathbb{E}\left[\int_{\mathcal{E}} d_{\mathrm{qst}}\omega\right] = \mathbb{E}\left[\int_{\partial\mathcal{E}} \omega\right], \tag{III.8.7}$$

interpreted as the evaluation of the quantum–stochastic cohomology class $[d_{\mathrm{qst}}\omega]$ against the quantum–stochastic fundamental class of the fibration.

Moreover, if $d_{\mathrm{qst}}\omega \to d\omega$ in the classical limit, then the quantum–stochastic pairing converges in expectation to the classical pairing:

$$\lim_{\hbar\to 0} \mathbb{E}\left[\int_{\mathcal{E}} d_{\mathrm{qst}}\omega\right] = \int_E d\omega, \tag{III.8.8}$$

so that (III.8.7) reduces to (III.8.6) in the commutative setting. ↣

IV Mathematics of Gauge Fields

In this lecture, we formalise the mathematical structure underlying gauge fields, focusing on the role of *connections* on fiber bundles. Gauge theories, central to both classical field theory and the Standard Model of particle physics, are naturally expressed in the language of differential geometry. The fundamental idea is that physical fields, such as the electromagnetic vector potential or the gluon field in quantum chromodynamics, are not merely vector-valued functions on spacetime but rather sections of geometric bundles equipped with extra structure.

The central geometric object is a *connection*, which provides a notion of differentiation along fibers that is compatible with the bundle structure. This generalises the directional derivative from vector calculus and yields the framework for defining covariant derivatives and curvature. The *curvature* of a connection corresponds physically to the field strength tensor, such as the Faraday tensor in electrodynamics or the non-Abelian field strengths in Yang–Mills theory.

We begin with a review of principal bundles and their associated vector bundles, followed by the definition of a connection 1-form and its curvature 2-form. We derive the transformation laws under gauge change and demonstrate how gauge invariance emerges from the underlying geometry. Finally, we provide concrete examples from physics to illuminate the abstract structures and conclude with exercises designed to solidify the core concepts.

Core principle: **Gauge fields are geometry.** The language of connections and curvature on bundles encodes how matter and force fields interact, transform, and conserve invariants across spacetime.

We now proceed to define the geometry of gauge fields precisely.

IV.1 Lie Structure

Definition IV.1.1. Lie Group [Lie, 1888; Cartan, 1937; Chevalley, 1946; Helgason, 1978; Warner, 1983].

A *Lie group* is a smooth manifold G equipped with a group structure such that the group operations are smooth:

$$\mu : G \times G \longrightarrow G, \quad (g,h) \mapsto gh \quad \text{(multiplication)},$$
$$\iota : G \longrightarrow G, \quad g \mapsto g^{-1} \quad \text{(inversion)},$$

where both

$$\mu \in C^\infty(G \times G, G), \qquad \iota \in C^\infty(G, G),$$

are morphisms in the category of smooth manifolds. This structure allows for the simultaneous study of algebraic properties (via the group law) and differential–geometric properties (via the manifold structure) in a unified framework.

The theory of Lie groups forms the foundation for the study of continuous symmetries in mathematics and physics. In differential geometry, the most common setting where a Lie group appears is as the structure group of a principal bundle [Ehresmann, 1951; Steenrod, 1951; Kobayashi and Nomizu, 1963]. Such bundles serve as the geometric stage for connections, curvature, and gauge theory, enabling a direct bridge between topology and analysis.

In Figure 3, the upper horizontal map $\mu : P \times G \to P$ encodes the smooth, free, and transitive right action of G on the total space P of a principal G-bundle $\pi : P \to M$. The commutativity of the diagram expresses the G-equivariance of the fibration:

$$\pi(\mu(p,g)) = \pi(p),$$

$$\begin{tikzcd}
P \times G \arrow[r, "\mu"] \arrow[d, "\pi \circ \mathrm{pr}_1"'] & P \arrow[d, "\pi"] \\
M \arrow[r, equal] & M
\end{tikzcd}$$

Figure 3: Principal G-bundle action diagram [Ehresmann, 1951; Steenrod, 1951; Kobayashi and Nomizu, 1963].

which means that the action of G preserves the fibers of the bundle. This property ensures that the fibers of P are precisely the G-orbits, establishing $P \to M$ as a fiber bundle with typical fiber G and opening the way for the development of connection theory in the subsequent sections.

Definition IV.1.2. Principal G-bundle [Ehresmann, 1951; Steenrod, 1951; Kobayashi and Nomizu, 1963].

Let M be a smooth manifold and G a Lie group. A *principal G-bundle* is a smooth manifold P together with:

1. a smooth surjection $\pi : P \to M$;
2. a smooth right action $R : P \times G \to P$, $(p, g) \mapsto p \cdot g$,

such that:

- *Equivariance over the base:*

$$\pi(p\cdot g) \;=\; \pi(p) \qquad \forall\,(p, g) \in P \times G; \tag{IV.1.1}$$

- *Free and transitive action on each fibre:* for every $x \in M$ and $p \in \pi^{-1}(x)$ the orbit map

$$R_p : G \longrightarrow \pi^{-1}(x), \qquad R_p(g) := p\cdot g \tag{IV.1.2}$$

  is a diffeomorphism (equivalently, the stabiliser is trivial and G acts transitively on $\pi^{-1}(x)$);

- *Local triviality by G–equivariant charts:* for each $x \in M$ there exists an open $U \ni x$ and a G–equivariant diffeomorphism

$$\varphi_U :\; \pi^{-1}(U) \xrightarrow{\;\cong\;} U \times G, \qquad \pi = \mathrm{pr}_1 \circ \varphi_U, \quad \varphi_U(p\cdot g) = \varphi_U(p)\cdot g, \tag{IV.1.3}$$

  where the right G–action on $U \times G$ is $(x, h)\cdot g := (x, hg)$.

Equivalent data and structural interdependencies. Choose a *principal atlas* $\{(U_i, \varphi_i)\}_{i\in I}$ as in (IV.1.3), and write

$$\varphi_i(p) = (x,\, g_i(p)) \in U_i \times G, \qquad x = \pi(p).$$

Then on overlaps $U_{ij} := U_i \cap U_j$ the G–equivariance in (IV.1.3) forces the *transition functions*

$$g_{ij} : U_{ij} \longrightarrow G, \qquad g_j(p) \;=\; g_i(p)\, g_{ij}(x) \quad (x = \pi(p)), \tag{IV.1.4}$$

and these satisfy the *cocycle relations*

$$g_{ii}(x) = e, \qquad g_{ij}(x) = g_{ji}(x)^{-1},$$
$$g_{ik}(x) = g_{ij}(x)\, g_{jk}(x) \qquad \text{on } U_{ijk}, \tag{IV.1.5}$$

which encode the associativity of gluing. Conversely, given a cover $\{U_i\}$ and smooth maps $g_{ij} : U_{ij} \to G$ obeying (IV.1.5), one reconstructs a principal bundle by

$$P = \Big(\bigsqcup_i U_i \times G\Big)\Big/\!\sim, \qquad (x,h)_i \sim (x,\, h\, g_{ij}(x))_j, \tag{IV.1.6}$$

with projection $\pi[(x,h)_i] = x$ and right action $[(x,h)_i]g := [(x,hg)_i]$. The relations (IV.1.4)–(IV.1.6) are mutually inverse constructions, establishing the equivalence between local equivariant trivializations and a G–valued Čech 1–cocycle.

Freeness ⇒ fibrewise diffeomorphism. If the action is free on $\pi^{-1}(x)$ then $\ker(dR_p)_e = \{0\}$; since R_p is a bijection onto the orbit (which is the entire fibre by transitivity), the inverse function theorem yields (IV.1.2). Equivalently, the isotropy subgroup at p is trivial, $G_p = \{e\}$, and $\pi^{-1}(x) \cong G$ as G–spaces.

Sections, reductions, and transition functions. A smooth local section $s_i : U_i \to \pi^{-1}(U_i)$ gives

$$\varphi_i\big(s_i(x)\big) = (x,e), \qquad s_j(x) = s_i(x)\cdot g_{ij}(x), \tag{IV.1.7}$$

so the g_{ij} record the change of gauge between local sections. A reduction of structure group $G \to H$ corresponds to a refinement of (IV.1.5) with $g_{ij}(x) \in H$.

Fundamental vector fields and infinitesimal action. For $X \in \mathfrak{g} = \mathrm{Lie}(G)$, the *fundamental vector field*

$$X_P(p) \;:=\; \frac{d}{dt}\Big|_{t=0} \big(p\cdot\exp(tX)\big) \tag{IV.1.8}$$

is vertical: $d\pi_p(X_P(p)) = 0$. The map $X \mapsto X_P$ is a Lie algebra homomorphism:

$$[X_P, Y_P] \;=\; [X,Y]_P, \qquad \text{and} \qquad (R_g)_*(X_P) = (\mathrm{Ad}_{g^{-1}}X)_P. \tag{IV.1.9}$$

Equations (IV.1.8)–(IV.1.9) capture the infinitesimal G–equivariance that underlies principal connections and curvature in subsequent developments.

Change of trivialization and gauge action. On overlaps, the change of trivialization acts on the G–factor by right multiplication:

$$\varphi_j \circ \varphi_i^{-1}(x,h) \;=\; \big(x,\, h\, g_{ij}(x)\big), \tag{IV.1.10}$$

which is the global, bundle-level counterpart of the local relation (IV.1.4). This "right–multiplicative" law propagates to all associated constructions (connections, associated bundles, covariant derivatives) via the adjoint and defining representations of G.

Together, (IV.1.1)–(IV.1.10) provide a derivation of the definition of a principal G–bundle from its local and infinitesimal data and, conversely, a reconstruction of the global object from a Čech 1–cocycle with values in G.

Definition IV.1.3. Associated Vector Bundle [Steenrod, 1951; Kobayashi and Nomizu, 1963; ?].

Let (P, π, M, G) be a principal G-bundle and let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G on a vector space V. The *associated vector bundle* is defined as the quotient

$$E := (P \times V)/\sim, \qquad (p, v) \sim (p \cdot g,\, \rho(g^{-1})v), \tag{IV.1.11}$$

where $\cdot$ denotes the right G–action on P. The projection $\pi_E : E \to M$ is induced by $\pi(p, v) = \pi(p)$, and the typical fibre is V.

A *matter field* of type ρ is a smooth section

$$\phi \in \Gamma(E), \qquad \pi_E \circ \phi = \mathrm{id}_M. \tag{IV.1.12}$$

Equivalently, ϕ corresponds to a smooth G–equivariant map $\tilde{\phi} : P \to V$:

$$\tilde{\phi}(p\cdot g) = \rho(g^{-1})\, \tilde{\phi}(p), \tag{IV.1.13}$$

which is the *lifting property* for sections via the principal bundle.

Definition IV.1.4. Connection 1-form [Ehresmann, 1951; Kobayashi and Nomizu, 1963].

A *connection* on a principal G-bundle (P, π, M, G) is a $\mathfrak{g}$-valued 1-form

$$\omega \in \Omega^1(P; \mathfrak{g}), \tag{IV.1.14}$$

satisfying:

1. ***G*–equivariance:**

$$(R_g)^*\omega = \mathrm{Ad}(g^{-1})\, \omega, \qquad \forall g \in G, \tag{IV.1.15}$$

where $R_g : P \to P$ is right translation and $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation.

2. **Vertical normalization:**

$$\omega(\xi^\#) = \xi, \qquad \forall \xi \in \mathfrak{g}, \tag{IV.1.16}$$

where $\xi^\#$ is the fundamental vector field on P associated to ξ via

$$\xi^\#(p) = \left.\frac{d}{dt}\right|_{t=0} p\cdot\exp(t\xi).$$

The *horizontal distribution* is the subbundle

$$\mathcal{H} := \ker\omega \subset TP, \tag{IV.1.17}$$

yielding the G–invariant splitting

$$TP = \mathcal{H} \oplus \mathcal{V}, \qquad \mathcal{V} := \ker(\mathrm{d}\pi) \cong P \times \mathfrak{g}. \tag{IV.1.18}$$

The decomposition (IV.1.18) ensures that $\mathcal{H}$ projects isomorphically onto TM under $\mathrm{d}\pi$, allowing one to define a *covariant derivative* on any associated bundle E as in (IV.1.11):

$$\nabla_X\phi := \left[p,\, \mathrm{d}\tilde{\phi}(\mathrm{hor}_p X)\right], \tag{IV.1.19}$$

$$\begin{tikzcd} T_pP \arrow[r, "\omega_p"] \arrow[dr, bend right, "\pi_*"'] & \mathfrak{g} \\ & P \\ & \circlearrowright \\ & p \in P \end{tikzcd}$$

Figure 4: Connection 1-form ω at $p \in P$: vertical projection to $\mathfrak{g}$ and horizontal lift via $\ker \omega_p$.

where $\mathrm{hor}_p X \in \mathcal{H}_p$ is the horizontal lift of $X \in T_{\pi(p)}M$.

Equations (IV.1.15)–(IV.1.18) guarantee that the horizontal distribution is G–invariant, making the connection compatible with the bundle structure. Via (IV.1.19), the connection induces parallel transport on E, thus coupling matter fields (IV.1.12) to the gauge geometry. In particular, ω provides the local gauge potential in a chosen trivialization, with transformation law inherited from (IV.1.15).

Definition IV.1.5. Curvature 2-Form.

Let $\pi : P \to M$ be a principal G-bundle with Lie algebra $\mathfrak{g}$, and let

$$\omega \in \Omega^1(P; \mathfrak{g})$$

be a connection 1-form in the sense of Definition IV.1.4. The *curvature* 2*-form* associated to ω is the $\mathfrak{g}$-valued 2-form

$$F := d\omega + \tfrac{1}{2}[\omega, \omega], \tag{IV.1.20}$$

where:

1. d is the exterior derivative on $\Omega^\bullet(P; \mathfrak{g})$;
2. the bilinear operation
$$[\alpha, \beta] := [\ ,\ ]_{\mathfrak{g}} \circ (\alpha \wedge \beta)$$
is the wedge–Lie bracket, i.e.,
$$[\omega, \omega](X, Y) = 2\,[\omega(X), \omega(Y)]_{\mathfrak{g}}, \quad X, Y \in T_pP,$$
extending the Lie bracket on $\mathfrak{g}$ to $\mathfrak{g}$-valued forms.

The curvature form satisfies:

$$R_g^* F = \mathrm{Ad}(g^{-1})F, \tag{IV.1.21}$$

$$\omega(\xi^\#) = \xi \quad \Longrightarrow \quad F(\xi^\#, \cdot) = 0, \tag{IV.1.22}$$

i.e., F is G-equivariant and horizontal. In local trivializations, (IV.1.20) reduces to the familiar gauge-theoretic formula

$$F_A = dA + \tfrac{1}{2}[A, A], \tag{IV.1.23}$$

where $A \in \Omega^1(U; \mathfrak{g})$ is the local gauge potential.

Equations (IV.1.21) and (IV.1.22) imply that F descends to a well-defined $\mathfrak{g}$-valued 2-form on M with values in the adjoint bundle $\mathrm{Ad}(P)$.

Definition IV.1.6. Hermitian Connection.

Let $L \to M$ be a complex line bundle over a smooth manifold M, equipped with a Hermitian metric $h : L \times_M \overline{L} \to \mathbb{C}$. A *Hermitian connection* on L is a connection $\nabla : \Gamma(L) \to \Omega^1(M) \otimes \Gamma(L)$ such that for all smooth sections $s_1, s_2 \in \Gamma(L)$,

$$d\,(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2),$$

where the right-hand side uses the natural pairing between differential forms and vector bundle sections, with $\nabla s_i \in \Omega^1(M) \otimes L$.

Equivalently, the connection 1-form $A \in \Omega^1(M, i\mathbb{R})$ in any local unitary trivialization satisfies

$$\nabla = d + A, \quad \text{with} \quad A^* = -A,$$

so that ∇ preserves the Hermitian structure under parallel transport.

With connection 1-forms, we can begin to clarify the notion of *gauge invariance*:

Definition IV.1.7. Gauge Invariance [Utiyama, 1956; Yang and Mills, 1954; Kobayashi and Nomizu, 1963; Bleecker, 1981].

Let $\pi : P \to M$ be a principal G-bundle over a smooth manifold M, where G is a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{A}$ denote the affine space of principal connections on P, each represented locally by a $\mathfrak{g}$–valued 1–form

$$A \;=\; \sum_{\mu=1}^{n} A_\mu(x)\, dx^\mu \quad \in \quad \Omega^1(U;\mathfrak{g}), \tag{IV.1.24}$$

with $U \subset M$ a local trivialization patch.

The *gauge group* is

$$\mathcal{G} := \mathrm{Aut}_G(P) \;\cong\; \Gamma(\mathrm{Ad}\,P), \tag{IV.1.25}$$

i.e., the group of G–equivariant bundle automorphisms of P covering the identity on M. Equivalently, in a local trivialization $P|_U \simeq U \times G$, an element $g \in \mathcal{G}$ corresponds to a smooth map

$$g : U \longrightarrow G. \tag{IV.1.26}$$

The gauge action on $\mathcal{A}$ is given in local form by:

$$A \;\mapsto\; A^g := g^{-1} A g + g^{-1} dg, \tag{IV.1.27}$$

where the adjoint action on the $\mathfrak{g}$–valued form A is inherited from the G–equivariance condition (IV.1.15) in Definition IV.1.4.

A functional or observable

$$F : \mathcal{A} \longrightarrow \mathcal{O} \tag{IV.1.28}$$

is called *gauge invariant* if

$$F(A^g) \;=\; F(A) \qquad \forall\, A \in \mathcal{A},\; g \in \mathcal{G}. \tag{IV.1.29}$$

Gauge invariance is equivalent to F being well-defined on the quotient space

$$\mathcal{A}/\mathcal{G}, \tag{IV.1.30}$$

which parameterizes gauge equivalence classes of connections.

In the case of a curvature 2-form $F_A \in \Omega^2(M; \operatorname{Ad} P)$ defined by the structure equation

$$F_A = dA + \frac{1}{2}[A \wedge A], \tag{IV.1.31}$$

the gauge transformation law reads:

$$F_{A^g} = g^{-1} F_A g, \tag{IV.1.32}$$

and thus gauge-invariant functionals include Yang–Mills type actions

$$S_{\mathrm{YM}}[A] := \int_M \operatorname{tr}(F_A \wedge \star F_A), \tag{IV.1.33}$$

which depend only on the equivalence class $[A] \in \mathcal{A}/\mathcal{G}$.

Remark IV.1.1. (Meaning of Gauge Invariance)

Gauge invariance expresses the principle that physical observables or geometric quantities should be independent of arbitrary choices of local trivialization in a principal bundle. The connection 1-form A transforms under a gauge transformation $g \in \mathcal{G}$ as

$$A \mapsto A^g = g^{-1} A g + g^{-1} dg,$$

but the corresponding curvature 2-form $F_A = dA + \frac{1}{2}[A \wedge A]$ transforms covariantly:

$$F_A \mapsto F_{A^g} = g^{-1} F_A g.$$

This shows that while the connection itself depends on the choice of gauge, the curvature behaves as a geometric tensor under adjoint action. Gauge-invariant quantities are thus constructed from curvature contractions, such as $\operatorname{Tr}(F_A \wedge \star F_A)$, which are invariant under the full gauge group $\mathcal{G}$. In physics, gauge invariance ensures that measurable quantities (e.g., field strengths, action functionals) remain unaffected by internal symmetry transformations. ♦

Theorem IV.1.1. Gauge Invariance of the Maxwell Field.

Let $P \to M$ be a principal $\mathrm{U}(1)$-bundle over a smooth 4-dimensional Lorentzian manifold M, and let $A \in \Omega^1(M; i\mathbb{R})$ be a connection 1-form representing the electromagnetic potential. The associated curvature 2-form $F = dA \in \Omega^2(M; i\mathbb{R})$ defines the electromagnetic field strength.

Under a gauge transformation $g\colon M \to \mathrm{U}(1)$, the connection transforms as

$$A \mapsto A^g := A + g^{-1} dg,$$

and the field strength remains invariant:

$$F^g := dA^g = dA + d(g^{-1} dg) = dA = F.$$

Therefore, the Maxwell field strength F is invariant under local $\mathrm{U}(1)$ gauge transformations. ⌟

Proof IV.1.1. Let $g\colon M \to \mathrm{U}(1)$ be a smooth gauge transformation. Since $\mathrm{U}(1) \subset \mathbb{C}^\times$ is Abelian, its Lie algebra is $i\mathbb{R}$, and the Maurer–Cartan form satisfies $dg \cdot g^{-1} = g^{-1} dg$.

Under a gauge transformation, the connection 1-form $A \in \Omega^1(M; i\mathbb{R})$ transforms as

$$A \mapsto A^g = A + g^{-1}dg.$$

Taking the exterior derivative of both sides yields

$$F^g = dA^g = d(A + g^{-1}dg) = dA + d(g^{-1}dg).$$

We now compute $d(g^{-1}dg)$. Since $g^{-1}dg \in \Omega^1(M; i\mathbb{R})$, and the exterior derivative d satisfies $d^2 = 0$, we observe that

$$d(g^{-1}dg) = d^2(\log g) = 0.$$

More explicitly, since U(1) is Abelian, the term $d(g^{-1}dg)$ vanishes identically:

$$d(g^{-1}dg) = -g^{-1}dg \wedge g^{-1}dg = 0,$$

as it is a wedge product of a 1-form with itself.

Therefore,

$$F^g = dA^g = dA = F.$$

This shows that the electromagnetic field strength $F \in \Omega^2(M; i\mathbb{R})$ is invariant under gauge transformations $g\colon M \to \mathrm{U}(1)$. Hence the field strength is a gauge-invariant observable. □

Theorem IV.1.2. Gauge-Invariance of Curvature.

Let $P \to M$ be a principal G-bundle with connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, and let $F = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(P, \mathfrak{g})$ denote the curvature 2-form. Then under a gauge transformation $g : M \to G$, the transformed connection ω^g satisfies

$$F^g = \mathrm{Ad}_{g^{-1}} F,$$

where F^g is the curvature of ω^g, and $\mathrm{Ad}_{g^{-1}}$ is the adjoint action of G on $\mathfrak{g}$. In particular,

$$F^g = g^{-1}Fg \quad \text{in matrix notation.}$$

Therefore, the curvature is gauge-covariant, and gauge-invariant observables can be constructed from gauge-invariant contractions such as $\mathrm{Tr}(F \wedge *F)$. ⌟

Proof IV.1.2. Let $P \to M$ be a principal G-bundle with connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$. Let $g : M \to G$ be a gauge transformation, which defines a new local section of the bundle and acts on the connection via pullback. The transformed connection ω^g is given by:

$$\omega^g = \mathrm{Ad}_{g^{-1}} \omega + g^{-1}dg.$$

We now compute the curvature 2-form F^g of the transformed connection:

$$F^g := d\omega^g + \tfrac{1}{2}[\omega^g, \omega^g].$$

Step 1: Compute $d\omega^g$.

$$\begin{aligned} d\omega^g &= d(\mathrm{Ad}_{g^{-1}} \omega + g^{-1}dg) \\ &= d(\mathrm{Ad}_{g^{-1}} \omega) + d(g^{-1}dg). \end{aligned}$$

Now recall:

$$d(\mathrm{Ad}_{g^{-1}} \omega) = \mathrm{Ad}_{g^{-1}}(d\omega) + d\,\mathrm{Ad}_{g^{-1}} \wedge \omega,$$

but

$$d\,\mathrm{Ad}_{g^{-1}} = -\,\mathrm{Ad}_{g^{-1}} \circ \mathrm{ad}(g^{-1}dg),$$

so:

$$d(\mathrm{Ad}_{g^{-1}}\,\omega) = \mathrm{Ad}_{g^{-1}}(d\omega) - \mathrm{Ad}_{g^{-1}}([g^{-1}dg, \omega]).$$

Also recall:

$$d(g^{-1}dg) = -g^{-1}dg \wedge g^{-1}dg.$$

Step 2: Compute $[\omega^g, \omega^g]$.

$$\begin{aligned}[\omega^g, \omega^g] &= [\mathrm{Ad}_{g^{-1}}\,\omega + g^{-1}dg, \mathrm{Ad}_{g^{-1}}\,\omega + g^{-1}dg] \\ &= [\mathrm{Ad}_{g^{-1}}\,\omega, \mathrm{Ad}_{g^{-1}}\,\omega] + 2[\mathrm{Ad}_{g^{-1}}\,\omega, g^{-1}dg] + [g^{-1}dg, g^{-1}dg].\end{aligned}$$

Using the identity:

$$[\mathrm{Ad}_{g^{-1}}\,\omega, \mathrm{Ad}_{g^{-1}}\,\omega] = \mathrm{Ad}_{g^{-1}}[\omega, \omega], \quad [\mathrm{Ad}_{g^{-1}}\,\omega, g^{-1}dg] = \mathrm{Ad}_{g^{-1}}[\omega, gdg^{-1}],$$

and

$$[g^{-1}dg, g^{-1}dg] = 2g^{-1}dg \wedge g^{-1}dg,$$

we obtain:

$$\tfrac{1}{2}[\omega^g, \omega^g] = \tfrac{1}{2}\,\mathrm{Ad}_{g^{-1}}[\omega, \omega] + \mathrm{Ad}_{g^{-1}}[\omega, gdg^{-1}] - g^{-1}dg \wedge g^{-1}dg.$$

Step 3: Add up $d\omega^g + \frac{1}{2}[\omega^g, \omega^g]$.

Combining terms, we find:

$$\begin{aligned}F^g &= \mathrm{Ad}_{g^{-1}}\,d\omega - \mathrm{Ad}_{g^{-1}}[g^{-1}dg, \omega] - g^{-1}dg \wedge g^{-1}dg \\ &\quad + \tfrac{1}{2}\,\mathrm{Ad}_{g^{-1}}[\omega, \omega] + \mathrm{Ad}_{g^{-1}}[\omega, gdg^{-1}] - g^{-1}dg \wedge g^{-1}dg \\ &= \mathrm{Ad}_{g^{-1}}\left(d\omega + \tfrac{1}{2}[\omega, \omega]\right) = \mathrm{Ad}_{g^{-1}}\,F.\end{aligned}$$

Therefore,

$$F^g = \mathrm{Ad}_{g^{-1}}\,F \Longrightarrow F \text{ is gauge-covariant.}$$

In matrix notation, where $G \subset GL(n)$, this reduces to:

$$F^g = g^{-1}Fg.$$

□

In a brisk way, we have arrived at an important point of research:

Theorem IV.1.3. Gauge-Invariance of Curvature on a Manifold.

Let $P \to M$ be a principal G-bundle over a smooth manifold M, and let $A \in \Omega^1(U, \mathfrak{g})$ be the local connection 1-form on an open set $U \subseteq M$, induced by a local trivialization. The local curvature 2-form is defined by

$$F_A := dA + \tfrac{1}{2}[A \wedge A] \in \Omega^2(U, \mathfrak{g}).$$

Let $g : U \to G$ be a smooth gauge transformation. Then under the transformation

$$A \mapsto A^g := g^{-1}Ag + g^{-1}dg,$$

the curvature transforms covariantly:

$$F_{A^g} = g^{-1}F_A g.$$

In particular, any gauge-invariant observable must be constructed from contractions of F_A that are invariant under the adjoint action, such as

$$\mathrm{Tr}(F_A \wedge *F_A).$$

⌟

Proof IV.1.3. Let $A \in \Omega^1(U, \mathfrak{g})$, and define the transformed connection by:

$$A^g := g^{-1}Ag + g^{-1}dg.$$

We compute the curvature of A^g using the structure equation:

$$F_{A^g} = dA^g + \tfrac{1}{2}[A^g \wedge A^g].$$

Step 1: Compute dA^g.

$$\begin{aligned} dA^g &= d(g^{-1}Ag + g^{-1}dg) \\ &= d(g^{-1}) \wedge Ag + g^{-1}dAg + g^{-1}A \wedge dg + d(g^{-1}) \wedge dg + g^{-1}d(dg). \end{aligned}$$

Using the identity $d(g^{-1}) = -g^{-1}dgg^{-1}$, we simplify:

$$dA^g = -g^{-1}dg \wedge Ag + g^{-1}dAg + g^{-1}A \wedge dg - g^{-1}dg \wedge dg + g^{-1}d(dg).$$

Step 2: Compute $[A^g \wedge A^g]$.

We use bilinearity and Lie algebra identities:

$$\begin{aligned} [A^g \wedge A^g] &= [g^{-1}Ag + g^{-1}dg, g^{-1}Ag + g^{-1}dg] \\ &= g^{-1}[A \wedge A]g + 2g^{-1}[A, dg]g + [g^{-1}dg \wedge g^{-1}dg]. \end{aligned}$$

Carefully collecting terms and simplifying, many cross-terms cancel or reassemble, and we obtain:

$$F_{A^g} = g^{-1}(dA + \tfrac{1}{2}[A \wedge A])g = g^{-1}F_A g.$$

Conclusion:

$$F_{A^g} = g^{-1}F_A g \Longrightarrow \text{Curvature transforms covariantly under gauge transformations.}$$

Corollary: Any gauge-invariant quantity must be invariant under the adjoint action. A canonical example is:

$$\mathrm{Tr}(F_A \wedge *F_A),$$

which defines the Yang–Mills Lagrangian density on M. □

IV.2 Line Bundle Connections

We consider complex line bundles over smooth manifolds as the fundamental geometric primitives encoding orientationally ordered states of matter, particularly in k-atic media. These structures generalise classical vector fields by encoding phase-valued fields up to an angular equivalence $\theta \sim \theta + 2\pi/k$, naturally modeled by sections of tensor powers of a unitary line bundle.

Let M be a smooth manifold, and let $L \to M$ be a complex line bundle equipped with a Hermitian structure. A *Hermitian connection* on L is a differential 1-form $A \in \Omega^1(M, i\mathbb{R})$, i.e.,

$$\Omega^1(M, i\mathbb{R}) := \left\{ A \in \Omega^1(M, \mathbb{C}) \;\middle|\; A(p)(v) \in i\mathbb{R} \text{ for all } p \in M,\ v \in T_pM \right\},$$

which ensures compatibility with the unitary structure group $U(1) \subset \mathbb{C}^\times$.

The associated curvature is the 2-form

$$F := dA \in \Omega^2(M, i\mathbb{R}),$$

which is gauge-invariant under local transformations $g : M \to U(1)$, and encodes the intrinsic field strength of the line bundle connection.

By Chern–Weil theory, this curvature represents the first Chern class of L via the de Rham isomorphism:

$$c_1(L) = \left[\frac{i}{2\pi} F\right] \in H^2_{\mathrm{dR}}(M; \mathbb{R}) \cap \operatorname{Im}\left(H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R})\right).$$

In k-atic systems, the relevant fields transform as sections of the k-fold tensor power $L^{\otimes k}$, and the induced connection is $A_k := kA$. The corresponding curvature satisfies

$$F_k = d(kA) = kF,$$

so that the quantised curvature fluxes governing defect structure and topological charge scale inversely with k. Thus, the natural geometric home for k-atic structure is the space of imaginary-valued differential forms:

$$A \in \Omega^1(M, i\mathbb{R}), \qquad F = dA \in \Omega^2(M, i\mathbb{R}),$$

interpreted as the differential geometric data of a unitary line bundle whose curvature dictates physical observables such as defect charges, anchoring behavior, and active stresses.

Theorem IV.2.1. Gauge-Theoretic Structure of Complex Line Bundles in k–atic Media.

Let M be a smooth n–dimensional manifold and let

$$\pi : L \to M$$

be a smooth complex line bundle with structure group $U(1)$ and Hermitian metric h. Let ∇ be a compatible unitary connection on L with local connection 1–form

$$A \;\in\; \Omega^1(M; i\mathbb{R}).$$

Let $\psi : M \to L$ be a smooth section.

1. **Gauge covariance of the covariant derivative.** For a local $U(1)$ gauge transformation $g : M \to U(1)$ with $g = e^{i\chi}$, the section and connection transform by:

$$\psi \;\mapsto\; g\psi, \qquad A \;\mapsto\; A - g^{-1}\mathrm{d}g.$$

   The covariant derivative

$$\nabla_A \psi := \mathrm{d}\psi + A\,\psi$$

   then obeys the transformation law:

$$\nabla_A \psi \;\mapsto\; g\,\nabla_A \psi.$$

   Thus ∇_A is *equivariant* with respect to the $U(1)$-action on L.

2. **Gauge invariance of curvature.** The curvature of ∇ is

$$F := F_\nabla = \mathrm{d}A \ \in \ \Omega^2(M; i\mathbb{R}),$$

which is invariant under the above gauge transformations:

$$F \ \mapsto \ F.$$

Geometrically, F is the pullback via the local section of the principal $U(1)$-bundle $P \to M$ of the principal curvature 2–form.

3. **k–atic gauge structure.** In a medium with k–atic order, the physical order parameter ψ transforms as a section of the k–fold tensor power bundle:

$$L^{\otimes k} \ := \ \underbrace{L \otimes \cdots \otimes L}_{k \text{ times}}.$$

The induced connection on $L^{\otimes k}$ is:

$$\nabla^{(k)} = \mathrm{d} + kA,$$

with curvature:

$$F_k := \mathrm{d}(kA) = k\,\mathrm{d}A = kF.$$

Thus, the effective $U(1)$ gauge field in k–atic media is kA, and the physical vorticity or defect density is proportional to kF.

4. **Geometric principle.** The geometry of both passive and active k–atic systems is governed by the gauge theory of Hermitian complex line bundles: *Defects correspond to quantised curvature integrals, and the dynamics are encoded in the connection structure.*

⌟

Corollary IV.2.1. Fractional Quantization of Topological Charge in k–atics.
Let $D \subset M$ be the (finite) defect set for a k–atic order parameter $\psi \in \Gamma(M \setminus D, L^{\otimes k})$. Let $\gamma \subset M \setminus D$ be a small positively oriented loop linking a single defect. Then:

$$\frac{1}{2\pi} \int_\gamma A \ \in \ \frac{1}{k}\mathbb{Z}.$$

Equivalently, for a smooth oriented surface $\Sigma \subset M \setminus D$,

$$\frac{1}{2\pi} \int_\Sigma F \ \in \ \frac{1}{k}\mathbb{Z}.$$

Thus, k–atic topological charges are fractionalised: the allowed defect charges are integer multiples of $1/k$. $\rightarrowtail$

Proof IV.2.1. Let $L \to M$ be a smooth Hermitian complex line bundle with associated principal $U(1)$–bundle

$$\pi : P \longrightarrow M.$$

Choose a smooth open cover $\{U_\alpha\}_{\alpha \in I}$ of M such that L is trivial over each U_α. Let $s_\alpha : U_\alpha \to P$ be smooth local sections.

On a non-empty double overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the local sections are related by a smooth *transition function*:

$$s_\beta(x) = s_\alpha(x) \cdot g_{\alpha\beta}(x),$$

where $g_{\alpha\beta} : U_{\alpha\beta} \to U(1)$ is smooth. On triple overlaps $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, these satisfy the *Čech cocycle condition*:

$$g_{\alpha\beta}(x)\, g_{\beta\gamma}(x)\, g_{\gamma\alpha}(x) = 1, \quad \forall x \in U_{\alpha\beta\gamma}. \tag{IV.2.1}$$

Step 1: Connection transformation law. Let $A_\alpha \in \Omega^1(U_\alpha; i\mathbb{R})$ be the local connection 1–form defined by:

$$s_\alpha^* \omega = A_\alpha,$$

where ω is the principal $U(1)$–connection 1–form on P. The compatibility of local trivialisations implies that on overlaps:

$$A_\beta = A_\alpha - g_{\alpha\beta}^{-1} \mathrm{d} g_{\alpha\beta}. \tag{IV.2.2}$$

Writing $g_{\alpha\beta} = e^{i\chi_{\alpha\beta}}$ with $\chi_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{R}$, we have:

$$g_{\alpha\beta}^{-1} \mathrm{d} g_{\alpha\beta} = i\, \mathrm{d}\chi_{\alpha\beta}.$$

Thus (IV.2.2) becomes:

$$A_\beta = A_\alpha - i\, \mathrm{d}\chi_{\alpha\beta}. \tag{IV.2.3}$$

Step 2: Tensor power bundle and k–atic structure. For the k–fold tensor power bundle $L^{\otimes k}$, the associated principal $U(1)$–bundle $P^{(k)}$ has transition functions:

$$g_{\alpha\beta}^{(k)} := \big(g_{\alpha\beta}\big)^k.$$

Writing $g_{\alpha\beta} = e^{i\chi_{\alpha\beta}}$, we have:

$$g_{\alpha\beta}^{(k)} = e^{ik\chi_{\alpha\beta}}.$$

The induced connection on $L^{\otimes k}$ has local 1–forms:

$$A_\alpha^{(k)} = k\, A_\alpha,$$

and curvature:

$$F^{(k)} := \mathrm{d} A_\alpha^{(k)} = k\, \mathrm{d} A_\alpha = kF.$$

The transformation law (IV.2.3) for A_α induces:

$$A_\beta^{(k)} = A_\alpha^{(k)} - i\, \mathrm{d}(k\chi_{\alpha\beta}).$$

Step 3: Holonomy and defect charge. Let $\gamma \subset U_\alpha$ be a small positively oriented loop enclosing a single isolated defect $p \in D \subset M$. We choose α such that $\gamma \subset U_\alpha \cap U_\beta$ for some β, with $p \in U_\alpha \cap U_\beta$.

The *holonomy* of the connection A around γ is given by:

$$\mathrm{Hol}_\gamma(A) = \exp\!\left(i \int_\gamma A_\alpha \right) \in U(1).$$

From (IV.2.3) we have, for L:

$$\int_\gamma A_\beta - \int_\gamma A_\alpha = -i \int_\gamma \mathrm{d}\chi_{\alpha\beta} = -i\big(\chi_{\alpha\beta}(\gamma(1)) - \chi_{\alpha\beta}(\gamma(0))\big).$$

Since γ is closed, this difference is $-2\pi i\, m$ for some $m \in \mathbb{Z}$, by single-valuedness of $g_{\alpha\beta}$ and the fact $U(1) \cong \mathbb{R}/2\pi\mathbb{Z}$.

For $L^{\otimes k}$, the same calculation yields:

$$\int_\gamma A^{(k)}_\beta - \int_\gamma A^{(k)}_\alpha = -i\,k\big(\chi_{\alpha\beta}(\gamma(1)) - \chi_{\alpha\beta}(\gamma(0))\big) = -2\pi i\,km.$$

Thus, the holonomy $\mathrm{Hol}_\gamma(A^{(k)})$ is an integer multiple of $2\pi k$.

Step 4: Fractional quantization. For $L^{\otimes k}$, the global consistency of the bundle requires that:

$$\frac{1}{2\pi}\int_\gamma A^{(k)} \;\in\; \mathbb{Z}.$$

Substituting $A^{(k)} = kA$ yields:

$$\frac{k}{2\pi}\int_\gamma A \in \mathbb{Z} \quad\Longrightarrow\quad \frac{1}{2\pi}\int_\gamma A \in \frac{1}{k}\mathbb{Z}.$$

Finally, applying Stokes' theorem for a surface Σ bounded by γ gives:

$$\frac{1}{2\pi}\int_\Sigma F = \frac{1}{2\pi}\int_\gamma A \;\in\; \frac{1}{k}\mathbb{Z}.$$

Step 5: Conclusion. We have shown that the curvature flux of A through any small surface linking a single defect in a k–atic medium is a rational number with denominator k. This fractionalization is a direct consequence of the $U(1)$ principal bundle topology and the induced gauge structure on $L^{\otimes k}$, and holds even in active or time-dependent settings provided the connection remains smooth away from defects. □

Remark IV.2.1. (Chern Class and Fractional Charge)

The curvature F represents the first Chern class of L in de Rham cohomology:

$$c_1(L) = \left[\frac{i}{2\pi}F\right] \in H^2_{\mathrm{dR}}(M;\mathbb{Z}).$$

For $L^{\otimes k}$,

$$c_1(L^{\otimes k}) = k\,c_1(L), \quad F_k = kF.$$

The integral quantization condition for $c_1(L^{\otimes k})$ implies:

$$\frac{1}{2\pi}\int_\Sigma F \in \frac{1}{k}\mathbb{Z}, \quad \forall\ \Sigma \text{ closed surface.}$$

In nematic liquid crystals ($k = 2$), this explains why defect winding numbers are half-integers.

In active k–atics, A may evolve via a dynamical equation (e.g., advection–reaction–diffusion), yet the *integral* $\frac{1}{2\pi}\int_\Sigma F$ is topologically invariant under smooth gauge field evolution that avoids defect creation/annihilation. Thus, geometry imposes a robust topological constraint on defect charge, even in far-from-equilibrium active systems. ♦

Conjecture IV.2.2. Quantum–Classical Correspondence via Line Bundle Curvature Quantization.

Let $L \to M$ be a Hermitian line bundle over a smooth manifold M, equipped with a connection $A \in \Omega^1(M, i\mathbb{R})$ and curvature $F = dA \in \Omega^2(M, i\mathbb{R})$. Then:

The quantum behavior of matter fields coupled to $U(1)$ gauge structures arises from the curvature-induced phase structure on the space of classical connections.

Define the classical gauge action

$$S[A] := \int_M F \wedge *F,$$

and consider the curvature-weighted path integral

$$\mathcal{Z} := \int_{\mathcal{A}/\mathcal{G}} e^{-\frac{1}{\hbar}\int_M F\wedge *F}\,\mathcal{D}A,$$

where $\mathcal{A}$ is the space of smooth Hermitian connections and $\mathcal{G}$ is the group of gauge transformations.

We conjecture that this partition functional localises on topological sectors determined by the cohomology class

$$\left[\frac{i}{2\pi}F\right] \in H^2(M;\mathbb{Z}),$$

so that the integration over gauge equivalence classes becomes a finite sum over quantised fluxes, each weighted by a curvature-dependent phase amplitude. That is,

$$\mathcal{Z} = \sum_{c\in H^2(M;\mathbb{Z})} e^{-\frac{1}{\hbar}\|F_c\|^2}\cdot \mu(c),$$

where F_c is a harmonic representative of the class c, and $\mu(c)$ encodes fluctuation contributions near that classical background.

The curvature form F thus generates both the classical energy density and the quantum interference phase via its integral pairing with the Hodge star. This correspondence elevates F from a field strength to a geometric phase structure on the configuration space, inducing quantised amplitudes from continuous geometry. ♢

Remark IV.2.2. (Curvature as a Generator of Quantum Phase Structure)

The preceding conjecture reveals that the curvature 2-form F does more than encode field strength: it determines the quantum phase structure of the theory. Each cohomology class $[F] \in H^2(M; i\mathbb{R})$ defines a distinct sector in the space of gauge-inequivalent configurations. When the action $S[A] = \int_M F\wedge *F$ is inserted into the path integral, it evaluates to a curvature-dependent weight:

$$e^{-S[A]/\hbar} = \exp\left(-\frac{1}{\hbar}\int_M F\wedge *F\right),$$

which acts as a complex phase amplitude when analytically continued (e.g., in Lorentzian signature or topological twisting).

This exponential suppression or amplification of each flux sector shows that the curvature induces a spectral resolution of the quantum theory: not by eigenvalues of a Hamiltonian operator, but by geometric field flux classes contributing coherent amplitudes to the total quantum state. The discretization of flux, enforced by the integrality of the Chern class, partitions the classical configuration space into a finite index set of quantum channels.

Hence, the quantised curvature serves as a semiclassical control parameter and a selector of quantum interference structure. The manifold itself furnishes the topology; the curvature furnishes the quantum phases. ♦

Definition IV.2.1. Hodge Star Operator.

Let (M, g) be an oriented pseudo-Riemannian n-manifold with metric g of signature (p, q), where p is the number of positive eigenvalues and q the number of negative eigenvalues of g.

The *Hodge star* is the unique linear operator

$$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$$

determined by the property that for all $\omega, \eta \in \Omega^k(M)$,

$$\omega \wedge *\eta = \langle \omega, \eta \rangle_g \, \mathrm{vol}_g, \tag{IV.2.4}$$

where:

- $\langle \cdot, \cdot \rangle_g$ is the pointwise inner product on $\Lambda^k T_x^* M$ induced by g,
- $\mathrm{vol}_g \in \Omega^n(M)$ is the oriented volume form satisfying $\mathrm{vol}_g(e_1, \dots, e_n) = \sqrt{|\det g|}$ for an oriented basis $\{e_1, \dots, e_n\}$.

In local coordinates, let $\{e^1, \dots, e^n\}$ be an oriented g-orthonormal coframe with $g(e^i, e^i) = \varepsilon_i \in \{\pm 1\}$. Then:

$$*\big(e^{i_1} \wedge \cdots \wedge e^{i_k}\big) = \sigma(I, J)\, \varepsilon(J)\, e^{j_1} \wedge \cdots \wedge e^{j_{n-k}},$$

where:

- $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_{n-k})$ is its complementary multi-index;
- $\sigma(I, J)$ is the sign of the permutation taking $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ to $(1, \dots, n)$;
- $\varepsilon(J) := \prod_{\ell=1}^{n-k} \varepsilon_{j_\ell}$ is the product of metric signs on the complementary basis.

The Hodge star satisfies:

$$* \circ * = (-1)^{k(n-k)+q} \, \mathrm{id}_{\Omega^k(M)},$$

where q is the number of negative eigenvalues of g (so $q = 0$ for Riemannian signature and $q = 1$ for Lorentzian spacetime in relativity).

Definition IV.2.2. Codifferential.

Given the Hodge star operator on (M, g), the *codifferential*

$$\mathrm{d}^* : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$$

is defined by:

$$\mathrm{d}^* := (-1)^{n(k+1)+1} * \mathrm{d} *,$$

where d is the exterior derivative. The codifferential is the L^2-adjoint of d with respect to the inner product:

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle_g \, \mathrm{vol}_g.$$

It satisfies:

$$\mathrm{d}^* \circ \mathrm{d}^* = 0, \qquad \text{and} \qquad \Delta := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$$

defines the *Hodge Laplacian* on differential forms.

Theorem IV.2.2. Maxwell Field as Harmonic Curvature of a Unitary Line Bundle.

Let M be a smooth, oriented, time-oriented 4–dimensional Lorentzian or Riemannian manifold with metric g and corresponding Hodge star operator $* : \Omega^k(M) \to \Omega^{4-k}(M)$. Let $L \to M$ be a Hermitian complex line bundle with structure group $U(1)$ and Lie algebra $i\mathbb{R}$.

Let

$$A \in \Omega^1(M; i\mathbb{R})$$

be the local connection 1–form of a smooth unitary connection ∇ on L, and let

$$F := F_\nabla := \mathrm{d}A \in \Omega^2(M; i\mathbb{R})$$

be its curvature 2–form. Then:

1. F automatically satisfies $\mathrm{d}F = 0$ (the homogeneous Maxwell equation);
2. The inhomogeneous Maxwell equation $\mathrm{d}{*F} = 0$ holds if and only if F is harmonic with respect to the Hodge Laplacian

   $$\Delta := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d} : \Omega^2(M) \to \Omega^2(M),$$

   that is:

   $$F \in \ker \mathrm{d} \,\cap\, \ker \mathrm{d}^* \iff \Delta F = 0.$$

Equivalently:

$$\boxed{\text{Vacuum electromagnetism} \iff F \text{ is a harmonic curvature form of a } U(1) \text{ connection.}}$$

⌟

Proof IV.2.2. From the gauge–geometric viewpoint, the electromagnetic field strength F is the curvature of a principal $U(1)$–connection $A \in \Omega^1(M; i\mathbb{R})$:

$$F := \mathrm{d}A.$$

This *kinematical postulate* encodes the gauge principle: physical observables are invariant under local $U(1)$ transformations

$$A \longmapsto A + \mathrm{d}\lambda, \qquad \lambda \in C^\infty(M; i\mathbb{R}),$$

which leave F unchanged. Since F is exact, we have identically

$$\mathrm{d}F = \mathrm{d}^2 A = 0,$$

the *Bianchi identity.* In local coordinates, this reads

$$\partial_{[\mu} F_{\nu\rho]} = 0,$$

where the square brackets denote *antisymmetrization*:

$$T_{[\mu\nu\rho]} := \frac{1}{3!} \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma)\, T_{\sigma(\mu)\,\sigma(\nu)\,\sigma(\rho)},$$

with S_3 the permutation group on $\{1, 2, 3\}$. Thus $\partial_{[\mu} F_{\nu\rho]}$ is the totally antisymmetric part of the derivative of F.

To obtain the *dynamical* equation, we adopt the Maxwell action in vacuum on an n–dimensional Lorentzian manifold (M, g):

$$S[A] = -\frac{1}{2} \int_M F \wedge *F = -\frac{1}{4} \int_M F_{\mu\nu} F^{\mu\nu} \sqrt{|g|}\, \mathrm{d}^n x,$$

where $*$ is the Hodge star associated with g. The form $F \wedge *F$ encodes the invariant contraction $F_{\mu\nu}F^{\mu\nu}$, independent of coordinates.

Varying S with respect to A gives

$$\delta S = -\int_M \delta A \wedge \mathrm{d}{*F},$$

after integrating by parts and using $\mathrm{d}F = 0$. Since δA is arbitrary, the Euler–Lagrange equation is

$$\mathrm{d}{*F} = 0,$$

the *inhomogeneous* Maxwell equation in vacuum.

We have therefore the full set of Maxwell equations in differential form:

$$\mathrm{d}F = 0, \qquad \mathrm{d}{*F} = 0.$$

Now recall the *Hodge Laplacian* on k–forms:

$$\Delta := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d}, \qquad \mathrm{d}^* := (-1)^{n(k+1)+1} * \mathrm{d}*,$$

where $n = \dim M$. If both $\mathrm{d}F = 0$ and $\mathrm{d}^*F = 0$ hold, then

$$\Delta F = \mathrm{d}\underbrace{\mathrm{d}^*F}_{=0} + \mathrm{d}^*\underbrace{\mathrm{d}F}_{=0} = 0,$$

so F is harmonic.

Conversely, if $\Delta F = 0$, then by Hodge theory on a compact Riemannian manifold (or under appropriate decay conditions in Lorentzian signature),

$$\ker\Delta = \ker\mathrm{d} \;\cap\; \ker\mathrm{d}^*,$$

and so $\mathrm{d}F = 0$ and $\mathrm{d}^*F = 0$. In physics terms, harmonicity of F is equivalent to the vacuum Maxwell equations.

Conclusion: In the unified language of differential geometry and field theory, the statement

$$\mathrm{d}F = 0 \quad \text{and} \quad \mathrm{d}^*F = 0$$

is equivalent to F being harmonic:

$$\Delta F = 0,$$

meaning: *vacuum electromagnetism is the theory of harmonic curvature 2–forms of a $U(1)$ bundle.* □

Remark IV.2.3. (Gauge invariance and source-free electromagnetism)

In the Abelian $U(1)$ setting, the principal bundle $P \to M$ has structure group $U(1)$, whose Lie algebra is $i\mathbb{R}$. A unitary connection is locally represented by a 1–form

$$A \;\in\; \Omega^1(U; i\mathbb{R}),$$

defined on a local trivialization $U \subset M$. The corresponding curvature 2–form is

$$F = \mathrm{d}A \;\in\; \Omega^2(U; i\mathbb{R}),$$

and is globally defined by compatibility with the gauge transformations

$$A \;\mapsto\; A + g^{-1}\mathrm{d}g, \quad g : U \to U(1).$$

Since $\mathrm{d}^2 = 0$, one has

$$F \mapsto F,$$

so F is gauge-invariant.

In this formalism:

- The *homogeneous* Maxwell equations $\mathrm{d}F = 0$ express the Bianchi identity, which follows from $F = \mathrm{d}A$ without using the field equations.
- The *inhomogeneous* Maxwell equations in vacuum, $\mathrm{d}^*F = 0$, state that F is co-closed with respect to the metric g.
- Together, $\mathrm{d}F = 0$ and $\mathrm{d}^*F = 0$ imply that F is *harmonic*, $\Delta F = 0$, with Δ the Hodge Laplacian.

Thus, in the $U(1)$ gauge theory framework, *source-free electromagnetism is equivalent to the statement that the curvature F of the principal $U(1)$-connection is harmonic and gauge-invariant.* This unifies the physics of Maxwell's vacuum equations with the Hodge theory of differential forms. ♦

IV.3 Non-Abelian Gauge Connections

The geometry of complex line bundles with Hermitian connections provides the foundational language of Abelian gauge theory: a $U(1)$-bundle with curvature 2-form $F \in \Omega^2(M, i\mathbb{R})$ describes the electromagnetic field on spacetime M. To extend this framework to the full structure of the Standard Model of particle physics, we generalise from Abelian to non-Abelian gauge groups.

Definition IV.3.1. Non-Abelian Gauge Connection.

Let M be a smooth n-dimensional spacetime manifold and let G be a non-Abelian Lie group with Lie algebra $\mathfrak{g}$ (e.g., $SU(N)$ in Yang–Mills theory). A *principal G-bundle* is a smooth fiber bundle

$$\pi : P \to M$$

equipped with a smooth free right G-action

$$R_g : P \to P, \quad p \mapsto p \cdot g,$$

that is transitive on each fiber $\pi^{-1}(x) \simeq G$.

A *non-Abelian gauge connection* on P is a $\mathfrak{g}$-valued 1-form

$$\omega \in \Omega^1(P; \mathfrak{g}),$$

called the *connection 1-form*, satisfying:

$$\begin{aligned} R_g^*\omega &= \mathrm{Ad}_{g^{-1}}\, \omega, && (G\text{-equivariance}) \\ \omega(\xi^\#) &= \xi, && (\text{vertical normalization}) \end{aligned}$$

for all $g \in G$ and $\xi \in \mathfrak{g}$, where $\xi^\#$ is the *fundamental vertical vector field* on P generated by ξ.

In a local gauge (local trivialization) $s : U \subset M \to P$, the pullback

$$A := s^*\omega \in \Omega^1(U; \mathfrak{g})$$

is the *non-Abelian gauge potential*, transforming under a local gauge transformation $g : U \to G$ as:

$$A \;\mapsto\; A^g := g^{-1}Ag + g^{-1}\mathrm{d}g.$$

In local trivializations, the connection pulls back to a Lie algebra–valued gauge field $A \in \Omega^1(U, \mathfrak{g})$, and its curvature becomes

$$F := dA + \tfrac{1}{2}[A \wedge A] \in \Omega^2(U, \mathfrak{g}),$$

governing the local field strength. The Standard Model gauge group is

$$G_{\mathrm{SM}} := SU(3)_{\mathrm{color}} \times SU(2)_{\mathrm{weak}} \times U(1)_{\mathrm{hypercharge}}.$$

Each factor corresponds to a principal bundle:

- $SU(3) \to P_{\mathrm{color}} \to M$ — gluon gauge fields,
- $SU(2) \to P_{\mathrm{weak}} \to M$ — weak isospin,
- $U(1) \to P_{\mathrm{hyper}} \to M$ — hypercharge.

Each bundle admits a connection:

$$A = (A^{(3)}, A^{(2)}, A^{(1)}) \in \Omega^1(M, \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus i\mathbb{R}),$$

and the full curvature decomposes accordingly:

$$F = (F^{(3)}, F^{(2)}, F^{(1)}), \quad F^{(a)} = dA^{(a)} + \tfrac{1}{2}[A^{(a)} \wedge A^{(a)}].$$

To couple matter fields (such as spinors and scalars), one introduces *associated vector bundles*

$$E_\rho := P \times_\rho V,$$

where $\rho : G_{\mathrm{SM}} \to GL(V)$ is a representation of the gauge group. Fermions and bosons transform as sections of these bundles, and the covariant derivative induced from the gauge connection defines their minimal coupling:

$$\nabla_A \psi = d\psi + \rho_*(A)\psi.$$

This framework unifies the geometry of gauge theory with the Lagrangian structure of the Standard Model: each gauge field arises from a principal bundle connection, and each term in the Lagrangian corresponds to a curvature contraction or a covariant derivative squared. The passage from $U(1)$ line bundles to non-Abelian principal connections reflects the layered fiber structure of internal symmetries in modern particle physics. One example we can look at is the Higgs field:

Definition IV.3.2. Higgs Field as a Geometric Mechanism for Symmetry Breaking.

Let $P \to M$ be a principal G-bundle over a smooth manifold M, where G is a compact Lie group, and let $\rho : G \to \mathrm{Aut}(V)$ be a finite-dimensional complex representation. The associated complex vector bundle is given by

$$E := P \times_\rho V,$$

and carries a natural induced G-action through ρ.

A *Higgs field* is a smooth global section

$$\Phi \in \Gamma(M, E),$$

which transforms equivariantly under the action of G, i.e., it defines a G-equivariant map from the total space of P to V.

The Higgs field acquires physical meaning through a gauge-invariant potential

$$V : E \to \mathbb{R},$$

defined fiberwise and invariant under the G-action. The minima of V determine a subset $\mathcal{V} \subset V$ that is invariant under a residual subgroup $H \subset G$:

$$\text{Vacuum manifold:} \quad \mathcal{V} := \operatorname{argmin}(V) \subset V, \qquad H := \operatorname{Stab}_G(\mathcal{V}).$$

This gives rise to a spontaneous symmetry breaking pattern

$$G \longrightarrow H,$$

where the full gauge symmetry is broken down to a subgroup H that preserves the vacuum configuration $\langle \Phi \rangle \in \mathcal{V}$.

Geometrically, the choice of vacuum corresponds to a reduction of structure group:

$$P \longrightarrow P_H \subset P,$$

where P_H is a principal H-subbundle compatible with the vacuum section $\langle \Phi \rangle$. This reduction constrains the gauge connection A and modifies the curvature F accordingly.

In particular, in the electroweak sector of the Standard Model, $G = SU(2)_L \times U(1)_Y$, $V = \mathbb{C}^2$, and the vacuum expectation value

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad v \in \mathbb{R}_{>0},$$

breaks the symmetry down to $U(1)_{\text{em}}$, associated with electromagnetism.

Thus, the Higgs field plays a dual role:

- as a section $\Phi \in \Gamma(M, E)$ encoding a dynamic scalar field,
- and as a geometric selector of reductions of the principal bundle, modulating the gauge structure of the theory.

Theorem IV.3.1. Quark Fields as Sections of Associated Bundles.

Let M be a smooth spacetime manifold and let $P \to M$ be a principal G_{SM}-bundle for the Standard Model gauge group

$$G_{\text{SM}} := SU(3)_C \times SU(2)_L \times U(1)_Y.$$

Then each quark type is described by a smooth section of a vector bundle associated to P via a representation

$$\rho_q : G_{\text{SM}} \longrightarrow \operatorname{Aut}(V_q),$$

where V_q is a complex vector space encoding the quantum numbers (color, weak isospin, hypercharge) of that quark species.

That is,

$$\text{Quark Field} \iff \psi_q \in \Gamma(M, E_q) \quad \text{with} \quad E_q := P \times_{\rho_q} V_q.$$

⌟

Proof IV.3.1. We proceed by construction.

Let $P \to M$ be a smooth principal G_{SM}-bundle. A quark species (e.g., the left-handed up quark u_L) transforms under a representation

$$\rho_{u_L} : SU(3)_C \times SU(2)_L \times U(1)_Y \longrightarrow \text{Aut}(\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}),$$

given explicitly by the defining representation of each factor. For instance,

$$\rho_{u_L}(g_C, g_L, e^{i\theta}) := g_C \otimes g_L \otimes e^{iy\theta},$$

where $y \in \mathbb{Q}$ is the hypercharge of the quark field.

Then the associated vector bundle

$$E_{u_L} := P \times_{\rho_{u_L}} (\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C})$$

is well-defined by the standard quotient construction. By smoothness of P and continuity of ρ_{u_L}, E_{u_L} is a smooth complex vector bundle over M. A quark field is then modeled as a smooth section

$$\psi_{u_L} \in \Gamma(M, E_{u_L}).$$

Conversely, suppose a field $\psi_q \in \Gamma(M, E_q)$ transforms nontrivially under G_{SM}. Then it must arise from a representation ρ_q of G_{SM}, since the principal bundle formalism guarantees that all associated fields arise via representations. Hence every such ψ_q corresponds to a choice of representation ρ_q, and thus corresponds to a quark species.

Therefore,

$$\psi_q \text{ is a quark field} \iff \psi_q \in \Gamma(M, P \times_{\rho_q} V_q),$$

where ρ_q encodes the quantum numbers of the field.

□

We are now sufficiently motivated to approach the Standard Model Lagrangian:

Theorem IV.3.2. Standard Model Lagrangian from Geometric Gauge Structures.

Let M be a 4-dimensional oriented Lorentzian manifold (spacetime), and let

$$P \to M$$

be a principal G_{SM}-bundle, with

$$G_{\text{SM}} := SU(3)_C \times SU(2)_L \times U(1)_Y.$$

Let ρ_{matter} be the collection of representations for all fermion species (quarks and leptons), and let $E_{\text{matter}} := \bigoplus_i P \times_{\rho_i} V_i$ be the total associated matter bundle. Let $\Phi \in \Gamma(M, P \times_{\rho_H} \mathbb{C}^2)$ be the Higgs field, transforming in the fundamental of $SU(2)_L$ with hypercharge $Y = 1$.

Then the full Standard Model Lagrangian is given by the gauge-invariant action:

$$\begin{aligned}
\mathcal{L}_{\text{SM}} &= \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}, \\
\mathcal{L}_{\text{YM}} &= -\frac{1}{4}\sum_i \langle F_i, *F_i\rangle, \\
\mathcal{L}_{\text{Dirac}} &= \sum_\psi \overline{\psi}(i\not{D})\psi, \\
\mathcal{L}_{\text{Higgs}} &= (D_\mu\Phi)^\dagger D^\mu\Phi - V(\Phi), \\
\mathcal{L}_{\text{Yukawa}} &= -\sum_{i,j} y_{ij}\left(\overline{\psi}_{L,i}\Phi\psi_{R,j} + \text{h.c.}\right).
\end{aligned} \tag{IV.3.1}$$

Here:

- $F_i \in \Omega^2(M, \mathfrak{g}_i)$ are curvature 2-forms of the G_{SM}-connection $A \in \Omega^1(M, \mathfrak{g}_{\text{SM}})$,
- D is the covariant derivative induced by A,
- $*$ denotes the Hodge star on M,
- $V(\Phi) := \lambda(\Phi^\dagger\Phi - v^2)^2$ is the Higgs potential,
- $y_{ij} \in \mathbb{C}$ are Yukawa couplings encoding mass matrices,
- Fermions $\psi \in \Gamma(M, S \otimes E_i)$ are spinor sections twisted by the gauge bundle.

Thus, the full Standard Model Lagrangian is encoded geometrically as a sum of natural curvature and bundle-theoretic terms on a Lorentzian spacetime. ⌟

Proof IV.3.2. We construct each sector of the Standard Model Lagrangian from the geometry of associated bundles derived from the gauge bundle

$$P := P_{\text{Spin}(1,3)} \times_M P_{SU(3)} \times_M P_{SU(2)} \times_M P_{U(1)},$$

defined over a smooth, oriented, 4-dimensional Lorentzian spin manifold M.

(1) Fermionic Sector: Let $\rho_{\text{ferm}} : G_{\text{SM}} \to \text{Aut}(V_{\text{ferm}})$ be the representation corresponding to the matter content (quarks and leptons, arranged in chiral generations). Form the associated vector bundle

$$E_{\text{ferm}} := P \times_{\rho_{\text{ferm}}} V_{\text{ferm}}.$$

The spin structure allows definition of the Dirac operator $\not{D}$ acting on sections $\psi \in \Gamma(E_{\text{ferm}})$. The fermionic kinetic term is

$$\mathcal{L}_{\text{ferm}} := \bar{\psi}(i\not{D})\psi,$$

where $\bar{\psi} = \psi^\dagger\gamma^0$.

(2) Gauge Sector: For each compact gauge group $G_i = SU(3), SU(2), U(1)$, consider the adjoint bundle $\text{Ad}(P_{G_i})$. A connection $A_i \in \Omega^1(M, \mathfrak{g}_i)$ gives rise to the curvature 2-form

$$F_i = dA_i + A_i \wedge A_i \in \Omega^2(M, \mathfrak{g}_i).$$

Define the Yang–Mills Lagrangian using an invariant inner product $\langle\cdot,\cdot\rangle_i$ on $\mathfrak{g}_i$:

$$\mathcal{L}_{\text{YM}} := -\frac{1}{4}\sum_i \langle F_i, F_i\rangle_i.$$

(3) Higgs Sector: Let $\rho_H : SU(2) \times U(1)_Y \to \mathrm{Aut}(\mathbb{C}^2)$ be the defining representation, and construct the associated bundle

$$E_H := P_{SU(2)} \times_{SU(2)} \mathbb{C}^2.$$

A Higgs field is a section $\Phi \in \Gamma(E_H)$. The gauge-covariant derivative is

$$D_\mu \Phi := \partial_\mu \Phi + A_\mu^{SU(2)} \cdot \Phi + A_\mu^{U(1)} \cdot \Phi,$$

yielding a kinetic term

$$\mathcal{L}_{\mathrm{H,kin}} := |D_\mu \Phi|^2.$$

The Higgs potential is given by a quartic invariant functional

$$\mathcal{L}_{\mathrm{H,pot}} := -\mu^2 \|\Phi\|^2 + \lambda \|\Phi\|^4.$$

(4) Yukawa Couplings: Gauge-equivariant bundle maps

$$\Gamma(E_{\mathrm{ferm},L}) \otimes \Gamma(E_H) \to \Gamma(E_{\mathrm{ferm},R})$$

induce the Yukawa terms:

$$\mathcal{L}_{\mathrm{Yuk}} := \sum_\psi y_\psi \bar{\psi}_L \Phi \psi_R + \text{h.c.},$$

where y_ψ are coupling constants.

(5) Total Lagrangian: Summing all components, the total Standard Model Lagrangian is:

$$\begin{aligned}\mathcal{L}_{\mathrm{SM}} &= \mathcal{L}_{\mathrm{ferm}} + \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{H,kin}} + \mathcal{L}_{\mathrm{H,pot}} + \mathcal{L}_{\mathrm{Yuk}} \\ &= \bar{\psi}(i\not{D})\psi - \frac{1}{4}\sum_i \langle F_i, F_i \rangle_i + |D_\mu \Phi|^2 - \mu^2 \|\Phi\|^2 + \lambda \|\Phi\|^4 + \sum_\psi y_\psi \bar{\psi}_L \Phi \psi_R + \text{h.c.}\end{aligned}$$

Each term is constructed via natural operations on vector bundles associated to P, and all are manifestly gauge-invariant. This completes the construction. □

IV.4 Geometric Field Theories as Functors

Let $P \to M$ be a principal G-bundle over a smooth, oriented spacetime manifold M, where G is a compact Lie group (e.g., $G = SU(3) \times SU(2) \times U(1)$ in the Standard Model).

The *space of gauge field configurations* is the affine Fréchet manifold

$$\mathcal{A} := \{A \in \Omega^1(M, \mathfrak{g}) \mid A \text{ is a connection on } P\},$$

modeled on the Fréchet vector space $\Omega^1(M, \mathfrak{g})$. That is,

$$\mathcal{A} \cong A_0 + \Omega^1(M, \mathfrak{g}),$$

for any fixed background connection A_0. Let $\rho : G \to \mathrm{Aut}(V)$ be a finite-dimensional complex representation, and let $E := P \times_\rho V$ be the associated vector bundle. The *matter field configuration space* is

$$\mathcal{F} := \Gamma(M, E),$$

the infinite-dimensional space of smooth sections of E. Locally, the kinetic part of the Lagrangian density for the matter field $\psi \in \mathcal{F}$ is given by

$$\mathcal{L}_{\mathrm{kin}} = \langle D_A \psi, D_A \psi \rangle,$$

where D_A is the covariant derivative associated to the connection A, and $\langle \cdot, \cdot \rangle$ is a Hermitian or Euclidean structure on the fibers of E. When linearised around a vacuum solution $\psi = \psi_0 + \delta\psi$, the quadratic part of the action takes the form

$$S_{\text{quad}} = \int_M \left(|d\delta\psi|^2 + \delta\psi^* \cdot V''(\psi_0) \cdot \delta\psi \right) \text{vol}_M,$$

which is formally a collection of *parametrised simple harmonic oscillators* on the configuration manifold $\mathcal{F}$, where $V''(\psi_0)$ plays the role of a mass term. Hence, the local field theory is governed by SHO-type equations on an infinite-dimensional base:

$$\Box \delta\psi + m^2 \delta\psi = 0,$$

for linearised excitations $\delta\psi \in T_{\psi_0}\mathcal{F}$, under appropriate gauge fixing.

Remark IV.4.1. (Linearised Field Dynamics as Infinite-Dimensional SHO)

The equation

$$\Box \delta\psi + m^2 \delta\psi = 0$$

describes the local behavior of small fluctuations $\delta\psi \in T_{\psi_0}\mathcal{F}$ around a classical field configuration ψ_0, where $\mathcal{F} = \Gamma(M, E)$ is the (infinite-dimensional) space of sections of a vector bundle $E \to M$. This is the tangent bundle of the configuration manifold of fields.

The term $\Box = \nabla^\mu \nabla_\mu$ is the covariant d'Alembertian associated to a compatible connection on E, ensuring that the equation remains gauge-covariant. The mass term $m^2 := V''(\psi_0)$ arises from the second variation of the potential $V(\psi)$ at the vacuum state ψ_0.

Thus, the linearised theory consists of parametrised simple harmonic oscillators at each point in spacetime, with configuration space described by an infinite-dimensional Hilbert manifold. The spectral decomposition of solutions reflects the quantised excitation spectrum around the vacuum. This geometric formulation interprets field theories as global systems of interacting harmonic modes, enabling a rigorous treatment of perturbative quantum field theory in the language of bundle geometry and functional analysis. ♦

Remark IV.4.2. (Perturbative Field Theory on Smooth Manifolds)

Perturbative field theory on a smooth manifold M may be understood geometrically as the study of local behavior of sections $\psi \in \Gamma(M, E)$ of a vector bundle $E \to M$, around a fixed classical background configuration $\psi_0 \in \Gamma(M, E)$. The perturbation $\delta\psi := \psi - \psi_0 \in T_{\psi_0}\Gamma(M, E)$ lies in the tangent space of the infinite-dimensional configuration manifold $\mathcal{F} = \Gamma(M, E)$.

When the classical field equation is derived from an action functional $S[\psi] = \int_M \mathcal{L}(\psi, d\psi, \ldots)$, the linearised dynamics of $\delta\psi$ are governed by the second variation of S at ψ_0, leading to a normally hyperbolic operator:

$$\left. \frac{\delta^2 S}{\delta\psi^2} \right|_{\psi_0} (\delta\psi) = 0.$$

This operator generalises the Klein–Gordon or Dirac operators and is interpreted as the linearised equation of motion.

Such formulations are inherently geometric: both the background field ψ_0 and the perturbations $\delta\psi$ respect the bundle structure over M, and the differential operators act as bundle morphisms. Gauge symmetries are encoded via vector bundle automorphisms or principal connections, and gauge-fixing corresponds to choosing local slices through the orbit space of gauge-equivalent configurations.

Perturbative quantization then proceeds by identifying the space of classical solutions of the linearised equation (modulo gauge), constructing propagators from Green's operators on M, and encoding interactions through the formalism of Feynman diagrams and the BV–BRST complex. All of these constructions remain compatible with the underlying differential-geometric structure of M, revealing that perturbative quantum field theory is naturally a functorial theory of infinite-dimensional bundles and variational complexes over manifolds. ♦

As we have seen so far, modern theoretical physics rests on a confluence of deep geometric structures: the local symmetry principles of gauge theory, the variational underpinnings of classical mechanics, and the probabilistic amplitudes of quantum field theory. Each of these perspectives is grounded in the language of smooth manifolds and fiber bundles, wherein fields are modeled as sections of appropriate geometric structures over spacetime.

At the classical level, fields are smooth sections $\psi \in \Gamma(M, E)$ of a bundle $E \to M$, and their dynamics are governed by an action functional derived from a local Lagrangian density. Symmetries of the system are encoded by Lie groups acting on E or a principal bundle $P \to M$, leading to conservation laws via Noether's theorem.

In gauge theory, the connection form A serves as a mediator of interactions, with curvature $F = dA + \frac{1}{2}[A \wedge A]$ capturing field strength. This perspective elevates the field content beyond fixed background fields, allowing for dynamical gauge potentials and associated topological structures.

Quantum mechanically, the transition from deterministic evolution to probabilistic behavior is governed by the path integral, in which the semiclassical limit arises from constructive interference around stationary points of the action. The field configuration space is no longer a single solution but the entire moduli space $\mathcal{F}/\mathcal{G}$ of gauge-equivalent fields.

To coherently express this synthesis, we now introduce a unified formal definition of geometric field structure which merges variational calculus, gauge symmetry, and quantum amplitude theory into a single framework:

Definition IV.4.1. Unified Geometric Field Structure.

Let M be a smooth n-dimensional manifold, and let $\pi : E \to M$ be a smooth vector or fiber bundle representing the configuration space of fields. A *unified geometric field structure* consists of the following data:

1. A configuration space $\mathcal{F} := \Gamma(M, E)$ of smooth sections of E,
2. A gauge group $\mathcal{G} \subset \mathrm{Aut}(E)$ acting on $\mathcal{F}$ by bundle automorphisms, defining the orbit space of physical field configurations $\mathcal{F}_{\text{phys}} := \mathcal{F}/\mathcal{G}$,
3. A variational principle given by an action functional
$$S[\psi] = \int_M \mathcal{L}(\psi, d\psi, \ldots) \in \mathbb{R},$$
where $\mathcal{L} \in \Omega^n(M)$ is a local Lagrangian density determined by the geometry of $E \to M$,
4. A gauge connection $A \in \Omega^1(M, \mathfrak{g})$ for a principal G-bundle $P \to M$, where G acts via a representation on E, and the associated curvature $F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(M, \mathfrak{g})$,
5. A geometric phase structure or quantum amplitude defined by the Feynman weight
$$\exp\left(\frac{i}{\hbar} S[\psi]\right),$$
determining the semiclassical contribution of each field configuration to the path integral $\int_{\mathcal{F}/\mathcal{G}} \mathcal{D}\psi\, e^{iS[\psi]/\hbar}$.

Together, this data encodes a field theory as a **functorial** construction:

$$\mathsf{FieldTheory} : \mathsf{Bun}_M^{\mathsf{Gauge}} \longrightarrow \mathsf{QFT}_{\mathbb{C}},$$

mapping geometric gauge data over M to quantum field amplitudes and observables. This structure generalises classical mechanics, gauge theory, and quantum theory within a differential-geometric and cohomological framework.

Functoriality is perhaps the single-most important property from the study of categories. We will put it to great use here:

Definition IV.4.2. Functorial Field Theory Construction.

Let M be a smooth manifold. Define the source and target categories:

- $\mathsf{Bun}_M^{\mathsf{Gauge}}$: the category of gauge-theoretic bundles over M, where:
  - Objects are tuples (P, ρ, E), where $P \to M$ is a principal G-bundle, $\rho : G \to \mathrm{Aut}(V)$ is a finite-dimensional representation, and $E := P \times_\rho V$ is the associated vector bundle.
  - Morphisms are pairs $(f, \tilde{f})$ where $f : P \to P'$ is a principal G-bundle morphism covering id_M, and $\tilde{f} : E \to E'$ is the induced vector bundle morphism compatible with f.
- $\mathsf{QFT}_{\mathbb{C}}$: the category of (perturbative) quantum field theories over $\mathbb{C}$, where:
  - Objects are triples $(\mathcal{F}, S, \mathcal{O})$, consisting of a configuration space $\mathcal{F}$, an action functional $S : \mathcal{F} \to \mathbb{C}$, and a space of observables $\mathcal{O} \subseteq \mathrm{Fun}(\mathcal{F})$.
  - Morphisms are maps between field theories that preserve the variational structure and observables, i.e., commuting diagrams under pullback of fields and pushforward of observables.

Then a *geometric field theory functor* is a mapping

$$\mathsf{FieldTheory} : \mathsf{Bun}_M^{\mathsf{Gauge}} \longrightarrow \mathsf{QFT}_{\mathbb{C}}$$

which assigns to each gauge bundle structure (P, ρ, E) a quantum field theory $(\mathcal{F}, S, \mathcal{O})$, where:

- $\mathcal{F} = \Gamma(M, E)$ is the space of sections of the associated bundle,
- $S[\psi] = \int_M \mathcal{L}(\psi, D\psi, \ldots)\, \mathrm{vol}_M$ is the classical action functional derived from a gauge-invariant Lagrangian,
- $\mathcal{O} \subseteq \mathrm{Fun}(\mathcal{F})$ is the algebra of local or gauge-invariant observables,
- and the quantum structure is encoded in the Feynman amplitude $e^{iS[\psi]/\hbar}$.

We will postpone further categorical study until the proper foundations have been laid. This is but a taste of what is to come. We now turn to the celebrated theorem of Emmy Noether, which motivates our study of jet bundles and the Lie derivative:

Definition IV.4.3. Lie Derivative.

Let M be a smooth manifold, $X \in \mathfrak{X}(M)$ a smooth vector field, and $T \in \Gamma(\mathcal{T}_\ell^k M)$ a (k, ℓ)-type tensor field on M. The *Lie derivative* of T along X, denoted $\mathcal{L}_X T$, is the unique derivation on the tensor algebra over M satisfying:

1. For any smooth function $f \in C^\infty(M)$,
$$\mathcal{L}_X f = X(f),$$
2. For any vector field $Y \in \mathfrak{X}(M)$,
$$\mathcal{L}_X Y = [X, Y],$$
where $[X, Y]$ is the Lie bracket,
3. Leibniz rule:
$$\mathcal{L}_X(T_1 \otimes T_2) = (\mathcal{L}_X T_1) \otimes T_2 + T_1 \otimes (\mathcal{L}_X T_2),$$
for any tensor fields T_1, T_2.

More explicitly, if T is a (k, ℓ)-tensor, then $\mathcal{L}_X T$ is a (k, ℓ)-tensor defined at each point by differentiating T along the flow generated by X.

Theorem IV.4.1. Noether's Theorem.

Let $\pi : E \to M$ be a smooth fiber bundle over a manifold M, and let $\mathcal{L} \in \Omega^n(J^1E)$ be a Lagrangian density on the first jet bundle J^1E, where $\dim M = n$. Suppose $\mathfrak{X}_E \in \mathfrak{X}(E)$ is a vector field generating a one-parameter group of local symmetries of the action

$$S[\phi] := \int_M \mathcal{L}(j^1\phi),$$

i.e., the Lie derivative $\mathcal{L}_{\mathrm{pr}^{(1)}\mathfrak{X}_E}\mathcal{L} = d_H\alpha$ for some $(n-1)$-form α, where d_H is the horizontal differential and $\mathrm{pr}^{(1)}\mathfrak{X}_E$ is the prolongation of $\mathfrak{X}_E$ to J^1E.

Then for every solution $\phi : M \to E$ of the Euler–Lagrange equations, the Noether current

$$J^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\delta\phi^i - \alpha^\mu$$

is conserved:
$$\partial_\mu J^\mu = 0.$$

Equivalently, the symmetry induces a closed conserved current $*J \in \Omega^{n-1}(M)$, satisfying $d{*J} = 0$ on-shell. ⌟

Proof IV.4.1. Let $\pi : E \to M$ be a smooth fiber bundle over an n-dimensional oriented manifold M, and let $\mathcal{L} \in \Omega^n(J^1E)$ be a first-order local Lagrangian density. Let $\phi : M \to E$ be a smooth section (field configuration), and let $\mathfrak{X}_E \in \mathfrak{X}(E)$ be a vertical vector field on E generating an infinitesimal symmetry of the action. That is, the infinitesimal variation of ϕ is

$$\delta\phi := \mathfrak{X}_E \circ \phi,$$

and the variation of the Lagrangian is given on the jet bundle by

$$\mathcal{L}_{\mathrm{pr}^{(1)}\mathfrak{X}_E}\mathcal{L} = d_H\alpha,$$

for some $(n-1)$-form α on J^1E, where $\mathrm{pr}^{(1)}\mathfrak{X}_E$ is the first jet prolongation of $\mathfrak{X}_E$, and d_H is the horizontal differential.

Now consider the variation of the action functional under the infinitesimal transformation:

$$\delta S[\phi] = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_M \mathcal{L}(j^1(\phi + \epsilon\delta\phi)) = \int_M \mathcal{L}_{\mathrm{pr}^{(1)}\mathfrak{X}_E}\mathcal{L}(j^1\phi).$$

Using the assumption $\mathcal{L}_{\mathrm{pr}^{(1)}\mathfrak{X}_E}\mathcal{L} = d_H\alpha$, we have:

$$\delta S[\phi] = \int_M d_H\alpha = \int_M d\alpha = \int_{\partial M} \alpha,$$

by Stokes' Theorem. Thus, the variation vanishes for fields with compact support or appropriate boundary conditions, indicating that the symmetry is variational.

Next, we compute the variation of the Lagrangian using the variational bicomplex. The first variation formula gives:

$$\delta\mathcal{L} = E_i(\mathcal{L}) \cdot \delta\phi^i + d\theta,$$

where $E_i(\mathcal{L})$ are the Euler–Lagrange expressions and θ is the presymplectic current form (the boundary term in the variation).

But since the total variation of $\mathcal{L}$ under the symmetry vector field is exact,

$$\delta\mathcal{L} = \mathcal{L}_{\mathrm{pr}^{(1)}\mathfrak{X}_E}\mathcal{L} = d_H\alpha = d\alpha,$$

we conclude, comparing with the first variation,

$$E_i(\mathcal{L}) \cdot \delta\phi^i + d\theta = d\alpha.$$

Subtracting both sides and using the definition $J := \theta - \alpha$, we obtain:

$$dJ = -E_i(\mathcal{L}) \cdot \delta\phi^i.$$

Now, if ϕ is a solution of the Euler–Lagrange equations, i.e., $E_i(\mathcal{L})(\phi) = 0$, then $dJ = 0$. Thus, the $(n-1)$-form $J \in \Omega^{n-1}(M)$, which depends on the field ϕ and the symmetry $\mathfrak{X}_E$, defines a conserved current:

$$dJ = 0 \quad \Longrightarrow \quad \partial_\mu J^\mu = 0,$$

in local coordinates. This completes the proof of the conservation law associated to a variational symmetry of the Lagrangian. □

Conjecture IV.4.1. Functorial Geometric–Variational Field Calculus.

Let Geom_n denote the category of smooth, oriented n-manifolds equipped with geometric structure sheaves (e.g., vector bundles, principal bundles, metrics, connections, group actions), and let $\mathsf{Alg}_{\mathrm{var},d}$ denote the category of differential-graded variational algebras with horizontal and vertical differentials (d_H, d_V), equipped with jet prolongations, variational bicomplexes, and exterior operators.

Then there exists a covariant functor

$$\mathsf{GeoCalc} : \mathsf{Geom}_n \longrightarrow \mathsf{Alg}_{\mathrm{var},d}$$

such that:

1. For each geometrised n-manifold $(M, \mathcal{E}) \in \mathsf{Geom}_n$, where $\mathcal{E} \to M$ is a smooth fiber bundle of fields, the image $\mathsf{GeoCalc}(M, \mathcal{E})$ is the variational bicomplex of differential forms on the infinite jet bundle $J^\infty(\mathcal{E})$, structured as a bigraded complex with operators (d_H, d_V) satisfying $d = d_H + d_V$ and $d^2 = 0$.

2. Variational principles, Euler–Lagrange operators, symmetries, and Noether currents arise as natural transformations within this functorial image, and satisfy:

$$\delta S = 0 \quad \Longleftrightarrow \quad \mathrm{EL}(\mathcal{L}) = 0,$$

where $\mathcal{L} \in \Omega^{n,0}(J^\infty(\mathcal{E}))$ is a horizontal Lagrangian form and EL is the Euler–Lagrange operator obtained by vertical variation.

3. For every local symmetry group G acting fiberwise on $\mathcal{E} \to M$, the associated infinitesimal generator $\xi \in \mathfrak{g}$ induces a variational vector field $\hat{\xi} \in \mathfrak{X}(J^\infty(\mathcal{E}))$, and Noether's first theorem holds:

$$\hat{\xi} \lrcorner\, d_V \mathcal{L} = d_H J_\xi,$$

where $J_\xi \in \Omega^{n-1,0}(J^\infty(\mathcal{E}))$ is the conserved Noether current.

4. Quantum amplitudes are realised via a secondary functor

$$\mathsf{Quant} : \mathsf{Alg}_{\mathrm{var},d} \longrightarrow \mathsf{QFT}_{\mathbb{C}},$$

sending variational complexes to quantum field theories by assigning to each Lagrangian form $\mathcal{L}$ the formal path integral measure $\exp\left(\frac{i}{\hbar}\int_M \mathcal{L}\right)\mathcal{D}\psi$, defined on the moduli stack of fields modulo gauge equivalence.

5. The total composite structure

$$\mathsf{QFT}_\hbar := \mathsf{Quant} \circ \mathsf{GeoCalc}$$

defines a global functorial field theory from geometrised manifolds to quantum amplitude assignments, incorporating both classical and quantum structure, and respecting naturality with respect to smooth maps, fibered pullbacks, and symmetry reductions.

Hence, the calculus of differential geometry, field variation, and quantum integration arises as a canonical and functorial extension of the geometry of n-manifolds, where local-to-global coherence is mediated through jet prolongation, variational bicomplexes, and stacky gauge reductions. $\Diamond$

Remark IV.4.3. (Categorical–Analytic Geometry of Fields)

The functorial correspondence

$$\mathsf{GeoCalc} : \mathsf{Geom}_n \longrightarrow \mathsf{Alg}_{\mathrm{var},d} \longrightarrow \mathsf{Func}_\infty^{\delta\mathrm{dga}}$$

may be understood as encoding the *geometry of physical theories* in a simultaneously **algebraic**, **geometric**, and **analytic** framework. Each smooth n-manifold $M \in \mathsf{Geom}_n$, together with a geometric structure (e.g., a bundle $\mathcal{E} \to M$, a connection, or a jet prolongation), determines a sheaf $\mathcal{J}^\infty(\mathcal{E})$ of differential graded algebras, locally modeling the *variational bicomplex*.

This bicomplex canonically encodes:

- The *horizontal differential* d_H, i.e., the de Rham differential on spacetime variables, enabling coordinate-free formulations of conservation laws;
- The *vertical differential* δ_V, i.e., the Euler–Lagrange operator, characterizing the stationarity condition of an action;
- The **Noether complex**, expressing symmetries as cycles and conserved currents as cohomological representatives;
- The *variational bracket* structure, including presymplectic and Poisson forms, compatible with derived symplectic geometry.

This formalism bridges the gap between differential geometry and functional analysis: differential equations become structured morphisms between sheaves of functionals, and analytical estimates, such as energy bounds or coercivity, can be interpreted categorically via the properties of morphisms in the image of $\mathsf{GeoCalc}$. In this sense, *functional analytic regularity theory* is a shadow of deeper **functorial geometric regularity**, constrained by cohomology, curvature, and variational finiteness. ♦

Definition IV.4.4. Stationarity of the Action.

Let $\mathcal{C}$ be a smooth configuration space of fields $\phi\colon M \to N$, and let

$$S[\phi] := \int_M \mathcal{L}(\phi, \partial\phi, \ldots)\, |d^n x|$$

denote the action functional defined by a Lagrangian density $\mathcal{L}\colon J^k(\mathcal{C}) \to \Lambda^n M$, where $J^k(\mathcal{C})$ is the k-jet bundle of fields and their derivatives.

We say that a field configuration $\phi_0 \in \mathcal{C}$ is *stationary* if the first variation of S vanishes under all compactly supported smooth variations $\delta\phi \in T_{\phi_0}\mathcal{C}$, i.e.,

$$\frac{d}{d\varepsilon} S[\phi_0 + \varepsilon\delta\phi]\bigg|_{\varepsilon=0} = 0.$$

Such configurations satisfy the Euler–Lagrange equations associated with $\mathcal{L}$.

This notion of action stationarity leads us naturally to the jet stack:

Definition IV.4.5. Jet Stack.

Let $\pi\colon E \to M$ be a smooth fiber bundle over a manifold M, with sections $\Gamma(E)$. For each integer $k \geq 0$, the k-jet bundle $J^k E \to M$ is the bundle whose fiber over $x \in M$ consists of equivalence classes of sections $\phi \in \Gamma(E)$ under the relation:

$$\phi \sim_k \psi \quad \Longleftrightarrow \quad j^k_x \phi = j^k_x \psi,$$

i.e., all partial derivatives of ϕ and ψ at x up to order k agree in some local trivialization. The *jet stack* $\mathcal{J}^\infty(E)$ is the colimit (formal inductive limit) of all finite-order jet bundles:

$$\mathcal{J}^\infty(E) := \varinjlim_k J^k E.$$

It defines a sheaf (or stack) on the site of open sets $U \subseteq M$ assigning to each U the space of infinite jets of sections over U, with structure maps given by jet prolongations. In the functorial formalism, $\mathcal{J}^\infty(E)$ is the sheaf (or higher stack) of infinite-order differential operators on E, representing the functor:

$$U \mapsto \{\text{Formal Taylor expansions at each point } x \in U \text{ of sections } \phi \in \Gamma(E|_U)\}.$$

Remark IV.4.4. (Gauge Theory, Action Principle, and Jet Formalism)

In modern geometric field theory, gauge fields, variational principles, and jet bundles interact as a unified formal structure. A gauge field is modeled as a connection A on a principal G-bundle $P \to M$, with curvature $F_A = dA + \frac{1}{2}[A \wedge A]$. The physical theory is encoded via an action functional

$$S[A] = \int_M \mathcal{L}(A, F_A),$$

where $\mathcal{L}$ is a Lagrangian density invariant under the gauge group $\mathcal{G} := \mathrm{Aut}_G(P)$.

To rigorously define variational derivatives of S, one lifts the configuration space of fields to the infinite jet bundle $\mathcal{J}^\infty(\mathcal{A})$, where $\mathcal{A}$ denotes the space of gauge connections. The jet stack encodes all derivatives of fields as independent coordinates, enabling a geometric derivation of the Euler–Lagrange equations and conservation laws via the variational bicomplex.

In this setting, gauge invariance manifests as a symmetry of the action on $\mathcal{J}^\infty(\mathcal{A})$, and Noether's theorems acquire a cohomological interpretation. Thus, the gauge field theory is not merely defined by a Lagrangian but by a triple: the space of fields $\mathcal{F}$, the action functional $S\colon \mathcal{F} \to \mathbb{R}$, and the symmetry group acting on $\mathcal{J}^\infty(\mathcal{F})$ preserving the variational structure. ⧫

Before I discuss too much more on geometry, I should make a remark on that-which-shall-not-be-named but rhymes with PU.

Remark IV.4.5. (On unified geometry; why Weinstein's framework falls short)

Classical symplectic and Poisson geometry, as developed by Weinstein and others, provides a powerful framework for finite-dimensional Hamiltonian systems. However, it falls short when applied to the geometry of field theories involving gauge symmetry, infinite-dimensional configuration spaces, and variational principles.

Key limitations include:

- **Lack of jet-theoretic structure:** Symplectic geometry does not natively encode derivatives of fields as independent geometric data. Jet bundles, by contrast, make differential dependence explicit.
- **Inadequate treatment of gauge transformations:** Traditional approaches model gauge symmetry via Lie group actions on phase space, but fail to capture the sheaf-theoretic and stack-theoretic subtleties of local-to-global gauge descent.
- **No intrinsic variational bicomplex:** Symplectic mechanics encodes evolution via Hamiltonian flows, but not via Lagrangian densities or Euler–Lagrange equations derived geometrically from horizontal forms.

A true unified geometric formalism must integrate:

1. Jet bundles for infinitesimal field data,
2. Stacky or groupoid actions for gauge symmetries,
3. The variational bicomplex for Lagrangian and conservation structure,
4. Cohomological tools to encode Noether currents, topological charges, and higher field observables.

While Weinstein's contributions laid the groundwork for the geometry of classical mechanics, **a unified theory of gauge-invariant field dynamics requires a jet–gauge–variational geometry**, not reducible to finite-dimensional symplectic structures. ⧫

IV.5 Gauge Field–Cohomology Correspondence

A major classical result that assists in the study of gauge fields is the Chern–Weil Theorem:

Theorem IV.5.1. Chern–Weil Theorem.

Let $P \to M$ be a principal G-bundle over a smooth manifold M, with G a Lie group and $\mathfrak{g}$ its Lie algebra. Let $A \in \Omega^1(P; \mathfrak{g})$ be a connection 1-form on P, and let $F_A = dA + \frac{1}{2}[A \wedge A]$ denote its curvature.

Let $f \in \mathrm{Inv}(\mathfrak{g}^{\otimes k})$ be an invariant symmetric polynomial of degree k on $\mathfrak{g}$. Then the form

$$\omega_{2k}(A) := f(F_A^{\wedge k}) \in \Omega^{2k}(M)$$

is closed:

$$d\,\omega_{2k}(A) = 0,$$

and its de Rham cohomology class

$$[\omega_{2k}(A)] \in H^{2k}_{\mathrm{dR}}(M)$$

is independent of the choice of connection A. That is, $[\omega_{2k}(A)]$ is a characteristic class of the principal bundle P, naturally determined by f. ⌟

Proof IV.5.1. Let A_0 and A_1 be two connections on a principal G-bundle $P \to M$, and define the affine interpolation $A_t := (1-t)A_0 + tA_1$ for $t \in [0,1]$. This defines a smooth path A_t in the space of connections. Let F_t denote the curvature of A_t, and observe that $\dot{A}_t := \frac{d}{dt}A_t = A_1 - A_0$ is constant in t.

Define the transgression form

$$T_{2k-1}(A_0, A_1) := k\int_0^1 f\left(\dot{A}_t \wedge F_t^{\wedge(k-1)}\right)\, dt,$$

where $f \in \mathrm{Inv}^k(\mathfrak{g})$ is an invariant symmetric polynomial on the Lie algebra $\mathfrak{g}$. By standard transgression arguments (see, e.g., Bott–Tu or Kobayashi–Nomizu), one obtains

$$\omega_{2k}(A_1) - \omega_{2k}(A_0) = d\,T_{2k-1}(A_0, A_1),$$

where $\omega_{2k}(A) := f(F_A^{\wedge k})$ is the Chern–Weil form associated to the connection A.

This shows that the differential form $\omega_{2k}(A)$ is closed and that its de Rham cohomology class $[\omega_{2k}(A)] \in H^{2k}_{\mathrm{dR}}(M)$ is independent of the choice of connection A. In particular, it depends only on the principal bundle P and the choice of invariant polynomial f.

Finally, the closedness of $\omega_{2k}(A)$ follows from the Bianchi identity $d_A F_A = 0$, together with the Ad-invariance and symmetry of f, which implies

$$d\,f(F_A^{\wedge k}) = k \cdot f(d_A F_A \wedge F_A^{\wedge(k-1)}) = 0.$$

Thus $\omega_{2k}(A)$ is a closed form, and its de Rham class is well-defined. □

This approach provides an exceptional basis in the analysis of line bundles:

Lemma IV.5.1. Gauge–Cohomology Correspondence for Line Bundles.

Let $L \to M$ be a complex line bundle over a smooth manifold M, equipped with a Hermitian structure and a unitary connection ∇. Then:

1. The curvature 2-form $F_\nabla \in \Omega^2(M; i\mathbb{R})$ satisfies $dF_\nabla = 0$.
2. The cohomology class
$$\left[\frac{F_\nabla}{2\pi i}\right] \in H^2_{\mathrm{dR}}(M;\mathbb{R})$$
is closed and independent of the choice of connection, and corresponds under the real de Rham isomorphism to the first Chern class:
$$c_1(L) \otimes \mathbb{R} = \left[\frac{F_\nabla}{2\pi i}\right].$$

3. If the periods of $F_\nabla/2\pi i$ are integral over all closed surfaces $\Sigma \subset M$, then

$$\left[\frac{F_\nabla}{2\pi i}\right] \in H^2(M;\mathbb{Z}) \subset H^2_{\mathrm{dR}}(M;\mathbb{R})$$

and recovers the integral Chern class of L.

4. The triple (L, ∇, F_∇) defines a class in differential cohomology:

$$\widehat{c}_1(L,\nabla) \in \widehat{H}^2(M;\mathbb{Z}),$$

which refines both the topological class $c_1(L)$ and the curvature form F_∇.

$\rightarrowtail$

Remark IV.5.1. (Integral Cohomology and Thermodynamic Quantization)

In thermodynamic systems with gauge symmetry—such as superconductors, superfluids, or charged fluids—the geometry of line bundles $L \to M$ encodes topologically distinct physical sectors.

The first Chern class $c_1(L) \in H^2(M;\mathbb{Z})$ serves as a topological invariant classifying quantised observables:

- In the presence of a U(1)-connection ∇, the curvature F_∇ satisfies

$$\left[\frac{F_\nabla}{2\pi i}\right] = c_1(L) \otimes \mathbb{R} \in H^2_{\mathrm{dR}}(M).$$

- The integrality condition on flux,

$$\int_\Sigma \frac{F_\nabla}{2\pi i} \in \mathbb{Z} \quad \text{for all } \Sigma \in H_2(M;\mathbb{Z}),$$

reflects conservation laws that are immune to continuous deformations—i.e., thermal fluctuations cannot alter the integral cohomology class.

- In condensed matter systems, these integral classes correspond to quantised charges, magnetic monopoles, vortex numbers, or holonomy defects. Thus, integral cohomology underlies the topological quantization of thermodynamic observables.

Hence, the gauge field–cohomology correspondence provides a rigorous mathematical mechanism by which discrete topological invariants manifest as conserved quantities in macroscopic thermodynamic phases. ◆

Example IV.1 (Abrikosov Vortex Lattices and Integral Cohomology)**.** Consider a type-II superconductor in an external magnetic field. In the Ginzburg–Landau theory, the superconducting phase is described by a complex scalar field $\psi : M \to \mathbb{C}$, defined over a spatial manifold M (often $\mathbb{R}^2$ or a torus T^2), coupled to a U(1)-gauge connection A.

- The scalar field ψ defines a complex line bundle $L \to M$, and the gauge field $A \in \Omega^1(M; i\mathbb{R})$ defines a connection on L.
- The curvature $F = dA \in \Omega^2(M; i\mathbb{R})$ corresponds physically to the local magnetic field penetrating the superconductor.

- In the presence of quantised vortices, the zeros of ψ define a discrete set of singularities $\{p_i\} \subset M$, each carrying an integer winding number (vorticity). Around each vortex core, the phase of ψ winds by $2\pi n_i$, corresponding to nontrivial holonomy.
- The total magnetic flux is quantised:

$$\frac{1}{2\pi i}\int_M F = \sum_i n_i \in \mathbb{Z},$$

  and represents the first Chern class $c_1(L) \in H^2(M;\mathbb{Z})$. Hence, the entire vortex configuration defines an integral cohomology class.
- In equilibrium, the vortices arrange into a *lattice structure* (Abrikosov lattice) minimizing the free energy. This lattice is a macroscopic thermodynamic realization of a nontrivial cohomology class in $H^2(M;\mathbb{Z})$, realised geometrically as a divisor of the zero set of a section of L.

Thus, Abrikosov vortex lattices serve as a physical instantiation of the gauge–cohomology correspondence: discrete, topologically conserved flux quanta emerge from the integral structure of the line bundle underlying the superconducting state.

IV.6 Boundary–Bulk Interactions

Definition IV.6.1. Winding Number.

Let $\gamma : S^1 \to \mathbb{R}^2 \setminus \{0\}$ be a smooth closed loop in the punctured plane, and let $\theta : S^1 \to \mathbb{R}$ be a smooth lift of the phase of a complex-valued function $\psi = |\psi|e^{i\theta}$ along γ. Then the *winding number* of ψ along γ is defined as

$$\mathrm{wind}_\gamma(\psi) := \frac{1}{2\pi}\int_\gamma d\theta \in \mathbb{Z}.$$

Equivalently, in terms of the complex logarithmic derivative,

$$\mathrm{wind}_\gamma(\psi) = \frac{1}{2\pi i}\int_\gamma \frac{d\psi}{\psi}.$$

If ψ has an isolated zero at a point $p \in \mathbb{R}^2$, and γ is a positively oriented loop enclosing p and no other zero, then $\mathrm{wind}_\gamma(\psi)$ equals the multiplicity of the zero at p.

Definition IV.6.2. Pontryagin Index.

Let M be a closed oriented 4-manifold and let $P \to M$ be a principal G-bundle with compact Lie group G, equipped with a connection A and corresponding curvature 2-form $F \in \Omega^2(M;\mathfrak{g})$. Then the *first Pontryagin number* (or Pontryagin index) of the bundle is defined as

$$p_1(P) := -\frac{1}{8\pi^2}\int_M \mathrm{tr}(F \wedge F) \in \mathbb{Z},$$

where tr is a suitably normalised invariant bilinear form on the Lie algebra $\mathfrak{g}$, such as the Killing form.

Properties:

- $p_1(P)$ is a topological invariant of the bundle P, independent of the choice of connection A.

- If $G = \mathrm{SU}(2)$, this integral computes the **instanton number** or *second Chern class* $c_2(P) \in H^4(M;\mathbb{Z})$.
- It counts the homotopy class of maps $M \to G$, measuring the degree of a gauge field configuration in 4D.

Conjecture IV.6.1. Boundary–Bulk Interactions in Topological Gauge Flow.

Let M be a smooth, oriented manifold of dimension $d \in \{2,3,4\}$, possibly with smooth boundary ∂M, and let

$$\pi : P \to M$$

be a principal G–bundle with G a compact Lie group, Lie algebra $\mathfrak{g}$, and an Ad–invariant bilinear form $\mathrm{tr} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ (e.g. the negative of the Killing form for semisimple G). Let

$$A \in \Omega^1(P;\mathfrak{g})$$

be a smooth connection 1–form with curvature

$$F := dA + A \wedge A \ \in \ \Omega^2(P;\mathfrak{g}),$$

horizontal and Ad–equivariant.

Let $\psi : M \to \mathbb{C}$ be a smooth complex scalar field or spinor order parameter, interpreted as a section of an associated complex line bundle $E := P \times_\rho \mathbb{C}$ for some representation $\rho : G \to \mathrm{GL}(1,\mathbb{C})$.

- For a smooth, closed, oriented loop $\gamma \subset \partial M$, the *winding number* of ψ along γ is

$$\mathrm{wind}_\gamma(\psi) := \frac{1}{2\pi}\int_\gamma d\arg(\psi) \ \in \ \mathbb{Z}.$$

  This counts the number of times the phase $\arg(\psi)$ winds around the unit circle S^1 as γ is traversed once in the positive direction.

- For $d = 4$, the *Pontryagin index* of A is

$$\mathcal{P}(A) := -\frac{1}{8\pi^2}\int_M \mathrm{tr}(F \wedge F) \ \in \ \mathbb{Z}$$

  when M is closed, by the Chern–Weil theorem. On manifolds with boundary, $\mathcal{P}(A)$ need not be integer–valued unless supplemented with a boundary correction term given by the Chern–Simons form

$$\mathrm{CS}(A) := \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right).$$

- The *topological action* associated to the curvature is

$$S_{\mathrm{top}}[A] := \int_M \mathcal{L}_{\mathrm{top}}(A), \quad \mathcal{L}_{\mathrm{top}}(A) := \mathrm{tr}(F \wedge F).$$

We say that the configuration $(M, \partial M, A, \psi)$ exhibits a *boundary–bulk topological coupling* if

$$\boxed{\mathrm{wind}_{\partial M}(\psi) \ = \ \mathcal{P}(A) \ = \ \frac{1}{8\pi^2}\int_M \mathrm{tr}(F \wedge F)}$$

where $\mathrm{wind}_{\partial M}(\psi)$ denotes the sum of winding numbers over a basis of $H_1(\partial M;\mathbb{Z})$.

This equality expresses a *cohomological interpolation*:

boundary phase windings inject quantised topological charge into the bulk,

mediated by the curvature F and encoded in the Chern–Weil representative of the Pontryagin class. Such relations naturally arise in effective field theories featuring Wess–Zumino–Witten terms, Chern–Simons actions, and gauge anomaly inflow. ♢

Theorem IV.6.1. Boundary–Bulk Interpolation and Coupling Theorem.
Let M be a compact, oriented smooth 4–manifold with smooth boundary ∂M, and let

$$\pi : P \to M$$

be a principal G–bundle, where G is a compact, connected Lie group with Lie algebra $\mathfrak{g}$ and an Ad–invariant bilinear form tr. Let $A \in \Omega^1(M; \mathfrak{g})$ be a smooth connection with curvature

$$F := dA + A \wedge A \ \in \ \Omega^2(M; \mathfrak{g}),$$

and suppose that A is gauge–equivalent to the trivial connection on ∂M, i.e.

$$A|_{\partial M} = g^{-1}\, dg$$

for some smooth $g : \partial M \to G$.

(1) Bulk Pontryagin number. The normalised Pontryagin number

$$\mathcal{P}(A) := -\frac{1}{8\pi^2} \int_M \mathrm{tr}(F \wedge F)$$

is an integer, representing the image of the second Chern class (for $G = \mathrm{SU}(n)$) under the natural map $H^4(M; \mathbb{Z}) \to \mathbb{Z}$, hence defining a class in the relative group $H^4(M, \partial M; \mathbb{Z})$.

(2) Chern–Simons transgression. The Chern–Simons 3–form

$$\mathrm{CS}(A) := \mathrm{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \Big)$$

satisfies

$$d\,\mathrm{CS}(A) = \mathrm{tr}(F \wedge F),$$

and therefore by Stokes' theorem:

$$\int_M \mathrm{tr}(F \wedge F) = \int_{\partial M} \mathrm{CS}(A).$$

Equivalently,

$$\boxed{\mathcal{P}(A) = -\frac{1}{8\pi^2} \int_{\partial M} \mathrm{CS}(A)} \quad \in \mathbb{Z}.$$

(3) Degree formula. If $A|_{\partial M} = g^{-1}dg$, then $\mathrm{CS}(g^{-1}dg)$ is the canonical bi–invariant 3–form on G generating $H^3(G; \mathbb{Z})$. When $\partial M \cong S^3$, one has

$$\frac{1}{24\pi^2} \int_{\partial M} \mathrm{CS}(g^{-1}dg) = \deg(g) \ \in \ \pi_3(G) \cong \mathbb{Z},$$

and thus

$$\boxed{\mathcal{P}(A) = \deg(g)}.$$

(4) Abelian specialization. If $G = U(1)$ and $\psi : \partial M \to \mathbb{C} \setminus \{0\}$ is a nowhere–vanishing section of the associated line bundle, then in the gauge where $A|_{\partial M} = \psi^{-1} d\psi$ one has

$$\mathrm{CS}(A) = A \wedge dA$$

and

$$-\frac{1}{8\pi^2} \int_{\partial M} \mathrm{CS}(A) = \frac{1}{2\pi} \int_{\partial M} \psi^*(d \arg \psi) =: \mathrm{wind}(\psi) \;\in\; \mathbb{Z}.$$

In this case the identity becomes

$$\boxed{\mathrm{wind}(\psi) = \mathcal{P}(A)},$$

expressing that the integer winding number of the boundary phase field equals the Pontryagin index of the bulk $U(1)$ connection.

Interpretation. In full generality, the boundary Chern–Simons invariant encodes the same topological charge as the bulk Pontryagin class, with opposite sign in the transgression formula. In the Abelian case, this reduces to the statement that boundary phase windings inject quantised topological charge into the bulk. This identity underlies anomaly inflow, instanton–defect correspondence, and topological phase matching in gauge theory. ⌟

Proof IV.6.1. Let $(M, \partial M)$ be a compact, oriented smooth 4–manifold with smooth boundary, and let $\pi : P \to M$ be a principal G–bundle for a compact, connected Lie group G with Lie algebra $\mathfrak{g}$ and an Ad–invariant bilinear form $\mathrm{tr} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Fix a smooth connection $A \in \Omega^1(M; \mathfrak{g})$ with curvature

$$F := \mathrm{d}A + A \wedge A \;\in\; \Omega^2(M; \mathfrak{g}).$$

Assume A is gauge–equivalent to the trivial connection on the boundary, i.e.

$$A|_{\partial M} \;=\; g^{-1}\,\mathrm{d}g \qquad \text{for some smooth } g : \partial M \to G.$$

Step 1: Chern–Weil integrality (bulk). Let $p(X) := \mathrm{tr}(X \wedge X)$ for $X \in \Omega^2(M; \mathfrak{g})$. By Chern–Weil theory, the 4–form $p(F) = \mathrm{tr}(F \wedge F)$ is closed:

$$\mathrm{d}\,\mathrm{tr}(F \wedge F) \;=\; 0 \quad\Longleftrightarrow\quad \big[\mathrm{tr}(F \wedge F)\big] \in H^4_{\mathrm{dR}}(M).$$

With the normalization

$$\mathcal{P}(A) \;:=\; -\frac{1}{8\pi^2} \int_M \mathrm{tr}(F \wedge F),$$

one has $\mathcal{P}(A) \in \mathbb{Z}$ (e.g. for $G = \mathrm{SU}(n)$ this is the image of $c_2(P)$ under $H^4(M; \mathbb{Z}) \to \mathbb{Z}$). Equivalently,

$$\mathcal{P}(A) \in \mathbb{Z} \quad\Longleftrightarrow\quad \frac{1}{8\pi^2}\big[\mathrm{tr}(F \wedge F)\big] \text{ is integral.}$$

Step 2: Transgression identity (local computation). Define the Chern–Simons 3–form

$$\mathrm{CS}(A) \;:=\; \mathrm{tr}\!\left(A \wedge \mathrm{d}A \;+\; \frac{2}{3}\, A \wedge A \wedge A \right) \;\in\; \Omega^3(M).$$

Using $F = \mathrm{d}A + A \wedge A$, the graded Leibniz rule, $\mathrm{tr}([X, Y]\cdot) = 0$ (Ad–invariance), and the Bianchi identity $\mathrm{d}_A F := \mathrm{d}F + [A, F] = 0$, one obtains

$$\boxed{\mathrm{d}\,\mathrm{CS}(A) \;=\; \mathrm{tr}(F \wedge F)} \quad\Longleftrightarrow\quad \big[\mathrm{tr}(F \wedge F)\big] \;=\; \delta\big([\mathrm{CS}(A)]\big) \;\in\; H^4(M, \partial M),$$

where $\delta : H^3(\partial M) \to H^4(M, \partial M)$ is the connecting homomorphism for the pair $(M, \partial M)$.

Step 3: Stokes' theorem (bulk $\Longleftrightarrow$ boundary). Integrating the identity in Step 2 and using Stokes' theorem,

$$\int_M \operatorname{tr}(F \wedge F) \;=\; \int_M \mathrm{d}\,\mathrm{CS}(A) \;=\; \int_{\partial M} \mathrm{CS}(A).$$

Equivalently,

$$\boxed{\mathcal{P}(A) \;=\; -\frac{1}{8\pi^2}\int_{\partial M} \mathrm{CS}(A)} \quad\Longleftrightarrow\quad -\frac{1}{8\pi^2}\,[\mathrm{CS}(A)] \text{ represents the same relative class as } \mathcal{P}(A).$$

Step 4: Explicit boundary trivialization and pure gauge formula. Set $\vartheta := g^{-1}\,\mathrm{d}g \in \Omega^1(\partial M; \mathfrak{g})$ (the left Maurer–Cartan form on the boundary). The gauge transformation law for Chern–Simons reads

$$\mathrm{CS}(A^g) \;=\; \mathrm{CS}(A) \;-\; \mathrm{d}\,\operatorname{tr}(\vartheta \wedge A) \;+\; \frac{1}{3}\operatorname{tr}\big(\vartheta \wedge \vartheta \wedge \vartheta\big), \qquad A^g := g^{-1}Ag + g^{-1}\,\mathrm{d}g.$$

Taking $A \equiv 0$ yields the *pure gauge* identity

$$\boxed{\mathrm{CS}(\vartheta) \;=\; \frac{1}{3}\operatorname{tr}\big(\vartheta \wedge \vartheta \wedge \vartheta\big)} \quad\Longleftrightarrow\quad \mathrm{d}\,\mathrm{CS}(\vartheta) \;=\; 0 \text{ (the Maurer–Cartan equation).}$$

Since $A|_{\partial M} = \vartheta$, we have

$$\int_{\partial M} \mathrm{CS}(A) \;=\; \int_{\partial M} \mathrm{CS}(\vartheta) \;=\; \frac{1}{3}\int_{\partial M} \operatorname{tr}\big(\vartheta \wedge \vartheta \wedge \vartheta\big).$$

Combining with Step 3,

$$\int_M \operatorname{tr}(F \wedge F) \;=\; \frac{1}{3}\int_{\partial M} \operatorname{tr}\big(\vartheta \wedge \vartheta \wedge \vartheta\big) \quad\Longleftrightarrow\quad \boxed{\mathcal{P}(A) \;=\; -\frac{1}{24\pi^2}\int_{\partial M} \operatorname{tr}\big(\vartheta \wedge \vartheta \wedge \vartheta\big)}.$$

Step 5: Identification with the degree for $\partial M \cong S^3$. For $\partial M \cong S^3$, the bi-invariant 3–form

$$\Omega_3 \;:=\; \frac{1}{24\pi^2}\operatorname{tr}\big(\vartheta \wedge \vartheta \wedge \vartheta\big) \;\in\; \Omega^3(G)$$

represents a generator of $H^3(G; \mathbb{Z})$ under the chosen normalization. Hence

$$\deg(g) \;=\; \int_{S^3} g^*\Omega_3 \;=\; \frac{1}{24\pi^2}\int_{S^3} \operatorname{tr}\big(\vartheta \wedge \vartheta \wedge \vartheta\big).$$

Using Step 4,

$$\boxed{\mathcal{P}(A) \;=\; \deg(g)}.$$

This proves the degree formula and, a fortiori, the boundary–bulk identity of Step 3, while Step 1 gives integrality.

Abelian specialization and dimensional remark. If $G = \mathrm{U}(1)$, then $\pi_3(\mathrm{U}(1)) = 0$ and $H^3(\mathrm{U}(1); \mathbb{Z}) = 0$, so for $\partial M \cong S^3$ the degree is zero and, for the *pure gauge* boundary potential $A|_{\partial M} = \mathrm{d}\phi$, one has $\mathrm{CS}(A) = A \wedge \mathrm{d}A = \mathrm{d}(\phi\,\mathrm{d}A)$ with $\mathrm{d}A = 0$ on ∂M, hence $\int_{\partial M} \mathrm{CS}(A) = 0$. The winding interpretation naturally arises for a 2–manifold bulk with 1–dimensional boundary, where $\frac{1}{2\pi}\int_{\partial M} \mathrm{d}\arg(\psi) \in \mathbb{Z}$ is the boundary winding number of a phase field ψ.

All equivalences used above can be summarised as

$$\mathrm{d}\,\mathrm{CS}(A) = \mathrm{tr}(F \wedge F) \iff \big[\,\mathrm{tr}(F \wedge F)\big] = \delta\big([\mathrm{CS}(A)]\big) \iff \int_M \mathrm{tr}(F \wedge F) = \int_{\partial M} \mathrm{CS}(A),$$

and, under $A|_{\partial M} = g^{-1}\,\mathrm{d}g$,

$$-\frac{1}{8\pi^2}\int_{\partial M} \mathrm{CS}(A) \iff -\frac{1}{24\pi^2}\int_{\partial M} \mathrm{tr}\big(\vartheta^{\wedge 3}\big) \iff \deg(g),$$

which yields $\mathcal{P}(A) \iff \deg(g)$ and completes the proof. □

This statement is classical and can be found as a corollary of the Chern–Simons transgression formula and the fact that $\pi_3(G) \cong \mathbb{Z}$ for compact simple G. We attempted a more physical intuition to the mathematics.

V Mathematics of Quantum Mechanics I

This section develops the mathematical framework of nonrelativistic quantum mechanics in the language of functional analysis and operator theory. Our aim is to present the postulates as precise mathematical statements while maintaining contact with their physical interpretation. Throughout, $\mathcal{H}$ will denote a complex Hilbert space, serving as the state space of the quantum system.

V.1 Introduction

Definition V.1.1. State space.

Let $\mathcal{H}$ be a complex Hilbert space, that is, a complete complex vector space equipped with an inner product

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

that is linear in the first argument, conjugate-symmetric $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$, and positive-definite $\langle \psi, \psi \rangle > 0$ for all $\psi \neq 0$. The norm $\|\psi\| := \sqrt{\langle \psi, \psi \rangle}$ induces the topology in which $\mathcal{H}$ is complete.

A *state* of a quantum system is mathematically represented by a nonzero vector $\psi \in \mathcal{H}$. Since physical measurements depend only on the direction of ψ in $\mathcal{H}$ and not on its complex phase or overall scaling, the true space of physical states is the *projective Hilbert space*

$$\mathbb{P}(\mathcal{H}) := \{ \ [\psi] \mid \psi \in \mathcal{H} \setminus \{0\} \ \},$$

where the equivalence class (or *ray*) is defined by

$$[\psi] := \{\lambda\psi \mid \lambda \in \mathbb{C} \setminus \{0\}\}.$$

Two vectors ψ and ϕ represent the same physical state if and only if they differ by a nonzero complex scalar multiple. For convenience, states are often represented by *normalized vectors* satisfying $\|\psi\| = 1$, in which case $\langle \psi, \psi \rangle = 1$.

The probabilistic interpretation of quantum mechanics is formulated in terms of these normalized representatives: given an orthogonal projection $P \in \mathcal{L}(\mathcal{H})$ corresponding to a measurement outcome, the probability of obtaining that outcome when the system is in state ψ is given by $\|P\psi\|^2$.

Remark V.1.1. (Superposition and probabilistic interpretation)

The Hilbert space framework embodies two fundamental features of quantum theory.

(1) Superposition principle. Because $\mathcal{H}$ is a complex vector space, if $\psi_1, \psi_2 \in \mathcal{H}$ are two (normalized) state vectors representing physically realizable states, then for any $\alpha, \beta \in \mathbb{C}$ the linear combination

$$\psi = \alpha\psi_1 + \beta\psi_2$$

is also a valid state vector (nonzero, in general) of the system. This expresses the principle that quantum states form a linear space: the system can be prepared in coherent superpositions of distinct states, with α and β encoding both relative magnitude and relative phase. The geometry of $\mathcal{H}$ makes phase physically significant in interference phenomena, even though global phase is unobservable.

(2) Probabilistic interpretation. The norm $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$ plays a central role in encoding the statistical structure of quantum theory. Upon normalization ($\|\psi\| = 1$), the squared modulus of the inner product

$$|\langle \phi, \psi \rangle|^2$$

is interpreted as the probability of obtaining the outcome associated with state ϕ when measuring a system prepared in state ψ. More generally, for a projection operator P representing a measurement event, the Born rule states

$$\mathrm{Prob}_{\psi}(P) = \|P\psi\|^2.$$

This ensures that probabilities lie in $[0, 1]$ and sum to unity over a complete set of mutually orthogonal projectors.

(3) Projectivization. Since multiplication of ψ by any $\lambda \in \mathbb{C} \setminus \{0\}$ does not alter these probabilities, physical states are identified with equivalence classes

$$[\psi] = \{\lambda\psi \mid \lambda \neq 0\} \in \mathbb{P}(\mathcal{H}),$$

the *projective Hilbert space*. This projectivization removes the physically irrelevant global phase and isolates the true degrees of freedom of the quantum state. It also equips the state space with a natural Kähler geometry via the Fubini–Study metric, which quantifies the statistical distinguishability between pure states.

In summary, the linear structure of $\mathcal{H}$ enables coherent superposition, the norm and inner product define the probabilistic rules, and projectivization reflects the fact that only relative phase and amplitude carry physical content. ♦

Definition V.1.2. Observables.

An *observable* is a densely-defined self-adjoint operator $A : \mathcal{D}(A) \to \mathcal{H}$. The spectral theorem guarantees the existence of a projection-valued measure E_A on the Borel subsets of $\mathbb{R}$ such that

$$A = \int_{\mathbb{R}} \lambda \, dE_A(\lambda).$$

The possible outcomes of measuring A lie in $\sigma(A)$, the spectrum of A.

Remark V.1.2. (Role of self-adjointness)

Self-adjointness, not merely symmetry, ensures that the spectrum is real and that time evolution generated by A (when A is the Hamiltonian) is unitary. This property is essential for the conservation of probability and the physical consistency of the theory. ♦

Theorem V.1.1. Spectral theorem for self-adjoint operators.

Let A be a densely-defined self-adjoint operator on a complex Hilbert space $\mathcal{H}$. Then there exists a unique projection-valued measure

$$E_A : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$$

such that:

1. For every $\psi \in \mathcal{H}$, the map $\Delta \mapsto \langle \psi, E_A(\Delta)\psi \rangle$ defines a countably additive measure on the Borel σ–algebra $\mathcal{B}(\mathbb{R})$.
2. The operator A can be represented as the (strong) spectral integral

$$A = \int_{\mathbb{R}} \lambda \, dE_A(\lambda),$$

meaning that for each $\psi \in \mathcal{D}(A)$,

$$\langle \psi, A\psi \rangle = \int_{\mathbb{R}} \lambda \, d\langle \psi, E_A(\lambda)\psi \rangle.$$

3. For every bounded Borel function $f : \mathbb{R} \to \mathbb{C}$, the operator

$$f(A) = \int_{\mathbb{R}} f(\lambda)\, dE_A(\lambda)$$

is bounded on $\mathcal{H}$ and satisfies $\|f(A)\| \leq \sup_{\lambda \in \mathbb{R}} |f(\lambda)|$.

The map $f \mapsto f(A)$ defines the *functional calculus* for A, extending the usual polynomial calculus and preserving algebraic relations: $(fg)(A) = f(A)g(A)$ and $\overline{f}(A) = f(A)^*$. ⌟

Proof V.1.1. We outline the main steps.

Step 1: Bounded normal operator case. If T is a bounded normal operator on $\mathcal{H}$, the Riesz–Markov representation theorem applied to the Gelfand transform of the C^*–algebra generated by T yields a spectral measure E_T on the compact spectrum $\sigma(T) \subset \mathbb{C}$ such that

$$T = \int_{\sigma(T)} z\, dE_T(z).$$

For T self-adjoint and bounded, $\sigma(T) \subset \mathbb{R}$, so the integral is over $\mathbb{R}$.

Step 2: Reduction of the unbounded case to the bounded case. Let A be unbounded self-adjoint. Define the Cayley transform

$$U := (A - iI)(A + iI)^{-1}.$$

This is a unitary operator with dense range, and its spectrum lies on the unit circle $\mathbb{T} \subset \mathbb{C}$. The mapping $\lambda \mapsto \frac{\lambda - i}{\lambda + i}$ is a bijection from $\mathbb{R}$ onto $\mathbb{T} \setminus \{1\}$, with inverse

$$\lambda = i\frac{1+z}{1-z}.$$

Step 3: Apply the spectral theorem to U. By Step 1, U admits a projection-valued spectral measure E_U on $\mathbb{T}$ such that

$$U = \int_{\mathbb{T}} z\, dE_U(z).$$

Via the inverse Cayley transform, we push E_U forward to a projection-valued measure E_A on $\mathbb{R}$.

Step 4: Recover A and the functional calculus. For bounded Borel $f : \mathbb{R} \to \mathbb{C}$, define

$$f(A) := \int_{\mathbb{R}} f(\lambda)\, dE_A(\lambda).$$

In particular, with $f(\lambda) = \lambda$, we recover

$$A = \int_{\mathbb{R}} \lambda\, dE_A(\lambda)$$

in the strong sense on $\mathcal{D}(A)$. The construction guarantees uniqueness of E_A and the multiplicativity of the functional calculus.

This completes the proof. □

Remark V.1.3. (Spectral measure and Born rule)

The spectral measure E_A assigns to each Borel set $\Delta \subset \mathbb{R}$ the orthogonal projection onto the subspace of states with measurement outcomes in Δ. For a normalized state ψ, the probability of obtaining a result in Δ is $\|E_A(\Delta)\psi\|^2$. This is the rigorous expression of the Born rule. ♦

Definition V.1.3. Time evolution.

Let H be the Hamiltonian of the system, a self-adjoint operator on $\mathcal{H}$. The *time evolution operator* is

$$U(t) := e^{-iHt/\hbar},$$

defined via the spectral theorem as

$$U(t) = \int_{\mathbb{R}} e^{-i\lambda t/\hbar}\, dE_H(\lambda).$$

Then $U(t)$ is unitary and satisfies the Schrödinger equation

$$i\hbar \frac{d}{dt}\psi(t) = H\psi(t)$$

for all $\psi(t) \in \mathcal{D}(H)$.

Remark V.1.4. (Unitarity and probability conservation)

The unitarity of $U(t)$ expresses the conservation of total probability: $\|\psi(t)\| = \|\psi(0)\|$ for all $t \in \mathbb{R}$. Geometrically, $U(t)$ moves the state vector along the unit sphere in $\mathcal{H}$ without changing its length. ♦

Definition V.1.4. Commutator and compatibility.

Given two densely-defined operators A, B on $\mathcal{H}$ with common dense domain $\mathcal{D}$, the *commutator* is

$$[A, B] := AB - BA.$$

Two observables are *compatible* if their spectral measures commute, which is equivalent to the existence of a joint spectral resolution and a common orthonormal basis of eigenvectors.

Remark V.1.5. (Compatibility and simultaneous measurement)

Compatibility corresponds to the ability to measure two observables simultaneously with arbitrary precision. Non-commutativity is the mathematical origin of the uncertainty principle. ♦

Theorem V.1.2. Heisenberg uncertainty principle.

Let A and B be self-adjoint operators with common dense domain and let $\psi \in \mathcal{H}$ be normalized. Define

$$\Delta A = \|(A - \langle A\rangle_\psi)\psi\|, \quad \Delta B = \|(B - \langle B\rangle_\psi)\psi\|.$$

Then

$$\Delta A\, \Delta B \;\geq\; \frac{1}{2}\left|\langle \psi, [A, B]\psi\rangle\right|.$$

⌟

Proof V.1.2. Apply the Cauchy–Schwarz inequality to the vectors $(A-\langle A\rangle_\psi)\psi$ and $(B-\langle B\rangle_\psi)\psi$, then use the polarization identity to relate the imaginary part of the inner product to the expectation value of the commutator. □

Remark V.1.6. (Canonical position–momentum case)

On $L^2(\mathbb{R})$, with Q the position operator and P the momentum operator satisfying $[Q, P] = i\hbar I$, the inequality becomes

$$\Delta Q\, \Delta P \;\geq\; \frac{\hbar}{2}.$$

This expresses the intrinsic spread in simultaneous position and momentum measurements, arising entirely from the Hilbert space and operator framework. ⧫

Remark V.1.7. (Dirac bra–ket formalism and operator matrix elements)

The *Dirac bracket notation* offers a compact and conceptually precise way to represent states, linear functionals, and operators in Hilbert space. Given a Hilbert space $\mathcal{H}$, a *ket* $|\psi\rangle$ denotes a vector $\psi \in \mathcal{H}$, while the corresponding *bra* $\langle\psi|$ denotes the conjugate-linear functional

$$\langle\psi| : \mathcal{H} \to \mathbb{C}, \quad \langle\psi|\phi\rangle = \langle\psi, \phi\rangle_{\mathcal{H}}.$$

This identification between $\mathcal{H}$ and its dual $\mathcal{H}^*$ is canonical in finite dimensions, and extends rigorously via the Riesz representation theorem in the infinite-dimensional setting.

Given a bounded operator $X \in \mathcal{L}(\mathcal{H})$, the expression

$$\langle\phi|X|\psi\rangle$$

is defined as the scalar $\langle\phi, X\psi\rangle$, i.e., the image of the ket $X|\psi\rangle$ under the bra $\langle\phi|$. This is the *matrix element* of X between the states ϕ and ψ. In a fixed orthonormal basis $\{|e_j\rangle\}_{j\in J}$, these elements form the entries $X_{ij} := \langle e_i|X|e_j\rangle$ of the operator's matrix representation.

For any complete orthonormal basis $\{|n\rangle\}_{n\in J}$, we have the *resolution of the identity*

$$I = \sum_{n\in J} |n\rangle\langle n|,$$

with convergence in the strong operator topology. This expansion allows one to insert completeness relations in operator computations, yielding

$$\langle\phi|X|\psi\rangle = \sum_{n\in J}\langle\phi|X|n\rangle\langle n|\psi\rangle = \sum_{m,n\in J} \langle\phi|m\rangle\langle m|X|n\rangle\langle n|\psi\rangle,$$

which mirrors the usual double-sum in matrix multiplication.

In multi-particle systems, the Hilbert space takes the form of a tensor product $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$. A computational basis for a qubit system ($\mathbb{C}^2$ per subsystem) consists of vectors $|b_1 b_2 \dots b_k\rangle$ with each $b_j \in \{0, 1\}$. For example,

$$|001\rangle = |0\rangle \otimes |0\rangle \otimes |1\rangle$$

denotes the pure state with the first two qubits in the $|0\rangle$ state and the third in $|1\rangle$. Matrix elements in such spaces take the form

$$\langle a_1 a_2 \dots a_k|X|b_1 b_2 \dots b_k\rangle$$

and are computed with respect to the product basis.

The formalism extends naturally to unbounded operators with appropriate domain considerations. For example, in the position representation $\mathcal{H} = L^2(\mathbb{R}^d)$, the operator X of multiplication by x has matrix elements

$$\langle\phi|X|\psi\rangle = \int_{\mathbb{R}^d} \overline{\phi(x)}\, x\, \psi(x)\, dx,$$

while the momentum operator $P = -i\hbar\nabla$ has

$$\langle\phi|P|\psi\rangle = -i\hbar \int_{\mathbb{R}^d} \overline{\phi(x)}\, \nabla\psi(x)\, dx.$$

These expressions make clear that $\langle\phi|X|\psi\rangle$ generalizes the classical expectation $\mathbb{E}[X]$ to the quantum setting: when $\phi = \psi$ and $\|\psi\| = 1$, this is precisely the expectation value $\langle X\rangle_\psi$.

In finite-dimensional systems, one can move seamlessly between the operator picture and its matrix representation via the completeness relation, while in infinite-dimensional systems, careful attention to domains and convergence is required. Nevertheless, the Dirac notation remains an indispensable computational and conceptual tool in both pure and applied quantum theory. ♦

We now turn to some machinery we need to approach the n-dimensional quantum harmonic oscillator:

Definition V.1.5. Configuration space and Hilbert space.

For $n \in \mathbb{N}$, the *configuration space* of a nonrelativistic particle in $\mathbb{R}^n$ is $\mathbb{R}^n$ equipped with its standard Lebesgue measure dx. The associated Hilbert space of pure states in the position representation is

$$\mathcal{H} := L^2(\mathbb{R}^n, dx) = \left\{ \psi : \mathbb{R}^n \to \mathbb{C} \;\middle|\; \int_{\mathbb{R}^n} |\psi(x)|^2 \, dx < \infty \right\},$$

with inner product

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}^n} \overline{\phi(x)} \, \psi(x) \, dx.$$

Definition V.1.6. Position and momentum operators.

For each $j \in \{1, \ldots, n\}$, the *position operator* X_j acts by

$$(X_j \psi)(x) := x_j \, \psi(x),$$

and the *momentum operator* P_j acts by

$$(P_j \psi)(x) := -\, i\hbar \, \frac{\partial \psi}{\partial x_j}(x).$$

These are densely defined on the Schwartz space $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, and satisfy the canonical commutation relations

$$[X_j, P_k] = i\hbar \, \delta_{jk} I, \qquad [X_j, X_k] = [P_j, P_k] = 0$$

on $\mathcal{S}(\mathbb{R}^n)$.

Definition V.1.7. Anisotropic n-dimensional quantum harmonic oscillator.

Fix $m > 0$ and frequencies $\omega_j > 0$ for $j = 1, \ldots, n$. The *harmonic oscillator Hamiltonian* is the symmetric operator H on $\mathcal{S}(\mathbb{R}^n)$ defined by

$$H := \sum_{j=1}^{n} \left(\frac{P_j^2}{2m} + \frac{1}{2} m \omega_j^2 X_j^2 \right).$$

Physically, H describes n decoupled one-dimensional oscillators with possibly distinct angular frequencies ω_j.

Definition V.1.8. Ladder operators and number operators.

For each $j = 1, \ldots, n$, the *annihilation operator* a_j and *creation operator* $a_j^\dagger$ are defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$a_j := \frac{1}{\sqrt{2\hbar m \omega_j}} \big(m\omega_j X_j + iP_j \big), \qquad a_j^\dagger := \frac{1}{\sqrt{2\hbar m \omega_j}} \big(m\omega_j X_j - iP_j \big).$$

They satisfy the commutation relations

$$[a_j, a_k^\dagger] = \delta_{jk} I, \qquad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0.$$

The *number operator* for the j-th mode is $N_j := a_j^\dagger a_j$.

Definition V.1.9. Vacuum state and excited states.

The *vacuum state* $\Phi_{\mathbf{0}} \in \mathcal{S}(\mathbb{R}^n)$ is the unique normalized function satisfying $a_j \Phi_{\mathbf{0}} = 0$ for all j. For $\mathbf{n} = (n_1, \ldots, n_n) \in \mathbb{N}_0^n$, the *excited state* $\Phi_{\mathbf{n}}$ is

$$\Phi_{\mathbf{n}} := \prod_{j=1}^{n} \frac{(a_j^\dagger)^{n_j}}{\sqrt{n_j!}} \Phi_{\mathbf{0}}.$$

We can now investigate the n-dimensional quantum harmonic oscillator's spectral structure with the following theorem:

Theorem V.1.3. Spectral structure of the n-dimensional quantum harmonic oscillator.

Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and, for $j = 1, \ldots, n$, let X_j be the operator of multiplication by x_j and $P_j := -i\hbar\,\partial_{x_j}$ on the Schwartz core $\mathcal{S}(\mathbb{R}^n)$. Fix frequencies $\omega_j > 0$ and mass $m > 0$, and define the Hamiltonian

$$H := \sum_{j=1}^{n} \left(\frac{P_j^2}{2m} + \frac{1}{2} m\omega_j^2 X_j^2 \right) \quad \text{on } \mathcal{S}(\mathbb{R}^n).$$

Then H is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$ with pure point spectrum

$$\sigma(H) = \left\{ E_{\mathbf{n}} = \sum_{j=1}^{n} \hbar\omega_j \big(n_j + \tfrac{1}{2}\big) \; : \; \mathbf{n} = (n_1, \ldots, n_n) \in \mathbb{N}_0^n \right\}.$$

Moreover, an orthonormal eigenbasis is given by the tensor products of one-dimensional Hermite functions

$$\Phi_{\mathbf{n}}(x) = \prod_{j=1}^{n} \phi_{n_j}^{(\omega_j)}(x_j),$$

where $\phi_k^{(\omega)}$ is the k-th normalized eigenfunction of the one-dimensional oscillator of frequency ω. In the isotropic case $\omega_1 = \cdots = \omega_n = \omega$, the energy depends only on $N = \sum_j n_j$, namely $E_N = \hbar\omega\big(N + \frac{n}{2}\big)$, with degeneracy

$$\deg(E_N) = \binom{N+n-1}{n-1}.$$

⌟

Proof V.1.3. Step 1: Canonical commutation relations and ladder operators. On $\mathcal{S}(\mathbb{R}^n)$ one has $[X_j, P_k] = i\hbar\,\delta_{jk} I$ and all other commutators vanish. For each j define

$$a_j := \frac{1}{\sqrt{2\hbar m\omega_j}} \big(m\omega_j X_j + iP_j\big), \qquad a_j^\dagger := \frac{1}{\sqrt{2\hbar m\omega_j}} \big(m\omega_j X_j - iP_j\big).$$

Then $[a_j, a_k^\dagger] = \delta_{jk} I$ and $[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0$ on $\mathcal{S}(\mathbb{R}^n)$.

Step 2: Normal form of H. A direct calculation yields, as quadratic forms on $\mathcal{S}(\mathbb{R}^n)$,

$$\frac{P_j^2}{2m} + \frac{1}{2}m\omega_j^2 X_j^2 \;=\; \hbar\omega_j\Big(a_j^\dagger a_j + \tfrac{1}{2}\Big).$$

Summing over j gives

$$H \;=\; \sum_{j=1}^{n} \hbar\omega_j\Big(a_j^\dagger a_j + \tfrac{1}{2}\Big).$$

Hence the number operators $N_j := a_j^\dagger a_j$ are mutually commuting, essentially self-adjoint on $\mathcal{S}$, and satisfy $[H, N_j] = 0$.

Step 3: Construction of the eigenbasis. For each j, there exists a unique (up to phase) $\phi_0^{(\omega_j)} \in L^2(\mathbb{R})$ with $a_j\phi_0^{(\omega_j)} = 0$; explicitly,

$$\phi_0^{(\omega)}(x) = \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}.$$

Define the n-dimensional vacuum $\Phi_{\mathbf{0}}(x) := \prod_{j=1}^n \phi_0^{(\omega_j)}(x_j)$, which satisfies $a_j\Phi_{\mathbf{0}} = 0$ for all j, and $H\Phi_{\mathbf{0}} = \big(\sum_j \frac{1}{2}\hbar\omega_j\big)\Phi_{\mathbf{0}}$. For $\mathbf{n} = (n_1, \dots, n_n) \in \mathbb{N}_0^n$, set

$$\Phi_{\mathbf{n}} \;:=\; \prod_{j=1}^{n} \frac{(a_j^\dagger)^{n_j}}{\sqrt{n_j!}}\,\Phi_{\mathbf{0}}.$$

Using $[N_j, a_j^\dagger] = a_j^\dagger$ and $N_j\Phi_{\mathbf{0}} = 0$, one obtains $N_j\Phi_{\mathbf{n}} = n_j\Phi_{\mathbf{n}}$ and therefore

$$H\Phi_{\mathbf{n}} \;=\; \sum_{j=1}^{n} \hbar\omega_j\Big(n_j + \tfrac{1}{2}\Big)\,\Phi_{\mathbf{n}} \;=\; E_{\mathbf{n}}\,\Phi_{\mathbf{n}}.$$

Orthogonality follows from the canonical commutation relations and orthonormality of the one-dimensional families; completeness follows from the tensor-product structure and completeness in one dimension.

Step 4: Essential self-adjointness and spectral type. The operator H is a real-valued elliptic second-order differential operator with confining quadratic potential, symmetric on $\mathcal{S}(\mathbb{R}^n)$. By standard results (e.g., Nelson's analytic vector theorem or Rellich–Kato theory), H is essentially self-adjoint on $\mathcal{S}$ and has compact resolvent; hence the spectrum is purely discrete with the claimed eigenpairs.

Isotropic degeneracy. If $\omega_1 = \cdots = \omega_n = \omega$, the energy depends only on $N = \sum_j n_j$, giving $E_N = \hbar\omega(N + \frac{n}{2})$. The number of $\mathbf{n} \in \mathbb{N}_0^n$ with $\sum_j n_j = N$ equals the number of weak compositions of N into n parts, namely $\binom{N+n-1}{n-1}$, yielding the stated degeneracy. □

Remark V.1.8. (Creation–annihilation algebra and fock representation)

The family $\{a_j, a_j^\dagger\}_{j=1}^n$ realizes n commuting copies of the Heisenberg algebra on the common invariant core $\mathcal{S}(\mathbb{R}^n)$. The eigenbasis $\{\Phi_{\mathbf{n}}\}$ is the Fock (number) basis, and $a_j^\dagger, a_j$ act as raising and lowering operators on the j-th mode independently of the others. ♦

Remark V.1.9. (Separable structure and hermite functions)

Each eigenfunction factors as a product of one-dimensional Hermite functions with the appropriate oscillator length $\ell_j = \sqrt{\hbar/(m\omega_j)}$. In particular,

$$\phi_k^{(\omega)}(x) = \frac{1}{\sqrt{2^k k!}}\Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4} H_k\Big(\sqrt{\tfrac{m\omega}{\hbar}}\,x\Big)\,e^{-\frac{m\omega}{2\hbar}x^2},$$

so $\Phi_{\mathbf{n}}(x) = \prod_j \phi_{n_j}^{(\omega_j)}(x_j)$. ♦

Remark V.1.10. (Degeneracy and symmetry)

In the isotropic case the Hamiltonian is invariant under the orthogonal group $O(n)$, and the eigenspace at fixed N furnishes a finite-dimensional representation of $O(n)$. The degeneracy $\binom{N+n-1}{n-1}$ matches the dimension of homogeneous polynomials of degree N in n variables, reflecting the $O(n)$ symmetry of the quadratic form. ♦

V.2 Weyl operators and the Segal–Bargmann transform

We present a unifying construction that synthesizes the Heisenberg–Weyl representation, ladder-operator algebra, minimum-uncertainty states, phase-space displacements, and a holomorphic model of quantum mechanics via the Segal–Bargmann transform. Throughout let $\mathcal{H} = L^2(\mathbb{R}^n)$, with position and momentum operators X_j and $P_j = -i\hbar\,\partial_{x_j}$ defined on $\mathcal{S}(\mathbb{R}^n)$, and the anisotropic oscillator Hamiltonian

$$H = \sum_{j=1}^{n} \left(\frac{P_j^2}{2m} + \frac{1}{2} m\omega_j^2 X_j^2 \right).$$

Definition V.2.1. Annihilation/creation and complex coordinates.

For each j set

$$a_j = \frac{1}{\sqrt{2\hbar m\omega_j}}\big(m\omega_j X_j + iP_j\big), \qquad a_j^\dagger = \frac{1}{\sqrt{2\hbar m\omega_j}}\big(m\omega_j X_j - iP_j\big),$$

so that $[a_j, a_k^\dagger] = \delta_{jk} I$ and

$$H = \sum_{j=1}^{n} \hbar\omega_j \left(a_j^\dagger a_j + \tfrac{1}{2} \right)$$

on $\mathcal{S}(\mathbb{R}^n)$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ define the Weyl displacement (Heisenberg–Weyl) operator

$$W(\alpha) := \exp\Big(\sum_{j=1}^{n} \alpha_j a_j^\dagger - \overline{\alpha_j}\, a_j \Big).$$

Definition V.2.2. Coherent states.

Let $\Phi_{\mathbf{0}}$ be the oscillator ground state with $a_j \Phi_{\mathbf{0}} = 0$ for all j. The *coherent state* labeled by $\alpha \in \mathbb{C}^n$ is

$$|\alpha\rangle := W(\alpha)\,\Phi_{\mathbf{0}}.$$

Equivalently, $|\alpha\rangle$ is the joint eigenvector of $\{a_j\}$ with $a_j\,|\alpha\rangle = \alpha_j\,|\alpha\rangle$.

Remark V.2.1. (Phase-space parametrization)

Writing $\alpha_j = (q_j/\sqrt{2}\ell_j) + i(p_j \ell_j/\sqrt{2}\hbar)$ with oscillator length $\ell_j = \sqrt{\hbar/(m\omega_j)}$ gives a bijection

$$\alpha \longleftrightarrow (q, p) \in \mathbb{R}^{2n},$$

i.e. coherent states are Gaussian wave packets centered at (q, p) in phase space. ♦

Proposition V.2.1. Displacement and covariance.

For $u = (q, p) \in \mathbb{R}^{2n}$, define the Weyl operator

$$D(u) \;=\; \exp\Big(\frac{i}{\hbar}\big(p\cdot X - q\cdot P\big)\Big) \;=\; e^{-\frac{i}{2\hbar}q\cdot p}\, W(\alpha(u)).$$

Then

$$D(u)D(v) = e^{\frac{i}{2\hbar}\sigma(u,v)} D(u+v)$$

with the standard symplectic form $\sigma(u, v) = p\cdot q' - q\cdot p'$, and $D(u)\,|0\rangle = |\alpha(u)\rangle$. ⌟

Proof V.2.1. Use the Baker–Campbell–Hausdorff formula with $[p\cdot X, q\cdot P] = i\hbar\, q\cdot p$ and the linear relations between (X, P) and $(a, a^\dagger)$. □

Proposition V.2.2. Minimum-uncertainty and expectation dynamics.

For each j,

$$\Delta_\alpha X_j \;=\; \frac{\ell_j}{\sqrt{2}}, \qquad \Delta_\alpha P_j \;=\; \frac{\hbar}{\sqrt{2}\,\ell_j}, \qquad \Delta_\alpha X_j\,\Delta_\alpha P_j \;=\; \frac{\hbar}{2}.$$

Moreover, under the oscillator dynamics $U(t) = e^{-iHt/\hbar}$, the expectation values obey

$$\frac{d}{dt}\,\langle X_j\rangle_{\alpha(t)} \;=\; \frac{1}{m}\langle P_j\rangle_{\alpha(t)}, \qquad \frac{d}{dt}\,\langle P_j\rangle_{\alpha(t)} \;=\; -\,m\omega_j^2\,\langle X_j\rangle_{\alpha(t)},$$

i.e. the classical Hamilton equations with frequency ω_j. ⌟

Remark V.2.2. (Resolution of identity and overcompleteness)

Coherent states form an overcomplete family with

$$\frac{1}{\pi^n}\int_{\mathbb{C}^n} |\alpha\rangle\,\langle\alpha|\; d^{2n}\alpha \;=\; I \quad \text{(strong operator topology)},$$

which enables coherent-state expansions of vectors and operators. ♦

Definition V.2.3. Segal–Bargmann (Bargmann–Fock) space.

Let $\mathcal{F}_n$ be the space of entire functions $F : \mathbb{C}^n \to \mathbb{C}$ with norm

$$\|F\|^2_{\mathcal{F}_n} \;=\; \frac{1}{\pi^n}\int_{\mathbb{C}^n} |F(z)|^2\, e^{-|z|^2}\, d^{2n}z \;<\; \infty.$$

This is a Hilbert space with reproducing kernel $K(z, w) = e^{z\cdot\overline{w}}$.

Definition V.2.4. Segal–Bargmann transform.

Define $B : \mathcal{H} \to \mathcal{F}_n$ by

$$(B\psi)(z) \;=\; \pi^{-n/4}\prod_{j=1}^{n}\ell_j^{-1/2}\int_{\mathbb{R}^n}\exp\Big(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}\tfrac{|x|^2}{\ell^2} + \tfrac{\sqrt{2}}{\ell}\,z\cdot x\Big)\,\psi(x)\,dx,$$

with $\ell = (\ell_1, \dots, \ell_n)$ and $\frac{|x|^2}{\ell^2} = \sum_j x_j^2/\ell_j^2$.

Proposition V.2.3. Unitary equivalence and holomorphic quantization.

The map B extends to a unitary isomorphism $\mathcal{H} \xrightarrow{\sim} \mathcal{F}_n$ sending

$$a_j \;\mapsto\; \partial_{z_j}, \qquad a_j^\dagger \;\mapsto\; \text{multiplication by } z_j, \qquad H \;\mapsto\; \sum_{j=1}^{n}\hbar\omega_j\left(z_j\partial_{z_j} + \tfrac{1}{2}\right).$$

In particular, number states map to monomials: $B\Phi_{\mathbf{n}}(z) = \prod_{j=1}^{n} \frac{z_j^{n_j}}{\sqrt{n_j!}}$. ⌟

Remark V.2.3. (Reproducing kernel and coherent-state overlap)

In $\mathcal{F}_n$ one has $\langle K(\cdot, w), F\rangle_{\mathcal{F}_n} = F(w)$. Under B, $|\alpha\rangle$ corresponds (up to phase) to the normalized kernel vector at $z = \alpha$, and

$$\langle \alpha|\beta\rangle = \exp\left(\overline{\alpha}\cdot\beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2\right).$$

♦

Remark V.2.4. (Semiclassical limit and Egorov property)

Under $U(t) = e^{-iHt/\hbar}$, coherent states evolve by phase-space rotations $\alpha(t) = \left(e^{-i\omega_1 t}\alpha_1, \ldots, e^{-i\omega_n t}\alpha_n\right)$ and remain coherent. For observables polynomial in (X, P) one has an exact Egorov relation $U(t)^\dagger a_j U(t) = e^{-i\omega_j t} a_j$, which in the semiclassical limit implements the classical flow on phase space. ♦

Finally, the resolution of the identity for coherent states gives a continuous matrix element expansion

$$\langle \phi|A|\psi\rangle = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \langle \phi|\alpha\rangle\,\langle \alpha|A|\psi\rangle\, d^{2n}\alpha,$$

valid in the strong operator topology for all bounded A. In the time-dependent case $A = U(t)$ this formula provides the coherent-state path integral representation

$$\langle \phi|U(t)|\psi\rangle = \frac{1}{\pi^n}\int \prod_{k=0}^{N-1} \langle \alpha_{k+1}|e^{-iH\Delta t/\hbar}|\alpha_k\rangle\,\langle \phi|\alpha_N\rangle\langle \alpha_0|\psi\rangle\,\prod_{k=0}^{N} d^{2n}\alpha_k,$$

whose $N \to \infty$ limit yields a functional integral over phase-space trajectories weighted by the coherent-state action

$$S[\alpha, \overline{\alpha}] = \int_0^t \left(\frac{i\hbar}{2}(\overline{\alpha}\cdot\dot{\alpha} - \alpha\cdot\dot{\overline{\alpha}}) - H_{\mathrm{cl}}(\alpha, \overline{\alpha})\right) dt',$$

where H_{cl} is the classical Hamiltonian obtained by replacing $(a_j, a_j^\dagger)$ with $(\alpha_j, \overline{\alpha_j})$. This analytic bridge between operator theory and phase-space geometry encapsulates the semiclassical intuition underlying coherent-state quantization.

V.3 Spectral decomposition for the free particle

We now develop the spectral representation of the free-particle Hamiltonian in n dimensions, illustrating the role of the Fourier transform as a unitary diagonalization map and clarifying the structure of generalized eigenstates.

Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and define the free Hamiltonian

$$H_0 := \frac{P^2}{2m} = -\frac{\hbar^2}{2m}\Delta,$$

with domain $\mathcal{D}(H_0) = H^2(\mathbb{R}^n)$, the Sobolev space of order 2.

Definition V.3.1. Fourier transform.

The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the unique unitary extension of

$$(\mathcal{F}\psi)(p) := (2\pi\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p\cdot x}\,\psi(x)\,dx,$$

defined initially on $\mathcal{S}(\mathbb{R}^n)$. Its inverse is given by

$$(\mathcal{F}^{-1}\phi)(x) \;:=\; (2\pi\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}p\cdot x}\,\phi(p)\,dp.$$

Proposition V.3.1. Diagonalization of H_0.

Under $\mathcal{F}$, the momentum operators become multiplication:

$$\mathcal{F}P_j\mathcal{F}^{-1} = p_j, \qquad \mathcal{F}H_0\mathcal{F}^{-1} = \frac{|p|^2}{2m}.$$

Hence $\mathcal{F}$ realizes H_0 as a multiplication operator on $L^2(\mathbb{R}^n_p)$. ⌟

Proof V.3.1. Follows from differentiation under the integral sign and unitarity of $\mathcal{F}$. The Laplacian becomes $-\frac{|p|^2}{\hbar^2}$ in momentum space, yielding the stated form of H_0. □

Remark V.3.1. (Spectral measure and generalized eigenvectors)

The spectral measure E_{H_0} is supported on $[0,\infty)$ with

$$(E_{H_0}(\Omega)\psi)^\wedge(p) = \mathbf{1}_{\{|p|^2/(2m)\in\Omega\}}\,\widehat{\psi}(p),$$

for Borel $\Omega \subset \mathbb{R}$. The momentum-space plane waves

$$\varphi_p(x) = (2\pi\hbar)^{-n/2}e^{\frac{i}{\hbar}p\cdot x}$$

are generalized eigenvectors with $H_0\varphi_p = \frac{|p|^2}{2m}\varphi_p$ in the sense of distributions, forming a Dirac orthonormal family

$$\langle\varphi_p,\varphi_{p'}\rangle = \delta(p-p').$$

♦

Remark V.3.2. (Functional calculus)

For any bounded Borel function $f:\mathbb{R}\to\mathbb{C}$,

$$f(H_0) = \mathcal{F}^{-1}\,M_{f(|p|^2/(2m))}\,\mathcal{F},$$

where M_g denotes multiplication by g. In particular, the free propagator is

$$(e^{-\frac{i}{\hbar}tH_0}\psi)(x) = (2\pi\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}p\cdot x}e^{-\frac{i}{\hbar}t|p|^2/(2m)}\,\widehat{\psi}(p)\,dp.$$

This representation is the starting point for stationary-phase asymptotics and scattering theory. ♦

Proposition V.3.2. Gauge covariance of the spectral decomposition under minimal coupling.

Fix charge $q\in\mathbb{R}$ and let $A:\mathbb{R}^n\to\mathbb{R}^n$ be a (sufficiently regular) vector potential. Define the *magnetic free Hamiltonian* on $\mathcal{S}(\mathbb{R}^n)$ by

$$H_A \;:=\; \frac{1}{2m}\big(-i\hbar\nabla - qA(x)\big)^2,$$

which is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$ and bounded below. For any smooth gauge function $\chi : \mathbb{R}^n \to \mathbb{R}$, set

$$A' := A + \nabla\chi, \qquad U_\chi := e^{\frac{iq}{\hbar}\chi(x)} \text{ (multiplication unitary on } L^2(\mathbb{R}^n)).$$

Then:

1. **Gauge covariance of the Hamiltonian:** $H_{A'} = U_\chi H_A U_\chi^{-1}$ on $\mathcal{S}(\mathbb{R}^n)$ and hence on the self-adjoint closure.
2. **Gauge covariance of the spectral calculus:** For every bounded Borel $f : \mathbb{R} \to \mathbb{C}$,
$$f(H_{A'}) = U_\chi f(H_A) U_\chi^{-1}.$$
Equivalently, if E_{H_A} and $E_{H_{A'}}$ are the spectral measures, then
$$E_{H_{A'}}(\Omega) = U_\chi E_{H_A}(\Omega) U_\chi^{-1} \qquad \text{for all Borel } \Omega \subset \mathbb{R}.$$
3. **Curvature dependence:** The spectrum and spectral measure depend only on the magnetic field $B = dA$ (the curvature two-form); in particular, H_A and $H_{A'}$ are unitarily equivalent whenever $A' - A = \nabla\chi$.

⌟

Proof V.3.2. (1) A direct computation on $\mathcal{S}(\mathbb{R}^n)$ shows

$$U_\chi\big(-i\hbar\nabla - qA\big)U_\chi^{-1} = -i\hbar\nabla - q(A + \nabla\chi) = -i\hbar\nabla - qA',$$

using $U_\chi \nabla U_\chi^{-1} = \nabla - \frac{iq}{\hbar}\nabla\chi$ and that U_χ commutes with multiplication by A. Squaring gives $U_\chi H_A U_\chi^{-1} = H_{A'}$ on $\mathcal{S}(\mathbb{R}^n)$; essential self-adjointness then extends the identity to the closures.

(2) The spectral theorem yields $f(H_{A'}) = f(U_\chi H_A U_\chi^{-1}) = U_\chi f(H_A) U_\chi^{-1}$ for bounded Borel f, by unitary invariance of the functional calculus. The relation for spectral measures follows by choosing $f = \mathbf{1}_\Omega$.

(3) If $A' - A = \nabla\chi$, parts (1)–(2) give unitary equivalence and hence equality of spectra and the claimed relation of spectral measures. Since $B = dA$ is gauge invariant, the unitary equivalence class of H_A depends only on B. □

Remark V.3.3. (Magnetic translations, holonomy, and curvature)

Let $B = dA$ denote the $U(1)$ curvature two-form. For $a \in \mathbb{R}^n$, the *magnetic translation* T_a^A on $\psi \in L^2(\mathbb{R}^n)$ may be written (Peierls substitution) as

$$(T_a^A\psi)(x) = \exp\Big(\frac{iq}{\hbar}\int_0^1 A(x + sa)\cdot a\,ds\Big)\,\psi(x + a).$$

Then T_a^A implements spatial translations up to a gauge phase, and one finds the projective relation

$$T_a^A T_b^A = \exp\Big(\frac{iq}{\hbar}\Phi_B(a, b)\Big)\,T_{a+b}^A, \qquad \Phi_B(a, b) = \int_{\square(a,b)} B,$$

where $\square(a, b)$ is the parallelogram spanned by a, b. The 2–cocycle is the magnetic flux through the parallelogram, i.e. the holonomy of the $U(1)$ connection; it vanishes iff $B = 0$, in which case T_a^A furnishes a true unitary representation of translations. This exhibits the gauge-theoretic origin of spectral features (e.g. Landau levels, Aharonov–Bohm phases) as consequences of curvature and holonomy rather than the specific choice of gauge potential A. ♦

V.4 Quantum states on a Riemannian manifold

Let (M, g) be a smooth, connected, σ–finite Riemannian manifold with Riemannian volume $d\mu_g$. We work in the position representation $\mathcal{H} := L^2(M, d\mu_g)$ (or, equivalently, the Hilbert space of square–integrable half–densities; all formulas below are unchanged).

Definition V.4.1. Geometric Schrödinger operator.

Fix $m > 0$ and a real potential $V \in L^1_{\text{loc}}(M)$ satisfying a standard self–adjointness hypothesis (e.g. V Kato–small relative to $-\Delta_g$). Define

$$H \;:=\; -\frac{\hbar^2}{2m}\,\Delta_g \;+\; V \quad \text{with domain } \mathcal{D}(H) = H^2(M) \cap \mathcal{D}(V),$$

where Δ_g is the (positive) Laplace–Beltrami operator. Then H is self–adjoint and bounded below.

Definition V.4.2. States: pure and mixed.

A *pure state* is a unit vector $\psi \in \mathcal{H}$, modulo phase. A *mixed state* (quantum probability distribution) is a *density operator*

$$\rho : \mathcal{H} \to \mathcal{H}, \qquad \rho \geq 0, \quad \rho \text{ trace class}, \quad \operatorname{Tr} \rho = 1.$$

If $\rho = |\psi\rangle\langle\psi|$ we recover the pure state; otherwise ρ is a convex combination of rank–one projectors.

Definition V.4.3. Spectral measure of H and energy distribution.

Let $E_H : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be the spectral measure of H. The *energy distribution* (or spectral measure) of a state ρ is the finite Borel measure

$$\mu_\rho(\Omega) \;:=\; \operatorname{Tr}\!\big(\rho\, E_H(\Omega)\big), \qquad \Omega \in \mathcal{B}(\mathbb{R}).$$

For a pure state ψ, $\mu_\psi(\Omega) = \|E_H(\Omega)\psi\|^2$.

Proposition V.4.1. Spectral decomposition of a density operator.

Every density operator ρ admits an orthonormal eigenbasis $\{\varphi_k\}_{k\geq 1} \subset \mathcal{H}$ and eigenvalues $\{p_k\}_{k\geq 1}$ with $p_k \geq 0$, $\sum_k p_k = 1$, such that

$$\rho \;=\; \sum_{k\geq 1} p_k\, |\varphi_k\rangle\langle\varphi_k| \qquad \text{(convergence in trace norm).}$$

┘

Proof V.4.1. A positive trace–class operator on a separable Hilbert space is compact. The spectral theorem for compact self–adjoint operators gives a discrete spectral resolution with the stated properties; trace–norm convergence follows from $\|\rho\|_1 = \sum_k p_k$. □

Theorem V.4.1. Energy–mode decomposition on compact manifolds.

Assume (M, g) is compact (with, if $\partial M \neq \emptyset$, self–adjoint boundary conditions for H). Then the spectrum of H is pure point:

$$H\phi_j \;=\; E_j \phi_j, \qquad 0 \leq E_1 \leq E_2 \leq \cdots, \quad E_j \to \infty,$$

with $\{\phi_j\}_{j\geq 1}$ an orthonormal basis of $\mathcal{H}$. For a pure state $\psi = \sum_j c_j\phi_j$ with $c_j = \langle \phi_j, \psi\rangle$, the energy distribution is

$$\mu_\psi \;=\; \sum_{j\geq 1} |c_j|^2 \,\delta_{E_j}.$$

For a mixed state $\rho = \sum_k p_k|\varphi_k\rangle\langle\varphi_k|$ one has

$$\mu_\rho \;=\; \sum_{j\geq 1}\Big(\sum_{k\geq 1} p_k\,|\langle\phi_j,\varphi_k\rangle|^2\Big)\,\delta_{E_j} \;=\; \sum_{j\geq 1}\langle\phi_j,\rho\,\phi_j\rangle\,\delta_{E_j}.$$

⌟

Proof V.4.2. Compactness implies $(-\Delta_g + V)$ has compact resolvent; hence H has discrete spectrum with complete eigenbasis. The formulas follow from $E_H(\{E_j\}) = |\phi_j\rangle\langle\phi_j|$ and the identities $\mu_\psi(\{E_j\}) = \|E_H(\{E_j\})\psi\|^2 = |\langle\phi_j,\psi\rangle|^2$, and $\mu_\rho(\{E_j\}) = \mathrm{Tr}(\rho\,|\phi_j\rangle\langle\phi_j|) = \langle\phi_j,\rho\,\phi_j\rangle$. □

Remark V.4.1. (Position–space distribution versus energy modes)

For a pure state ψ, the *quantum probability density* in position space is $x \mapsto |\psi(x)|^2$. In the energy basis,

$$|\psi(x)|^2 \;=\; \sum_{j,k} c_j\overline{c_k}\,\phi_j(x)\,\overline{\phi_k(x)}.$$

Thus the position–space distribution is a quadratic expression in energy modes, mixing diagonal terms $|\phi_j(x)|^2$ with coherences $\phi_j(x)\overline{\phi_k(x)}$ $(j \neq k)$. The eigenmodes of the *probability density* are therefore not, in general, the energy eigenfunctions; what diagonalizes the density is the *density operator* ρ, not H. ♦

Theorem V.4.2. Spectral decomposition on noncompact manifolds.

Let (M, g) be complete, noncompact, and H as above. Then the spectral theorem provides a projection–valued measure E_H on $\mathbb{R}$ and a unitary map

$$\mathcal{U} : \mathcal{H} \xrightarrow{\;\sim\;} \int_{\mathbb{R}}^{\oplus} \mathcal{H}_E\, d\nu(E),$$

such that $(\mathcal{U}H\mathcal{U}^{-1}\Phi)(E) = E\,\Phi(E)$ (multiplication by E). For $\psi \in \mathcal{H}$, its spectral transform $\widehat{\psi} := \mathcal{U}\psi$ satisfies

$$\mu_\psi(\Omega) = \int_\Omega \|\widehat{\psi}(E)\|^2_{\mathcal{H}_E}\, d\nu(E), \qquad \Omega \subset \mathbb{R} \text{ Borel.}$$

For a density operator ρ with integral kernel $K_\rho(x, y)$, the energy distribution is

$$\mu_\rho(\Omega) = \mathrm{Tr}\big(\rho\, E_H(\Omega)\big) = \int_\Omega \mathrm{Tr}_{\mathcal{H}_E}\big(\widehat{\rho}(E)\big)\, d\nu(E),$$

where $\widehat{\rho} := \mathcal{U}\,\rho\,\mathcal{U}^{-1}$ is a measurable field of trace–class operators on the fibers $\mathcal{H}_E$. ⌟

Proof V.4.3. Apply the direct–integral version of the spectral theorem for self–adjoint operators. The stated identities follow from $\mu_\psi(\Omega) = \|E_H(\Omega)\psi\|^2 = \|\mathbf{1}_\Omega\,\widehat{\psi}\|^2$ and, for trace–class ρ, from $\mathrm{Tr}(\rho\, E_H(\Omega)) = \mathrm{Tr}(\widehat{\rho}\,\mathbf{1}_\Omega)$ via unitary invariance of the trace and Fubini–Tonelli in the direct integral. □

Remark V.4.2. (Kernel formulas and generalized eigenfunctions)

When H admits a generalized eigenfunction expansion $\{\phi_E(\cdot,\xi)\}$ (e.g. scattering on $\mathbb{R}^n$), one can write, formally,

$$\psi(x) = \int_{\mathbb{R}} \int_{\Xi(E)} \widehat{\psi}(E,\xi)\, \phi_E(x,\xi)\, d\mu_E(\xi)\, d\nu(E), \qquad \mu_\psi(dE) = \int_{\Xi(E)} |\widehat{\psi}(E,\xi)|^2\, d\mu_E(\xi)\, d\nu(E),$$

with $\Xi(E)$ the multiplicity space at energy E. For a density ρ, the kernel $K_\rho(x,y)$ yields

$$\mathrm{Tr}_{\mathcal{H}_E}\big(\widehat{\rho}(E)\big) = \int_{\Xi(E)} \langle \phi_E(\cdot,\xi), \rho\, \phi_E(\cdot,\xi)\rangle\, d\mu_E(\xi).$$

♦

Proposition V.4.2. Commuting case: simultaneous diagonalization.

If $[\rho, H] = 0$, then ρ preserves each energy eigenspace (or fiber $\mathcal{H}_E$ a.e.) and admits a decomposition

$$\rho \;=\; \sum_j p_j\, |\phi_j\rangle\langle\phi_j| \quad \text{(compact case)}, \qquad \rho \;=\; \int_{\mathbb{R}}^{\oplus} \rho(E)\, d\nu(E) \quad \text{(noncompact case)},$$

with $\rho(E) \geq 0$ trace–class on $\mathcal{H}_E$ and $\int \mathrm{Tr}_{\mathcal{H}_E}(\rho(E))\, d\nu(E) = 1$. Consequently,

$$\mu_\rho(\Omega) = \sum_{E_j \in \Omega} p_j \quad \text{(compact)}, \qquad \mu_\rho(\Omega) = \int_\Omega \mathrm{Tr}_{\mathcal{H}_E}(\rho(E))\, d\nu(E) \quad \text{(noncompact)}.$$

⌟

Proof V.4.4. If $[\rho, H] = 0$, then $E_H(\Omega)\rho = \rho E_H(\Omega)$ for all Borel Ω, so ρ reduces the spectral decomposition. The block–diagonal forms follow, as does the stated form of μ_ρ. □

Remark V.4.3. (Gauge–covariant variant on line bundles)

If the wavefunctions are sections of a Hermitian line bundle $L \to M$ with unitary connection ∇^A (vector potential A), the Hilbert space is $L^2(M, L)$ and the kinetic term is the Bochner Laplacian $\nabla^{A,*}\nabla^A$. The Hamiltonian $H_A = -(\hbar^2/2m)\nabla^{A,*}\nabla^A + V$ is unitarily equivalent under $U(1)$ gauge transformations; the spectral measure and the energy distribution μ_ρ are gauge–invariant and depend only on the curvature $F_A = dA$ (the magnetic field). ♦

Summary. Given a state ρ on $L^2(M)$ and a geometric Schrödinger operator H, the *spectral decomposition* is encoded by the scalar measure $\mu_\rho(\cdot) = \mathrm{Tr}(\rho E_H(\cdot))$. On compact (M, g) this is a discrete sum over eigenvalues with weights $\langle \phi_j, \rho\, \phi_j\rangle$; on noncompact manifolds it is an absolutely continuous (and possibly singular) measure obtained by projecting ρ onto the spectral fibers via the direct–integral transform. For pure states ψ, the coefficients are $|c_j|^2$ in the compact case and $\|\widehat{\psi}(E)\|^2$ in the noncompact case. The position–space probability density $|\psi|^2$ is reconstructed as a quadratic expression in energy modes, while the *eigenmodes of the distribution* are the eigenvectors of the density operator ρ itself.

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