Cosmología
Unidad I: Ecs. de Einstein y métrica FLRW

Alejandro Avilés (CONACyT/ININ)

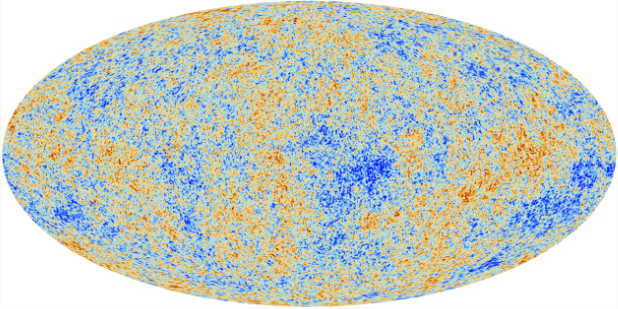
semestre 2022-1

Curso PCF-UNAM

Unidad I clase 1

11 de agosto de 2021

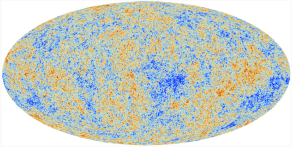
The universe is highly homogeneous and isotropic



$$\frac{\Delta T}{T} \sim 10^{-5}, \qquad T = 2.725\,\mathrm{K}$$

Cosmology

$$\text{Universe}: \quad \underbrace{\text{Universe}^{(0)}}_{\text{homogeneous and isotropic}} \quad + \quad \underbrace{\text{Universe}^{(1)}}_{\text{statistically homogeneous and isotropic}} \quad + \cdots$$



Background Cosmology

Homogeneous and isotropic

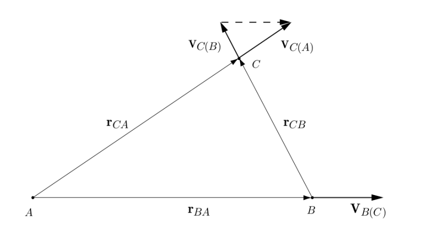
- The cosmological principle holds exactly.
  This is true when averaged over regions larger than 100-150 Mpc
  ($1\,\mathrm{Mpc} \simeq 3.26 \times 10^6$ light years $\simeq 3.08 \times 10^{24}$ cm)
  The observable patch of the universe is about $3000\,h^{-1}$Mpc.

- The Universe is filled with a collection of fluids with certain properties. e.g., they are perfectly homogeneous and isotropic.

- The Universe expands according to the Hubble law.

Hubble's law: The velocity of observer B relative to observer A is

$$\boldsymbol{v}_{B(A)} = H(t)\boldsymbol{r}_{BA}$$



The relative velocities of a third observer C with respect to A is

$$\boldsymbol{v}_{C(A)} = H(t)\boldsymbol{r}_{CA}.$$

By adding vectors, the relative velocity of C with respect to B is

$$\boldsymbol{v}_{C(B)} = \boldsymbol{v}_{C(A)} - \boldsymbol{v}_{B(A)} = H(t)(\boldsymbol{r}_{CA} - \boldsymbol{r}_{BA}) = H(t)\boldsymbol{r}_{CB},$$

which is again Hubble's law.

Actually, Hubble's law is the only form of expansion compatible with homogeneity and isotropy of the Universe

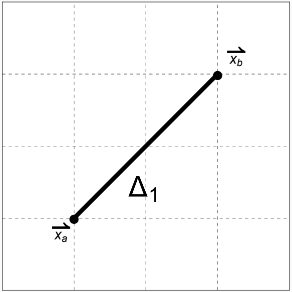
The scale factor

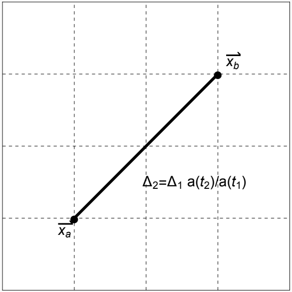
One can integrate Hubble's law and get the distance $\boldsymbol{r}$ between two points

$$\boldsymbol{r} = a(t)\boldsymbol{x}_{com}$$

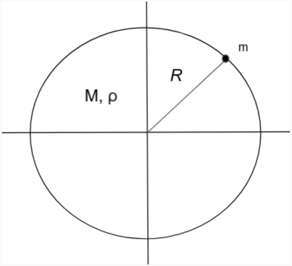
where $\boldsymbol{x}_{com}$ a constant vector, named the comoving distance, and $a(t)$ is the scale factor. Such that

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}.$$





Evolution of the scale factor



$$E = \frac{1}{2}mv^2 - \frac{mMG}{R}$$

Consider a mass element m at a position $R(t) = a(t)x$. The total mass M enclosed within the sphere at R is given in terms of the mass (or energy) density ρ:

$$M = \frac{4\pi}{3}\rho R^3$$

The velocity of the mass element is

$$v = \frac{\dot{a}}{a}R(t)$$

Hence,

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{\text{constant}}{a^2}, \qquad \text{Friedmann equation}$$

Energy density evolution

- The total energy inside the sphere $\mathcal{E} = \rho V$ is not conserved if there exist pressure support p, but
$$d\mathcal{E} = -p dV.$$

- Using $d\mathcal{E} = \rho dV + V d\rho$, and $V \propto a^3$ which implies $dV/V = 3da/a$, we have
$$d\rho = -3(\rho + p)\frac{da}{a}$$

- Or
$$\dot{\rho} + 3H(\rho + p) = 0, \qquad \text{Continuity equation}$$

Acceleration equation

Taking the derivative of the Friedmann equation with respect to time, and using the continuity equation, one arrives to

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

which is commonly called the second Friedmann equation.

- The universe expansion will be decelerating at least there is a fluid with negative pressure

$$p < -\frac{1}{3}\rho$$

Acceleration equation

Taking the derivative of the Friedmann equation with respect to time, and using the continuity equation, one arrives to

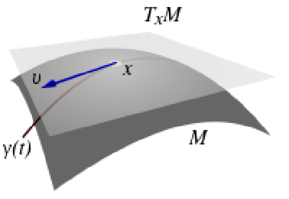
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

which is commonly called the second Friedmann equation.

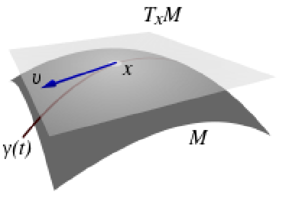
- The universe expansion will be decelerating at least there is a fluid with negative pressure
$$p < -\frac{1}{3}\rho$$
- The universe is speeding up $\implies$ dark energy

GR summary

Tensors

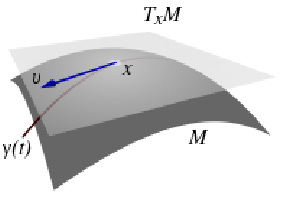


A Riemannian space is a differentiable manifold equipped with a metric.



A Riemannian space is a differentiable manifold equipped with a metric.

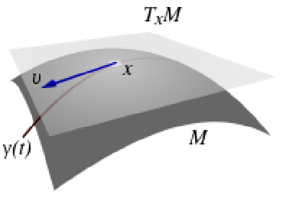
General relativity is formulated in a four-dimensional Riemannian space in which points are labelled by a general coordinate system (x_0, x_1, x_2, x_3), often written as x^μ, $\mu = 0, 1, 2, 3$.



A Riemannian space is a differentiable manifold equipped with a metric.

General relativity is formulated in a four-dimensional Riemannian space in which points are labelled by a general coordinate system (x_0, x_1, x_2, x_3), often written as x^μ, $\mu = 0, 1, 2, 3$.

Several coordinate patches may be necessary to cover the whole of space-time. The space has three spatial- and one time-like dimension.



A Riemannian space is a differentiable manifold equipped with a metric.

General relativity is formulated in a four-dimensional Riemannian space in which points are labelled by a general coordinate system (x_0, x_1, x_2, x_3), often written as $x^\mu,\ \mu = 0, 1, 2, 3$.

Several coordinate patches may be necessary to cover the whole of space-time. The space has three spatial- and one time-like dimension.

Against the change of coordinates $x^\mu \to \bar{x}^\mu(\boldsymbol{x})$, a *contravariant* vector transform as

$$\bar{A}^\mu(\bar{\boldsymbol{x}}) = \frac{\partial \bar{x}^\mu}{\partial x^\nu} A^\nu(\boldsymbol{x}),$$

where sum over repeated indices is assumed.

Covariant vectors are defined as linear functions of contravariant vectors on its field (the real numbers)

$$\mathbf{B} : V_{\boldsymbol{p}} \longrightarrow \mathbb{R}$$

Covariant vectors are defined as linear functions of contravariant vectors on its field (the real numbers)

$$\mathbf{B} : V_{\boldsymbol{p}} \longrightarrow \mathbb{R}$$

Against the coordinate transformation $x^\mu \to \bar{x}^\mu(\boldsymbol{x})$, their components transform as

$$\bar{B}_\mu(\bar{\boldsymbol{x}}) = \frac{\partial x^\nu}{\partial \bar{x}^\mu} B_\nu(\boldsymbol{x}).$$

Covariant vectors are defined as linear functions of contravariant vectors on its field (the real numbers)

$$\mathbf{B} : V_{\boldsymbol{p}} \longrightarrow \mathbb{R}$$

Against the coordinate transformation $x^\mu \to \bar{x}^\mu(\boldsymbol{x})$, their components transform as

$$\bar{B}_\mu(\bar{\boldsymbol{x}}) = \frac{\partial x^\nu}{\partial \bar{x}^\mu} B_\nu(\boldsymbol{x}).$$

The collection of covariants vectors at a point $\boldsymbol{p}$ form a vector space, the dual vector space of $V_{\boldsymbol{p}}$, and denoted by $V_{\boldsymbol{p}}^*$.

A tensor **T** of rank (k, l) is a multilinear function

$$\mathbf{T} : \underbrace{V_{\boldsymbol{p}}^* \times \cdots \times V_{\boldsymbol{p}}^*}_{k \text{ times}} \times \underbrace{V_{\boldsymbol{p}} \times \cdots \times V_{\boldsymbol{p}}}_{l \text{ times}} \longrightarrow \mathbb{R}$$

A tensor $\mathbf{T}$ of rank (k, l) is a multilinear function

$$\mathbf{T} : \underbrace{V_{\boldsymbol{p}}^{*} \times \cdots \times V_{\boldsymbol{p}}^{*}}_{k \text{ times}} \times \underbrace{V_{\boldsymbol{p}} \times \cdots \times V_{\boldsymbol{p}}}_{l \text{ times}} \longrightarrow \mathbb{R}$$

Their components transform as a k-times covariant vector and l-times contravariant vector. For example, for a tensor of rank $(2, 1)$

$$\bar{T}^{\alpha}{}_{\beta\gamma}(\bar{\boldsymbol{x}}) = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\beta}} \frac{\partial x^{\rho}}{\partial \bar{x}^{\gamma}} T^{\mu}{}_{\nu\rho}(\boldsymbol{x})$$

Since V^* and V are vector spaces, their cartesian product is also a vector space.

Since V^* and V are vector spaces, their cartesian product is also a vector space.

The tensor product of the vector spaces V_1 and V_2 is the set of all linear (in both arguments) applications (mappings) from $V_1 \times V_2$ onto its field

Since V^* and V are vector spaces, their cartesian product is also a vector space.

The tensor product of the vector spaces V_1 and V_2 is the set of all linear (in both arguments) applications (mappings) from $V_1 \times V_2$ onto its field

Hence, a base of a tensor of rank (k, l) is given by the k-product of a base of the vector space V^* times the l-product of a base of the vector space V. Let's denote some bases as $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}^\alpha$. Such that we can write

$$\boldsymbol{A} = A^\alpha \boldsymbol{v}_\alpha, \qquad \boldsymbol{B} = B_\beta \boldsymbol{v}^\beta,$$

and more generally

$$\boldsymbol{T} = T^{\alpha_1 \cdots \alpha_l}{}_{\beta_1 \cdots \beta_k} \boldsymbol{v}_{\alpha_1} \otimes \cdots \otimes \boldsymbol{v}^{\beta_k}$$

A covariant vector takes a contravariant vector and returns a real number:

$$\boldsymbol{A}(\boldsymbol{B}) = A^{\alpha} B_{\beta} \boldsymbol{v}_{\alpha}(\boldsymbol{v}^{\beta})$$

The base $\boldsymbol{v}^{\alpha}$ is called dual to $\boldsymbol{v}_{\alpha}$ if

$$\boldsymbol{v}_{\alpha}(\boldsymbol{v}^{\beta}) = \delta_{\alpha}^{\beta}$$

If this is the case, the map of $\boldsymbol{A}$ maps $\boldsymbol{B}$ as

$$\boldsymbol{A}(\boldsymbol{B}) = A^{\mu} B_{\mu}$$

which is also called the contraction of $\boldsymbol{A}$ and $\boldsymbol{B}$.

A covariant vector takes a contravariant vector and returns a real number:

$$\boldsymbol{A}(\boldsymbol{B}) = A^{\alpha} B_{\beta} \boldsymbol{v}_{\alpha}(\boldsymbol{v}^{\beta})$$

The base $\boldsymbol{v}^{\alpha}$ is called dual to $\boldsymbol{v}_{\alpha}$ if

$$\boldsymbol{v}_{\alpha}(\boldsymbol{v}^{\beta}) = \delta_{\alpha}^{\beta}$$

If this is the case, the map of $\boldsymbol{A}$ maps $\boldsymbol{B}$ as

$$\boldsymbol{A}(\boldsymbol{B}) = A^{\mu} B_{\mu}$$

which is also called the contraction of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Note, the full contraction of tensors is a scalar:

$$\begin{aligned} \bar{A}^{\mu} \bar{B}_{\mu} &= \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}} B_{\beta} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}} A^{\alpha} B_{\beta} \\ &= \frac{\partial x^{\beta}}{\partial x^{\alpha}} A^{\alpha} B_{\beta} = \delta_{\beta}^{\alpha} A^{\alpha} B_{\beta} = A^{\mu} B_{\mu} \end{aligned}$$

The metric tensor

Roughly speaking, the metric tensor $g_{\mu\nu}$ is a function which tells how to compute the distance between any two points in a given space. Its components can be viewed as multiplication factors which must be placed in front of infinitesimal displacements dx^μ in a generalized Pythagorean theorem

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

The metric tensor

Roughly speaking, the metric tensor $g_{\mu\nu}$ is a function which tells how to compute the distance between any two points in a given space. Its components can be viewed as multiplication factors which must be placed in front of infinitesimal displacements dx^μ in a generalized Pythagorean theorem

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

The metric has some properties

- It is symmetric $g(\boldsymbol{x}_1, \boldsymbol{x}_2) = g(\boldsymbol{x}_2, \boldsymbol{x}_1)$ or $g_{\mu\nu} = g_{\nu\mu}$.
- Is is non-degenerate, meaning that if $g(\boldsymbol{q}, \boldsymbol{x}_1) = 0$ for all $\boldsymbol{q}$ in V, then $\boldsymbol{x}_1 = 0$.

The metric tensor

Roughly speaking, the metric tensor $g_{\mu\nu}$ is a function which tells how to compute the distance between any two points in a given space. Its components can be viewed as multiplication factors which must be placed in front of infinitesimal displacements dx^μ in a generalized Pythagorean theorem

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

The metric has some properties

- It is symmetric $g(\boldsymbol{x}_1, \boldsymbol{x}_2) = g(\boldsymbol{x}_2, \boldsymbol{x}_1)$ or $g_{\mu\nu} = g_{\nu\mu}$.
- Is is non-degenerate, meaning that if $g(\boldsymbol{q}, \boldsymbol{x}_1) = 0$ for all $\boldsymbol{q}$ in V, then $\boldsymbol{x}_1 = 0$.

At each point $\boldsymbol{p}$, a metric g is a tensor of type (0,2) over V_p, i.e. a multilinear map from $V_p \times V_p \to \mathbb{R}$. However, we can also view g as a linear map from V_p into V_p^* via

$$\boldsymbol{A} \to \tilde{\boldsymbol{A}} = g(\boldsymbol{A}, \cdot)$$

given that the metric is nondegenerate, this map is one-to-one.

The components of $\tilde{\boldsymbol{A}}$ are

$$\begin{aligned} A_\alpha &= \tilde{\boldsymbol{A}}(\boldsymbol{v}_\alpha) = g_{\mu\nu}(\boldsymbol{v}^\mu \otimes \boldsymbol{v}^\nu)\big(A^\beta \boldsymbol{v}_\beta, \boldsymbol{v}_\alpha\big) \\ &= A^\beta g_{\mu\nu} \boldsymbol{v}^\mu(\boldsymbol{v}_\beta)\boldsymbol{v}^\nu(\boldsymbol{v}_\alpha) = A^\beta g_{\mu\nu}\delta^\mu_\alpha \delta^\nu_\beta \\ &= g_{\alpha\beta}A^\beta. \end{aligned}$$

Hence, the metric serves to lower indices.

The inverse of the metric g^{-1} is a (2,0) tensor, such that

$$g^{-1}g = gg^{-1} = I$$

and it is easy to show that

$$(g^{-1})^{\mu\nu} = g^{\mu\nu},$$

That is

$$g^{\mu\alpha}g_{\alpha\nu} = g^\mu{}_\nu = \delta^\mu_\nu$$

It follows that the metric also raises indices:

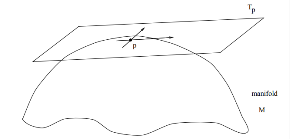
$$A^\alpha = g^{\alpha\beta}A_\beta.$$

Clase 2

16 de agosto de 2021

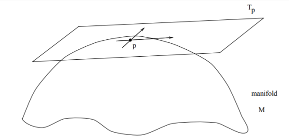
Vectors again

How can we construct the tangent space T_p of a Manifold M at a point p?



Vectors again

How can we construct the tangent space T_p of a Manifold M at a point p?



Consider the *space* of all curves of the form

$$\gamma : \mathbb{R} \to M,$$

such that $\gamma(\lambda) = p$.

In a coordinate system x^μ, any curve defines the set of numbers $dx^\mu/d\lambda$ in $\mathbb{R}^n$. One can try to define T_p as the collection of tangent vectors $dx^\mu/d\lambda$ at p. But this map is coordinate dependent, so it is not inherent to the manifold.

Vectors again

Let $\mathcal{F}$ to be the space of all smooth functions on M

$$f : M \to \mathbb{R}.$$

Each curve through p defines an operator on this space, the directional derivative, that maps

$$f \to \left.\frac{df}{d\lambda}\right|_p$$

1. The space of directional derivatives is a vector space
2. It has dimension n and gives the notion of a vector pointing in a certain direction

Consider two curves $x^\mu(\lambda)$ and $x^\mu(\eta)$ through p. Clearly the scaled sum

$$a\frac{d}{d\lambda} + b\frac{d}{d\eta}$$

is a new operator.

Consider two curves $x^\mu(\lambda)$ and $x^\mu(\eta)$ through p. Clearly the scaled sum

$$a\frac{d}{d\lambda} + b\frac{d}{d\eta}$$

is a new operator.

Now, this is also a directional derivative operator, since

$$\begin{aligned}\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)(fg) &= af\frac{dg}{d\lambda} + bf\frac{dg}{d\eta} + ag\frac{df}{d\lambda} + bg\frac{df}{d\eta} \\ &= g\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)f + f\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)g.\end{aligned}$$

i.e. it obeys Leibniz rule.

Consider two curves $x^\mu(\lambda)$ and $x^\mu(\eta)$ through p. Clearly the scaled sum

$$a\frac{d}{d\lambda} + b\frac{d}{d\eta}$$

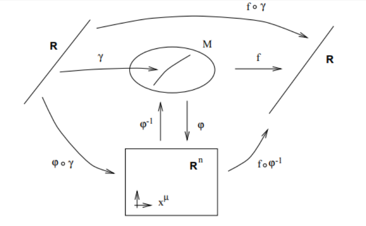
is a new operator.

Now, this is also a directional derivative operator, since

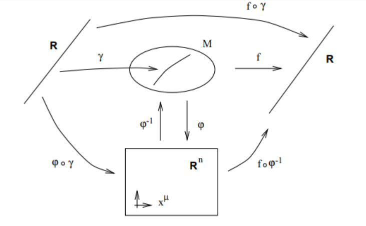
$$\begin{aligned}\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)(fg) &= af\frac{dg}{d\lambda} + bf\frac{dg}{d\eta} + ag\frac{df}{d\lambda} + bg\frac{df}{d\eta} \\ &= g\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)f + f\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)g.\end{aligned}$$

i.e. it obeys Leibniz rule.

Hence, the space of directional derivatives is a vector space.



coordinate chart	$\varphi : M \to \mathbb{R}^n$
curve	$\gamma : \mathbb{R} \to M$
function	$f : M \to \mathbb{R}$

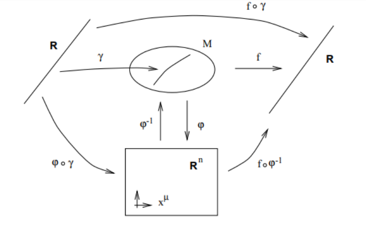


coordinate chart $\varphi : M \to \mathbb{R}^n$

curve $\gamma : \mathbb{R} \to M$

function $f : M \to \mathbb{R}$

$$\frac{df}{d\lambda} = \frac{d(f \circ \gamma)}{d\lambda} = \frac{d\left[(f \circ \varphi^{-1}) \circ (\varphi \circ \gamma)\right]}{d\lambda} = \frac{d(\varphi \circ \gamma)^{\mu}}{d\lambda} \frac{\partial(f \circ \varphi^{-1})}{\partial x^{\mu}}$$
$$= \frac{dx^{\mu}}{d\lambda} \partial_{\mu} f$$

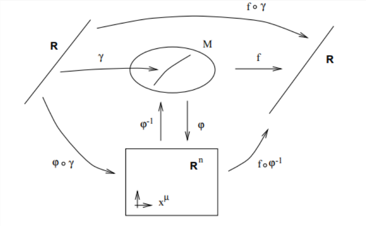


coordinate chart	$\varphi : M \to \mathbb{R}^n$
curve	$\gamma : \mathbb{R} \to M$
function	$f : M \to \mathbb{R}$

$$\frac{df}{d\lambda} = \frac{d(f \circ \gamma)}{d\lambda} = \frac{d\left[(f \circ \varphi^{-1}) \circ (\varphi \circ \gamma)\right]}{d\lambda} = \frac{d(\varphi \circ \gamma)^\mu}{d\lambda} \frac{\partial (f \circ \varphi^{-1})}{\partial x^\mu}$$
$$= \frac{dx^\mu}{d\lambda} \partial_\mu f$$

Since f is arbitrary

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu .$$



coordinate chart $\varphi : M \to \mathbb{R}^n$

curve $\gamma : \mathbb{R} \to M$

function $f : M \to \mathbb{R}$

$$\frac{df}{d\lambda} = \frac{d(f \circ \gamma)}{d\lambda} = \frac{d\big[(f \circ \varphi^{-1}) \circ (\varphi \circ \gamma)\big]}{d\lambda} = \frac{d(\varphi \circ \gamma)^\mu}{d\lambda} \frac{\partial (f \circ \varphi^{-1})}{\partial x^\mu}$$
$$= \frac{dx^\mu}{d\lambda} \partial_\mu f$$

Since f is arbitrary

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu .$$

Hence, the $\{\partial_\mu\}$ are a basis for the vector space of directional derivatives, which can be identified with the tangent space T_p

These particular bases $\boldsymbol{v}_\alpha = \partial_\alpha$ are called *coordinate bases*. Changing to other coordinate system

$$\partial_{\bar{\mu}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}}\,\partial_\mu.$$

Clearly, a vector

$$\boldsymbol{A} = A^\mu \partial_\mu = A^{\bar{\mu}} \partial_{\bar{\mu}}$$

is coordinate independent.

These particular bases $\boldsymbol{v}_\alpha = \partial_\alpha$ are called *coordinate bases*. Changing to other coordinate system

$$\partial_{\bar{\mu}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \, \partial_\mu .$$

Clearly, a vector

$$\boldsymbol{A} = A^\mu \partial_\mu = A^{\bar{\mu}} \partial_{\bar{\mu}}$$

is coordinate independent.

Notice that vectors at p live in the tangent space T_p, one cannot move them freely through the manifold M.

one-forms (or covariant vectors)

The cotangent space T_p^* is the set of linear functions $w : T_p \to \mathbb{R}$. It is a result of linear algebra that this is a vector space with the same dimension n. The canonical example of a one-form is the gradient of a function.

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu.$$

Indeed, the dx^μ constitute a basis for T_p^*, and it is dual to ∂_μ

$$dx^\mu(\partial_\nu) = \delta^\mu_\nu.$$

one-forms (or covariant vectors)

The cotangent space T_p^* is the set of linear functions $w : T_p \to \mathbb{R}$. It is a result of linear algebra that this is a vector space with the same dimension n. The canonical example of a one-form is the gradient of a function.

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu.$$

Indeed, the dx^μ constitute a basis for T_p^*, and it is dual to ∂_μ

$$dx^\mu(\partial_\nu) = \delta^\mu_\nu.$$

A general tensor can be written as

$$\boldsymbol{T} = T^{\alpha_1 \cdots \alpha_l}{}_{\beta_1 \cdots \beta_k}\, \partial_{\alpha_1} \otimes \cdots \otimes dx^{\beta_k}$$

$$\text{with} \qquad T^{\alpha_1 \cdots \alpha_l}{}_{\beta_1 \cdots \beta_k} = \boldsymbol{T}(dx^{\alpha_1}, \ldots, \partial_{\beta_k})$$

The metric again

The metric is a (0,2) tensor:

$$\boldsymbol{g} = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

where $dx^{\mu} dx^{\nu}$ is a shorthand notation for $dx^{\mu} \otimes dx^{\nu}$.

The metric again

The metric is a (0,2) tensor:

$$\boldsymbol{g} = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

where $dx^{\mu} dx^{\nu}$ is a shorthand notation for $dx^{\mu} \otimes dx^{\nu}$.

It is often called the interval

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}.$$

The metric again

The metric is a (0,2) tensor:

$$\boldsymbol{g} = g_{\mu\nu} dx^\mu dx^\nu$$

where $dx^\mu dx^\nu$ is a shorthand notation for $dx^\mu \otimes dx^\nu$.

It is often called the interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu .$$

The metric is symmetric, hence (in $n = 4$) it has 10 degrees of freedom.

The metric again

The metric is a (0,2) tensor:

$$\boldsymbol{g} = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

where $dx^{\mu} dx^{\nu}$ is a shorthand notation for $dx^{\mu} \otimes dx^{\nu}$.

It is often called the interval

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} .$$

The metric is symmetric, hence (in $n = 4$) it has 10 degrees of freedom.

In GR, the metric replaces the Newtonian gravitational potential.

Local flatness

There are coordinates x^μ such that a point $p \in M$

$$
\begin{aligned}
g_{\mu\nu}|_p &= \eta_{\mu\nu}, \\
\partial_\alpha g_{\mu\nu}|_p &= 0, \\
\partial_\alpha \partial_\beta g_{\mu\nu}|_p &\neq 0,
\end{aligned}
$$

hence, at $\boldsymbol{p}$

$$g_{\mu\nu}|_p = \eta_{\mu\nu} + \mathcal{O}_2(x^\mu)$$

Local flatness

There are coordinates x^μ such that a point $p \in M$

$$
\begin{aligned}
g_{\mu\nu}|_p &= \eta_{\mu\nu}, \\
\partial_\alpha g_{\mu\nu}|_p &= 0, \\
\partial_\alpha \partial_\beta g_{\mu\nu}|_p &\neq 0,
\end{aligned}
$$

hence, at $\boldsymbol{p}$

$$g_{\mu\nu}|_p = \eta_{\mu\nu} + \mathcal{O}_2(x^\mu)$$

Space-time is locally Minkowsky for Free-falling observers

The laws of physics are given by the Standard Model for free-falling observers

Covariant derivative

A *covariant operator* ∇ is a map from (k, l) tensors to $(k, l+1)$ tensors which obeys

1. Linearity: $\nabla(V_1 + V_2) = \nabla V_1 + \nabla V_2$
2. Leibniz rule: $\nabla(V_1 \otimes V_2) = V_1 \otimes \nabla V_2 + (\nabla V_1) \otimes V_2$

Covariant derivative

A *covariant operator* ∇ is a map from (k, l) tensors to $(k, l+1)$ tensors which obeys

1. Linearity: $\nabla(V_1 + V_2) = \nabla V_1 + \nabla V_2$
2. Leibniz rule: $\nabla(V_1 \otimes V_2) = V_1 \otimes \nabla V_2 + (\nabla V_1) \otimes V_2$

- For scalar functions $\nabla = \partial_\mu$ is a covariant derivative. This option leads to

$$\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f, \qquad \text{for all } f \in \mathcal{F}.$$

  This is a torsion-free covariant derivative

For vectors (tensors (1,0)) the naive option ∂_μ does not work, since $\partial_\nu V^\mu$ is not a tensor, bcs it is coordinate dependent. However the quantity

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\sigma} A^\sigma$$

is a (1,1) tensor if the *connection coefficients* $\boldsymbol{\Gamma}$ transform as

$$\Gamma^{\bar{\nu}}_{\bar{\mu}\bar{\sigma}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \frac{\partial x^\sigma}{\partial x^{\bar{\sigma}}} \frac{\partial x^{\bar{\nu}}}{\partial x^\nu} \Gamma^\nu_{\mu\sigma} - \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \frac{\partial x^\sigma}{\partial x^{\bar{\sigma}}} \frac{\partial^2 x^{\bar{\nu}}}{\partial x^\mu \partial x^\sigma}$$

For vectors (tensors (1,0)) the naive option ∂_μ does not work, since $\partial_\nu V^\mu$ is not a tensor, bcs it is coordinate dependent. However the quantity

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\sigma} A^\sigma$$

is a (1,1) tensor if the *connection coefficients* $\boldsymbol{\Gamma}$ transform as

$$\Gamma^{\bar{\nu}}_{\bar{\mu}\bar{\sigma}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \frac{\partial x^\sigma}{\partial x^{\bar{\sigma}}} \frac{\partial x^{\bar{\nu}}}{\partial x^\nu} \Gamma^\nu_{\mu\sigma} - \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \frac{\partial x^\sigma}{\partial x^{\bar{\sigma}}} \frac{\partial^2 x^{\bar{\nu}}}{\partial x^\mu \partial x^\sigma}$$

Given a covariant ∇, the operator $\tilde{\nabla}$

$$\tilde{\nabla}_\mu A^\alpha = \nabla_\mu A^\alpha - C^\alpha_{\mu\nu} A^\nu$$

is a covariant derivative if $C^\alpha_{\mu\nu}$ is a tensor symmetric in its lower indices.

What is the corresponding covariant derivative of a co-vector $\boldsymbol{B}$?

$$\text{Use} \quad \nabla_\mu (A^\mu B_\mu) = \partial_\mu (A^\mu B_\mu)$$

and obtain

$$\nabla_\mu B_\nu = \partial_\mu B_\nu - \Gamma^\sigma_{\mu\nu} B_\sigma$$

!

What is the corresponding covariant derivative of a co-vector $\boldsymbol{B}$?

$$\text{Use} \quad \nabla_\mu (A^\mu B_\mu) = \partial_\mu (A^\mu B_\mu)$$

and obtain

$$\nabla_\mu B_\nu = \partial_\mu B_\nu - \Gamma^\sigma_{\mu\nu} B_\sigma$$

!

For a general tensor add a $+\Gamma$ for each upper index, and a $-\Gamma$ for each lower index. e.g.:

$$\nabla_\mu S^\alpha_{\ \beta} = \partial_\mu S^\alpha_{\ \beta} + \Gamma^\alpha_{\mu\sigma} S^\sigma_{\ \beta} - \Gamma^\sigma_{\mu\beta} S^\alpha_{\ \sigma}$$

Parallel transport

Given a curve γ on M with tangent vector $t^\alpha = dx^\alpha(\lambda)/d\lambda$, and a covariant derivative we can define the notion of *parallel transport* of a vector along the curve γ.

A^α is parallelly transported along γ if

$$t^\mu \nabla_\mu A^\alpha = 0 \quad = \frac{dA^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\nu} t^\mu A^\nu$$

This equation has a unique solution. So, given a curve γ connecting two points p and q in M, we can map the tangent spaces T_p and T_q.

Parallel transport

Given a curve γ on M with tangent vector $t^\alpha = dx^\alpha(\lambda)/d\lambda$, and a covariant derivative we can define the notion of *parallel transport* of a vector along the curve γ.

A^α is parallelly transported along γ if

$$t^\mu \nabla_\mu A^\alpha = 0 \quad = \frac{dA^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\nu} t^\mu A^\nu$$

This equation has a unique solution. So, given a curve γ connecting two points p and q in M, we can map the tangent spaces T_p and T_q. A desire property of this map is that it preserves the inner product. That is

$$t^\alpha \nabla_\alpha (g_{\mu\nu} A^\mu B^\nu) = 0,$$

for all parallel transported vector fields A and B, and any curve γ.

This implies

$$\nabla_\alpha g_{\mu\nu} = 0.$$

A covariant derivative with this property is called *metric compatible*. This derivative is unique, its connections are given by the *Christoffel symbols*

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

Often, the Christoffel symbols are written as $\left\{ {\alpha \atop \mu\nu} \right\}$.

Exercise:

Compute the Christoffel symbols of the following metric

$$\boldsymbol{g} = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

as you know, this is called the FLRW metric and describes the homogeneous and isotropic space-time with zero spatial curvature.

Exercise:

Compute the Christoffel symbols of the following metric

$$\boldsymbol{g} = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

as you know, this is called the FLRW metric and describes the homogeneous and isotropic space-time with zero spatial curvature.

Solution:

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{0j} = \Gamma^i_{00} = \Gamma^i_{jk} = 0,$$

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij}, \qquad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$$

Geodesics

Let γ a curve on M with tangent vector $T^\alpha = dx^\alpha(\lambda)/d\lambda$

- Definition 1: A geodesic is a curve that parallel transport its tangent vector: $T^\mu \nabla_\mu T^\alpha = 0$:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0.$$

Geodesics

Let γ a curve on M with tangent vector $T^\alpha = dx^\alpha(\lambda)/d\lambda$

- Definition 1: A geodesic is a curve that parallel transport its tangent vector: $T^\mu \nabla_\mu T^\alpha = 0$:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0.$$

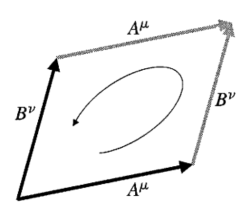
- Definition 2: A geodesic is a curve that extremizes the length between two points

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \quad \text{and} \quad \nabla \boldsymbol{g} = 0.$$

Clase 3

18 de agosto de 2021

Riemann curvature

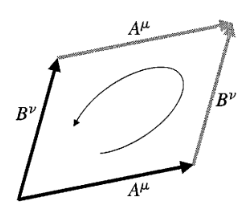


We parallel transport a vector V around a closed infinitesimal loop, constructed by two infinitesimal vectors A and B.

We expect there exists a tensor that quantifies the change on the vector when it comes back to its starting point

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$

Riemann curvature



We parallel transport a vector V around a closed infinitesimal loop, constructed by two infinitesimal vectors A and B.

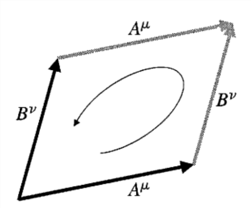
We expect there exists a tensor that quantifies the change on the vector when it comes back to its starting point

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$

Where the Riemann tensor is given by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

Riemann curvature



We parallel transport a vector V around a closed infinitesimal loop, constructed by two infinitesimal vectors A and B.

We expect there exists a tensor that quantifies the change on the vector when it comes back to its starting point

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$

Where the Riemann tensor is given by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

The Riemann tensor also quantifies the failure of covariant derivatives to commute. e.g., on a vector

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$$

Symmetries of the Riemann tensor

- Invariant under the interchange of the two first pair of indices with the second

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$

- Antisymmetric in its first two indices

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$$

- Antisymmetric in its last two indices

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$$

- The sum of cyclic permutations of its last three indices vanishes:

$$R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} + R_{\mu\gamma\alpha\beta} = 0$$

Symmetries of the Riemann tensor

- Invariant under the interchange of the two first pair of indices with the second

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$

- Antisymmetric in its first two indices

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$$

- Antisymmetric in its last two indices

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$$

- The sum of cyclic permutations of its last three indices vanishes:

$$R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} + R_{\mu\gamma\alpha\beta} = 0$$

The Riemman tensor has 4^4 components, but these symmetries imply only 20 of these are independent

Einstein Tensor

Bianchi identity: $\nabla_{[\gamma} R_{\rho\sigma]\mu\nu} = 0$

The Ricci tensor is the contraction of the Riemann tensor

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$$

and the Ricci scalar is the trace of the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}$$

Einstein Tensor

Bianchi identity: $\nabla_{[\gamma} R_{\rho\sigma]\mu\nu} = 0$

The Ricci tensor is the contraction of the Riemann tensor

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$$

and the Ricci scalar is the trace of the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}$$

We can construct the *Einstein tensor*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$$

with the property

$$\nabla^{\mu} G_{\mu\nu} = 0.$$

Ricci scalar and maximally symmetric space-times

Ricci scalar in flat space: $R = 0$

Ricci scalar and maximally symmetric space-times

Ricci scalar in flat space: $R = 0$

dem: In Minkowski space there is a coordinate system where the metric components are $g_{\alpha\beta} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Thus, in this coordinate system the Christoffel symbols vanish, and hence all the components of the Riemann tensor vanish as well. Then R becomes zero in this coordinate system. But R is a scalar, so its value is coordinate independent. $\Rightarrow R = 0$.

Ricci scalar in flat-FLRW:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

Using the Christoffel symbols $\Gamma^i_{j0} = \Gamma^i_{0j} = H\delta^i_j$, and $\Gamma^0_{ij} = a\dot{a}\delta_{ij}$, and the rest are zero, one arrives after a lot of algebra to

$$R = 6(\dot{H} + 2H^2)$$

(Remind $H = \dot{a}/a$.)

Hence, FLRW is a curved space-time.

Are there solutions with constant curvature? Notice that if H is constant, that is, if $a(t) = \exp(H_0 t)$, then $R = 12H_0^2$ = constant

Spacetimes with constant curvature are called maximally symmetric spacetimes. They have the maximum number of independent isometries (diffeomorphisms that leave the metric invariant). For a manifold of n dimensions this number is $n + \frac{1}{2}n(n-1) = n(n+1)/2$. For our spacetime, this is 10.

Spacetimes with constant curvature are called maximally symmetric spacetimes. They have the maximum number of independent isometries (diffeomorphisms that leave the metric invariant). For a manifold of n dimensions this number is $n + \frac{1}{2}n(n-1) = n(n+1)/2$. For our spacetime, this is 10.

I am not going to enter into mathematical details here, but each symmetry (continuous with the identity symmetry) of the spacetime generates an isometry. In FLRW we can foliate the spacetime in space-like hypersurfaces of equal constant coordinate time (this can be the time where we have written the FLRW metric). By construction we have isotropy (3-rotations) and homogeneity (3-translations) over this foliation. Hence FLRW have 9 symmetries. If we further demand a time elapsed symmetry we have have an additional isometry, yielding a total of 10 symmetries, and then $R =$ constant.

Spacetimes with constant curvature are called maximally symmetric spacetimes. They have the maximum number of independent isometries (diffeomorphisms that leave the metric invariant). For a manifold of n dimensions this number is $n + \frac{1}{2}n(n-1) = n(n+1)/2$. For our spacetime, this is 10.

I am not going to enter into mathematical details here, but each symmetry (continuous with the identity symmetry) of the spacetime generates an isometry. In FLRW we can foliate the spacetime in space-like hypersurfaces of equal constant coordinate time (this can be the time where we have written the FLRW metric). By construction we have isotropy (3-rotations) and homogeneity (3-translations) over this foliation. Hence FLRW have 9 symmetries. If we further demand a time elapsed symmetry we have have an additional isometry, yielding a total of 10 symmetries, and then $R =$ constant.

$$\begin{aligned} R = \text{constant} < 0 &\qquad \text{Anti-de Sitter (AdS)} \\ R = 0 &\qquad \text{Minkowski} \\ R = \text{constant} > 0 &\qquad \text{de Sitter (dS)} \end{aligned}$$

According to our most successful cosmological theory, we live in a universe that is asymptotically de Sitter.

In a maximally symmetric manifold, the geometry looks the same in all directions and positions. Then, the curvature tensor should look the same in all directions and positions. So, it should be constructed out of the metric, the Kronecker delta and the curvature scalar (that should be a constant). Using the symmetries of the Riemann tensor, there is only one option

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$$

This equation holds for any maximally symmetric Riemannian manifold, in any number of dimensions n, at any point, in any coordinate system.

velocity

The 4-velocity is the invariant generalization of the velocity of a particle through space-time,

$$U^\mu = \frac{dx^\mu}{d\tau}$$

where τ is its proper time. Consider flat spacetime, and small velocities, then $U^\mu = (1, \boldsymbol{v})$, and one recovers the idea of velocity. It is immediate to prove that

$$g_{\mu\nu} U^\mu U^\nu = -1$$

velocity

The 4-velocity is the invariant generalization of the velocity of a particle through space-time,

$$U^\mu = \frac{dx^\mu}{d\tau}$$

where τ is its proper time. Consider flat spacetime, and small velocities, then $U^\mu = (1, \boldsymbol{v})$, and one recovers the idea of velocity. It is immediate to prove that

$$g_{\mu\nu} U^\mu U^\nu = -1$$

The 4-velocity is tangent to the trajectories of time-like worldlines. In particular, for a time-like geodesic γ

$$\frac{dU^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0$$

where the affine parameter and the proper time are linealy related $\lambda = a\tau + b$

geodesics in FLRW I

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

geodesics in FLRW I

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$x^0 = t : \qquad \frac{d^2t}{d\tau^2} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{d^2t}{d\tau^2} + a\dot{a}\frac{dx^i}{d\tau}\frac{dx_i}{d\tau} = 0,$$

$$x^i = \boldsymbol{x} : \qquad \frac{d^2x^i}{d\tau^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{d^2x^i}{d\tau^2} + 2H\frac{dx^i}{d\tau}\frac{dt}{d\tau} = 0$$

geodesics in FLRW I

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$x^0 = t: \qquad \frac{d^2t}{d\tau^2} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{d^2t}{d\tau^2} + a\dot{a}\frac{dx^i}{d\tau}\frac{dx_i}{d\tau} = 0,$$

$$x^i = \boldsymbol{x}: \qquad \frac{d^2x^i}{d\tau^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{d^2x^i}{d\tau^2} + 2H\frac{dx^i}{d\tau}\frac{dt}{d\tau} = 0$$

A solution is

$$t = \tau, \qquad x^i = \text{constants}$$

The free-fall observers have constant comoving coordinates x^i and their proper time τ coincides with the cosmic time t.

These are the observers that see an isotropic and homogeneous universe.

momentum

For a time-like particle we multiply the four-velocity by its mass m

$$P^\mu = mU^\mu \qquad \text{four-momentum.}$$

momentum

For a time-like particle we multiply the four-velocity by its mass m

$$P^\mu = mU^\mu \qquad \text{four-momentum.}$$

Its components are given by

$$P^\mu = (E, P^i)$$

For a non-relativistic particle in flat space $P^\mu = (m, m\boldsymbol{v} = \boldsymbol{p})$. Notice that

$$g_{\mu\nu}P^\mu P^\nu = -m^2$$

momentum

For a time-like particle we multiply the four-velocity by its mass m

$$P^\mu = mU^\mu \qquad \text{four-momentum.}$$

Its components are given by

$$P^\mu = (E, P^i)$$

For a non-relativistic particle in flat space $P^\mu = (m, m\boldsymbol{v} = \boldsymbol{p})$. Notice that

$$g_{\mu\nu} P^\mu P^\nu = -m^2$$

This definition can be extended to non time-like worldlines:

$$g_{\mu\nu} P^\mu P^\nu = 0 \qquad \text{(for massless particles)}$$

It is convenient to go to a Lorentz frame in flat space-time

$$P^\mu = (\gamma m, \gamma m v), \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

e.g., for small v, $E = p^0 = m + \frac{1}{2} m v^2$.

More generally $P_\mu P^\mu = -m^2$, or $E = \sqrt{p^2 + m^2}$, with $p^2 = g_{ij} P^i P^j$

It is convenient to go to a Lorentz frame in flat space-time

$$P^\mu = (\gamma m, \gamma m v), \qquad \gamma = \frac{1}{\sqrt{1 - v^2}}$$

e.g., for small v, $E = p^0 = m + \frac{1}{2} m v^2$.

More generally $P_\mu P^\mu = -m^2$, or $E = \sqrt{p^2 + m^2}$, with $p^2 = g_{ij} P^i P^j$

The energy of P^μ, measured by observer V^μ is

$$E = g_{\mu\nu} P^\mu V^\nu = -P_\mu V^\mu$$

e.g. A photon has energy $2\pi\nu$ (we are using units where $\hbar = 1$), as measured by a reference frame. That is, $P^\mu = (2\pi\nu, 2\pi\nu)$. A second observer has a four velocity $V^\mu = (\gamma, \gamma v)$. Then, the energy of the photon as measured by this second observer is

$$E = -\eta_{\mu\nu} P^\mu V^\nu = -(-2\pi\nu\gamma + 2\pi\nu\gamma v) = 2\pi\nu(1 - v)\gamma = 2\pi\nu\sqrt{\frac{1 - v}{1 + v}}$$

Longitudinal relativistic Doppler effect

geodesics FLRW again

From $P^\alpha = \frac{dx^\alpha}{d\lambda}$, $\quad \frac{d}{d\lambda} = \frac{dx^0}{d\lambda}\frac{d}{dx^0} = E\frac{d}{dt}$

Geodesic equations become

$$E\frac{dP^\mu}{dt} = -\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta.$$

geodesics FLRW again

From $P^\alpha = \frac{dx^\alpha}{d\lambda}$, $\quad \frac{d}{d\lambda} = \frac{dx^0}{d\lambda}\frac{d}{dx^0} = E\frac{d}{dt}$

Geodesic equations become

$$E\frac{dP^\mu}{dt} = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta.$$

For FLRW

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j,$$

Christoffel symbols are

$$\Gamma^0_{ij} = a\dot{a}\gamma_{ij}, \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = H\delta^i_j, \qquad \Gamma^i_{jk} = \frac{1}{2}\gamma^{il}(\partial_k\gamma_{jl} + \partial_j\gamma_{kl} - \partial_l\gamma_{jk})$$

geodesics FLRW again

From $P^\alpha = \frac{dx^\alpha}{d\lambda}$, $\quad \frac{d}{d\lambda} = \frac{dx^0}{d\lambda}\frac{d}{dx^0} = E\frac{d}{dt}$

Geodesic equations become

$$E\frac{dP^\mu}{dt} = -\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta.$$

For FLRW

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j,$$

Christoffel symbols are

$$\Gamma^0_{ij} = a\dot{a}\gamma_{ij}, \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = H\delta^i_j, \qquad \Gamma^i_{jk} = \frac{1}{2}\gamma^{il}(\partial_k\gamma_{jl} + \partial_j\gamma_{kl} - \partial_l\gamma_{jk})$$

$$E\frac{dP^\mu}{dt} = \left(-2\Gamma^\mu_{0j}P^0 - \Gamma^\mu_{ij}P^i\right)P^j$$

geodesics FLRW again

$$\underline{\mu = 0}: \qquad E\frac{dE}{dt} = -\frac{\dot{a}}{a}p^2, \qquad \text{with} \qquad p^2 = g_{ij}P^iP^j = a^2\gamma_{ij}P^iP^j$$

The components of the four momentum satisfy the constraint $g_{\mu\nu}P^\mu P^\nu = -m^2$, or $E^2 - p^2 = m^2$. That is, $EdE = pdp$. Hence, geodesic equation becomes

$$\frac{\dot{p}}{p} = -\frac{\dot{a}}{a}, \qquad \Rightarrow \qquad p \propto \frac{1}{a}$$

geodesics FLRW again

$$\underline{\mu = 0}: \qquad E\frac{dE}{dt} = -\frac{\dot{a}}{a}p^2, \qquad \text{with} \qquad p^2 = g_{ij}P^iP^j = a^2\gamma_{ij}P^iP^j$$

The components of the four momentum satisfy the constraint $g_{\mu\nu}P^\mu P^\nu = -m^2$, or $E^2 - p^2 = m^2$. That is, $EdE = pdp$. Hence, geodesic equation becomes

$$\frac{\dot{p}}{p} = -\frac{\dot{a}}{a}, \qquad \Rightarrow \qquad p \propto \frac{1}{a}$$

- For massive particles, this means that free-falling particles will converge onto the Hubble flow.
- For photons this means that particles will loose energy $p = E = \frac{E_0}{a}$. This is one of the most important consequences for observational Cosmology. Light emmitted with a frequency ν_1 at time t_1, will be measured at a later time t_2 with a lower frequency

$$\nu_2 = \frac{a(t_1)}{a(t_2)}\nu_1, \qquad \text{(Cosmological redshift)}$$

Clase 4

20 de agosto de 2021

Energy density

Assume a Minkowski space-time and two Lorentz frames. Consider a collection of particles with number density n and mass m, such that its density is $\rho = nm$. Under a boost transformation, the energy density transforms as

$$\rho \quad \longrightarrow \quad \frac{\rho}{1 - v^2}$$

One γ factor because of length contraction, a second γ factor because of energy.

Hence the energy density it is not a scalar, nor the component of a vector. One needs two factors $\frac{\partial x^{\bar{0}}}{\partial x^0} = \Lambda^{\bar{0}}_{0} = (1 - v^2)^{-1/2}$.

$$\boldsymbol{T}(dx^\mu, dx^\nu) = T^{\mu\nu} := \text{ flux of } \mu \text{ momentum across a surface of constant of } x^\nu$$

$$\text{Dust:} \qquad \boldsymbol{T} = \rho \boldsymbol{U} \otimes \boldsymbol{U}.$$

Continuous matter distribution are described by a symmetric tensor $T_{\alpha\beta}$ called the energy momentum tensor. For an observer with a four velocity v^α.

$$T_{\alpha\beta}v^\alpha v^\beta = \rho|_v \qquad \text{as measured by } v^\alpha$$

this means that in the fluid's rest reference frame $T^{00} = \rho = T_{\alpha\beta}u^\alpha u^\beta$ with u^α the 4-velocity of the fluid.

The energy momentum tensor is conserved:

$$\nabla_\mu T^{\mu\nu} = 0.$$

A *perfect fluid* is defined to be a continuous distribution of matter with energy momentum of the form

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + \mathcal{P}(g_{\alpha\beta} + u_\alpha u_\beta).$$

where ρ is its energy density and $\mathcal{P}$ its pressure. Both are functions of space-time itself, but they are not scalars.

Continuous matter distribution are described by a symmetric tensor $T_{\alpha\beta}$ called the energy momentum tensor. For an observer with a four velocity v^α.

$$T_{\alpha\beta}v^\alpha v^\beta = \rho|_v \qquad \text{as measured by } v^\alpha$$

this means that in the fluid's rest reference frame $T^{00} = \rho = T_{\alpha\beta}u^\alpha u^\beta$ with u^α the 4-velocity of the fluid.

The energy momentum tensor is conserved:

$$\nabla_\mu T^{\mu\nu} = 0.$$

A *perfect fluid* is defined to be a continuous distribution of matter with energy momentum of the form

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + \mathcal{P}(g_{\alpha\beta} + u_\alpha u_\beta).$$

where ρ is its energy density and $\mathcal{P}$ its pressure. Both are functions of space-time itself, but they are not scalars.

$$3\mathcal{P} = T_{\alpha\beta}(g^{\alpha\beta} + u^\alpha u^\beta)$$

Assume a Lorentz frame ($\boldsymbol{g} = \boldsymbol{\eta}$) defined by the observer $v^\mu = (\gamma, \gamma v^i)$. The energy density of a fluid with $T_{\alpha\beta} = \rho u_\alpha u_\beta + \mathcal{P}(g_{\alpha\beta} + u_\alpha u_\beta)$ as measured by the observer v^μ is

$$\rho|_v = T_{\alpha\beta} v^\alpha v^\beta = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1).$$

(Notice we have used $u^\alpha = (1, 0, 0, 0)$, hence $u_\mu v^\mu = g_{\mu\nu} u^\mu v^\nu = -\gamma$. Also, since v^μ is a 4-velocity, $g_{\mu\nu} v^\mu v^\nu = -1$.)

Assume a Lorentz frame ($\boldsymbol{g} = \boldsymbol{\eta}$) defined by the observer $v^\mu = (\gamma, \gamma v^i)$. The energy density of a fluid with $T_{\alpha\beta} = \rho u_\alpha u_\beta + \mathcal{P}(g_{\alpha\beta} + u_\alpha u_\beta)$ as measured by the observer v^μ is

$$\rho|_v = T_{\alpha\beta} v^\alpha v^\beta = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1).$$

(Notice we have used $u^\alpha = (1, 0, 0, 0)$, hence $u_\mu v^\mu = g_{\mu\nu} u^\mu v^\nu = -\gamma$. Also, since v^μ is a 4-velocity, $g_{\mu\nu} v^\mu v^\nu = -1$.)

The pressure measured by v^μ becomes

$$3\mathcal{P}|_v = T_{\alpha\beta}(g^{\alpha\beta} + v^\alpha v^\beta) = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P}$$

Einstein’s Equation

$$\nabla^2 \phi_N = 4\pi G \rho$$

Einstein's Equation

$$\nabla^2 \phi_N = 4\pi G\rho$$

$$\rho \longrightarrow T_{\mu\nu}, \qquad \phi_N \longrightarrow g_{\mu\nu}$$

$$\text{Second order differential equation for the metric} = T_{\mu\nu}, \qquad + \quad \nabla_\mu T^{\mu\nu} = 0$$

Einstein's Equation

$$\nabla^2 \phi_N = 4\pi G \rho$$

$$\rho \longrightarrow T_{\mu\nu}, \qquad \phi_N \longrightarrow g_{\mu\nu}$$

$$\text{Second order differential equation for the metric} = T_{\mu\nu}, \qquad + \quad \nabla_\mu T^{\mu\nu} = 0$$

Einstein's equation

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Einstein's Equation

$$\nabla^2 \phi_N = 4\pi G \rho$$

$$\rho \longrightarrow T_{\mu\nu}, \qquad \phi_N \longrightarrow g_{\mu\nu}$$

$$\text{Second order differential equation for the metric} = T_{\mu\nu}, \qquad + \quad \nabla_\mu T^{\mu\nu} = 0$$

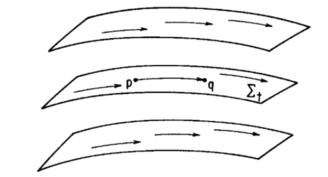
Einstein's equation

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

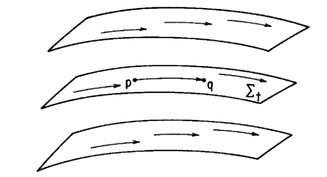
Homogeneity and isotropy

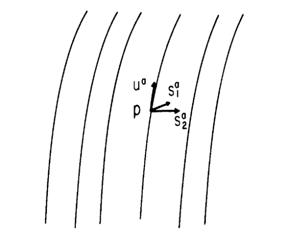
A spacetime is said to be spatially homogenous if there exists a ona-parameter family of spacelike hypersurfaces Σ_t foliating the spacetime such that for each t and for any points $p, q \in \Sigma_t$ there exists an isometry of the spacetime metric, $\boldsymbol{g}$, which takes p into q.



Homogeneity and isotropy

A spacetime is said to be spatially homogenous if there exists a ona-parameter family of spacelike hypersurfaces Σ_t foliating the spacetime such that for each t and for any points $p, q \in \Sigma_t$ there exists an isometry of the spacetime metric, $\boldsymbol{g}$, which takes p into q.





A spacetime is said to be spatially isotropic if there exists a congruence of timelike curves (observers), with tangents $\boldsymbol{v}^\alpha$ filling the spacetime and statisfying the following property. Given any point p and any two unit spatial tangent vectors $s_1^\mu, s_2^\mu \in V_p$ (i.e. vectors orthogonal to u^α) there exists an isometry of $\boldsymbol{g}$ which leaves p and u^α fixed, but rotates s_1^μ into s_2^μ

Consider the Riemann tensor constructed on Σ_t from h_{ab}, and raise one index: ${}^{(3)}R^{ab}{}_{cd}$. This can be seen as a linear map L of two-forms (completely antisymmetric (0,2) tensors) $L : W \to W$. Given the symmetries of the Riemann tensor, this map is symmetric. Then, we can choose a metric over which L is diagonal, and an orthonormal basis of eigenvectors of L. Furthermore all the eigenvalues should be the same, otherwise there would exist a preferred direction, violating isotropy. Hence L is of the form

$$L = KI$$

or

$${}^{(3)}R^{ab}{}_{cd} = K\delta^{[a}_c\delta^{b]}_d$$

or lowering indices

$${}^{(3)}R_{abcd} = K(h_{ac}h_{bd} - h_{ad}h_{bc})$$

The requirement of homogeneity means K is a constant on Σ_t.

Spaces with this property are called spaces of constant curvature.

The metric in the spacelike hypersurface Σ_t is the projection of the metric of the spacetime $g_{\mu\nu}$ into Σ_t

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu .$$

Notice $\boldsymbol{h}$ is indeed a projector:

$$\begin{aligned} \textit{(1)} \quad \boldsymbol{h}^2 = \boldsymbol{h} \ &: \qquad h_{\mu\sigma} h^{\sigma}{}_{\nu} = h_{\mu\nu} \\ \textit{(2)} \quad \boldsymbol{h}(\boldsymbol{u}, \cdot\,) = 0 \ &: \qquad h_{\mu\sigma} u^{\sigma} = 0 \end{aligned}$$

The metric in the spacelike hypersurface Σ_t is the projection of the metric of the spacetime $g_{\mu\nu}$ into Σ_t

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu .$$

Notice $\boldsymbol{h}$ is indeed a projector:

$$\begin{aligned} &\textit{(1)} \quad \boldsymbol{h}^2 = \boldsymbol{h} \; : \qquad && h_{\mu\sigma} h^{\sigma}{}_{\nu} = h_{\mu\nu} \\ &\textit{(2)} \quad \boldsymbol{h}(\boldsymbol{u}, \cdot\,) = 0 \; : \qquad && h_{\mu\sigma} u^{\sigma} = 0 \end{aligned}$$

The metric $\boldsymbol{h}$ induces a line element on Σ_t

$$d\ell^2 = h_{ab} dx^a dx^b$$

(h_{ij} is commonly called γ_{ij} in the literature.)

The metric in the spacelike hypersurface Σ_t is the projection of the metric of the spacetime $g_{\mu\nu}$ into Σ_t

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu.$$

Notice $\boldsymbol{h}$ is indeed a projector:

$$\begin{aligned} &\textit{(1)} \quad \boldsymbol{h}^2 = \boldsymbol{h} \ : \qquad && h_{\mu\sigma} h^\sigma{}_\nu = h_{\mu\nu} \\ &\textit{(2)} \quad \boldsymbol{h}(\boldsymbol{u}, \cdot\,) = 0 \ : \qquad && h_{\mu\sigma} u^\sigma = 0 \end{aligned}$$

The metric $\boldsymbol{h}$ induces a line element on Σ_t

$$d\ell^2 = h_{ab} dx^a dx^b$$

(h_{ij} is commonly called γ_{ij} in the literature.)

Clearly, I have the liberty to multiply h by any constant on Σ_t, that is, by any function of time t. Using

$$g_{\mu\nu} = -u_\mu u_\nu + h_{\mu\nu},$$

and $u^\alpha u_\alpha = -1$, we arrive at

$$ds^2 = -dt^2 + a^2(t) d\ell^2.$$

There are essentially three 3-spaces of constant curvature K/R_*^2. These can be seen as hypersurfaces embedded in a 4-dimensional space

$$\begin{aligned} x^2 + y^2 + z^2 + u^2 = R_*^2 : &\qquad S^3 \quad \text{if} \quad K = 1 \\ x + y + z + w = 0 : &\qquad E^3 \quad \text{if} \quad K = 0 \\ x^2 + y^2 + z^2 - w^2 = -R_*^2 : &\qquad H^3 \quad \text{if} \quad K = -1 \end{aligned}$$

There are essentially three 3-spaces of constant curvature K/R_*^2. These can be seen as hypersurfaces embedded in a 4-dimensional space

$$\begin{aligned} x^2 + y^2 + z^2 + u^2 = R_*^2 : &\qquad S^3 \quad \text{if} \quad K = 1 \\ x + y + z + w = 0 : &\qquad E^3 \quad \text{if} \quad K = 0 \\ x^2 + y^2 + z^2 - w^2 = -R_*^2 : &\qquad H^3 \quad \text{if} \quad K = -1 \end{aligned}$$

Hence the metric on the hypersurfaces Σ_t is either the metric of a flat spacetime, a 3-sphere or a 3-hyperboloid

$$d\ell^2 = \gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - Kr^2/R_*^2} + r^2 d\Omega, \qquad K = -1, 0, 1$$

The Friedmann-Lemaître-Robertson-Walker metric is

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega \right)$$

There are essentially three 3-spaces of constant curvature K/R_*^2. These can be seen as hypersurfaces embedded in a 4-dimensional space

$$\begin{aligned} x^2 + y^2 + z^2 + u^2 = R_*^2 : &\qquad S^3 \quad \text{if} \quad K = 1 \\ x + y + z + w = 0 : &\qquad E^3 \quad \text{if} \quad K = 0 \\ x^2 + y^2 + z^2 - w^2 = -R_*^2 : &\qquad H^3 \quad \text{if} \quad K = -1 \end{aligned}$$

Hence the metric on the hypersurfaces Σ_t is either the metric of a flat spacetime, a 3-sphere or a 3-hyperboloid

$$d\ell^2 = \gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - Kr^2/R_*^2} + r^2 d\Omega, \qquad K = -1, 0, 1$$

The Friedmann-Lemaître-Robertson-Walker metric is

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega\right)$$

To derive the FRLW metric we never used the Einstein's equation. This is determined by geometry and is valid in any metric theory of gravity.

Friedmann equations

- To preserve the symmetries of the metric, the energy momentum tensor must have the form of a perfect fluid

$$T_{\mu\nu} = \rho u_\mu u_\nu + \mathcal{P}(g_{\mu\nu} + u_\mu u_\nu)$$

with the energy density and pressure only a function of time (otherwise we break isotropy and/or homogenity).

Einstein's equation in FLRW

Friedmann equations

- To preserve the symmetries of the metric, the energy momentum tensor must have the form of a perfect fluid

$$T_{\mu\nu} = \rho u_\mu u_\nu + \mathcal{P}(g_{\mu\nu} + u_\mu u_\nu)$$

with the energy density and pressure only a function of time (otherwise we break isotropy and/or homogenity).

- 0-0 component EE $\Longrightarrow$ Friedmann Equation

$$G_{00} = 3\frac{\dot{a}^2}{a^2} + \frac{3k}{a^2} = 8\pi G\rho = 8\pi G T_{00}$$

Friedmann equation: $$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

- i-i component EE $\Longrightarrow$ Friedmann Equation 2

$$G_{**} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = 8\pi G\mathcal{P} = 8\pi G T_{**}$$

- i-i component EE $\Longrightarrow$ Friedmann Equation 2

$$G_{**} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = 8\pi G\mathcal{P} = 8\pi G T_{**}$$

It is usual to combine these two equations to get *the acceleration equation:*

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\mathcal{P}).$$

- i-i component EE $\Longrightarrow$ Friedmann Equation 2

$$G_{**} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = 8\pi G\mathcal{P} = 8\pi G T_{**}$$

It is usual to combine these two equations to get *the acceleration equation:*

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\mathcal{P}).$$

- $\nabla_\mu T^{\mu\nu} = 0$

$$\text{Continuity equation:} \qquad \dot{\rho} + 3H(\rho + \mathcal{P}) = 0$$

However, the Friedmann, Acceleration and Continuity equations are not independent.

Hence, the equations that the scale factor obeys are

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\dot{\rho} + 3H(\rho + P) = 0$$

Hence, the equations that the scale factor obeys are

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\dot{\rho} + 3H(\rho + P) = 0$$

We still need an equation to relate the energy density and the pressure (an equation of state) to close the system

$$\mathcal{P} = \mathcal{P}(\rho, ...) = w(\rho)\rho$$

Fluids for which the equation of state depends on the energy density only are called *barotropic*.

It is very common that the factor w is a constant.

Equations of state (dust)

Remind that given an observer $v^\mu = (\gamma, \gamma v^i)$ with $\gamma = \frac{1}{\sqrt{1-v^2}}$, the energy density and pressure transform as

$$\rho|_v = T_{\alpha\beta} v^\alpha v^\beta = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1)$$
$$3\mathcal{P}|_v = T_{\alpha\beta}(g^{\alpha\beta} + v^\alpha v^\beta) = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P}$$

- Dust: $\mathcal{P} = 0$.

$$\rho = mn, \qquad \rho|_v = \gamma^2 \rho$$

Equations of state (Vacuum energy)

Given an observer $v^\mu = (\gamma, \gamma v^i)$

$$\rho|_v = T_{\alpha\beta} v^\alpha v^\beta = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1)$$
$$3\mathcal{P}|_v = T_{\alpha\beta}(g^{\alpha\beta} + v^\alpha v^\beta) = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P}$$

Equations of state (Vacuum energy)

Given an observer $v^\mu = (\gamma, \gamma v^i)$

$$\rho|_v = T_{\alpha\beta} v^\alpha v^\beta = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1)$$
$$3\mathcal{P}|_v = T_{\alpha\beta}(g^{\alpha\beta} + v^\alpha v^\beta) = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P}$$

- We expect the density and pressure of vacuum energy is the same for all observers. That is

$$\rho|_v = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1) = \rho, \qquad \text{for all} \quad \gamma = \frac{1}{\sqrt{1 - v^2}}$$

Equations of state (Vacuum energy)

Given an observer $v^\mu = (\gamma, \gamma v^i)$

$$\rho|_v = T_{\alpha\beta} v^\alpha v^\beta = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1)$$
$$3\mathcal{P}|_v = T_{\alpha\beta}(g^{\alpha\beta} + v^\alpha v^\beta) = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P}$$

- We expect the density and pressure of vacuum energy is the same for all observers. That is

$$\rho|_v = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1) = \rho, \qquad \text{for all} \quad \gamma = \frac{1}{\sqrt{1 - v^2}}$$

$$\text{Hence} \qquad \mathcal{P} = -\rho, \qquad w = -1$$

Equations of state (Vacuum energy)

Given an observer $v^\mu = (\gamma, \gamma v^i)$

$$\rho|_v = T_{\alpha\beta} v^\alpha v^\beta = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1)$$
$$3\mathcal{P}|_v = T_{\alpha\beta}(g^{\alpha\beta} + v^\alpha v^\beta) = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P}$$

- We expect the density and pressure of vacuum energy is the same for all observers. That is

$$\rho|_v = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1) = \rho, \qquad \text{for all} \quad \gamma = \frac{1}{\sqrt{1 - v^2}}$$

$$\text{Hence} \qquad \mathcal{P} = -\rho, \qquad w = -1$$

of course $\mathcal{P}|_v = -\rho|_v$.

Equations of state (Vacuum energy)

Given an observer $v^\mu = (\gamma, \gamma v^i)$

$$\rho|_v = T_{\alpha\beta} v^\alpha v^\beta = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1)$$
$$3\mathcal{P}|_v = T_{\alpha\beta}(g^{\alpha\beta} + v^\alpha v^\beta) = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P}$$

- We expect the density and pressure of vacuum energy is the same for all observers. That is

$$\rho|_v = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1) = \rho, \qquad \text{for all} \quad \gamma = \frac{1}{\sqrt{1 - v^2}}$$

$$\text{Hence} \qquad \mathcal{P} = -\rho, \qquad w = -1$$

of course $\mathcal{P}|_v = -\rho|_v$. The vacuum energy has an energy-momentum tensor

$$T_{\mu\nu}^{\text{vacuum}} = -\rho g_{\mu\nu} = -\frac{\Lambda^{\text{vacuum}}}{8\pi G} g_{\mu\nu}$$

where the last equality is a definition of Λ^{vacuum}

Lorentz invariant equation of state

Given an observer $v^\mu = (\gamma, \gamma v^i)$

$$\rho|_v = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1) = \rho[\gamma^2 + w(\gamma^2 - 1)]$$
$$3\mathcal{P}|_v = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P} = \rho[(1 + w)(\gamma^2 - 1) + 3w]$$

Lorentz invariant equation of state

Given an observer $v^\mu = (\gamma, \gamma v^i)$

$$\rho|_v = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1) = \rho[\gamma^2 + w(\gamma^2 - 1)]$$
$$3\mathcal{P}|_v = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P} = \rho[(1 + w)(\gamma^2 - 1) + 3w]$$

- Under the boost v, the equation of state parameter w transform as

$$w|_v = \frac{\mathcal{P}|_v}{\rho|_v} = \frac{1}{3}\frac{(1 + w)(\gamma^2 - 1) + 3w}{\gamma^2 + w(\gamma^2 - 1)}$$

Lorentz invariant equation of state

Given an observer $v^\mu = (\gamma, \gamma v^i)$

$$\rho|_v = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1) = \rho[\gamma^2 + w(\gamma^2 - 1)]$$
$$3\mathcal{P}|_v = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P} = \rho[(1 + w)(\gamma^2 - 1) + 3w]$$

- Under the boost v, the equation of state parameter w transform as

$$w|_v = \frac{\mathcal{P}|_v}{\rho|_v} = \frac{1}{3}\frac{(1 + w)(\gamma^2 - 1) + 3w}{\gamma^2 + w(\gamma^2 - 1)}$$

The equation of state parameter is Lorentz invariant if $w|_v = w$ for all γ. This happens for the values $w = -1$ and $w = 1/3$. The former is vacuum energy. The latter corresponds to an equation of state

$$\mathcal{P} = \frac{1}{3}\rho.$$

Lorentz invariant equation of state

Given an observer $v^\mu = (\gamma, \gamma v^i)$

$$\rho|_v = \gamma^2 \rho + \mathcal{P}(\gamma^2 - 1) = \rho[\gamma^2 + w(\gamma^2 - 1)]$$
$$3\mathcal{P}|_v = (\rho + \mathcal{P})(\gamma^2 - 1) + 3\mathcal{P} = \rho[(1 + w)(\gamma^2 - 1) + 3w]$$

- Under the boost v, the equation of state parameter w transform as

$$w|_v = \frac{\mathcal{P}|_v}{\rho|_v} = \frac{1}{3}\frac{(1 + w)(\gamma^2 - 1) + 3w}{\gamma^2 + w(\gamma^2 - 1)}$$

The equation of state parameter is Lorentz invariant if $w|_v = w$ for all γ. This happens for the values $w = -1$ and $w = 1/3$. The former is vacuum energy. The latter corresponds to an equation of state

$$\mathcal{P} = \frac{1}{3}\rho.$$

Notice that the density and pressure do transform (unlike vacuum energy), although the equation of state remains the same.

Cosmología
Unidad 2: Modelos cosmológicos de FLRW y Energía oscura (de unidad 7)

Alejandro Avilés (CONACyT/ININ)
avilescervantes@gmail.com

semestre 2022-1

Curso PCF-UNAM

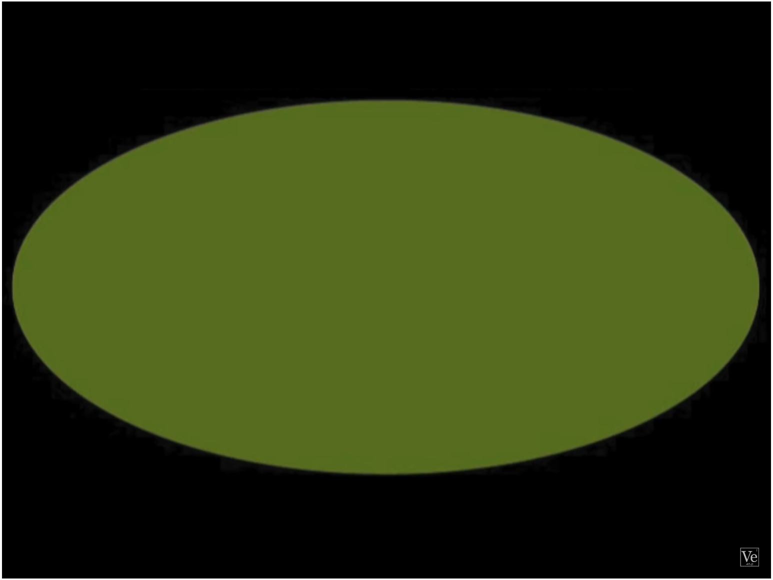
'Ustedes qué tanto saben da la vida, ¿conocen acaso la despiadada soledad de un profesor preguntando en clases por Zoom?'

Anónimo

Unidad 2	Modelos cosmológicos de Friedman-Robertson-Walker
2.1	Solución exacta para materia y radiación y fluido perfecto
2.2	Singularidad inicial
2.3	El corrimiento hacia el rojo y la determinación de distancias, el parámetro de Hubble
2.4	Horizontes de partículas y de eventos
2.5	La edad del Universo

7.2	Constante cosmológica: Motivación, evidencia observacional y problemas asociados
7.3	Modelos de Energía oscura: Modelos de campos escalares (e.g. potenciales inversos, exponencial), parametrizaciones como un fluido
7.4	Expansión acelerada debida a Gravedad Modificada: Parametrizaciones Observables para desviaciones de la Relatividad General

7.1 lo veremos más adelante



Unidad II, clase 1

23 de agosto de 2021

Units

Throughout this course we often use *Natural Units*, where $\hbar = 1$, $c = 1$ and $k_\mathrm{B} = 1$. (Also $\epsilon_0 = 1$.)

Since in SI units

$$\hbar = 6.62607004 \times 10^{-34}\,\mathrm{m}^2\,\mathrm{kg}\,\mathrm{s}^{-1}$$
$$c = 299792458\,\mathrm{m}\,\mathrm{s}^{-1}$$
$$k_B = 1.38064852 \times 10^{-23}\,\mathrm{m}^2\,\mathrm{kg}\,\mathrm{s}^{-2}\mathrm{K}^{-1},$$

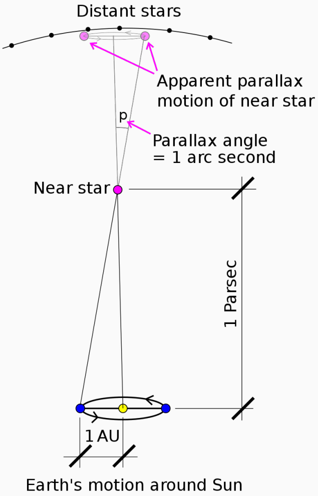
we can use $\hbar = c = k_\mathrm{B} = 1$ to eliminate all but one dimension.

$$[\text{distance}] = [\text{time}] = [\text{mass}]^{-1} = [\text{energy}]^{-1} = [\text{temperature}]^{-1}$$

e.g. $$T_\mathrm{CMB} = 2.725\,\mathrm{K} = 2.348 \times 10^{-4}\,\mathrm{eV} = \frac{1}{0.0845\,\mathrm{cm}}$$

A sense of scales

Cosmologists use a lot the *megaparsec* $\mathrm{Mpc} = 10^6\,\mathrm{pc}$ as a unit of distance.



$$1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$$

$$1\,\mathrm{pc} = 3.0857 \times 10^{16}\,\mathrm{m}$$

$$1\,\mathrm{Mpc} = 3.0857 \times 10^{22}\,\mathrm{m} = 3.26 \times 10^{6}\,\mathrm{ly}$$

- The closest stars to earth are about 1 parsec away
- Typical galaxies are 40 kpc across (the visible part)
- Typical distances between galaxies are of a few Mpc
- The baryon acoustic oscillation (BAO) standard ruler is about 150 Mpc

The scale factor

The FLRW metric is

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right].$$

Two objects separated at a physical distance X_{ini}^{phys} at time t_{ini} will be separated by a distance X_{fin}^{phys} at a time t_{fin}, with

$$X_{fin}^{phys} = \frac{a(t_{ini})}{a(t_{fin})} X_{ini}^{phys}.$$

A physical distance is given by

$$X_{phys} = a(t)X_{\text{com}}$$

with X_{com} the comoving distance. The units are carried by X_{com}, and can always be chosen to yield the physical distance today, at t_0. (It is very common in cosmology the use of a label "0" to refer to quantities evaluated nowadays.) That is, we can fix the value of the scale factor

$$a_0 = a(t_0) = 1.$$

That is, the physical content of the scale factor is encoded in its functional form and its value is not relevant. That is, it is defined up to a constant.

Hence, the Hubble function $H = \dot{a}/a$ contains the same cosmological information that a.

The redshift

Remind that due to the cosmological Doppler effect, a photon emitted a time t_{emit} with frequency $2\pi\nu_{\text{emit}}$, will be measured later (say nowadays, at t_0) with a frequency ν_0

$$\nu_0 = \frac{a(t_{\text{emit}})}{a(t_0)}\nu = a(t_{\text{emit}})\nu_{\text{emit}}.$$

or in terms of its wavelength λ

$$\frac{\lambda_0}{\lambda_{\text{emit}}} = \frac{1}{a_{\text{emit}}} \equiv 1 + z, \qquad \text{or} \quad 1 + z = \frac{1}{a}$$

where the *redshift* z is defined as the stretching factor.

Light emitted from objects that are receding from us will be see red-shifted, while light emitted from objects approaching us seem to be blue-shifted.

A quite common practice is to use the redshift instead of the scale factor. In fact, people often use the redshift as the time parameter, although to relate these quantities one needs to solve the background Cosmology. Notice, and remember, the following

$$z(a_0 = 1) = 0, \qquad z(a \to 0) \to \infty, \qquad z(a \to \infty) \to -1$$

The Hubble constant

The value of the Hubble function today is called the *Hubble constant* H_0 and is given by

$$H_0 = 100\,h\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = \frac{1}{2997.92\,h^{-1}\,\mathrm{Mpc}}$$

where $h = 0.65$-0.75 parametrizes our ignorance of its exact value.

H_0 is the most important parameter in Cosmology. The reason is because almost all cosmological scales depends on it

e.g., the age of the Universe is roughly $H_0^{-1} = 9.78\,\times 10^9\,h^{-1}$ years

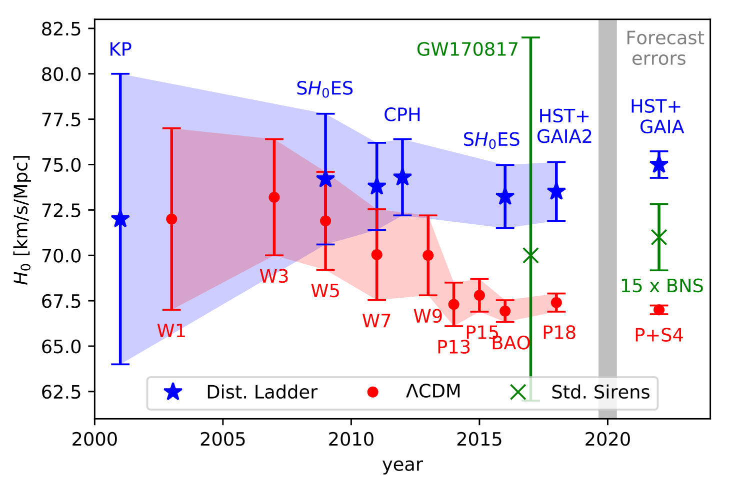
H_0 is so important, that

The most used units in Cosmology are $h^{-1}\,\mathrm{Mpc}$

e.g., the BAO peak in the correlation function of matter perturbations is located at a scale $r \approx 104\,h^{-1}\,\mathrm{Mpc}$, insensitive to the value of h.

The Hubble Tension

Nowadays, Cosmology suffers the huge problem that different cosmological observations measure different values of H_0



[From Ezquiaga & Zumalacarregui (arxiv:1807.09241)]

The Friedmann equations

Using Einstein's and continuity equation, one gets (we already derived them)

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_T + \frac{\Lambda}{3} - \frac{k}{a^2},$$

$$\dot{\rho} + 3H(\rho + \mathcal{P}) = 0.$$

where we sum over all the cosmic components

$$\rho_T = \sum_A \rho_A.$$

Each component, hopefully, has an equation of state relating ρ and $\mathcal{P}$ such that we can solve the above equations.

We can define

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}$$

and simply consider the cosmological constant as an additional fluid whose energy density is counted in ρ_T. It is not the best way to see Λ, but it is very practical.

$$H^2 = \frac{8\pi G}{3}\rho_T - \frac{k}{a^2}, \qquad \dot{\rho} + 3H(\rho + \mathcal{P}) = 0.$$

In the *Lambda Cold Dark Matter* (ΛCDM) model, the main cosmic components are

- Baryons (b)
- Cold Dark Matter (CDM, or sometimes in equations we use "c'')
- Photons (γ). CMB photons to be more precise
- Neutrinos (ν). These could be massive or massless
- Dark Energy. In ΛCDM this is given by the cosmological constant Λ

(When we refer to all relativistic particles (or ultra-relativistic) at once we use the letter "r".)

Critical density

From $H^2 = \frac{8\pi G}{3}\rho_T - \frac{k}{a^2}$, there is a value of the density ρ_T, such that $k = 0$. This is called the the critical density:

$$\rho_{cr}(a) = \frac{3H^2}{8\pi G}$$

If $\rho_T = \rho_{cr}$ then the spatial curvature must vanish. Notice ρ_{cr} is a function of time, or the scale factor (when $a(t)$ is monotonous in t).

Critical density

We further define the *cosmic abundances*

$$\Omega_A(a) = \frac{8\pi G}{3H^2}\rho_A = \frac{\rho_A}{\rho_{cr}}.$$

It is very important to notice that I wrote explicitly the argument a. We will see the reason for this. The Friedmann equation becomes

$$1 = \Omega_b(a) + \Omega_c(a) + \Omega_\Lambda(a) + \Omega_\gamma(a) + \Omega_\nu(a) + \Omega_k(a)$$

where

$$\Omega_k(a) = -\frac{k}{a^2 H^2}$$

.

Critical density

The critical density today (at $a = a_0 \equiv a(t_0) \equiv 1$) is given by

$$\rho_{cr}^0 = \frac{3H_0^2}{8\pi G}$$

To have an idea, its numerical value is

$$\rho_{cr}^0 = 1.88 \times 10^{-26} h^2 \,\mathrm{kg}\,\mathrm{m}^{-3} = 2.78 \times 10^{11} h^2 \, M_\odot \,\mathrm{Mpc}^{-3},$$

this is about four hydrogen atoms per cubic meter.

Critical density

One of the big questions in Cosmology is why the measured curvature of the Universe Ω_k is very close to zero, and hence the energy density of the Universe very close to ρ^0_{cr}.

Moreover, for normal energy components (e.g, thus having positive or zero pressure), the scale factor grows slower than $a(t) \propto t$ (decelerating expansion), and $H \propto t^{-1}$ approx. This means $\Omega_k(a) = k/a^2H^2$ was smaller in the past. Hence, a very close to flat space now, means a much closer to flat space in the past. Why?!

Unidad II, clase 2

25 de agosto de 2021

Densities evolve with the scale factor according to the continuity eqs.: $\rho_A(a) = \rho_{A0} f_A(a)$. This permits us to write the Friedmann equation as

$$\frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2} \sum_A \rho_A(a) = \sum_A \frac{\rho_{A0}}{\rho_{cr}^0} f_A(a)$$
$$\equiv \sum_A \Omega_A f_A(a)$$

where we defined the cosmic components today. For species A

$$\Omega_A = \frac{8\pi G}{3H_0^2} \rho_{A0} = \frac{\rho_{A0}}{\rho_{cr}^0}$$

and note

$$\rho_A(a) = \rho_{A0} f_A(a) = \rho_{cr}^0 \, \Omega_A \, f_A(a)$$

Clearly $\Omega_A = \Omega_A(t_0)$. But notice we do not use the label “0”, because these parameters are very important and we want to avoid cluttering. Although, some authors prefer to use “0”.

Indeed, the $\Omega_A(a)$s evolve as

$$\Omega_A(a) = \Omega_A \frac{H_0^2}{H^2} f_A(a), \qquad f_A(a) = \frac{\rho_A}{\rho_{A0}}$$

$$\Omega_K(a) = \Omega_K \frac{H_0^2}{H^2} f_K(a), \qquad f_K(a) = \frac{1}{a^2}$$

Functions $f_A(a)$ are largely specified by conservation arguments. For example, for dust $f_A(a) = 1/a^3 \propto V$. Then, it is convenient to write the Friedmann equation as

$$H(t) = H_0 \sqrt{\sum_A \Omega_A f_A(a) + \Omega_K a^{-2}}$$

Cosmic components

(simplified version)

Later we will discuss in more detail all the cosmic components. But for the moment let's simplify things so we can make some progress and plot some important things.

- Baryons have almost negligible pressure. CDM (by definition) has zero pressure. So let's use the symbol m for the combined CDM+b fluid.

$$\rho_m = \rho_c + \rho_b, \qquad \mathcal{P}_m = 0, \qquad \text{(so, this is dust).}$$

- Dark energy, in its simplest form Λ.

$$\rho_\Lambda = \text{constant}, \qquad \mathcal{P}_\Lambda = -\rho_\Lambda, \qquad \text{(so, this is like vacuum energy)}$$

- Radiation is composed of photons, neutrinos (for now, consider them massless).

$$\rho_r = \rho_\gamma + \rho_\nu, \qquad \mathcal{P}_r = \frac{1}{3}\rho_r, \qquad \text{(so, this is a Lorentz invariant fluid).}$$

- Let us consider flat space, hence $k = 0$.

In our simplified version, a **matter** fluid has zero pressure, then the continuity equation is

$$\dot{\rho}_m + 3H\rho_m = 0$$

or

$$\frac{d\rho_m}{\rho_m} = -3\frac{da}{a}$$

then the solution is

$$\rho_m(a) = \rho_m(a_0)\frac{a_0^3}{a^3} = \rho_{m0}\frac{1}{a^3}$$

with $\rho_{m0} \equiv \rho_m(a_0)$ and we have used the value $a_0 = 1$ for the scale factor today.

It should not be a surprise that the density of dust particles decays with a^3. This is volume dilution.

In terms of the redshift

$$\rho_m(z) = \rho_{m0}(1+z)^3$$

A **radiation** fluid has equation of state parameter $w = 1/3$, then the continuity equation is

$$\dot{\rho}_r + 3H\rho_r(1+w) = \dot{\rho}_r + 4H\rho_r = 0$$

or

$$\frac{d\rho_r}{\rho_r} = -4\frac{da}{a}$$

then the solution is

$$\rho_r(a) = \rho_{r0}\frac{1}{a^4}$$

with $\rho_{a0} \equiv \rho_m(r_0)$ and we have used the value $a_0 = 1$ for the scale factor today.

It should not be a surprise that the density of radiation particles decays with a^4: three powers come from the volume dilution, and an addition power because the wavelength of the photons are stretched with a and then their energy decays with a.

In terms of the redshift

$$\rho_r(z) = \rho_{r0}(1+z)^4$$

A **cosmological constant** "fluid" has equation of state parameter $w = -1$, then the continuity equation is

$$\dot{\rho}_\Lambda = -3H\rho_\Lambda(1+w) = 0$$

or

$$\rho_\Lambda(a) = \rho_\Lambda = \text{constant.}$$

It should not be a surprise that the density of a cosmological constant density fluid is constant because... there is nothing for it to depend on

Friedmann equation becomes

$$H(t) = H_0 \sqrt{\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda}$$

with (for massless neutrinos)

$$\Omega_r = \left[1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{eff}\right] \Omega_\gamma = 4.18 \times 10^{-5}\, h^{-2}$$

where N_{eff} is the effective number of fermionic relativistic degrees of freedom.

From Table 2 of *Planck 2018 results. VI. Cosmological parameters* [arxiv:1807.06209]

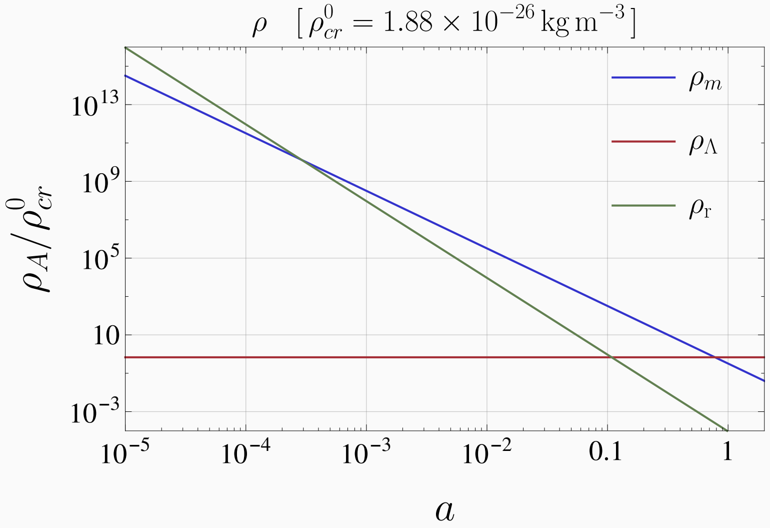
$$\begin{aligned} \Omega_m &= 0.3111 \pm 0.0056 \\ \Omega_\Lambda &= 0.6889 \pm 0.0056 \\ H_0 &= 66.88 \pm 0.92 \quad \mathrm{km\, s^{-1}\, Mpc^{-1}} \end{aligned}$$

From the same paper

$$\Omega_k = 0.001 \pm 0.002 \qquad \text{and} \qquad N_{eff} = 2.99 \pm 0.17,$$

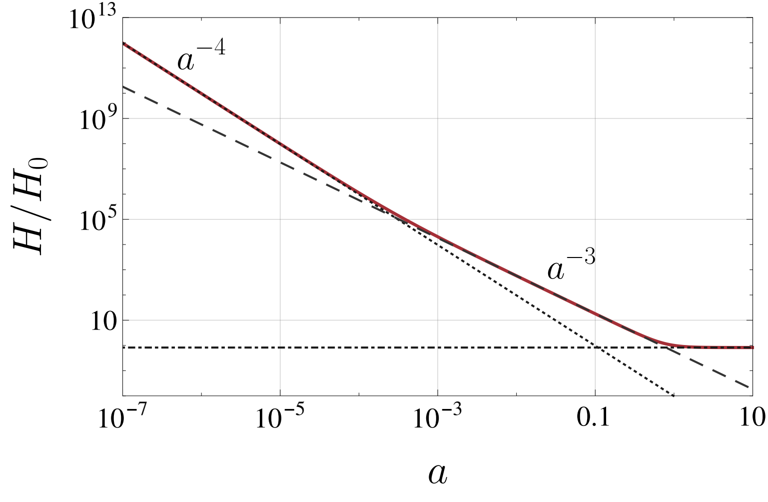
in agreement with the Standard Model prediction $N_{eff} = 3.046$

Cosmic densities

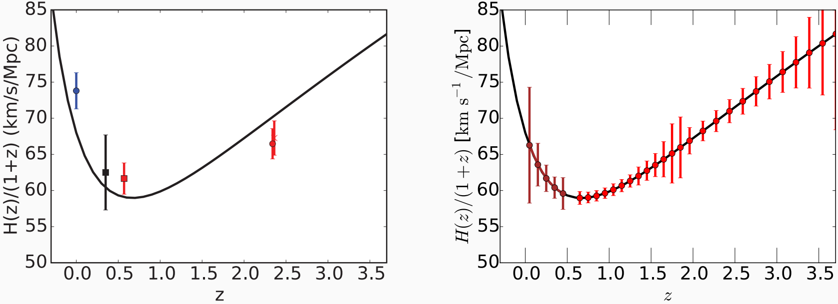


matter-radiation equality $\rho_r(z_{eq}) = \rho_m(z_{eq}) \Rightarrow 1 + z_{eq} = \dfrac{\Omega_m}{\Omega_r} = 3438$

Hubble function $H(a) = H_0\sqrt{\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda}$

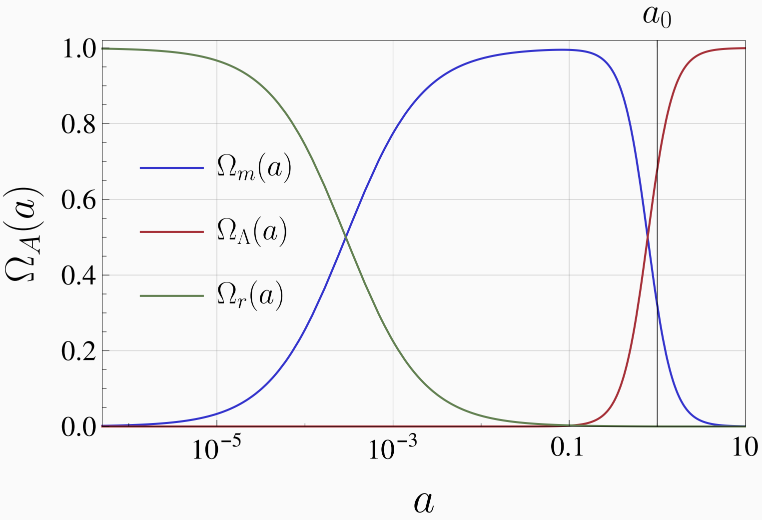


[From arXiv:1611.00036]



Expansion rate of the Universe as a function of redshift. In the left plot, the filled blue circle is the H_0 measurement of Riess 2011, the rest are measurements from SDSS I-III. The right plot shows projected DESI points.

Cosmic abundances



Elapsed time

From the definition of H

$$\frac{\dot{a}(t)}{a(t)} = H(a) \;\Rightarrow\; \frac{da}{aH(a)} = dt.$$

Hence

$$t(a) = \int_0^a \frac{da'}{a'H(a')}$$

assuming $t = 0$ at $a = 0$.

But what time is this one?

t is the proper time for comoving observers. So, $t(a)$ is the elapsed time from the "beginning of the universe" (also called "Big Bang") at $a = 0$, until the scale factor is equal to a.

For example, the *age of the universe* is

$$t_0 = \int_0^{a_0} \frac{da}{aH(a)} = \frac{1}{H_0} \int_0^1 \frac{da}{a\left[\Omega_\Lambda + \Omega_m a^{-3} + \Omega_r a^{-4}\right]^{1/2}}$$
$$= 0.9455\, H_0^{-1} \approx 13.83 \times 10^9 \text{ years}$$

where we used the best fit parameters in Planck 2018

In the absence of cosmological constant $\Omega_m = 1 - \Omega_r \simeq 1$

$$t_0^{\text{No}-\Lambda} = 9.75 \times 10^9 \text{ years}$$

This was a problem long time ago: the oldest stellar populations are 10-12 Gyr old! Of course, a consistent cosmological model must predict an age of the universe older than the objects that were formed in it. Fortunately, the cosmological constant solves this problem.

Friedmann equations for a single component

Consider a single cosmic components with equation-of-state (EoS) $\mathcal{P} = w\rho$, and EoS parameter w constant. Hence, from the continuity equation $\dot{\rho} + 3H\rho(1+w) = 0$ and Friedman equation:

$$\text{For } w > -1, \qquad \rho(a) = \rho_0 a^{-3(1+w)}$$

$$a(t) \propto t^{\frac{2}{3(1+w)}}$$

$$H(t) = \frac{2}{3(1+w)t}$$

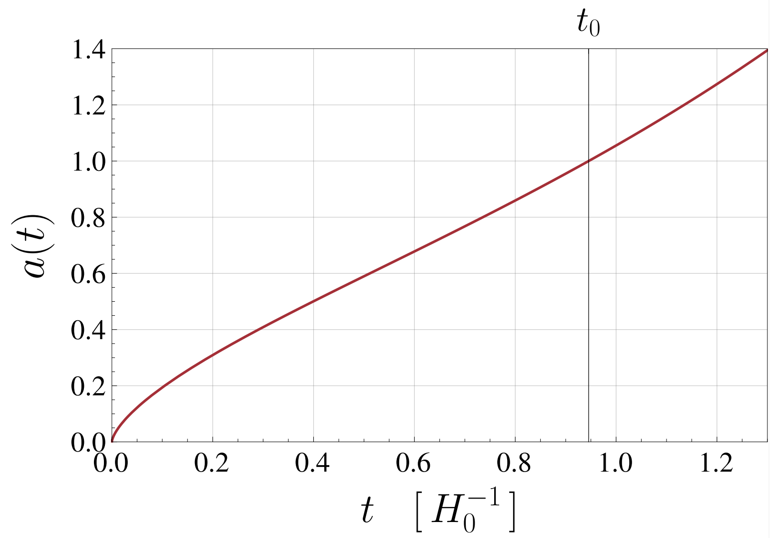
$$\text{For example:} \qquad \text{matter:} \quad a(t) \propto t^{2/3}, \qquad H(t) = \frac{2}{3t}$$

$$\text{radiation:} \quad a(t) \propto t^{1/2}, \qquad H(t) = \frac{1}{2t}$$

while for $w = -1$

$$\rho(a) = \rho_0 = \text{constant}, \quad a(t) \propto \exp(H_0 t), \quad H(t) = H_0 = \text{constant}$$

Hubble function $\frac{\dot{a}(t)}{a(t)} = H_0\sqrt{\Omega_r a(t)^{-4} + \Omega_m a(t)^{-3} + \Omega_\Lambda}$



Curvature

There is a mess in the literature with the symbols used for k and Ω_K. A lot of articles use

$$ds^2 = -dt^2 + a^2 \frac{\delta_{ij} dx^i dx^j}{1 - Kr^2}, \quad \text{with} \quad K = -1, 0, 1 \text{ !!!}$$

So K has no units! This is not wrong *per se*, since one can always rescale r and the scale factor absorbs the units. However, they define $\Omega_K(a) = -K/(a^2 H^2)$ (sometimes with a "+" instead of a "-" sign, but this doesn't matter).

This is quite reckless, at least for me, because today curvature abundance

$$\Omega_K = -\frac{K}{a_0^2 H_0^2}$$

has only 3 possible values $-1/a_0^2 H_0^2$, 0, $1/a_0^2 H_0^2$. But the curvature takes values in the continuum. I think this a common source of confusion regarding curved space.

(Notice in this case one cannot fix $a_0 = 1$, because the scale factor is no longer dimensionless.)

We have defined the curvature at the level of the metric in the factor $(1-kr^2)^{-1}$, with k having dimensions of [distance]$^{-2}$. And then $\Omega_K = -k/H_0^2$, from which

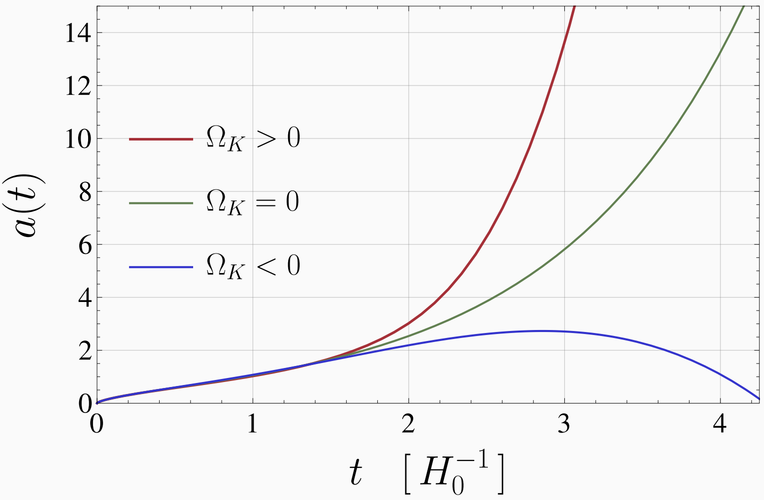
$$\Omega_K = 1 - \sum_A \Omega_A,$$

which is the definition of curvature in e.g. Dodelson *Modern Cosmology* textbook, or in observational papers (Pantheon , Planck, ...). With this definition the metric takes the form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+\Omega_K H_0^2 r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right].$$

Note the change in sign between Ω_K and k. Yes, it is annoying, but usual in observational papers. With this

$$\Omega_K < 0 \quad \text{(closed universe } S^3\text{)}, \qquad \Omega_K > 0 \quad \text{(open universe } H^3\text{)}$$



$\Omega_K > 0$
$\Omega_K = 0$
$\Omega_K < 0$
$a(t)$
$t \quad [H_0^{-1}]$
0
1
2
3
4
0
2
4
6
8
10
12
14

Unidad II, clase 3

27 de agosto de 2021

Cosmological distances

Comoving distance χ

The starting point for the calculation of distances is the comoving distance. Consider the comoving distance between a distant light source and us. In a small time interval dt, light travels a *comoving distance*

$$d\chi = \frac{dt}{a} \qquad \text{for} \quad d\chi = \frac{dr}{\sqrt{1-kr^2}}$$

such the total comoving distance traveled by light that began its journey from an object at time t when the scale factor was equal to a is

$$\chi(t) = \int_t^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')} = \int_0^z \frac{dz'}{H(z')}$$

Sometimes it is useful to think of the comoving distance as a function of the spatial coordinates. From $dt = a(1-kr^2)^{-1/2}dr$

$$\chi(r) = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \frac{1}{\sqrt{k}}\sin^{-1}\left[\sqrt{k}\,r\right] = \frac{1}{\sqrt{-\Omega_K}H_0}\sin^{-1}\left[\sqrt{-\Omega_K}\,H_0 r\right]$$

Hence, we can invert $\chi(r)$ and use it into the FLRW

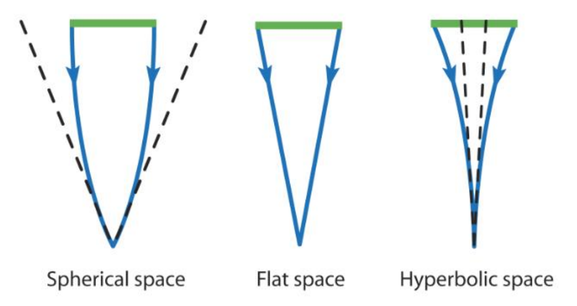
$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \\ &= -dt^2 + a^2(t)\left[d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \end{aligned}$$

$$\text{with} \qquad S_K(\chi) = \begin{cases} \frac{1}{\sqrt{-\Omega_K}H_0}\sinh\left[\sqrt{-\Omega_K}\,H_0\chi\right] & \Omega_K < 0 \\ \chi & \Omega_K = 0 \\ \frac{1}{\sqrt{\Omega_K}H_0}\sin\left[\sqrt{\Omega_K}\,H_0\chi\right] & \Omega_K > 0 \end{cases}$$

[It is more common the use of the symbol $f_K(\chi) = S_K(\chi)$.]

Let’s take a closer look to curvature in the new coordinates. The spatial part is

$$d\ell^2 = d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)$$



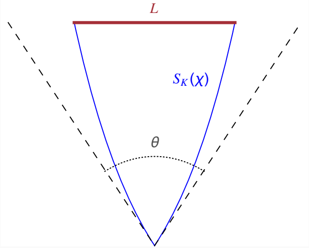
This means that spheres S^2 with comoving radius χ will have a proper area $4\pi S_K^2(\chi)$

Angular distance

A very common way to measure distances in Cosmology is through the subtended angle θ by an object of known size L (a "standard ruler"). In Euclidean space the distance d_A, L and θ are related by

$$d_A = \frac{L}{\theta}$$

The *angular diameter distance* is constructed with this idea. The comoving size of an object is $L_c = L/a$, and the *distance* (understood as the radius $R = (A/4\pi)^{1/2}$of the sphere where it lives and with us at the center) is $S_K(\chi)$.



Thus, the subtended angle is $\theta = (L/a)/S_K(\chi(z))$, and hence the diameter angular distance is given by

$$d_A(z) = \frac{1}{1+z} S_K(\chi(z)) = \frac{1}{1+z} \frac{1}{\sqrt{\Omega_K} H_0} \sin\left[\sqrt{\Omega_K}\, H_0 \chi(z)\right]$$

Luminosity distance

The luminosity distance to an object with absolute luminosity L_S (= energy emitted per second) is defined as

$$d_L^2 = \frac{L_S}{4\pi\mathcal{F}}$$

where $\mathcal{F}$ is the flux received by the observer (= energy received per unit time per unit area).

In a FLRW universe the flux received nowadays from the source with luminosity L_S at redshift z is altered because of three effects: *1)* the energy of the arriving photons is redshifted by a factor of $1/(1+z)$; *2)* the rate at which photons are emitted is larger than the rate at which are received, introducing another factor $1/(1+z)$; *3)* the proper area of a sphere drawn around the source and passing through the earth is $4\pi S_K^2(\chi)$. Hence the flux is

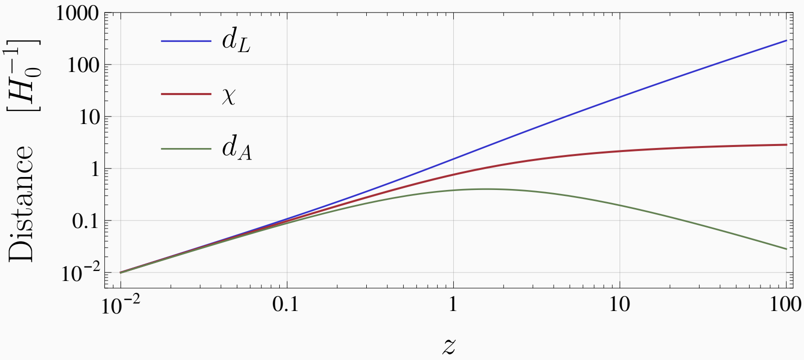
$$\mathcal{F} = \frac{L_S/(1+z)^2}{4\pi S_K^2(\chi)}.$$

The luminosity distance is

$$d_L(z) = (1+z)S_K(\chi(z))$$

Summary of distances

$$\chi(z) = \frac{c}{H_0}\int_0^z \frac{dz'}{H(z')/H_0}, \quad d_L(z) = (1+z)\chi(z), \quad d_A(z) = \frac{1}{1+z}\chi(z)$$

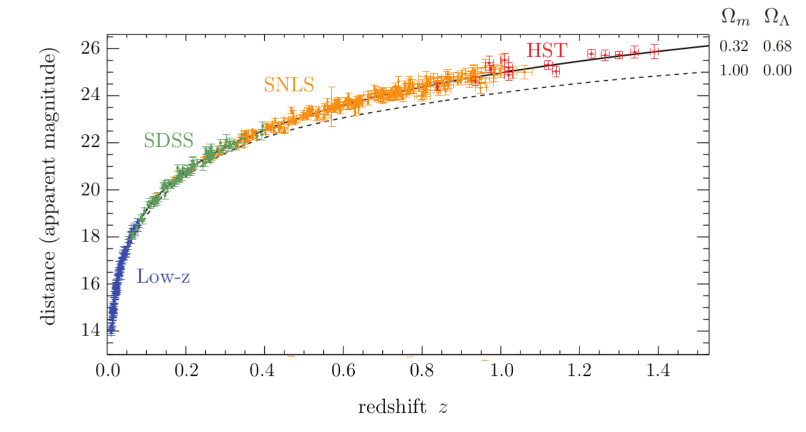


At small redshift, the three distances coincides: $d = H_0^{-1} z$, (or $z = H_0 d$)

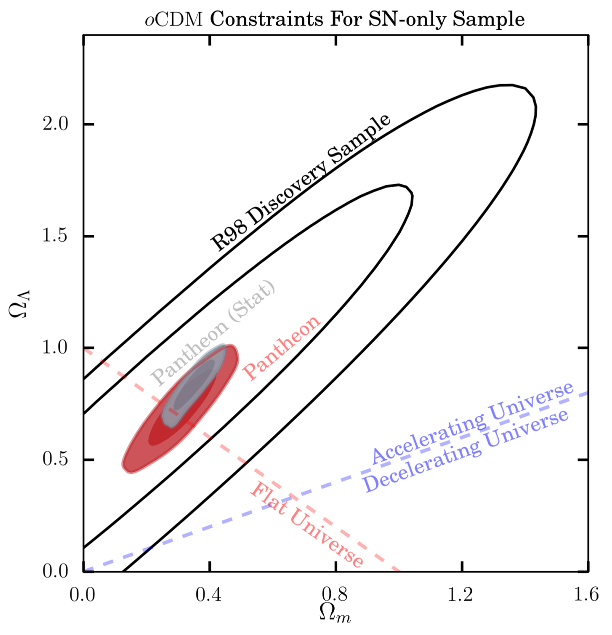
(Remind the longitudinal Doppler effect is $\lambda_0/\lambda_S = \sqrt{(1+v)/(1-v)} \approx 1 + v$.)

Type IA supernovae and the discovery of dark energy

$$m - M = 5\log_{10}\left[\frac{d_L(z;\Omega_A)}{1\,\mathrm{Mpc}}\right] + 25$$



[From Daniel Baumann Lecture Notes]



[From arXiv:1710.00845]

Particle horizon

The maximal comoving distance a signal could have propagated since the *beginning* of the universe until time t is

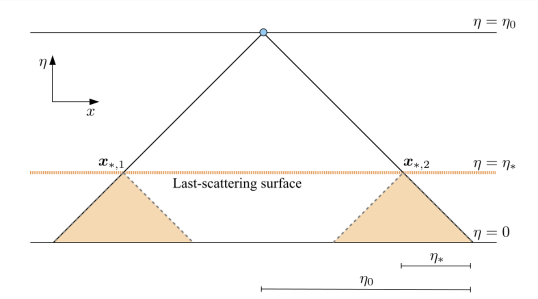
$$\eta(t) \equiv \int_0^t \frac{dt'}{a(t')}.$$

The reason this distance is so important is that no information could have propagated further on the coordinate grid than η since the *beginning* of time.

Therefore, if the above integral converge, regions (in the homogeneous and isotropic hypersurfaces of constant t) separated by distances greater than $\eta(t)$ had have no causal contact. Then, we can think of η as the comoving horizon.

The particle horizon at time t is $\eta(t)$

Notice that the particle horizon and the comoving distance are complementary
$\chi(t) = \eta(t_0) - \eta(t) \qquad (\int_t^{t_0} = \int_0^{t_0} - \int_0^t)$



Consider the last scattering surface at time t_*. Assuming there was only radiation and matter all the way back to the *beginning* of the universe, $\eta_* = 278\, h^{-1}$ Mpc.

The comoving distance between patches on the CMB sky today separated by an angle θ is (for small θ) $\chi(\theta) \approx \chi_* \theta = (\eta_0 - \eta_*)\theta$

Now, $\eta_0 \simeq 14146\, h^{-1}$ Mpc. Hence, two patches in the CMB separated by an angle

$$\theta > \tan^{-1} \frac{\eta_*}{\eta_0 - \eta_*} = 0.02 \approx 1.15^\circ$$

are causally disconnected. This is called *the horizon problem*.

Conformal time

One can use the particle horizon to define a time variable, called the *conformal time*

$$\eta = \int^{t} \frac{dt'}{a(t')}$$

The lower limit of the integral is not important, it is the zero of the measured time. Sometimes $\eta = 0$ at the *beginning* of the universe, sometimes it denotes the end of Inflation. Because of this, the conformal time should not be confused with the particle horizon. Unfortunately people use the same letter for both quantities.

Since $a(\eta)d\eta = dt$, we can write the FLRW metric as

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \gamma_{ij}dx^i dx^j\right]$$

The use of the above coordinates shows that the FLRW is conformally Minkowski if the spatial curvature is zero.

Unidad II, clase 4

30 de agosto de 2021

Cosmic components

CMB photons

The majority of the radiation contribution to the cosmic energy budget is in the form of the cosmic microwave background (CMB). Given its black-body nature, it has a Bose–Einstein phase-space distribution function,

$$f(\boldsymbol{p},\boldsymbol{x}) = f(p) = \frac{1}{e^{p/T}-1} = f^{(0)}(\boldsymbol{x},\boldsymbol{p}).$$

(At the background level, "$^{(0)}$", the dist funct cannot be a function of the position.)

The energy density is

$$\rho_\gamma = 2\int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{p/T}-1}$$

the p in the numerator comes from the energy equation for massless particles $E = p$. The $(2\pi)^3$ factor is actually the Planck constant to the third power and we use $\hbar = 1$. The factor of 2 outside the integral is the number of degrees of freedom $g_{eff} = 2$, accounting for the two spin states of photons.

Under the substitution $x = p/T$,

$$\rho_\gamma = \frac{8\pi T^4}{(2\pi)^3} \int dx \frac{x^3}{e^x - 1} = \frac{\pi^2}{15} T^4 = 4 \frac{a_{\mathrm{SB}}}{c} T^4$$

with the Stephen Boltzman constant $a_{\mathrm{SB}} = \frac{\pi^2 k_{\mathrm{B}}^4}{60\hbar^3 c^2}$

On the other hand, we know non-coherent radiation energy density (e.g. CMB radiation) decays as $1/a^4$. Hence, the CMB photons temperature is

$$T(a) = \frac{T_0}{a}$$

On the other hand, we measure the CMB radiation $T_{\mathrm{CMB}} = 2.725\,\mathrm{K}$, then

$$\Omega_\gamma h^2 = 2.47 \times 10^{-5}$$

Primordial neutrinos

Primordial neutrinos have not been observed directly yet. However, CMB observations are consistent with the theoretical expectations of three ($\times 2$) fermionic relativistic degrees of freedom with only one spin state: that is, they provide clear evidence of the existence of the primordial neutrinos expected from theoretical grounds.

Neutrinos are fermions, so their phase-space distribution is Fermi-Dirac

$$f_{\text{F-D}}(p) = \frac{1}{e^{p/T} + 1}$$

Actually, it is a little more complicated because neutrinos have tiny masses. So, one expects that the momentum in the exponential of the FD distribution would be replaced by $\sqrt{p^2 + m_\nu^2}$. However neutrinos are ultra-relativistic particles when created, and continue being ultra-relativistic when they decouple from the primordial plasma.

For the moment, let's just trust that neutrinos are relativistic until late times.

- Prior to decoupling, neutrinos were in thermal equilibrium with protons, neutrons and electrons, which was maintained through the weak interaction processes,e.g.

$$e^+ + e^- \longleftrightarrow \bar{\nu}_e + \nu_e$$

- As the universe expands, it cools as $T \propto 1/a$ until the weak interaction rate $\Gamma_\nu \sim G_{\mathrm{F}}^2 T^5$ falls below the expantion rate $H \sim G^{1/2} T^2$, and neutrinos decouple from the cosmic plasma. From $\Gamma_\nu \sim H$, the decoupling temperature is $T_{\mathrm{dec}} \sim 1\,\mathrm{MeV}$.

  Particle physics experiments put a limit on the neutrino masses $m_\nu \lesssim \mathcal{O}(1\,\mathrm{eV}) \ll T_{\mathrm{dec}}$. Hence, the neutrinos are still relativistic at decoupling.

After decoupling, the neutrinos do not interact anymore. Hence, they follow the same Fermi-Dirac distribution. T_ν is no longer a "true" temperature, but it becomes a parameter related to what the temperature was at large redshift. Assuming a microwave background temperature $T_\gamma^0 = 2.726\,\mathrm{K}$,

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma = 1.68 \times 10^{-4}(1+z)\,\mathrm{eV}$$

The photon temperature was enhanced by $e^+ + e^- \to 2\gamma$ annihilation, occurring at $T \sim 0.2\,\mathrm{MeV}$, just after neutrino decoupling.

Now, I want to show the above equation.

Before decoupling, let say at $a = a_1$, the neutrinos and radiation share a common temperature $T_1 = T_{\gamma,\nu}$, and the primordial plasma has an entropy density

$$\sigma \propto g_{eff} T^3_{\gamma,\nu}$$

the effective degrees of freedom at a_1 was

$$g_1 = \underbrace{2}_{\text{photons } \times 2 \text{ spins}} + \frac{7}{8}\underbrace{(4)}_{e^-, e^+ \times 2 \text{ spins}} + \frac{7}{8}\underbrace{(6)}_{\text{3 neutrinos + 3 antineutrinos}} = \frac{43}{4}$$

The factor $\frac{7}{8}$ comes from $\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{7}{8} \int_0^\infty \frac{x^3}{e^x + 1} dx$

Hence, before decoupling the entropy density was

$$\sigma_1 \propto \frac{43}{4} T^3_{\gamma,\nu}$$

After decoupling, at $a = a_2$, $e^+ + e^- \to 2\gamma$ annihilation occurs. Hence e^- and e^+ do not contribute to the entropy, instead they have putted more photons into play heating the photons bath. The entropy at a_2 is

$$\sigma_2 \propto g_{2,\nu} T_\nu^3 + g_{2,\gamma} T_\gamma^3 = \frac{7}{8} \times 6\, T_\nu^3 + 2\, T_\gamma^3$$

$\sigma_1 a_1^3 = \sigma_2 a_2^3 \Rightarrow$

$$\frac{43}{4}(a_1 T_{\gamma,\nu})^3 = \frac{21}{4} \times (a_2 T_\nu)^3 + 2(a_2 T_\gamma)^3$$

But once decoupled, the neutrinos follow the same temperature, i.e. $a_1 T_{\gamma,\nu} = a_2 T_\nu$. Hence

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

Hence, while still relativistic the neutrino density is

$$\rho_\nu = 3 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

Of course, the above computation is oversimplified.

A more rigorous derivation yields

$$\rho_\nu = N_{eff} \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

with

$$N_{eff} = 3.046$$

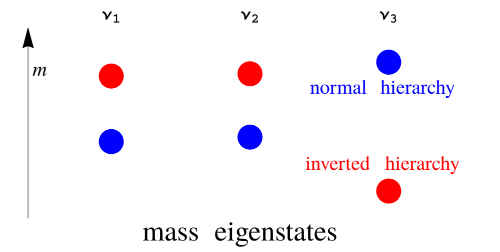
Neutrino masses

Energy eigenstates (ν_1, ν_2, ν_3): $\quad |\nu_i(t)\rangle = e^{-im_i^2 t/2E}|\nu_i(0)\rangle$

flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$: $\quad |\nu_\alpha\rangle = U_{\alpha i}|\nu_i\rangle$

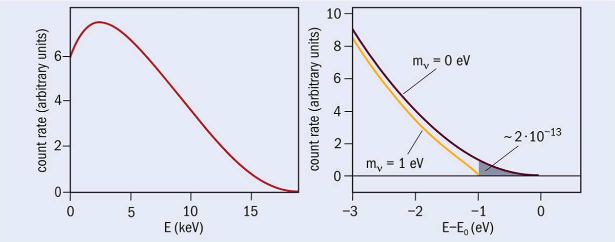
$$\mathcal{P}(\nu_\alpha \to \nu_\beta) = |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 = \sum_{ij} U_{\beta j}U^*_{\alpha j}U^*_{\beta i}U_{\alpha i}e^{-i(m_i^2-m_j^2)t/2E}$$

$$\Delta m^2_{21} = 7.4 \pm 0.2 \times 10^{-5}\,\mathrm{eV}^2, \qquad |\Delta m^2_{31}| = 2.52 \pm 0.03 \times 10^{-3}\,\mathrm{eV}^2$$



$$\sum_i m_{\nu,i} > 0.06\,\mathrm{eV}\ (\mathrm{NH}) \qquad \sum_i m_{\nu,i} > 0.098\,\mathrm{eV}\ (\mathrm{IH})$$

KArlsruhe TRitium Neutrino Experiment



KATRIN results: Aker et al, arxiv:1909.06048, 2105.08533

Tritium beta decay: $\quad {}^3_1\mathrm{H} \to {}^3_2\mathrm{He}^{+1} + e^- + \bar{\nu}_e$

$$0.06\,(0.10)\,\mathrm{eV} \quad \lesssim \quad \sum_i m_{\nu,\mathrm{i}} \quad \lesssim \quad 3 \times 0.9\,\mathrm{eV}$$

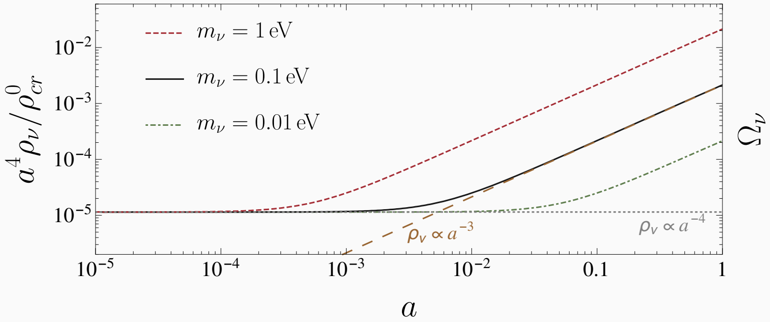
Neutrino oscillations — Tritium beta decay

Non-relativistic neutrinos

- Neutrinos become non-relativistic when the mean energy per particle in the relativistic limit,
$$\langle E \rangle = \frac{\int d^3p \; p \, f_{\text{F-D}}(p)}{\int d^3p \, f_{\text{F-D}}(p)} \approx 3.15 T_\nu ,$$
falls below their mass $m_{\nu,\mathrm{i}}$.
This occurs at redshift
$$1 + z_{\text{nr, i}} \approx 189 \left(\frac{m_{\nu,\mathrm{i}}}{0.1\,\mathrm{eV}} \right)$$

Neutrino Dark Matter

Energy density: $$\rho_{\nu,\mathrm{i}} = \frac{g}{(2\pi)^3}\int d^3p\sqrt{p^2+m_{\nu,i}^2}\,f_{\text{F-D}}(p)$$



$$\Omega_{\nu 0} = \frac{\rho_\nu^{z=0}}{\rho^0_{\mathrm{cr}}} = \frac{M_\nu}{93.14\,h^2\,\mathrm{eV}}, \qquad M_\nu \equiv \sum_{\mathrm{i}} m_{\nu,\mathrm{i}}$$

$M_\nu < 15\,\mathrm{eV}$ Gershtein & Zeldovich, 1966!

Dark Energy

Λ

The cosmological constant problem(s)

[S.Weinberg *The Cosmological Constant Problem*, Rev.Mod.Phys. 61 (1989) 1-23]

The cosmological constant problem*

Steven Weinberg

Theory Group, Department of Physics, University of Texas, Austin, Texas 78712

Astronomical observations indicate that the cosmological constant is many orders of magnitude smaller than estimated in modern theories of elementary particles. After a brief review of the history of this problem, five different approaches to its solution are described.

This is the so-called *the old CC problem*.

There are two components, as fare as we know unrelated, entering Einstein's equations and indistinguishable:

1. Λ_{cc}: Cosmological Constant, allowed into the Einstein equations because $\nabla^\mu g_{\mu\nu} = 0$
2. Λ_{vac} Vacuum energy from quantum fields. Contributing as $-\frac{\Lambda_{vac}}{8\pi G} g_{\mu\nu}$ to the total energy momentum tensor.

The Einstein's equations become

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_{cc} g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda_{vac} g_{\mu\nu}$$

or

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda_{obs} g_{\mu\nu}$$

where the observed cosmological constant is $\Lambda_{obs} = \Lambda_{cc} + \Lambda_{vac}$

Λ_{vac} is the vacuum energy from quantum fields. Contributing with an energy density $\rho_{vac} = \frac{\Lambda_{vac}}{8\pi G}$. Using standard tools of QFT,

$$\rho_{vac} = \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m^2} \qquad k^4 \quad \text{UV-divergent.}$$

(You can visualize it as the ground energy levels of harmonic oscillators $\frac{1}{2}\hbar\omega$ and use a dispersion relation $\omega = \sqrt{k^2 + m^2}$.)

The above integral diverges. But we don't expect our physical theories are valid up to arbitrary UV-scales. Assuming that QFT is valid up to the Planck scale (this can be relaxed by a lot, e.g. up to TeV, and the Cosm Const problem persists), such that we truncate the above integral at $k_{\text{max}} = M_{\text{Pl}} = G^{-1/2} \sim 10^{18}$ GeV, and we obtain

$$\rho_{vac} \sim \frac{c^5}{G^2\hbar} \sim 10^{76}\,\text{GeV}^4 \qquad \text{or} \qquad \Lambda_{vac} \sim 10^{40}\text{GeV}^2$$

The observed cosmological constant comes from Friedmann eq.

$$H_0^2 \sim \Lambda_{obs} \sim (10^{-33}\,\mathrm{eV})^2 = 10^{-84}\,\mathrm{GeV}^2$$

Hence, the two contributions Λ_{vac} and Λ_{cc} should cancel such that

$$\Lambda_{obs} = \Lambda_{vac} + \Lambda_{cc} \sim 10^{-84}\,\mathrm{GeV}^2,$$

or

$$\frac{\Lambda_{vac} + \Lambda_{cc}}{\Lambda_{vac}} \sim 10^{-124}\ !$$

What can we do?

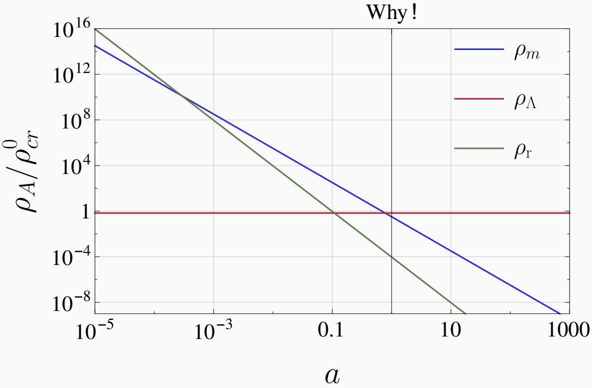
One way to ameliorate the “problem” is to assume that vacuum energy does not gravitate because some unknown mechanism. Hence $\Lambda_{obs} = \Lambda_{cc}$. So, we have to explain one small number, instead of why to very large numbers differ by $1/10^{120}$. Then we can either

- Accept that there exist such a small constant in Nature and we have measured it.
- Be unhappy with the existence of such a small constant in Nature, and search for alternatives.

It seems that the majority of cosmologists dislike the first option.

Hence, we can invent a fluid with very weird properties driving the accelerated expansion of the Universe (but remind that we have ignored the quantum fields vacuum contributions: Almost all dark energy/Modified Gravity models ignore this).

The coincidence problem



Scalar fields

What can we do?

Perhaps the most simply way to deal with the coincidence problem is to promote the cosmological constant to a function of "time", or to respect general covariance, to a dynamical scalar field:

$$\Lambda g_{\mu\nu} \longrightarrow T^{\phi}_{\mu\nu}$$

The simplest of this option is called quintessence, but there are several other realizations of dynamical scalar fields, e.g.,

Quintessence

K-essence

Kinetic braiding

Scalar Tensor Gravity

...

Unidad II, clase 5

1 de septiembre de 2021

Variational principles

The Einstein's field equation can be derived from a variational principle. In vacuum, the action is the Einstein-Hilbert action

$$S_{\mathrm{EH}} = \int d^4x \sqrt{-g}\,\frac{1}{16\pi G} R$$

EE are obtained by extremizing the action with respect to the metric. More precisely

$$\delta_g S = 0 : \qquad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}.$$

where the total action S is the sum of the E-H action and a second action related to the matter fields whose variation yields the energy momentum tensor

δ_g means that we are taking variations with respect to the **inverse** metric $g^{\mu\nu}$, and not to matter fields which are present in the matter action.

Variational methods

Consider an action $S[\boldsymbol{g};\Psi_i]$ depending on the metric $g^{\mu\nu}$ and possibly other fields Ψ_i

$$\delta_g S \equiv \int d^4x \frac{\delta S}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = -\frac{1}{2}\int d^4x\sqrt{-g}\left(-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu}$$

where $g = \det[g_{\mu\nu}]$ is the determinant of the metric.

Extremize the action:

$$\delta_g S = 0 \qquad \Rightarrow \qquad -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0$$

If the action represents matter fields, the energy momentum tensor is

$$T^{matter}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S^{matter}}{\delta g^{\mu\nu}}.$$

or

$$\delta_g S^{matter} = -\frac{1}{2}\int d^4x\sqrt{-g}T^{matter}_{\mu\nu}\delta g^{\mu\nu}$$

GR in vacuum from EH action

The action for GR in vacuum is the Einstein-Hilbert (EH) action

$$S_{\text{EH}} = \int d^4x \sqrt{-g}\, \frac{1}{16\pi G} R \qquad \text{with} \quad R = g^{\mu\nu} R_{\mu\nu}$$

Taking variations

$$\delta_g S_{\text{EH}} = \int d^4x \frac{1}{16\pi G} \left\{ \delta\left(\sqrt{-g}\right) g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\left(\delta g^{\mu\nu}\right) R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \right\}$$

We need $\delta\sqrt{-g} = -\frac{1}{2} g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}$

$$\delta_g S_{\text{EH}} = -\frac{1}{2} \int d^4 \sqrt{-g} \left[-\frac{1}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \right] \delta g^{\mu\nu}$$

where the term $\int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}$ becomes a boundary term (Gibbons-Hawking)

EE in vacuum are given by $\delta_g S_{\text{EH}} = 0$: $\qquad -\frac{1}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0.$

GR + matter

The total GR action is the sum of the EH action and the matter fields action

$$S = S_{\text{EH}}[g] + S_{matter}[g;\Psi_i]$$

Extremizing the action with respect to the metric. More precisely

$$-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0 : \qquad -\frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - T^{matter}_{\mu\nu} = 0$$

which are Einstein's field equations.

More generally, one can add a constant Λ to the Einstein-Hilbert action

$$S_{\text{EH}} = \int d^4x\sqrt{-g}\,\frac{1}{16\pi G}\,(R - 2\Lambda)$$

to obtain

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T^{matter}_{\mu\nu}$$

A scalar field

The most simple form of a matter field is given by a scalar field ϕ. The action is

$$S_\phi[g;\phi] = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right)$$

If you are not very familiar with this action, and only to get a sense of its meaning, consider the simple case of Minkowski space and imagine that ϕ depends only on *some Lorentz frame* time. Then $\partial_0\phi = \dot{\phi}$ and $\partial_i\phi = 0$. Further, $g^{00} = -1$. Hence, the Lagrangian density becomes

$$\mathcal{L}_\phi \; \rightarrow \; \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

This resembles a Classical Mechanics Lagrangian with kinetic energy $\frac{1}{2}\dot{\phi}^2$ and potential energy $V(\phi)$. With the caveat that ϕ is not a position of a particle, but it is a scalar field that permeates the whole space: ϕ is a function of the spacetime into the real numbers.

A free scalar field with mass m_ϕ has a potential $V(\phi) = \frac{1}{2}m_\phi^2\phi^2$.

E-M tensor of a scalar field

The energy momentum tensor of a scalar field is computed through

$$\delta_g S_\phi[g;\phi] = -\frac{1}{2}\int d^4x\sqrt{-g}T^\phi_{\mu\nu}\delta g^{\mu\nu}$$

but

$$\begin{aligned}\delta_g S_\phi[g;\phi] &= \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right)\\ &= -\frac{1}{2}\int d^4x\left[\nabla_\mu\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi - g_{\mu\nu}V(\phi)\right]\end{aligned}$$

or

$$T^\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi - g_{\mu\nu}V(\phi)$$

Dark Energy

Einstein's equation with standard model fields + CDM + a scalar field becomes

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left(T_{\mu\nu}^{\text{Standard Model + CDM}} + T_{\mu\nu}^{\phi} \right)$$

with $T_{\mu\nu}^{\phi}$ replacing the cosmological constant.

This is the simplest dark energy approach to explain the accelerated expansion of the Universe.

The hope is that we can find a scalar field such that $8\pi G T_{\mu\nu}^{\phi} \sim \Lambda g_{\mu\nu}$ nowadays.

Fortunately, we knew this was possible for a long time, since the first Inflation papers; and further later developments for late time cosmology by Bharat Ratra and James Peebles (1988).

We still need the evolution equation for the scalar field

Klein-Gordon equation

The evolution equation for the scalar field is obtained by taking variations of the metric $S_\phi[g;\phi]$ with respect to ϕ

$$\delta_\phi S_\phi[g;\phi] = \delta_\phi \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right) = 0$$

but

$$\begin{aligned}\delta_\phi S_\phi &= \int d^4x\sqrt{-g}\left(-g^{\mu\nu}\nabla_\mu\phi\nabla_\nu(\delta\phi) - V'(\phi)\delta\phi\right)\\ &= \int d^4x\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - V'(\phi)\right)\delta\phi\end{aligned}$$

or the *Klein-Gordon equation* is

$$\Box\phi - V'(\phi) = 0$$

with $\Box \equiv \nabla^\mu\nabla_\mu$ the D'Alambertian operator.

Quintessence

Consider a scalar field in a flat FLRW space-time. The pressure and energy density are

$$\rho_\phi = T_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$
$$\mathcal{P}_\phi = T_{ii} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

actually, one can show that the scalar field behaves as a perfect fluid at the background level only.

Then, the equation of state parameter is:

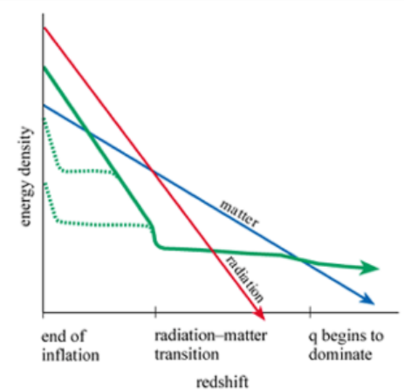
$$w_\phi \equiv = \frac{\mathcal{P}_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}$$
$$\simeq -1 \qquad \text{if the field is slow-rolling down the potential:} \qquad \dot{\phi}^2 \ll V$$

The evolution equation for the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi).$$

Tracker solutions area realized for some potentials. The most famous are

$$\text{Exponential:} \quad V(\phi) = V_0 \exp\left(-\frac{\phi}{M}\right)$$
$$\text{Power-Law:} \quad V(\phi) = \frac{M^{4+n}}{\phi^n}$$



Dark Energy Parametrizations

Do we have to test thousands of dark energy models with laborious analysis of supernovae, BAO, CMB, and so on? (Moreover, the vast majority of models are not very well motivated.)

Fortunately not: the energy-momentum tensor is completely general and is dictated by the symmetries of the FLRW spacetime. Hence, defining pressure via the equation of state $w_{\rm DE}(a)$, and given the continuity equation

$$\dot{\rho}_{\rm DE} + 3H\rho_{\rm DE}(1 + w_{\rm DE}(a)) = 0$$

whose solution is

$$\rho_{\rm DE}(a) = \rho_{\rm DE}^0 \exp\left[-3\int_1^a \frac{1 + w_{\rm DE}(a')}{a'} da'\right],$$

the effect of a general dark energy on the expansion history is completely determined by the function $w_{\rm DE}(a)$

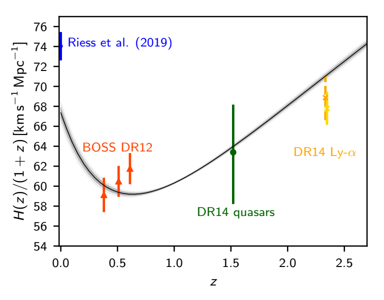
Actually, given an arbitrary Hubble flow $H(z)$, using the Friedmann eq. we have

$$H^2(z) = H_0^2 \left[\Omega_m (1+z)^3 + \Omega_{\rm DE} \exp\left(3 \int_0^z \frac{1 + w_{\rm DE}(z')}{1+z'} dz' \right) \right]$$

and we can find $w_{\rm DE}(z)$

$$w_{\rm DE}(z) = \frac{H^2(z) - (2/3) H(z)(1+z) dH(z)/dz}{H_0^2 \Omega_m (1+z)^3 + H^2(z)}.$$

Given a measured expansion history and a matter abundance, we can find $w_{\rm DE}(z)$.



wCDM

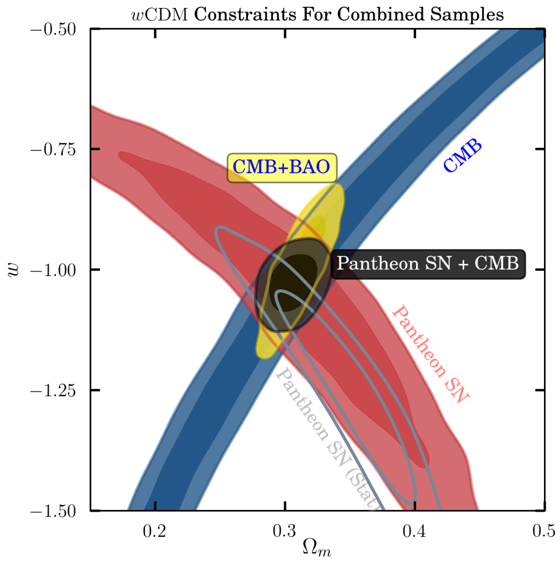
Of course, the most simple parametrization is to consider a constant EoS parameter:

$$w_{\mathrm{DE}}(z) = w = \text{constant}$$
$$\Rightarrow \quad \rho_{\mathrm{DE}} = \rho^0_{\mathrm{DE}} a^{-3(1+w)}$$

To provide acceleration w should be close to -1.

Friedmann equation becomes

$$H^2(z) = H_0^2 \left[\Omega_m (1+z)^3 + \Omega_{\mathrm{DE}} (1+z)^{3(1+w)}\right]$$



wCDM Constraints For Combined Samples
−0.50
−0.75
−1.00
−1.25
−1.50
w
0.2
0.3
0.4
0.5
Ω_m
CMB+BAO
CMB
Pantheon SN + CMB
Pantheon SN
Pantheon SN (Stat)

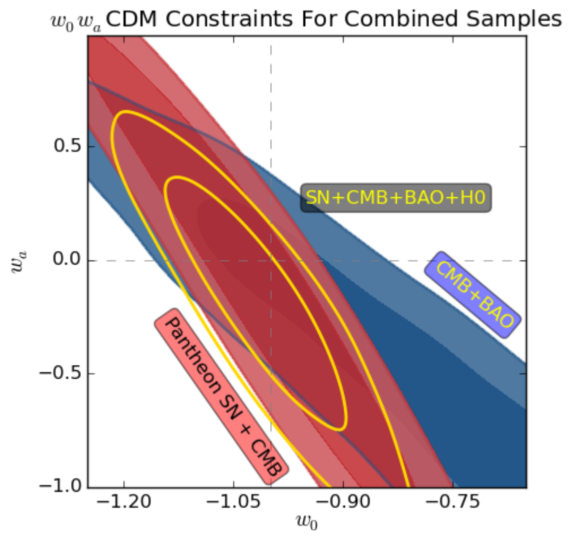
CPL parametrization

Chevalier-Polarski-Linder parametrization consists on expanding $w_{\rm DE}(a)$ around $a_0 = 1$ to linear order

$$w_{\rm DE}(a) = w_0 + (1-a)w_a \qquad \text{or} \qquad w_{\rm DE}(z) = w_0 + \frac{z}{1+z}w_a$$

Friedmann equation:

$$H^2(z) = H_0^2\left[\Omega_m(1+z)^3 + \Omega_{\rm DE}(1+z)^{3(1+w_0+w_a)}\exp\left(-w_a\frac{z}{1+z}\right)\right]$$



$w_0\,w_a$ CDM Constraints For Combined Samples
SN+CMB+BAO+H0
CMB+BAO
Pantheon SN + CMB
0.5
0.0
−0.5
−1.0
w_a
−1.20
−1.05
−0.90
−0.75
w_0

Warning:

The energy momentum tensor of DE with EoS parameter w_{DE} becomes

$$T_{\mu\nu} = \rho u_\mu u_\nu + \underbrace{w\rho}_{=\mathcal{P}}(g_{\mu\nu} + u_\mu u_\nu)$$

where u^μ are the comoving observers, in some coordinates $u^\mu = (1, 0, 0, 0)$. Consider a second observer in a Lorentz frame: $v^\mu = \gamma(1, \boldsymbol{v})$. The energy density measured for such observer is

$$\rho|_v = \rho \frac{1 + wv^2}{1 - v^2}$$

where we used $u_\mu v^\mu = \gamma = 1/\sqrt{1 - v^2}$.

If $w < -1$ and the relative velocity is

$$v^2 > -\frac{1}{w}$$

then

$$\rho|_v < 0$$

More parametrizations

There are dozens, if not hundreds, of dark energy parametrizations in the literature

$$w_{\mathrm{DE}} = w_{\mathrm{DE}}(z; \text{a few parameters})$$

Only two of them (as far as I know) are used in the observational collaboration papers.

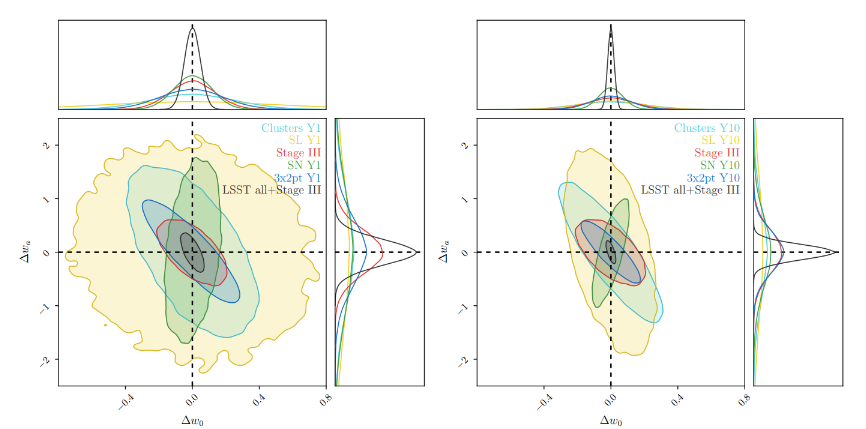
Some of them inspired in aspects of quintessence or other dark energy models. Some others with no real justification.

All of them share a suspicious property: the function

$$\frac{1 + w_{\mathrm{DE}}(z)}{1 + z}$$

has a primitive

Vera Rubin Observatory (LSST) forecasts for w_0-w_a



[From arxiv:1809.01669]

Modified Gravity

An alternative to dark energy is try to modified General Relativity.

For example:

$$S_{\mathrm{EH}} = \int d^4x\sqrt{-g}\,\frac{1}{16\pi G}\left(R - 2\Lambda\right) \to \int d^4x\sqrt{-g}\,\frac{1}{16\pi G}\left(R + f(R)\right)$$

where $f(R)$ is a function of the Ricci R, or more generally of other scalars as $R_{\mu\nu}R^{\mu\nu}$, or even of the torsion tensor.

- $f(R)$ is 4th order... in disguise. One can write the field equations as second order in the metric + an additional second order dynamical equation for a scalar field (this can be shown rigorously when $f'(R) \neq 0$). That is, only one extra degree of freedom is propagated.

Scalar tensor theories

Horndeski’s theory is the most general theory of gravity in four dimensions whose Lagrangian is constructed out of the metric tensor and a scalar field and leads to second order equations of motion

Action [edit]

Horndeski's theory can be written in terms of an action as[4]

$$S[g_{\mu\nu},\phi]=\int \mathrm{d}^4x\,\sqrt{-g}\left[\sum_{i=2}^{5}\frac{1}{8\pi G_{\mathrm{N}}}\mathcal{L}_i[g_{\mu\nu},\phi]\;+\mathcal{L}_{\mathrm{m}}[g_{\mu\nu},\psi_M]\right]$$

with the Lagrangian densities

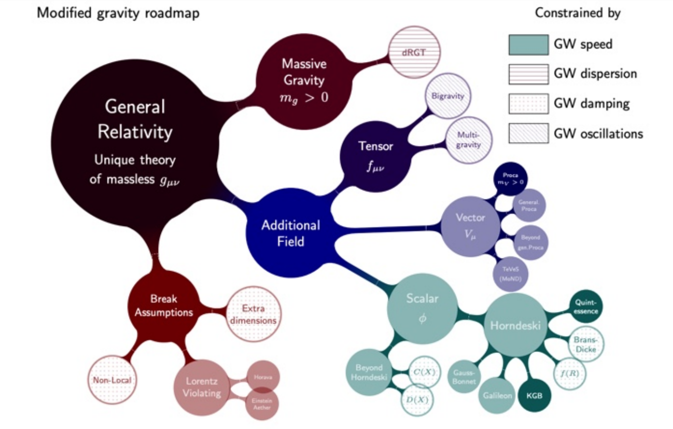
$$\mathcal{L}_2=G_2(\phi,\,X)$$

$$\mathcal{L}_3=G_3(\phi,\,X)\Box\phi$$

$$\mathcal{L}_4=G_4(\phi,\,X)R+G_{4,X}(\phi,\,X)\left[(\Box\phi)^2-\phi_{;\mu\nu}\phi^{;\mu\nu}\right]$$

$$\mathcal{L}_5=G_5(\phi,\,X)G_{\mu\nu}\phi^{;\mu\nu}-\frac{1}{6}G_{5,X}(\phi,\,X)\left[(\Box\phi)^3+2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu}-3\phi_{;\mu\nu}\phi^{;\mu\nu}\Box\phi\right]$$

Here G_N is Newton's constant, $\mathcal{L}_m$ represents the matter Lagrangian, G_2 to G_5 are generic functions of ϕ and X , $R, G_{\mu\nu}$ are the Ricci scalar and Einstein tensor, $g_{\mu\nu}$ is the Jordan frame metric, semicolon indicates covariant derivatives, commas indicate partial derivatives, $\Box\phi\equiv g^{\mu\nu}\phi_{;\mu\nu}$,$X\equiv -1/2g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}$ and repeated indices are summed over following Einstein's convention.



[From Ezquiaga & Zumalagárregui arxiv:1807.09241]

Parametrizations in MG?

At the background level, it doesn't have too much sense to parametrize the effects of MG. This is because an MG theory that predicts a Hubble slightly different than in ΛCDM, differs by a lot at the perturbative level.

This is because, in general, the two scalar gravitational potentials of the metric (Φ and $-\Psi$) are not equal in the absence of anisotropic stresses (e.g., at late times), as it happens in GR.

There are some MG theories tailored such that $-\Psi = \Phi$ at late times: *no-slip gravity*. [e.g: Eric Linder arXiv:1801.01503]

Cosmología

Unidad 5 y 7: Estadística en cosmología

Alejandro Avilés (CONACyT/ININ)
avilescervantes@gmail.com

semestre 2022-1

Curso PCF-UNAM

Temario

Unidad 5	Estadística en cosmología
5.1	Principio Cosmológico (Hipótesis estadística y realizaciones observacionales)
5.2	Función de correlación 2-puntos
5.3	Espectro de potencia: Teoría y parametrizaciones
5.4	Espectro de potencia esférica
5.5	Estadística a 3 y más puntos

NO existe !!!

5.4 -> Espectros angulares de potencia

7.1	Observaciones que sondean la aceleración actual: Supernovas, BAO, RSD
7.2	Constante cosmológica: Motivación, evidencia observacional y problemas asociados
7.3	Modelos de Energía oscura: Modelos de campos escalares (e.g. potenciales inversos, exponencial), parametrizaciones como un fluido
7.4	Expansión acelerada debida a Gravedad Modificada: Parametrizaciones Observables para desviaciones de la Relatividad General

Lo vimos en la unidad 2

Unidad 8	Cosmología observacional contemporánea
8.1	Lentes gravitacionales débiles

En este bloque veremos 5.1, 5.2, 5.3, 5.4, 5.5, 7.1 y 8.1

Y teoría de perturbaciones no lineales (estándar).

Unidad V & VII, clase 1

18 de octubre de 2021

Cosmological principle

Background: The universe is homogeneous and isotropic

Perturbations: The universe is *statistically* homogeneous and isotropic

Fourier Transform conventions

$$f(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}), \qquad f(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{k})$$

Hence

$$(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}') = \int d^3x\, e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}$$

For a real field $f(\boldsymbol{x})$

$$f(\boldsymbol{k}) = f^*(-\boldsymbol{k})$$

Statistical homogeneity

$$\text{Translation:} \qquad \hat{T}_{\boldsymbol{y}} f(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{y})$$

Fourier Space:

$$\hat{T}_{\boldsymbol{y}} f(\boldsymbol{k}) = \int d^3x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x} - \boldsymbol{y}) \overbrace{=}^{\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{y}} \int d^3x' \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}'} f(\boldsymbol{x}') e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}$$
$$= f(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}$$

Translation invariance

$$\langle f(\boldsymbol{k}) f(\boldsymbol{k}') \rangle = \langle \hat{T}_{\boldsymbol{y}} f(\boldsymbol{k}) \, \hat{T}_{\boldsymbol{y}} f(\boldsymbol{k}') \rangle$$

Then

$$\langle f(\boldsymbol{k}) f(\boldsymbol{k}') \rangle = \langle f(\boldsymbol{k}) f(\boldsymbol{k}') \rangle e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{y}} \qquad \text{(for any } \boldsymbol{y}\text{)}$$

Hence

$$\langle f(\boldsymbol{k}) f(\boldsymbol{k}') \rangle = (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k} + \boldsymbol{k}') P_f(\boldsymbol{k})$$

Statistical Isotropy

$$\text{Rotation:} \qquad \hat{R} f(\boldsymbol{x}) = f(\hat{R}^{-1}\boldsymbol{x})$$

Fourier Space:

$$\hat{R} f(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\hat{R}^{-1}\boldsymbol{x}) = \int d^3x\, e^{-i(\hat{R}^{-1}\boldsymbol{k})\cdot(\hat{R}^{-1}\boldsymbol{x})} f(\hat{R}^{-1}\boldsymbol{x})$$

$$\overbrace{=}^{\boldsymbol{x}' = \hat{R}^{-1}\boldsymbol{x}} \int d^3x'\, e^{-i(\hat{R}^{-1}\boldsymbol{k})\cdot\boldsymbol{x}'} f(\boldsymbol{x}') = f(\hat{R}^{-1}\boldsymbol{k})$$

Rotational invariance

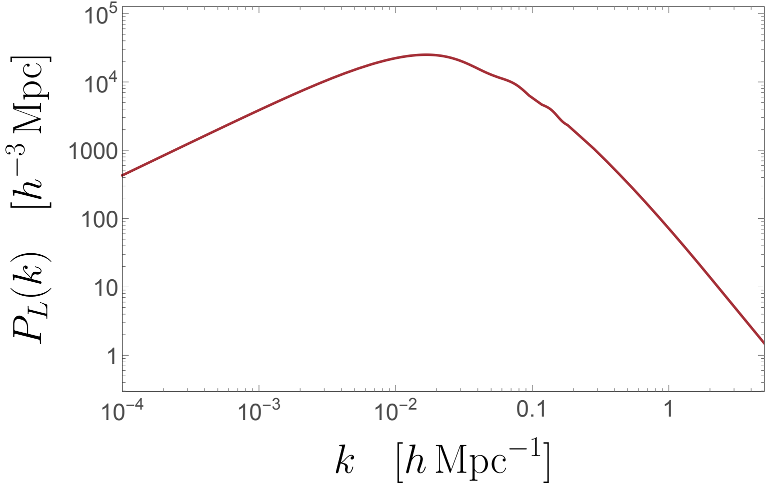
$$\begin{aligned} \langle f(\boldsymbol{k}) f(\boldsymbol{k}')\rangle &= \langle f(\hat{R}^{-1}\boldsymbol{k})\, f(\hat{R}^{-1}\boldsymbol{k}')\rangle \\ P_f(\boldsymbol{k})\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}') &= P_f(\hat{R}^{-1}\boldsymbol{k})\delta_{\rm D}(\hat{R}^{-1}\boldsymbol{k} + \hat{R}^{-1}\boldsymbol{k}') \qquad (\delta_{\rm D}(\hat{R}^{-1}\boldsymbol{k}) = \det\hat{R}\,\delta_{\rm D}(\boldsymbol{k}) = \delta_{\rm D}(\boldsymbol{k})) \\ &= P_f(\hat{R}^{-1}\boldsymbol{k})\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}') \qquad (\text{for any } \hat{R}) \end{aligned}$$

For statistically homogeneous and isotropic field f,

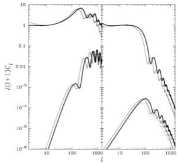
$$\boxed{\langle f(\boldsymbol{k}) f(\boldsymbol{k}')\rangle = (2\pi)^3 \delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}') P_f(k)}$$

Matter power spectrum

$$\langle \delta(\boldsymbol{k})\delta(\boldsymbol{k}') \rangle = (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k} + \boldsymbol{k}') P_L(k)$$



CAMB: https://camb.info/



Code for Anisotropies in the Microwave Background

by Antony Lewis and Anthony Challinor

Get help: [ ] Search Google Custom Search

Features:

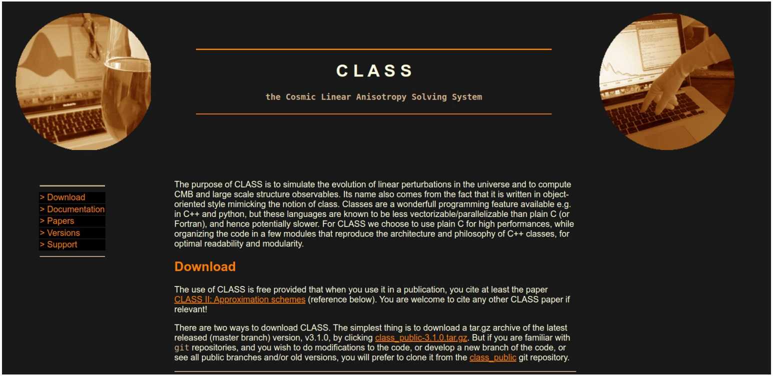
- Optimized Python and Fortran code
- Calculate CMB, lensing, source count and dark-age 21cm angular power spectra
- Matter transfer functions, power spectra, σ_8 and related quanties
- General background cosmology
- Support for closed, open and flat models
- Scalar, vector and tensor modes including polarization
- Fast computation to ~0.1% accuracy, with controllable accuracy level
- Object-oriented Python and easily-extensible modern Fortran 2008 classes
- Efficient support for massive neutrinos and arbitrary neutrino mass splittings
- Optional modelling of perturbed recombination and temperature perturbations
- Calculation of local primordial and CMB lensing bispectra (Fortran)

Download the source code and see:

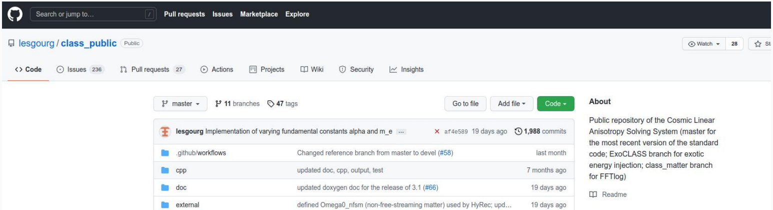
- Python documentation
- Fortran documentation

See CosmoCoffee for support, and the BibTex file for references. There are also theory derivations and CAMB notes describing some conventions and approximations. The full set of linear equations

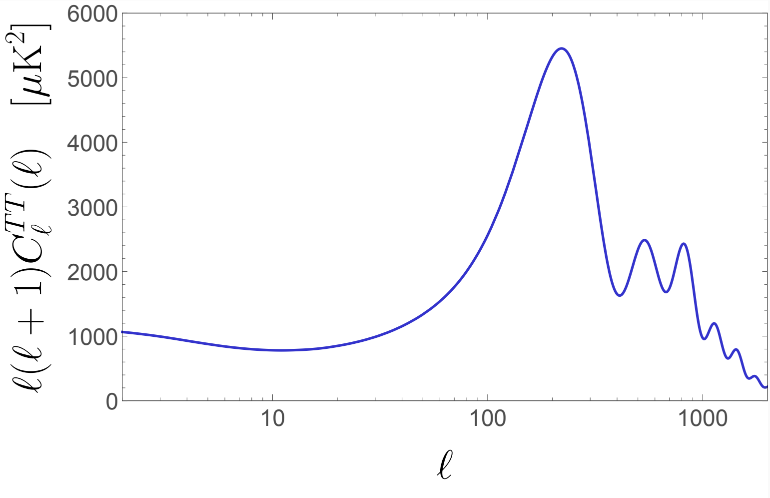
CLASS: lesgourg.github.io/class_public/class.html



CLASS: github.com/lesgourg/class_public



CMB angular TT power spectrum



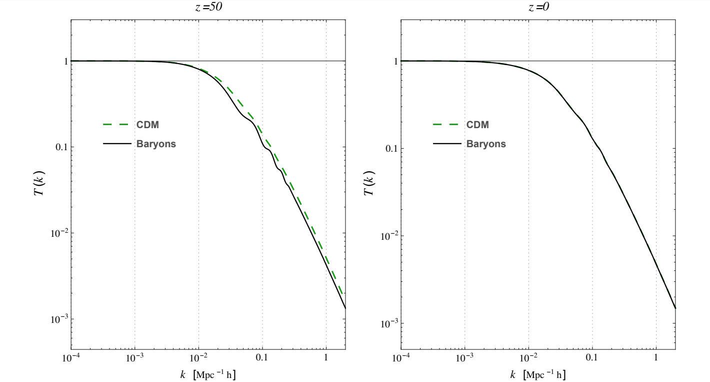
Unidad V & VII, clase 2

20 de octubre de 2021

Transfer function $T(k)$

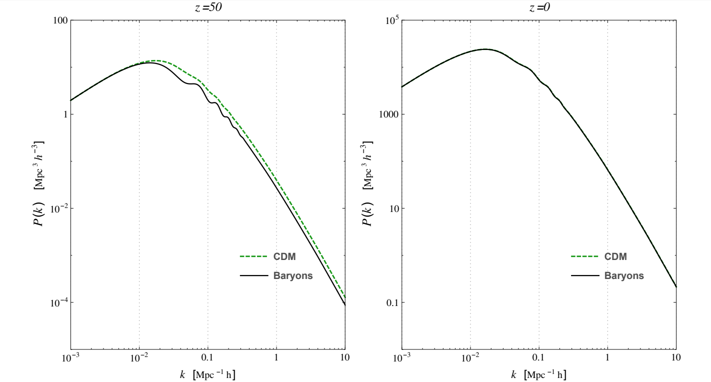
Warning: different definitions/conventions exist

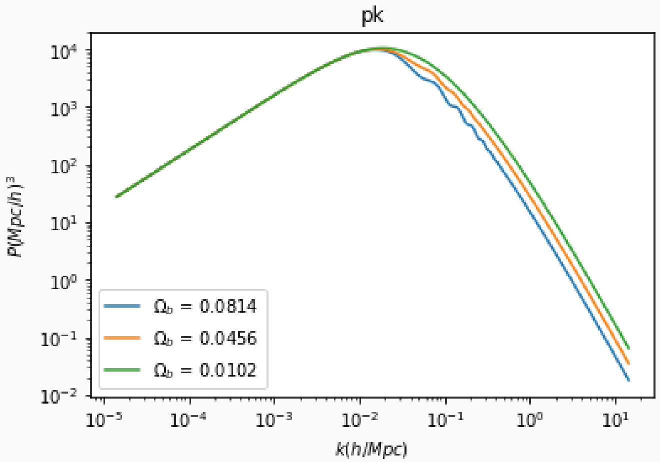
$$P_L(k) = \underbrace{A_s \left(\frac{k}{k_0}\right)^{n_s}}_{\text{Primordial pk}} \times \overbrace{T^2(k)}^{\text{Transfer function}}$$



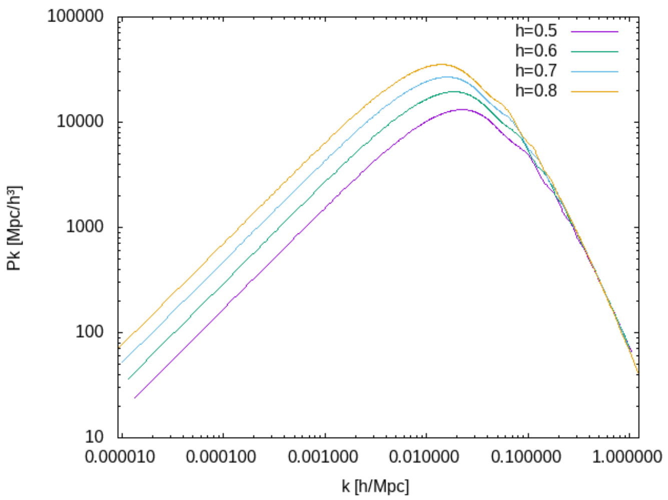
Power spectrum $P(k)$

$$P_L(k) = \underbrace{A_s \left(\frac{k}{k_0}\right)^{n_s}}_{\text{Primordial pk}} \times \overbrace{T^2(k)}^{\text{Transfer function}}$$

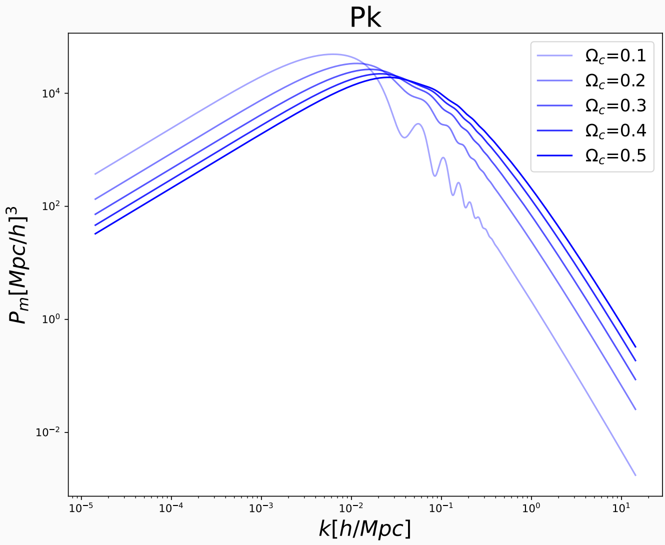




pk
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
$P(Mpc/h)^3$
Ω_b = 0.0814
Ω_b = 0.0456
Ω_b = 0.0102
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
$k(h/Mpc)$



100000
10000
1000
100
10
Pk [Mpc/h³]
0.000010
0.000100
0.001000
0.010000
0.100000
1.000000
k [h/Mpc]
h=0.5
h=0.6
h=0.7
h=0.8


Pk
Ω_c=0.1
Ω_c=0.2
Ω_c=0.3
Ω_c=0.4
Ω_c=0.5
$P_m[Mpc/h]^3$
10^4
10^2
10^0
10^{-2}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
$k[h/Mpc]$

Unidad V & VII, clase 3

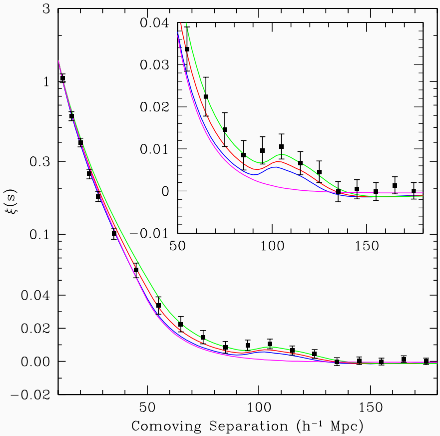
22 de octubre de 2021

The (2-point) correlation function

SUBMITTED TO *The Astrophysical Journal* 12/31/2004
Preprint typeset using LaTeX style emulateapj v. 11/12/01

DETECTION OF THE BARYON ACOUSTIC PEAK IN THE LARGE-SCALE CORRELATION FUNCTION OF SDSS LUMINOUS RED GALAXIES

DANIEL J. EISENSTEIN[1,2], IDIT ZEHAVI[1], DAVID W. HOGG[3], ROMAN SCOCCIMARRO[3], MICHAEL R.



Correlation function

We consider a point process $N(\boldsymbol{x})$ with mean number density $\bar{n}$. The number of particles N_1 inside a volume dV (located at a position $\boldsymbol{x}_1$) is

$$N_1 = dV_1 \bar{n}(1 + \delta_1)$$

where $\delta_1 = \delta(\boldsymbol{x}_1)$ is the overdensity in dV_1.

Over a very large volume V we have $N_T = \bar{n}V$.

Probability to find an object '1' inside a volume dV_1:

$$\mathcal{P}_1 = \frac{\langle N_1 \rangle}{\langle N_T \rangle} = \frac{\bar{n}\langle 1 + \delta(\boldsymbol{x}_1)\rangle dV}{\bar{n}V} = \frac{dV_1}{V}$$

Why $\langle \delta(\boldsymbol{x}_1) \rangle = 0$?

Because $\quad \bar{\rho}(t) = \big\langle \rho(\boldsymbol{x}, t) \big\rangle = \big\langle \bar{\rho}(t)(1 + \delta(\boldsymbol{x}, t)) \big\rangle = \bar{\rho}(t)\big\langle 1 + \delta(\boldsymbol{x}, t) \big\rangle$

Correlation function

Probability to find an object '2' inside a volume dV_2 at postition $\boldsymbol{x}_2$, given that we have an object '1' inside a volume dV_1 located at a position $\boldsymbol{x}_1$

$$\begin{aligned}\mathcal{P}_{2|1} = \frac{\langle N_1 N_2 \rangle}{\langle N_1 N_T \rangle} &= \frac{\bar{n}^2 \langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)\rangle dV_1 dV_2}{\bar{n}^2 dV_1 V \langle 1+\delta(\boldsymbol{x}_1)\rangle} \\ &= \big[1+\xi(\boldsymbol{x}_1,\boldsymbol{x}_2)\big]\frac{dV}{V}\end{aligned}$$

where we defined the correlation function

$$\xi(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \langle \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\rangle$$

Statistical homogeneity and isotropy:

$$\xi(\boldsymbol{x}_1,\boldsymbol{x}_2) = \xi(r), \qquad \mathbf{r} = \boldsymbol{x}_2 - \boldsymbol{x}_1$$

The correlation function is the Fourier Transform of the power spectrum

$$\xi(r) = \langle \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\rangle = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1 + i\boldsymbol{k}_2\cdot\boldsymbol{x}_2}\langle\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}_1 + i\boldsymbol{k}_2\cdot\boldsymbol{x}_2}(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}_2)P(k)$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\mathbf{r}}P(k)$$

Now, defining $\mu = \hat{k}\cdot\hat{x}$,

$$\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\mathbf{r}}P(k) = \frac{2\pi}{(2\pi)^3}\int_0^\infty dk\, k^2 P(k)\int_{-1}^1 d\mu\, e^{ikr\mu}$$

Then,

$$\xi(r) = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 P(k) j_0(kr)$$

with

$$j_0(x) = \frac{\sin x}{x}$$

the spherical Bessel function of order 0.

Numerical Issues

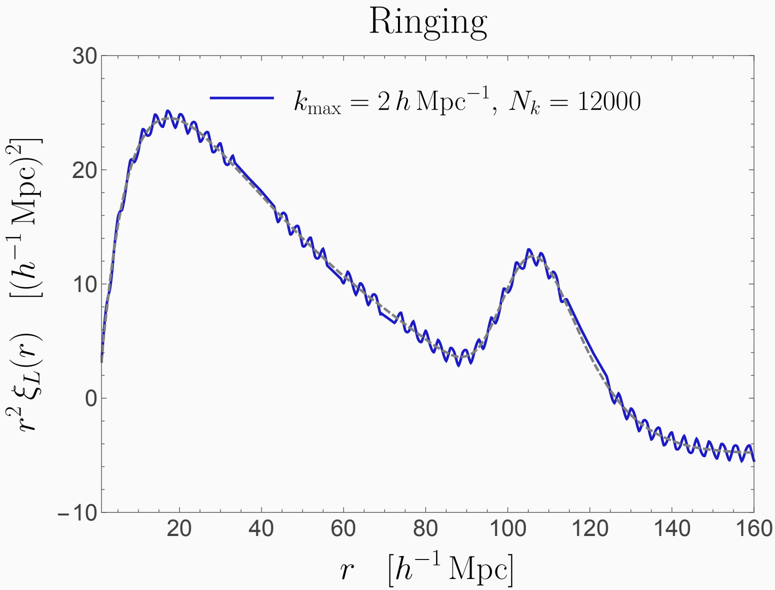
$$\xi(r) = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 P(k) j_0(kr)$$
$$= \frac{1}{2\pi^2}\int_{k_{\min}}^{k_{\max}} dk\, k^2 P(k) j_0(kr)$$

$$\xi(r) = \sum_{i=1}^{N_k} \frac{k_i^3}{2\pi^2} P(k_i) j_0(k_i r) \Delta(\log k_i)$$

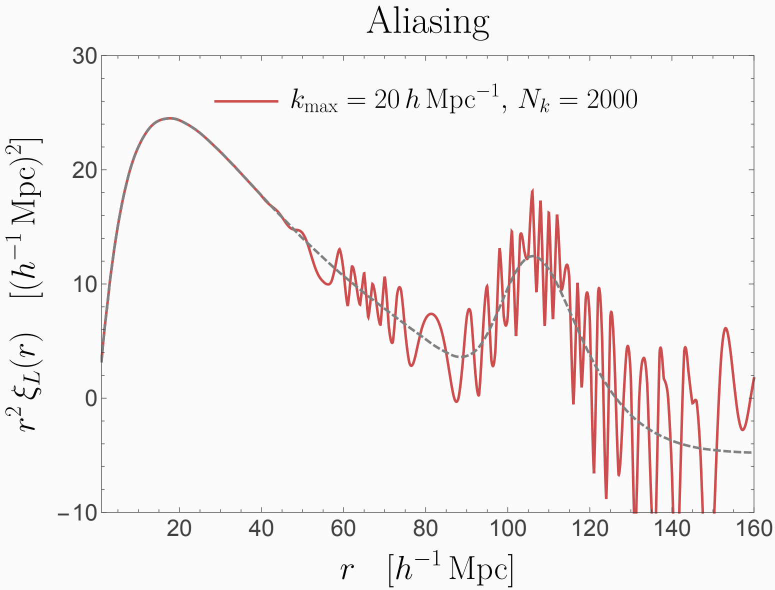
with

$$i = 1,\, 2, \cdots, N_k$$
$$k_i \in \{k_1 = k_{\min},\, k_2, \cdots, k_{N_k} = k_{\max}\}$$

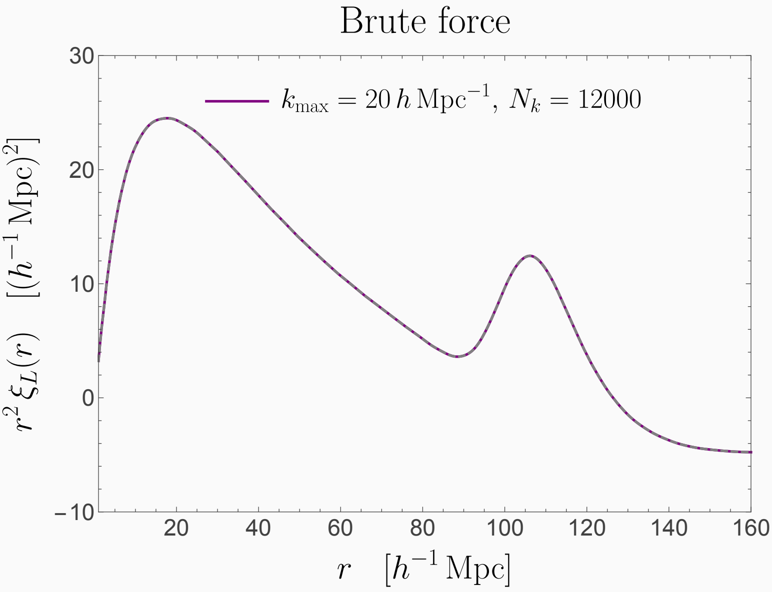
Ringing: cutting off high frequencies



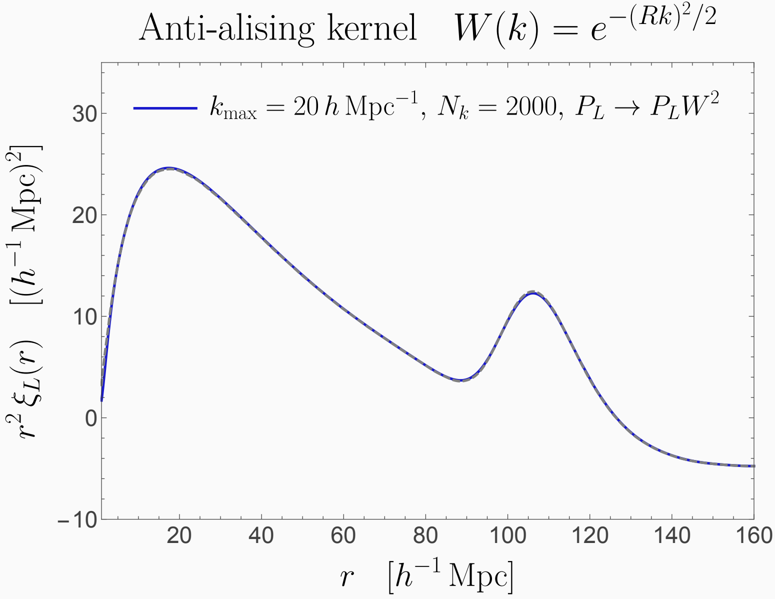
Aliasing: poor sampling



Brute force


Brute force
$k_{\max} = 20\,h\,\mathrm{Mpc}^{-1},\ N_k = 12000$
$r^2\,\xi_L(r) \quad [(h^{-1}\,\mathrm{Mpc})^2]$
30
20
10
0
−10
20
40
60
80
100
120
140
160
$r \quad [h^{-1}\,\mathrm{Mpc}]$

Damping (with $R = 1\,h^{-1}$ Mpc)



Vlasov Equation

From particles to fluids

Euler-Lagrangian equation

- The Lagrangian of a particle at position

$$\boldsymbol{r} = a(t)\boldsymbol{x}$$

under the gravitational potential ϕ_{N} is given by

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2}m(\dot{a}\boldsymbol{x} + a\dot{\boldsymbol{x}})^2 - m\phi_{\mathrm{N}}(\boldsymbol{x}, t),$$

where $\boldsymbol{x}$ is the comoving position of the particle and $a(t)$ is the scale factor.

- Adding a total derivative dg/dt to the Lagrangian, with $g = -ma\dot{a}\boldsymbol{x}^2/2$, we rewrite the Lagrangian as

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2}ma^2\dot{\boldsymbol{x}}^2 - m\Phi(\boldsymbol{x}, t),$$

where

$$\Phi = \phi_{\mathrm{N}} + \frac{1}{2}a\ddot{a}\boldsymbol{x}^2.$$

Poisson equation

$$\nabla_{\boldsymbol{r}}^2 \phi_N = 4\pi G\rho = 4\pi G\bar{\rho}(1+\delta)$$

where

$$\nabla_{\boldsymbol{r}} = \frac{\partial}{\partial \boldsymbol{r}} = \frac{1}{a}\frac{\partial}{\partial \boldsymbol{x}} = \frac{1}{a}\nabla_{\boldsymbol{x}} \equiv \frac{1}{a}\nabla$$

but $\phi_{\mathrm{N}} = \Phi - a\ddot{a}\boldsymbol{x}^2/2$, hence

$$4\pi G\bar{\rho}(1+\delta) = \frac{1}{a^2}\nabla^2\left(\Phi - \frac{1}{2}a\ddot{a}\boldsymbol{x}^2\right) = \frac{1}{a^2}\nabla^2\Phi - 3\frac{\ddot{a}}{a}$$

But Friedmann equations give

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho}$$

Then

$$\frac{1}{a^2}\nabla^2\Phi(\boldsymbol{x},t) = 4\pi G\bar{\rho}(t)\delta(\boldsymbol{x},t)$$

is the Poisson equation for the perturbed field.

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2} m a^2 \dot{\boldsymbol{x}}^2 - m\Phi(\boldsymbol{x}, t),$$

- The conjugate momentum $\boldsymbol{p}$ to $\boldsymbol{x}$ is then

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = m a^2 \dot{\boldsymbol{x}} = m a \boldsymbol{u}$$

where $\boldsymbol{u} = a\dot{\boldsymbol{x}} = d\boldsymbol{x}/d\tau$ is the peculiar velocity, i.e. the velocity of the particle with respect to the Hubble flow. The total velocity is $\boldsymbol{v}_T = a\boldsymbol{x}H + \boldsymbol{u}$.

- The equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} - \frac{\partial L}{\partial \boldsymbol{x}} = 0$$

becomes

$$\frac{d\boldsymbol{p}}{dt} = -m\nabla\Phi,$$

with $\nabla = \partial/\partial\boldsymbol{x}$.

Vlasov equation

The collisonless Boltzmann equation (or Vlasov equation) dictates the evolution of the phase-space particle distribution function $f(\boldsymbol{x}, \boldsymbol{p}, t)$

$$\begin{aligned} \frac{df}{dt}(\boldsymbol{x}, \boldsymbol{p}, t) &= \frac{\partial f}{\partial t} + \frac{\partial x^i}{\partial t}\frac{\partial f}{\partial x^i} + \frac{\partial p^i}{\partial t}\frac{\partial f}{\partial p^i} \\ &= \frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{ma^2} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - m\nabla\Phi \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0. \end{aligned}$$

It could be convenient to use the conformal time τ instead of cosmic time t, recall that $ad\tau = dt$, getting

$$\frac{df}{d\tau}(\boldsymbol{x}, \boldsymbol{p}, \tau) = \frac{\partial f}{\partial \tau} + \frac{\boldsymbol{p}}{ma} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - ma\nabla\Phi \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0$$

Moments of the distribution function

The comoving number density of particles is

$$n_C(\boldsymbol{x}, \tau) = \int d^3p f(\boldsymbol{x}, \boldsymbol{p}, \tau).$$

The number density is related as $n = n_C/a^3$.

Assuming all dark matter particles have the same mass m, we use $\rho = mn$ to get

$$\rho(\boldsymbol{x}, \tau) = \frac{m}{a^3} \int d^3p f(\boldsymbol{x}, \boldsymbol{p}, \tau).$$

In the following we will use the momentum average $\langle(\cdots)\rangle_p$ over the ensemble of matter particles. For a tensor $\mathbf{A}$

$$\langle \mathbf{A} \rangle_p = \frac{\int d^3p \mathbf{A} f}{\int d^3p f}, \quad \text{such that} \quad \rho \langle \mathbf{A} \rangle_p = \frac{m}{a^3} \int d^3p \mathbf{A} f$$

The mean velocity of particles is computed by taking $\langle p^i \rangle_p$ and the use of $\boldsymbol{p} = ma\boldsymbol{u}$

$$\rho v^i(\boldsymbol{x}, \tau) \equiv \rho \langle u^i \rangle_p = \frac{1}{am} \rho \langle p^i \rangle_p = \frac{1}{a^4} \int d^3p\, p^i f(\boldsymbol{x}, \boldsymbol{p}, \tau).$$

We define the velocity dispersion tensor as the square of difference between the peculiar velocities and the mean velocity of the particles averaged over the ensemble,

$$\sigma^{ij} = \langle (v^i - u^i)(v^j - u^j) \rangle_p .$$

Therefore, the second moment of the distribution function leads to

$$\rho \langle u^i u^j \rangle_p = \frac{\rho \langle p^i p^j \rangle_p}{m^2 a^2} = \frac{1}{m a^5} \int d^3p \, p^i p^j f = \rho (v^i v^j + \sigma^{ij}).$$

Higher rank tensors can be constructed, for example

$$\sigma^{ijk} \equiv \langle \Delta u^i \Delta u^j \Delta u^k \rangle_p = -\langle u^i u^j u^k \rangle_p + v^{\{i} \sigma^{jk\}} + v^i v^j v^k$$

where $\Delta \boldsymbol{u} = \boldsymbol{v} - \boldsymbol{u}$, and $T^{\{\cdots\}}$ indicate sum over cyclic permutations of indices. Note that

$$\rho \langle u^i u^j u^k \rangle_p = \frac{1}{m^2 a^6} \int d^3p \, p^i p^j p^k f.$$

Moments of the Boltzmann equation

We write the Boltzmann equation as

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 = 0,$$

with

$$\mathcal{B}_1 = \frac{\partial f}{\partial \tau}, \qquad \mathcal{B}_2 = \frac{p^i}{ma}\frac{\partial f}{\partial x_i}, \qquad \mathcal{B}_3 = -ma\frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial p^i},$$

and take moments:

$$\int d^3p\, p^{i_1} p^{i_2} \cdots p^{i_N} \mathcal{B} = 0$$

to obtain *the Boltzmann hierarchy*

Unidad V & VII, clase 4

25 de octubre de 2021

0-moment

The zero moment of the Boltzmann equation is $\frac{m}{a^3}\int d^3p\,\mathcal{B} = 0$

- $\mathcal{B}_1$:

$$\frac{m}{a^3}\int d^3p\,\mathcal{B}_1 = \frac{m}{a^3}\int d^3p\,\frac{\partial f}{\partial \tau} = \frac{m}{a^3}\frac{\partial}{\partial \tau}\int d^3p\,f = \frac{m}{a^3}\frac{\partial}{\partial \tau}\left(\frac{a^3}{m}\rho\right)$$
$$= \partial_\tau \rho + 3\mathcal{H}\rho$$

with $\mathcal{H} = \partial_\tau a/a = aH$ the conformal Hubble rate

- $\mathcal{B}_2$:

$$\frac{m}{a^3}\int d^3p\,\mathcal{B}_2 = \frac{m}{a^3}\int d^3p\,\frac{p^i}{ma}\frac{\partial f}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\frac{1}{a^4}\int d^3p\,p^i f\right)$$
$$= \partial_i(\rho v^i)$$

Note the second equality comes from the fact that comoving position $\boldsymbol{x}$ and its conjugate momentum p are independent variables in phase-space.

0-moment

- $\mathcal{B}_3$:

$$\frac{m}{a^3}\int d^3p\,\mathcal{B}_3 = \frac{m}{a^3}\int d^3p\left(-ma\frac{\partial f}{\partial p^i}\frac{\partial \Phi}{\partial x_i}\right) = -\frac{m^2}{a^2}\frac{\partial \Phi}{\partial x_i}\int d^3p\,\frac{\partial f}{\partial p^i} = 0$$

  The second equality holds because Φ does not depend on p^i. In the last term we assume f is zero at infinity momentum.

From the above equations we have the evolution of the zero order moment of the Boltzmann equation

$$\partial_\tau \rho + 3\mathcal{H}\rho + \partial_i(\rho v^i) = 0,$$

which is called *the continuity equation*.

1-moment

We make

$$\frac{1}{a^4}\int d^3p\,p\,p^i\mathcal{B}=0$$

- $\mathcal{B}_1$:

$$\begin{aligned}\frac{1}{a^4}\int d^3p\,p\,p^i\mathcal{B}_1 &= \frac{1}{a^4}\int d^3p\,p\,p^i\frac{\partial f}{\partial\tau}=\frac{1}{a^4}\frac{\partial}{\partial\tau}\int d^3p\,p\,p^i f=\frac{1}{a^4}\frac{\partial}{\partial\tau}(a^4\rho v^i)\\ &= \frac{\partial}{\partial\tau}(\rho v^i)+4\mathcal{H}\rho v^i,\end{aligned}$$

where the second equality follows because p^i and τ are independent variables.

- $\mathcal{B}_2$:

$$\begin{aligned}\frac{1}{a^4}\int d^3p\,p\,p^i\mathcal{B}_2 &= \frac{1}{a^4}\int d^3p\,p\,p^i\frac{p^j}{ma}\frac{\partial f}{\partial x^j}\\ &= \frac{\partial}{\partial x^j}\frac{1}{ma^5}\int d^3p\,p\,p^ip^jf=\partial_j\left[\rho(v^iv^j+\sigma^{ij})\right]\end{aligned}$$

1-moment

- $\mathcal{B}_3$:

$$
\begin{aligned}
\frac{1}{a^4}\int d^3p\, p^i \mathcal{B}_3 &= \frac{1}{a^4}\int d^3p\, p^i \left(-ma\frac{\partial \Phi}{\partial x_j}\frac{\partial f}{\partial p^j}\right) = -\frac{m}{a^3}\frac{\partial \Phi}{\partial x_j}\int d^3p\, p^i \frac{\partial f}{\partial p^j} \\
&= -\frac{m}{a^3}\frac{\partial \Phi}{\partial x_j}\int d^3p \left[\frac{\partial}{\partial p^j}(p^i f) - f\frac{\partial p^i}{\partial p^j}\right] = \frac{m}{a^3}\frac{\partial \Phi}{\partial x_j}\int d^3p\, f \delta^i_j \\
&= \frac{m}{a^3}\partial^i \Phi \frac{\rho a^3}{m} = \rho \partial^i \Phi
\end{aligned}
$$

Summing up $a^{-4}\int d^3p\,(p^i\mathcal{B}_1 + p^i\mathcal{B}_2 + p^i\mathcal{B}_3) = 0$, we have

$$
\partial_\tau(\rho v^i) + 4\mathcal{H}\rho v^i + \partial_j[\rho(v^i v^j + \sigma^{ij})] + \rho\partial^i\Phi = 0.
$$

With the use of the continuity equation we arrive to the more usual form of the *Euler equation*

$$
\partial_\tau v^i + \mathcal{H}v^i + v^j\partial_j v^i + \frac{1}{\rho}\partial_j(\rho\sigma^{ij}) + \partial^i\Phi = 0.
$$

2-moment

From

$$\frac{1}{ma^5}\int d^3p\, p^i p^j \mathcal{B} = 0$$

A long computation gives

$$\partial_\tau \sigma^{ij} + 2\mathcal{H}\sigma^{ij} + v^k\partial_k\sigma^{ij} + \sigma^{ik}\partial_k v^j + \sigma^{jk}\partial_k v^i = \frac{1}{\rho}\partial_k(\rho\sigma^{ijk}),$$

the velocity dispersion tensor (VDT) *equation*.

Hydrodynamical equations

We rewrite the continuity, Euler and VDT equations

$$\partial_\tau \rho + 3\mathcal{H}\rho + \partial_i(\rho v^i) = 0$$

$$\partial_\tau v^i + \mathcal{H}v^i + v^j\partial_j v^i + \partial^i\Phi = -\frac{1}{\rho}\partial_j(\rho\sigma^{ij})$$

$$\partial_\tau \sigma^{ij} + 2\mathcal{H}\sigma^{ij} + v^k\partial_k\sigma^{ij} + \sigma^{ik}\partial_k v^j + \sigma^{jk}\partial_k v^i = \frac{1}{\rho}\partial_k(\rho\sigma^{ijk}).$$

Using $\rho(\boldsymbol{x},t) = \bar{\rho}(t)(1+\delta(\boldsymbol{x},t))$, and $\partial_\tau\bar{\rho} + 3\mathcal{H}\bar{\rho} = 0$, we get the useful form

$$\partial_\tau \delta(\boldsymbol{x},\tau) + \partial_i((1+\delta)v^i) = 0,$$

$$\partial_\tau v^i(\boldsymbol{x},\tau) + \mathcal{H}v^i + v^j\partial_j v^i + \partial^i\Phi = -\frac{1}{(1+\delta)}\partial_j((1+\delta)\sigma^{ij})$$

$$\partial_\tau \sigma^{ij}(\boldsymbol{x},\tau) + 2\mathcal{H}\sigma^{ij} + v^k\partial_k\sigma^{ij}, + \sigma^{ik}\partial_k v^j + \sigma^{jk}\partial_k v^i = \frac{1}{(1+\delta)}\partial_k((1+\delta)\sigma^{ijk}),$$

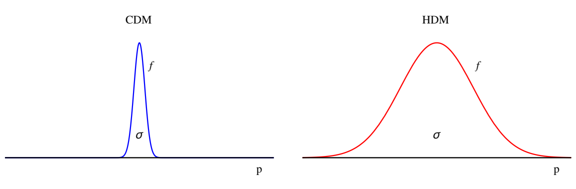
$$\cdots = \cdots$$

These equations are supplemented by the Poisson equation

$$\nabla^2\Phi = 4\pi G\bar{\rho}\delta = \frac{3}{2}\mathcal{H}^2\Omega_m(a)\delta.$$

Cold Dark Matter

- CDM is a pressureless fluid (as we already have assumed). Then it is described by the full (infinite) Boltzmann hierarchy.
- Notice $\sigma = \langle(\boldsymbol{v} - \boldsymbol{u})^2\rangle_p$ quantifies the failure of the particles to follow a *single stream*.



Velocity dispersions produce suppressions in clustering below the free-streaming scale: $\lambda_{\rm FS} \sim \sqrt{\sigma_0}/\mathcal{H}$. We have structures down to few tens of kiloparsecs. Hence, σ should be very small.

- CDM is a fluid with no velocity dispersions $\sigma = 0$.

Hydrodynamical equations

Using cosmic time t

$$\dot{\delta}(\boldsymbol{x},t)+\frac{1}{a}\partial_i v^i=-\frac{1}{a}\partial_i(\delta v^i),$$
$$\dot{v}^i(\boldsymbol{x},t)+Hv^i+\frac{1}{a}\partial^i\Phi=-\frac{1}{a}v^j\partial_j v^i.$$

with the Poisson equation

$$\frac{1}{a}\nabla^2\Phi(\boldsymbol{x},t)=4\pi G\bar{\rho}_m\delta=\frac{3}{2}\Omega_m(a)H^2\delta$$

Notice $\Omega_m(t)H^2=\Omega_m^0 H_0^2 a^{-3}$.

- At linear order

$$\dot{\delta}(\boldsymbol{x},t)+\frac{1}{a}\partial_i v^i=0,$$
$$\dot{v}^i(\boldsymbol{x},t)+Hv^i+\frac{1}{a}\partial^i\Phi=0$$

Velocity field is longitudinal

Take the rotational of the linear Euler equation

$$\partial_t \nabla \times \boldsymbol{v} + H \nabla \times \boldsymbol{v} + \frac{1}{a} \underbrace{\nabla \times \nabla \Phi}_{=0} = 0$$

hence $\boldsymbol{w} = \nabla \times \boldsymbol{v} \propto 1/a$. If we keep the VDT into the hydrodynamical equations, a term $\nabla \times (\nabla \cdot \boldsymbol{\sigma})$ becomes a source to Euler equation allowing the vorticity to grow, even though its contribution is still small.

On large scales we can safely characterize the velocity by its divergence

$$f\theta(\boldsymbol{x}, t) \equiv -\frac{\partial_i v^i}{aH}$$

where for convenience we introduced a function $f(t)$, for the moment arbitrary.

Warning: There are several notations for $\theta \propto \nabla \cdot \mathbf{v}$ in the literature.

Linear Standard Perturbation Theory

Linear theory

Taking the divergence of Euler's linear equation, and using $\partial_i v^i = -aHf\theta$,

$$H^{-1}\frac{\partial\delta}{\partial t}(\boldsymbol{x},t) - f\theta = 0,$$

$$H^{-1}\frac{\partial(f\theta)}{\partial t}(\boldsymbol{x},t) + \left(2 + \frac{\dot{H}}{H^2}\right) f\theta - \frac{3}{2}\Omega_m(a)\delta = 0$$

Combining these two equations we have (now in Fourier space)

$$\ddot{\delta}(\boldsymbol{k},t) + 2H\dot{\delta} - \frac{3}{2}\Omega_m(a)H^2\delta = 0.$$

This equation does not depend on $\boldsymbol{k}$, hence the solution can be separated $\delta(\boldsymbol{k},t) = D_+(t)A(\boldsymbol{k}) + D_-(t)B(\boldsymbol{k})$ with

$$\left(\frac{d^2}{dt^2} + 2H\frac{d}{dt} - \frac{3}{2}\Omega_m(a)H^2\right) D(t) = 0$$

The fastest growing solution is called the *linear growth function* D_+, then

$$\delta^{(1)}(\boldsymbol{k}, t) = D_+(t)\delta^{(1)}(\boldsymbol{k}, t_0)$$

where one normalize $D_+(t_0) = 1$. Typically one chooses t_0 to be the present time. The other solution is $D_- \propto H$.

Now, we choose f to be the *(logarithmic) growth factor*,

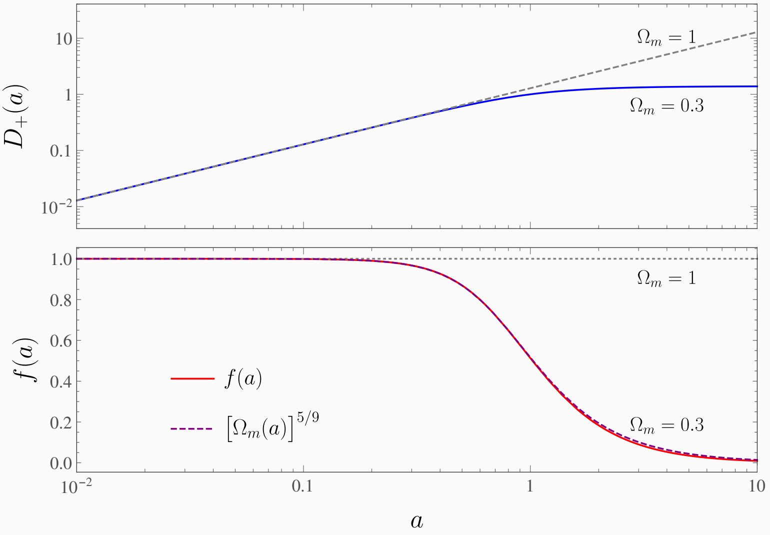
$$f(t) \equiv \frac{d \log D_+(t)}{d \log a(t)}.$$

Hence

$$\theta^{(1)}(\boldsymbol{k}, t) = \delta^{(1)}(\boldsymbol{k}, t)$$

Another popular notation defines $\theta = \nabla \cdot \boldsymbol{v}$, for which $\theta^{(1)} = -aHf\delta^{(1)}$

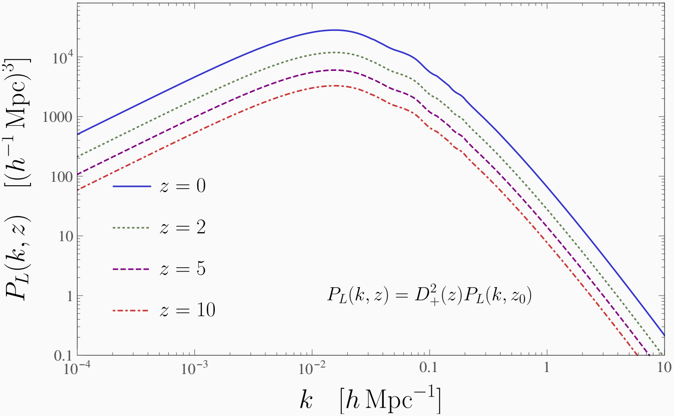
Linear growth function and growth rate

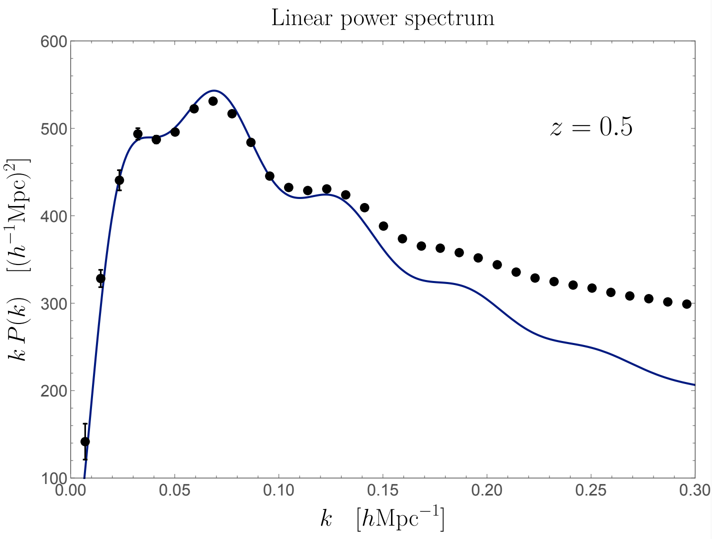


Linear power spectrum

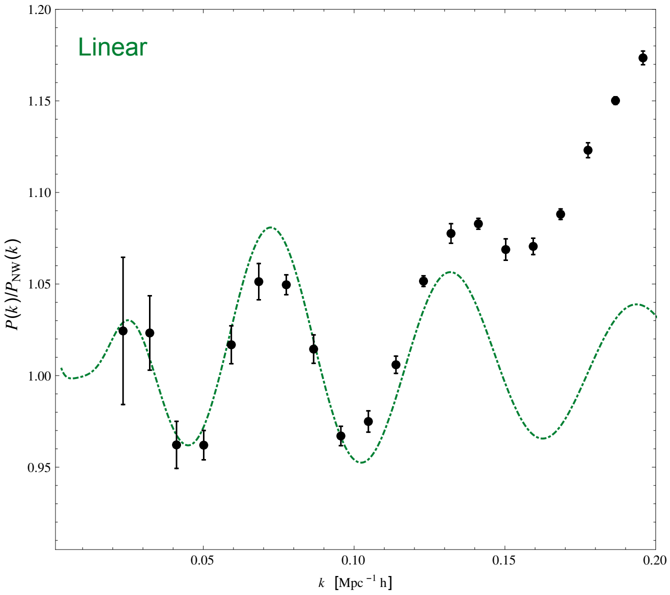
$$\langle \delta^{(1)}(\boldsymbol{k},t)\delta^{(1)}(\boldsymbol{k}',t)\rangle = D_+^2(t)\langle \delta^{(1)}(\boldsymbol{k},t_0)\delta^{(1)}(\boldsymbol{k}',t_0)\rangle$$

$$\Longrightarrow \qquad P_L(k,t) = D_+^2(t)P_L(k,t_0)$$

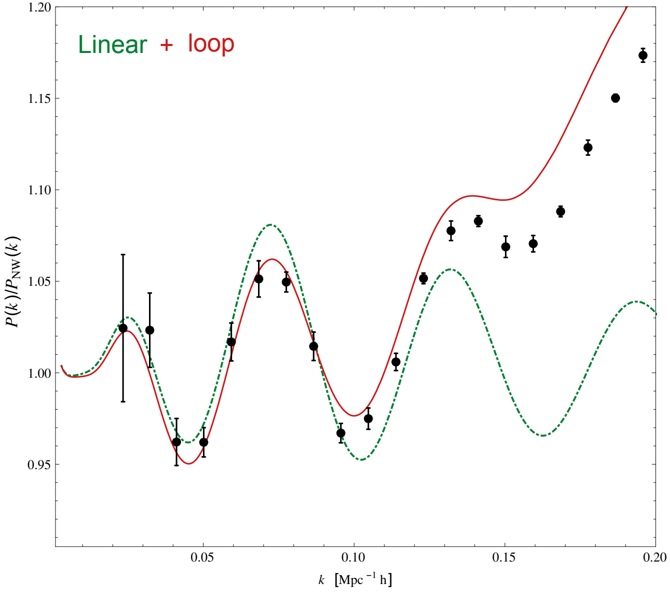




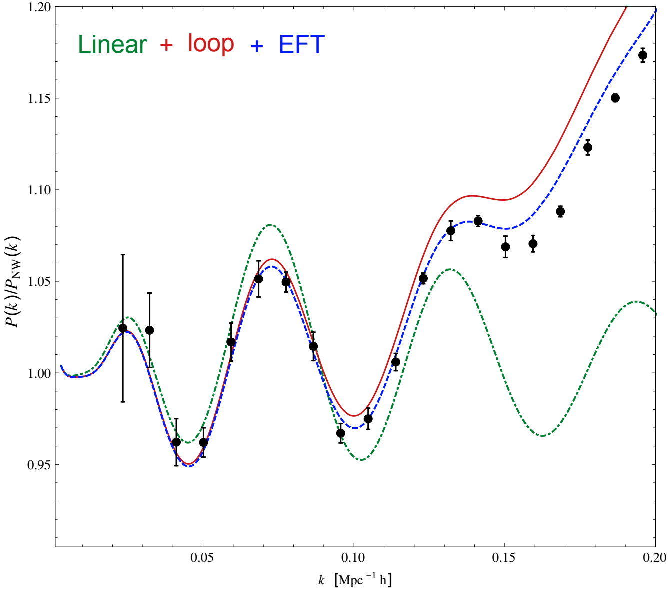
Linear power spectrum
$k\,P(k)\quad[(h^{-1}\mathrm{Mpc})^2]$
600
500
400
300
200
100
0.00
0.05
0.10
0.15
0.20
0.25
0.30
$k\quad[h\mathrm{Mpc}^{-1}]$
$z = 0.5$



Linear
1.20
1.15
1.10
1.05
1.00
0.95
$P(k)/P_{\rm NW}(k)$
0.05
0.10
0.15
0.20
k [Mpc^{-1} h]



Linear + loop
1.20
1.15
1.10
1.05
1.00
0.95
0.05
0.10
0.15
0.20
$P(k)/P_{\mathrm{NW}}(k)$
k [Mpc^{-1} h]



Linear + loop + EFT
1.20
1.15
1.10
1.05
1.00
0.95
0.05
0.10
0.15
0.20
$P(k)/P_{\rm NW}(k)$
k [Mpc^{-1} h]

Unidad V & VII, clase 5

27 de octubre de 2021

Non-linearities

Standard Perturbation Theory

Fourier Transform conventions

$$f(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}), \qquad f(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{k})$$

Hence

$$(2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k}) = \int d^3x\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\mathcal{FT}\big[\,\partial_i f(\boldsymbol{x})\,\big](\boldsymbol{k}) = ik_i\, \mathcal{FT}\big[\,f(\boldsymbol{x})\,\big](\boldsymbol{k}) = ik_i f(\boldsymbol{k})$$

$$\partial_i \longrightarrow ik_i, \qquad \nabla^2 \longrightarrow -k^2$$

Hydrodynamical equations

$$\frac{\partial}{\partial t}\delta(\boldsymbol{x},t) + \frac{1}{a}\partial_i v^i = -\frac{1}{a}\partial_i(v^i\delta),$$

$$\frac{\partial}{\partial t}(\partial_i v^i)(\boldsymbol{x},t) + H\partial_i v^i + \frac{1}{a}\nabla^2\Phi = -\frac{1}{a}\partial_i(v^j\partial_j v^i).$$

with the Poisson equation $\frac{1}{a^2}\nabla^2\Phi(\boldsymbol{x},t) = 4\pi G\bar{\rho}_m\delta = \frac{3}{2}\Omega_m(a)H^2(a)\delta$

In Fourier Space

$$\frac{\partial}{\partial t}\delta(\boldsymbol{k},t) - Hf\theta(\boldsymbol{k},t) = \mathcal{FT}\left[-\frac{1}{a}\partial_i(v^i\delta)\right](\boldsymbol{k},t),$$

$$-\frac{\partial}{\partial t}(aHf\theta(\boldsymbol{k},t)) - aH^2 f\theta(\boldsymbol{k},t) - a\frac{k^2}{a^2}\Phi(\boldsymbol{k},t) = \mathcal{FT}\left[-\frac{1}{a}\partial_i(v^j\partial_j v^i)\right](\boldsymbol{k},t)$$

Continuity equation NL:

$$\mathcal{FT}\left[-\frac{1}{a}\partial_i(v^i(\boldsymbol{x})\delta(\boldsymbol{x}))\right](\boldsymbol{k}) = -\frac{1}{a}\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\partial}{\partial x^i}\left(v^i(\boldsymbol{x})\delta(\boldsymbol{x})\right)$$

$$= -\frac{1}{a}\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\partial}{\partial x^i}\left[\int\frac{d^3k_1}{(2\pi)^3}e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}}v^i(\boldsymbol{k}_1)\int\frac{d^3k_2}{(2\pi)^3}e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}}\delta(\boldsymbol{k}_2)\right]$$

$$= -\frac{1}{a}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\int d^3x e^{-i(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}}i(k_1^i+k_2^i)v^i(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)$$

$$= -\frac{1}{a}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)i(k_1^i+k_2^i)v_i(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)$$

Using

$$\theta(\boldsymbol{x}) = -\frac{\partial_i v^i}{aHf} \qquad \Rightarrow \qquad \theta(\boldsymbol{k}) = -\frac{ik_i v^i(\boldsymbol{k})}{aHf} \qquad \Rightarrow \qquad v^i(\boldsymbol{k}) = i\frac{k^i}{k^2}aHf\theta(\boldsymbol{k})$$

We obtain

$$\mathcal{FT}\left[-\frac{1}{a}\partial_i(v^i(\boldsymbol{x})\delta(\boldsymbol{x}))\right](\boldsymbol{k}) = Hf\int_{\boldsymbol{k}_{12}=\boldsymbol{k}}\left(1+\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1^2}\right)\theta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)$$

Where we use the shorthand notation

$$\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} = \int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12}), \qquad \boldsymbol{k}_{12} = \boldsymbol{k}_1+\boldsymbol{k}_2$$

$$\int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} = \int\frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{1\cdots n}), \qquad \boldsymbol{k}_{1\cdots n} = \boldsymbol{k}_1+\cdots+\boldsymbol{k}_n$$

Euler's equation non-linear term:

$$
\begin{aligned}
&\mathcal{FT}\left[-\frac{1}{a}\partial_i\left(v^j(\boldsymbol{x})\partial_j v_i(\boldsymbol{x})\right)\right](\boldsymbol{k})\\
&\quad= -\frac{1}{a} ik_i \mathcal{FT}\left[v^j(\boldsymbol{x})\partial_j v_i(\boldsymbol{x})\right](\boldsymbol{k})\\
&\quad= -\frac{ik^i}{a}\int_{\boldsymbol{k}_{12}=\boldsymbol{k}}\left[\frac{ik_1^j aHf\theta(\boldsymbol{k}_1)}{k_1^2}\, ik_2^j\, \frac{ik_2^i aHf\theta(\boldsymbol{k}_2)}{k_2^2}\right]\\
&\quad= -aH^2f^2\int_{\boldsymbol{k}_{12}=\boldsymbol{k}}\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{k_1^2k_2^2}\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)\\
&\quad= \frac{1}{2}\left[(1\to 1, 2\to 2) + (1\to 2, 2\to 1)\right] \qquad \text{(symmetrizing indices)}\\
&\quad= -aH^2f^2\frac{1}{2}\left[\int_{\boldsymbol{k}_{12}=\boldsymbol{k}}\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{k_1^2k_2^2}\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2) + \int_{\boldsymbol{k}_{12}=\boldsymbol{k}}\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)}{k_2^2k_1^2}\theta(\boldsymbol{k}_2)\theta(\boldsymbol{k}_1)\right]\\
&\quad= -aH^2f^2\int_{\boldsymbol{k}_{12}=\boldsymbol{k}}\frac{\boldsymbol{k}\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{2k_1^2k_2^2}\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)\\
&\quad= -aH^2f^2\int_{\boldsymbol{k}_{12}=\boldsymbol{k}}\frac{k_{12}^2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{2k_1^2k_2^2}\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)
\end{aligned}
$$

Hydrodynamical equations - Fourier space

$$H^{-1}\frac{\delta(\boldsymbol{k},t)}{\partial t} - f\theta = f\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2),$$

$$H^{-1}\frac{\partial(f\theta)}{\partial t}(\boldsymbol{k},t) + \left(2+\frac{\dot{H}}{H^2}\right)f\theta - \frac{3}{2}\Omega_m(a)\delta = f^2\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \beta(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)$$

with

$$\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1+\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1^2}, \qquad \beta(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{k_{12}^2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{2k_1^2k_2^2}.$$

Remind the definition

$$\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\delta_{\mathrm{D}}(\boldsymbol{k}_1+\boldsymbol{k}_2)$$

Hydrodynamical equations - Perturbation Theory

Expand the fields

$$\delta = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \cdots, \qquad \theta = \theta^{(1)} + \theta^{(2)} + \theta^{(3)} + \cdots.$$

Solve order by order:

$$H^{-1}\frac{\partial \delta^{(n)}(\boldsymbol{k},t)}{\partial t} - f\theta^{(n)} = \sum_{m=1}^{n-1} f \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta^{(m)}(\boldsymbol{k}_1)\delta^{(n-m)}(\boldsymbol{k}_2),$$

$$H^{-1}\frac{\partial (f\theta^{(n)}(\boldsymbol{k},t))}{\partial t} + \left(2+\frac{\dot{H}}{H^2}\right) f\theta^{(n)} - \frac{3}{2}\Omega_m(a)\delta^{(n)} = \sum_{m=1}^{n-1} f^2 \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \beta(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta^{(m)}(\boldsymbol{k}_1)\theta^{(n-m)}(\boldsymbol{k}_2).$$

Propose solutions

$$\delta^{(n)}(\boldsymbol{k}) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1)\cdots\delta_L(\boldsymbol{k}_n),$$

$$\theta^{(n)}(\boldsymbol{k}) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1)\cdots\delta_L(\boldsymbol{k}_n).$$

At order n, each field is a (weighted) convolution of n linear density fields $\delta_L \equiv \delta^{(1)}$

$$F_1(\boldsymbol{k}) = 1, \qquad G_1(\boldsymbol{k}) = 1.$$

$$\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \alpha(\boldsymbol{k}_1, \boldsymbol{k}_2)\theta^{(m)}(\boldsymbol{k}_1)\delta^{(n-m)}(\boldsymbol{k}_2)$$

$$= \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \alpha(\boldsymbol{k}_{1\cdots m}, \boldsymbol{k}_{m+1\cdots n})G_m(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m)F_{n-m}(\boldsymbol{k}_{m+1}, \ldots \boldsymbol{k}_n)\delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n).$$

Tarea

Notation:

$$\alpha_{\bar{m},\bar{n}} = \alpha(\boldsymbol{k}_{1\cdots m}, \boldsymbol{k}_{m+1\cdots n}),$$

$$G_m(\bar{m}) = G_m(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m), \qquad F_{n-m}(\bar{n}) = F_{n-m}(\boldsymbol{k}_{m+1}, \ldots \boldsymbol{k}_n).$$

We obtain

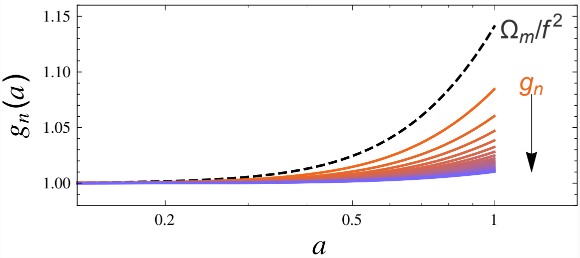
$$\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \alpha(\boldsymbol{k}_1, \boldsymbol{k}_2)\theta^{(m)}(\boldsymbol{k}_1)\delta^{(n-m)}(\boldsymbol{k}_2) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \alpha_{\bar{m},\bar{n}}G_m(\bar{m})F_{n-m}(\bar{n})\delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n)$$

Assuming time independent kernels

$$nF_n - G_n = \sum_{m=1}^{n-1} \alpha_{\bar{m},\bar{n}} G_m(\bar{m}) F_{n-m}(\bar{n})$$

$$\frac{1}{2} g_n(t)\,(2n+1)\,G_n - \frac{3}{2}\frac{\Omega_m(a)}{f^2} F_n = \sum_{m=1}^{n-1} \beta_{\bar{m},\bar{n}} G_m(\bar{m}) G_{n-m}(\bar{n})$$

with $$g_n(t) = \frac{2}{2n+1}\left[n + \frac{\dot{f}}{fH} + \frac{2}{f} + \frac{\dot{H}}{fH^2}\right]$$



EdS kernels: $\Omega_m(a) = 1$, $f = 1$ $\quad g_n = 1$

WARNING!: with the above equations one cannot obtain recursion relations for non-EdS kernels.

Kernels EdS

$$nF_n - G_n = \sum_{m=1}^{n-1} \alpha_{\bar{m}\bar{n}} G_m(\bar{m}) F_{n-m}(\bar{n})$$

$$\frac{1}{2}\left(2n+1\right) G_n - \frac{3}{2} F_n = \sum_{m=1}^{n-1} \beta_{\bar{m}\bar{n}} G_m(\bar{m}) G_{n-m}(\bar{n})$$

Solving for F_n and G_n:

$$F_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) = \sum_{m=1}^{n-1} \frac{G_m(\bar{m})}{(2n+3)(n-1)} \left[(2n+1)\alpha_{\bar{m}\bar{n}} F_{n-m}(\bar{n}) + 2\beta_{\bar{m}\bar{n}} G_{n-m}(\bar{n})\right]$$

$$G_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) = \sum_{m=1}^{n-1} \frac{G_m(\bar{m})}{(2n+3)(n-1)} \left[3\alpha_{\bar{m}\bar{n}} F_{n-m}(\bar{n}) + 2n\beta_{\bar{m}\bar{n}} G_{n-m}(\bar{n})\right]$$

n-th order fluctuations

$$\delta^{(n)}(\boldsymbol{k},t) = D_+^n(t) \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t_0)\cdots\delta_L(\boldsymbol{k}_n,t_0),$$

$$\theta^{(n)}(\boldsymbol{k},t) = D_+^n(t) \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t_0)\cdots\delta_L(\boldsymbol{k}_n,t_0).$$

with $F_1(\boldsymbol{k}) = G_1(\boldsymbol{k}) = 1,$ and

$$F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) = \sum_{m=1}^{n-1} \frac{G_m(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_m)}{(2n+3)(n-1)} \Big[(2n+1)\alpha(\boldsymbol{k}_{1\ldots m},\boldsymbol{k}_{m+1\cdots n})F_{n-m}(\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_n)$$
$$+ 2\beta(\boldsymbol{k}_{1\ldots m},\boldsymbol{k}_{m+1\cdots n})F_{n-m}(\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_n)\Big]$$

$$G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) = \sum_{m=1}^{n-1} \frac{G_m(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_m)}{(2n+3)(n-1)} \Big[3\alpha(\boldsymbol{k}_{1\ldots m},\boldsymbol{k}_{m+1\cdots n})F_{n-m}(\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_n)$$
$$+ 2n\beta(\boldsymbol{k}_{1\ldots m},\boldsymbol{k}_{m+1\cdots n})F_{n-m}(\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_n)\Big]$$

with $\quad \alpha(\boldsymbol{p}_1,\boldsymbol{p}_2) = 1 + \frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{p_1^2}, \qquad \beta(\boldsymbol{p}_1,\boldsymbol{p}_2) = \frac{p_{12}^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{2p_1^2p_2^2}$

Kernels F_2

Using $G_1 = F_1 = 1$

$$F_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{5}{7}\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2) + \frac{2}{7}\beta(\boldsymbol{k}_1, \boldsymbol{k}_2)$$
$$G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{3}{7}\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2) + \frac{4}{7}\beta(\boldsymbol{k}_1, \boldsymbol{k}_2)$$

Notice that $\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2)$ is not symmetric, but

$$\theta^{(2)}(\boldsymbol{k}) = \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} G_2(\boldsymbol{k}_1, \boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2) = \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} G_2^s(\boldsymbol{k}_1, \boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)$$

with $\quad G_2^s(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{1}{2}\big(G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) + G_2(\boldsymbol{k}_2, \boldsymbol{k}_1)\big)$

Symmetric F_2 and G_2 kernels

$$F_2^s(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{5}{14}\big(\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2) + \alpha(\boldsymbol{k}_2, \boldsymbol{k}_1)\big) + \frac{2}{7}\beta(\boldsymbol{k}_1, \boldsymbol{k}_2)$$
$$G_2^s(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{3}{14}\big(\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2) + \alpha(\boldsymbol{k}_2, \boldsymbol{k}_1)\big) + \frac{4}{7}\beta(\boldsymbol{k}_1, \boldsymbol{k}_2)$$

Kernels F_2 and G_2

Developing:

$$F_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{5}{7} + \frac{2}{7}\frac{(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)^2}{k_1^2 k_2^2} + \frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2}{2k_1 k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)$$
$$G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{3}{7} + \frac{4}{7}\frac{(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)^2}{k_1^2 k_2^2} + \frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2}{2k_1 k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)$$

Notice:

- $F_2, G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = F_2, G_2(k_1, k_2, x) = F_2, G_2(k_1/k_2, x)$ with $x = \hat{k}_1 \cdot \hat{k}_2$.
- $F_2, G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = 0$ for $\boldsymbol{k}_2 = -\boldsymbol{k}_1$
  This means that at very large scales $\delta^{(2)}(\boldsymbol{k} \to 0) \to 0$ and $\theta^{(2)}(\boldsymbol{k} \to 0) \to 0$.
- $F_2, G_2(\boldsymbol{p}, \boldsymbol{k} - \boldsymbol{p}) \propto k^2/p^2$ for $k \ll p$.
- $F_2, G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = F_2, G_2(\boldsymbol{k}_2, \boldsymbol{k}_1) = F_2, G_2(-\boldsymbol{k}_1, -\boldsymbol{k}_2)$

$$F_2(\boldsymbol{k},\boldsymbol{p}) = \frac{17}{21} + \frac{2}{7}\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\left(\hat{p}_i\hat{p}_j - \frac{1}{3}\delta_{ij}\right) + \frac{1}{2}\left(p_i\frac{k_i}{k^2} + k_i\frac{p_i}{p^2}\right)$$

$$\delta^{(2)}(\boldsymbol{k}) = \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)$$

- $$\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\delta(\boldsymbol{k}) \xrightarrow{\mathcal{FT}} \frac{2}{3a^2\Omega_m(a)H^2}\left(\partial_i\partial_j\Phi(\boldsymbol{x}) - \frac{1}{3}\delta_{ij}\nabla^2\Phi(\boldsymbol{x})\right)$$

- $$p_i\frac{k_i}{k^2}\delta^{(1)}(\boldsymbol{p})\delta^{(1)}(\boldsymbol{k}) \longrightarrow \left(\partial_i\delta^{(1)}(\boldsymbol{x})\right)\left(\frac{\partial_i}{\nabla^2}\delta^{(1)}(\boldsymbol{x})\right) = \Psi^i(\boldsymbol{x})\partial_i\delta^{(1)}(\boldsymbol{x})$$

where we used the *Lagrangian displacement at linear order* $\Psi_i(x) = \frac{\partial_i}{\nabla^2}\delta^{(1)}(\boldsymbol{x}) \propto \partial_i\Phi(\boldsymbol{x})$

In configuration space

$$\delta^{(2)}(\boldsymbol{x}) = \frac{17}{21}\left[\delta^{(1)}(\boldsymbol{x})\right]^2 + \Psi^i(\boldsymbol{x})\partial_i\delta^{(1)}(\boldsymbol{x}) + \frac{8}{63a^4\Omega_m(a)^2H^4}\left(\partial_i\partial_j\Phi(\boldsymbol{x}) - \frac{1}{2}\delta_{ij}\nabla^2\Phi(\boldsymbol{x})\right)^2$$

Notice $\qquad \Psi^i(\boldsymbol{x})\partial_i\delta^{(1)}(\boldsymbol{x}) \simeq \delta^{(1)}(\boldsymbol{x}+\Psi) - \delta^{(1)}(\boldsymbol{x})$

Unidad V & VII, clase 6

29 de octubre de 2021

F_3

$$F_3(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \frac{1}{18}\left[7\alpha(\boldsymbol{k}_1, \boldsymbol{k}_{23})F_2(\boldsymbol{k}_2, \boldsymbol{k}_3) + 2\beta(\boldsymbol{k}_1, \boldsymbol{k}_{23})G_2(\boldsymbol{k}_2, \boldsymbol{k}_3)\right] + \frac{G_2(\boldsymbol{k}_1, \boldsymbol{k}_2)}{18}\left[7\alpha(\boldsymbol{k}_{12}, \boldsymbol{k}_3) + 2\beta(\boldsymbol{k}_{12}, \boldsymbol{k}_3)\right]$$

$$F_3^s(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \frac{1}{3!}\Big[F_3(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) + \text{ permutations}\Big]$$

For fixed p: $\quad F_3^s(\boldsymbol{k}, \boldsymbol{p}, -\boldsymbol{p}) \propto \frac{k^2}{p^2}, \quad$ as $k \to 0$

EdS approximation

$$\delta^{(n)}(\boldsymbol{k},t) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t)\cdots\delta_L(\boldsymbol{k}_n,t),$$
$$\theta^{(n)}(\boldsymbol{k},t) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t)\cdots\delta_L(\boldsymbol{k}_n,t).$$

Using $\delta_L(k,t) = D_+(t)\delta_L(k,t_0)$

$$\delta^{(n)}(\boldsymbol{k},t) = D_+^n(t) \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t_0)\cdots\delta_L(\boldsymbol{k}_n,t_0),$$
$$\theta^{(n)}(\boldsymbol{k},t) = D_+^n(t) \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t_0)\cdots\delta_L(\boldsymbol{k}_n,t_0).$$

This is called the Einstein-de Sitter approximation. It uses EdS kernels, but ΛCDM linear growth functions $D_+(t)$.

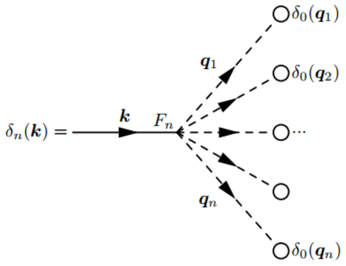
Perturbation Theories

Expansion of fields: $\phi = \lambda\phi^{(1)} + \lambda^2\phi^{(2)} + \lambda^3\phi^{(3)} + \cdots$

$$\phi^{(n)}_{\boldsymbol{k}} \sim [\phi^{(1)}_{\boldsymbol{k}_1} * \phi^{(1)}_{\boldsymbol{k}_2} * \cdots * \phi^{(1)}_{\boldsymbol{k}_n}]_{\sum \boldsymbol{k}_i = \boldsymbol{k}}$$

Linear modes interact to form higher order modes

$$\delta^{(n)}(\boldsymbol{k}) = \int \left(\prod_{m=1}^{n} \frac{d^3 q_m}{(2\pi)^3} \right) (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k} - \boldsymbol{q}_{1\cdots n}) F_n(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \delta_L(\boldsymbol{q}_1) \cdots \delta_L(\boldsymbol{q}_n)$$



Gaussian

- A random field is *Gaussian* if it is drawn from a Gaussian distribution function.

$$\langle f(\delta)\rangle = \int_{-\infty}^{\infty} d\delta \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\delta^2}{2\sigma^2}\right] f(\delta), \qquad \sigma^2 = \langle\delta^2\rangle$$

- Initial conditidions set by Inflation are highly Gaussian

For Gaussian fields X with zero mean $\langle X\rangle = 0$.

$$\langle X(\boldsymbol{x}_1)\cdots X(\boldsymbol{x}_n)\rangle = \sum_{\text{products}} \prod_{\text{pairs } i\neq j} \langle X(\boldsymbol{x}_i)X(\boldsymbol{x}_j)\rangle,$$

which is called *Wick's Theorem*, or *Isserlis' Theorem*. If n is odd, the above correlator vanishes.

$$\begin{aligned}\langle X(\boldsymbol{x}_1)X(\boldsymbol{x}_2)X(\boldsymbol{x}_3)X(\boldsymbol{x}_4)\rangle &= \langle X(\boldsymbol{x}_1)X(\boldsymbol{x}_2)\rangle\langle X(\boldsymbol{x}_3)X(\boldsymbol{x}_4)\rangle \\ &+ \langle X(\boldsymbol{x}_1)X(\boldsymbol{x}_3)\rangle\langle X(\boldsymbol{x}_2)X(\boldsymbol{x}_4)\rangle + \langle X(\boldsymbol{x}_1)X(\boldsymbol{x}_4)\rangle\langle X(\boldsymbol{x}_2)X(\boldsymbol{x}_3)\rangle\end{aligned}$$

Notice $\langle X_1\cdots X_n\rangle$ has $(n-1)!!$ terms.

Power spectrum

$$(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')P(k) = \langle\delta(\boldsymbol{k})\delta(\boldsymbol{k}')\rangle$$

- Assuming Gaussian linear order fields $\delta^{(s)}$'s

$$\begin{aligned}\langle\delta(\boldsymbol{k})\delta(\boldsymbol{k}')\rangle &= \langle(\delta^{(1)}(\boldsymbol{k})+\delta^{(2)}(\boldsymbol{k})+\delta^{(3)}(\boldsymbol{k})+\cdots)(\delta^{(1)}(\boldsymbol{k}')+\delta^{(2)}(\boldsymbol{k}')+\delta^{(3)}(\boldsymbol{k}')+\cdots)\rangle\\ &= (P_L(k)+2P^{(13)}(k)+P^{(22)}(k)+\cdots)(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')\\ &= (2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')P^{\rm SPT}_{\text{1-loop}}(k)+\cdots\end{aligned}$$

where $\qquad (2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')P^{(nm)}(\boldsymbol{k}) = \langle\delta^{(n)}(\boldsymbol{k})\delta^{(m)}(\boldsymbol{k}')\rangle$

- $P^{(12)} = \langle\delta^{(1)}\delta^{(2)}\rangle' \sim \int\langle\delta^{(1)}\delta^{(1)}\delta^{(1)}\rangle' = 0,$

No-PNG: The *bispectrum* is non-linear

$$B(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) \sim \langle\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(2)}(\boldsymbol{k}_3)\rangle' + \text{cyclic permutations}$$

P_{13}

$$P_{13}(k) \equiv P^{(13)}(k) + P^{(31)}(k) = 2P^{(13)}(k)$$

$$\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(3)}(\boldsymbol{k}')\right\rangle = \left\langle \delta^{(1)}(\boldsymbol{k}) \int_{\boldsymbol{k}_{123}=\boldsymbol{k}'} F_3(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle$$

$$= \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9}(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}' - \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)F_3(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\langle\delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\rangle$$

$$\begin{aligned}\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle =& \left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\right\rangle\left\langle \delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle \\ &+ \left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_2)\right\rangle\left\langle \delta^{(1)}(\boldsymbol{k}_3)\delta^{(1)}(\boldsymbol{k}_1)\right\rangle \\ &+ \left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_3)\right\rangle\left\langle \delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\right\rangle.\end{aligned}$$

But, inside the integral we can symmetrize $(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$: $F_3 \to F_3^s$ and

$$\begin{aligned}\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle &\to 3\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\right\rangle\left\langle \delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle \\ &= 3(2\pi)^6\delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}_1)\delta_{\mathrm{D}}(\boldsymbol{k}_2+\boldsymbol{k}_3)P_L(k)P_L(k_2)\end{aligned}$$

P_{13}

$$\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(3)}(\boldsymbol{k}')\right\rangle = 3(2\pi)^3 \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^3} \delta_{\mathrm{D}}(\boldsymbol{k}' - \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\delta_{\mathrm{D}}(\boldsymbol{k} + \boldsymbol{k}_1)\delta_{\mathrm{D}}(\boldsymbol{k}_2 + \boldsymbol{k}_3)$$
$$\times F_3^s(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) P_L(k) P_L(k_2)$$
$$= 3(2\pi)^3 \int \frac{d^3k_2}{(2\pi)^3} \delta_{\mathrm{D}}(\boldsymbol{k}' + \boldsymbol{k}) F_3^s(-\boldsymbol{k}, \boldsymbol{k}_2, -\boldsymbol{k}_2) P_L(k) P_L(k_2)$$
$$= (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k}' + \boldsymbol{k}) 3 P_L(k) \int \frac{d^3k_2}{(2\pi)^3} F_3^s(-\boldsymbol{k}, \boldsymbol{k}_2, -\boldsymbol{k}_2) P_L(k_2)$$
$$= (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k} + \boldsymbol{k}') 3 P_L(k) \int \frac{d^3p}{(2\pi)^3} F_3^s(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) P_L(p),$$

in the last equality we use $F_3^s(-\boldsymbol{k}, \boldsymbol{k}_2, -\boldsymbol{k}_2) = F_3^s(\boldsymbol{k}, -\boldsymbol{k}_2, \boldsymbol{k}_2)$ and define $\boldsymbol{p} = \boldsymbol{k}_2$.

Then

$$(2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k} + \boldsymbol{k}') P_{13}(k) \equiv \left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(3)}(\boldsymbol{k}')\right\rangle + \left\langle \delta^{(3)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}')\right\rangle$$

implies

$$\boxed{P_{13}(k) = 6 P_L(k) \int \frac{d^3p}{(2\pi)^3} F_3^s(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) P_L(p)}$$

P_{22}

$$(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')P_{22}(k) = \langle \delta^{(2)}(\boldsymbol{k})\delta^{(2)}(\boldsymbol{k}')\rangle$$

$$\left\langle \delta^{(2)}(\boldsymbol{k})\delta^{(2)}(\boldsymbol{k}')\right\rangle = \left\langle \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2) \int_{\boldsymbol{k}_{34}=\boldsymbol{k}'} F_2(\boldsymbol{k}_3,\boldsymbol{k}_4)\delta^{(1)}(\boldsymbol{k}_3)\delta^{(1)}(\boldsymbol{k}_4)\right\rangle$$

$$= \int \frac{d^3k_1 d^3k_2 d^3k_3 d^3k_4}{(2\pi)^6}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta_{\rm D}(\boldsymbol{k}'-\boldsymbol{k}_3-\boldsymbol{k}_4)F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)F_2(\boldsymbol{k}_3,\boldsymbol{k}_4)\langle\delta_1\delta_2\delta_3\delta_4\rangle$$

with $\delta_m \equiv \delta^{(1)}(\boldsymbol{k}_m, t)$.

$$\langle\delta_1\delta_2\delta_3\delta_4\rangle = \langle\delta_1\delta_2\rangle\langle\delta_3\delta_4\rangle + \langle\delta_1\delta_3\rangle\langle\delta_2\delta_4\rangle + \langle\delta_1\delta_4\rangle\langle\delta_2\delta_3\rangle$$

- The first correlator does not contribute since it yields $\langle\delta_1\delta_2\rangle\langle\delta_3\delta_4\rangle \ni \delta_{\rm D}(\boldsymbol{k}_1+\boldsymbol{k}_2)$; hence, $F_2(\boldsymbol{k}_1,\boldsymbol{k}_2) \to F_2(\boldsymbol{k}_1,-\boldsymbol{k}_1) = 0$.
- The integral is symmetric in k_3 and k_4, hence $\langle\delta_1\delta_3\rangle\langle\delta_2\delta_4\rangle + \langle\delta_1\delta_4\rangle\langle\delta_2\delta_3\rangle = 2\langle\delta_1\delta_3\rangle\langle\delta_2\delta_4\rangle$

Then

$$\langle\delta_1\delta_2\delta_3\delta_4\rangle \to 2\langle\delta_1\delta_3\rangle\langle\delta_2\delta_4\rangle = 2(2\pi)^3\delta_{\rm D}(\boldsymbol{k}_1+\boldsymbol{k}_3)\delta_{\rm D}(\boldsymbol{k}_2+\boldsymbol{k}_4)P_L(k_1)P_L(k_2)$$

P_{22}

$$
\begin{aligned}
\left\langle \delta^{(2)}(\boldsymbol{k})\delta^{(2)}(\boldsymbol{k}')\right\rangle &= 2(2\pi)^3 \int \frac{d^3k_1 d^3k_2 d^3k_3 d^3k_4}{(2\pi)^6} F_2(\boldsymbol{k}_1, \boldsymbol{k}_2) F_2(\boldsymbol{k}_3, \boldsymbol{k}_4) P_L(k_1) P_L(k_2) \\
&\quad \times \delta_{\mathrm{D}}(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\delta_{\mathrm{D}}(\boldsymbol{k}' - \boldsymbol{k}_3 - \boldsymbol{k}_4)\delta_{\mathrm{D}}(\boldsymbol{k}_1 + \boldsymbol{k}_3)\delta_{\mathrm{D}}(\boldsymbol{k}_2 + \boldsymbol{k}_4) \\
&= 2(2\pi)^3 \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} F_2(\boldsymbol{k}_1, \boldsymbol{k}_2) F_2(-\boldsymbol{k}_1, -\boldsymbol{k}_2) P_L(k_1) P_L(k_2) \delta_{\mathrm{D}}(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\delta_{\mathrm{D}}(\boldsymbol{k}' + \boldsymbol{k}_1 + \boldsymbol{k}_2) \\
&= 2(2\pi)^3 \int \frac{d^3k_1}{(2\pi)^6} F_2(\boldsymbol{k}_1, -\boldsymbol{k}' - \boldsymbol{k}_1) F_2(-\boldsymbol{k}_1, \boldsymbol{k}' + \boldsymbol{k}_1) P_L(k_1) P_L(|-\boldsymbol{k}' - \boldsymbol{k}_1|)\delta_{\mathrm{D}}(\boldsymbol{k} + \boldsymbol{k}') \\
&= (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k} + \boldsymbol{k}') 2 \int \frac{d^3p}{(2\pi)^6} \left[F_2(\boldsymbol{p}, \boldsymbol{k} - \boldsymbol{p})\right]^2 P_L(p) P_L(|\boldsymbol{k} - \boldsymbol{p}|)
\end{aligned}
$$

in the last equality we used $F_2(\boldsymbol{k}_1, -\boldsymbol{k}' - \boldsymbol{k}_1) = F_2(-\boldsymbol{k}_1, \boldsymbol{k}' + \boldsymbol{k}_1)$, used the Dirac delta to substitute $\boldsymbol{k}' = -\boldsymbol{k}$, and defined $\boldsymbol{p} = \boldsymbol{k}_1$.

Hence

$$
\boxed{P_{22}(k) = 2 \int \frac{d^3p}{(2\pi)^3} \left[F_2(\boldsymbol{p}, \boldsymbol{k} - \boldsymbol{p})\right]^2 P_L(p) P_L(|\boldsymbol{k} - \boldsymbol{p}|).}
$$

SPT power spectrum to 1-loop

$$P^{\text{SPT}}(k) = P_L(k) + P_{\text{1-loop}}(k) + \cdots$$
$$P_{\text{1-loop}}(k) = P_{22}(k) + P_{13}(k)$$

with
$$P_{22}(k) = 2\int \frac{d^3p}{(2\pi)^3}\left[F_2(\boldsymbol{p}, \boldsymbol{k}-\boldsymbol{p})\right]^2 P_L(p) P_L(|\boldsymbol{k}-\boldsymbol{p}|)$$
$$P_{13}(k) = 6 P_L(k) \int \frac{d^3p}{(2\pi)^3} F_3^s(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) P_L(p)$$

Since $P_L(k,t) \propto D_+^2(t)$, hence 1-loop corrections grow as $P_{\text{1-loop}}(k,t) = D_+^4(t) P_{\text{1-loop}}(k, t_0)$. At early times they are suppressed by linear growth.

We can write

$$P^{\text{SPT}}(k,t) = D_+^2(t) P_L(k, t_0) + D_+^4(t)\Big(P_{22}(k,t_0) + P_{13}(k,t_0)\Big) + \cdots$$

$$
\begin{aligned}
2\left[F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})\right]^2 &= 2\left[\frac{5}{7}+\frac{2}{7}\frac{\left(\boldsymbol{p}\cdot(\boldsymbol{k}-\boldsymbol{p})\right)^2}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}+\frac{(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{p}}{2|\boldsymbol{k}-\boldsymbol{p}|p}\left(\frac{p}{|\boldsymbol{k}-\boldsymbol{p}|}+\frac{|\boldsymbol{k}-\boldsymbol{p}|}{p}\right)\right]^2 \\
&= \frac{9}{98}q_1(\boldsymbol{k},\boldsymbol{p})+\frac{3}{7}q_2(\boldsymbol{k},\boldsymbol{p})+\frac{1}{2}q_3(\boldsymbol{k},\boldsymbol{p})
\end{aligned}
$$

with

$$
\begin{aligned}
q_1(\boldsymbol{k},\boldsymbol{p}) &= \left[1-\frac{\left(\boldsymbol{p}\cdot(\boldsymbol{k}-\boldsymbol{p})\right)^2}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}\right]^2 \\
q_2(\boldsymbol{k},\boldsymbol{p}) &= \frac{(\boldsymbol{k}\cdot\boldsymbol{p})\left(\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{p})\right)}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}\left[1-\frac{\left(\boldsymbol{p}\cdot(\boldsymbol{k}-\boldsymbol{p})\right)^2}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}\right] \\
q_3(\boldsymbol{k},\boldsymbol{p}) &= \frac{(\boldsymbol{k}\cdot\boldsymbol{p})^2\left(\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{p})\right)^2}{p^4|\boldsymbol{k}-\boldsymbol{p}|^4}
\end{aligned}
$$

Notice $q_{1,2,3}(\boldsymbol{k},\boldsymbol{p}) = q_{1,2,3}(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{p})$: A property inherited from the symmetric F_2.

$$6F_3(\boldsymbol{k},-\boldsymbol{p},\boldsymbol{p}) = \frac{10}{21} r_1(\boldsymbol{k},\boldsymbol{p}) + \frac{6}{7} r_2(\boldsymbol{k},\boldsymbol{p}) + r_3(\boldsymbol{k},\boldsymbol{p})$$

with

$$r_1(\boldsymbol{k},\boldsymbol{p}) = \frac{\big((\boldsymbol{k}\cdot\boldsymbol{p})\boldsymbol{k} - k^2\boldsymbol{p}\big)\cdot(\boldsymbol{k}-\boldsymbol{p})}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}\left[1 - \frac{\big(\boldsymbol{p}\cdot\boldsymbol{k}\big)^2}{p^2k^2}\right]$$

$$r_2(\boldsymbol{k},\boldsymbol{p}) = \frac{(\boldsymbol{k}\cdot\boldsymbol{p})\,\big(\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{p})\big)}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}\left[1 - \frac{\big(\boldsymbol{p}\cdot\boldsymbol{k}\big)^2}{p^2k^2}\right]$$

$$r_3(\boldsymbol{k},\boldsymbol{p}) = -\frac{(\boldsymbol{k}\cdot\boldsymbol{p})^2}{p^4}$$

$$6F_3(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) = \frac{10}{21} r_1(\boldsymbol{k}, \boldsymbol{p}) + \frac{6}{7} r_2(\boldsymbol{k}, \boldsymbol{p}) - \frac{(\boldsymbol{k} \cdot \boldsymbol{p})^2}{p^4}$$

$$\begin{aligned} P_{13}(k) \ni P_L(k) \int \frac{d^3p}{(2\pi)^3} \left(- \frac{(\boldsymbol{k} \cdot \boldsymbol{p})^2}{p^4} \right) P_L(p) &= -k^2 P_L(k) \int \frac{dp}{4\pi^2} P_L(p) \int_{-1}^{1} dx\, x^2 \\ &= -\sigma_\Psi^2 k^2 P_L(k) \end{aligned}$$

where

$$\sigma_\Psi^2 = \frac{1}{6\pi^2} \int dp P_L(p)$$

is the variance of linear Lagrangian displacements.

$$P_{22}(k) = 2\int \frac{d^3p}{(2\pi)^3}\left[F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})\right]^2 P_L(|\boldsymbol{k}-\boldsymbol{p}|)P_L(p) = \frac{9}{98}Q_1(k) + \frac{3}{7}Q_2(k) + \frac{1}{2}Q_3(k)$$

$$P_{13}(k) = 6P_L(k)\int \frac{d^3p}{(2\pi)^3}F_3(\boldsymbol{k},-\boldsymbol{p},\boldsymbol{p})P_L(p) = \frac{10}{21}R_1(k) + \frac{6}{7}R_2(k) - \sigma_\Psi^2 k^2 P_L(k)$$

with

$$Q_1(k) = \int \frac{d^3p}{(2\pi)^3} q_1(\boldsymbol{k},\boldsymbol{p})P_L(|\boldsymbol{k}-\boldsymbol{p}|)P_L(p)$$

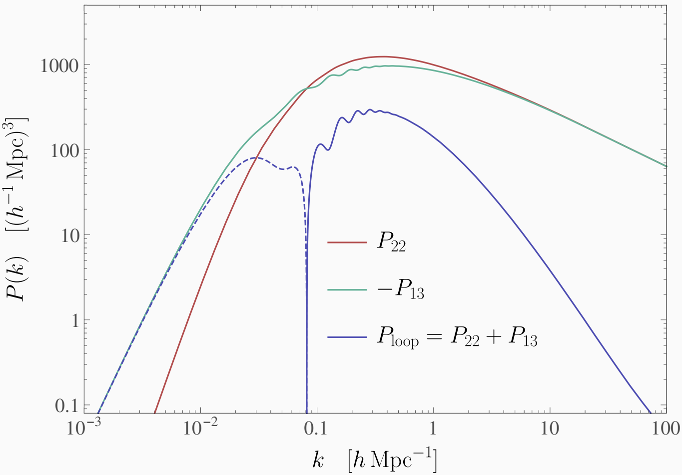
$$Q_2(k) = \int \frac{d^3p}{(2\pi)^3} q_2(\boldsymbol{k},\boldsymbol{p})P_L(|\boldsymbol{k}-\boldsymbol{p}|)P_L(p)$$

$$Q_3(k) = \int \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k},\boldsymbol{p})P_L(|\boldsymbol{k}-\boldsymbol{p}|)P_L(p)$$

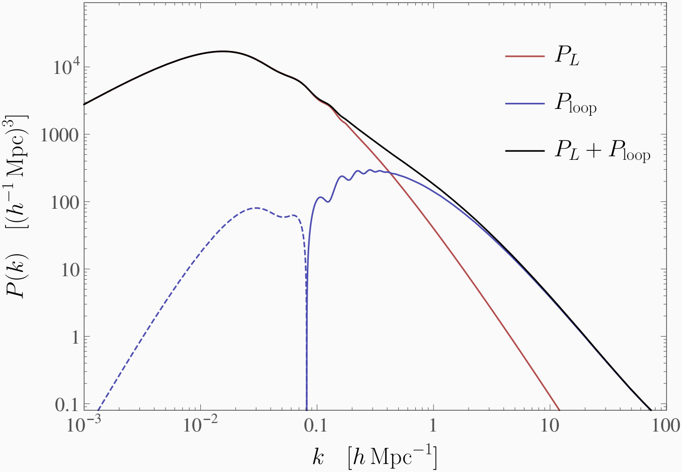
$$R_1(k) = P_L(k)\int \frac{d^3p}{(2\pi)^3} r_1(\boldsymbol{k},\boldsymbol{p})P_L(p)$$

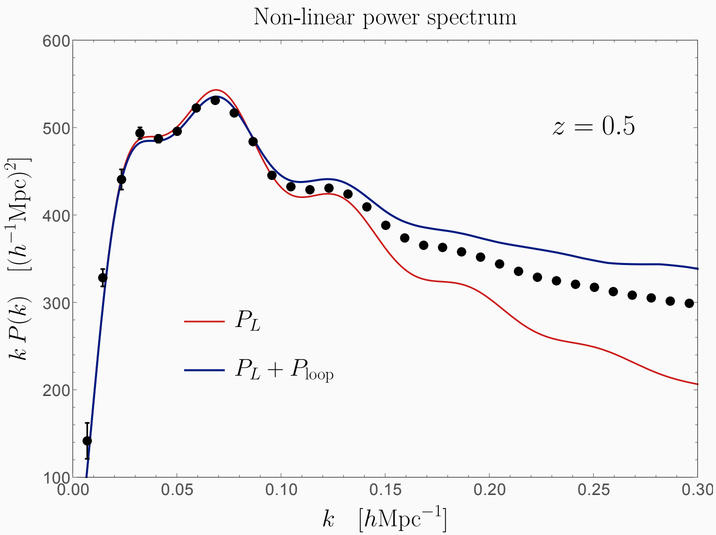
$$R_2(k) = P_L(k)\int \frac{d^3p}{(2\pi)^3} r_2(\boldsymbol{k},\boldsymbol{p})P_L(p)$$

Leading non-linear contributions



SPT power spectrum





Non-linear power spectrum
$z = 0.5$
P_L
$P_L + P_{\rm loop}$
$k\,P(k) \quad [(h^{-1}\mathrm{Mpc})^2]$
$k \quad [h\mathrm{Mpc}^{-1}]$
600
500
400
300
200
100
0.00
0.05
0.10
0.15
0.20
0.25
0.30

Unidad V & VII, clase 7

3 de noviembre de 2021

Effective Field Theory

So far, we have followed a standard PT approach. However, loop integrals are of the form $I(k) = \int_{\boldsymbol{p}} K(\boldsymbol{k}, \boldsymbol{p})$ and are computed over all internal momentum space, although $K(\boldsymbol{k}, \boldsymbol{p})$ does not hold at all scales, particularly for high internal momentum.

Though these kernels are typically suppressed for regions $p \gg k$, such that small scales do not affect considerably the $I(k)$ functions at moderate, quasilinear scales, they pose a fundamentally wrong UV behaviour —in particular P_{13}.

The Effective Field Theory for Large Scale Structure (EFT, Baumman et al, arxiv:1004.2488) formalism cuts-off the loop integrals, by directly smoothing the overdensity fields by an arbitrary scale, and introduces counterterms necessary to remove the cut-off dependence on the final expressions.

$$\bar{I}(k) = \int_{\boldsymbol{p}} K(\boldsymbol{k}, \boldsymbol{p}) \quad \longrightarrow \quad I_{reg}(k, \Lambda) = \int_{p<\Lambda} K(\boldsymbol{k}, \boldsymbol{p})$$

$$\longrightarrow \quad I(k) = I_{reg}(k, \Lambda) + I_{ct}(k, \Lambda)$$

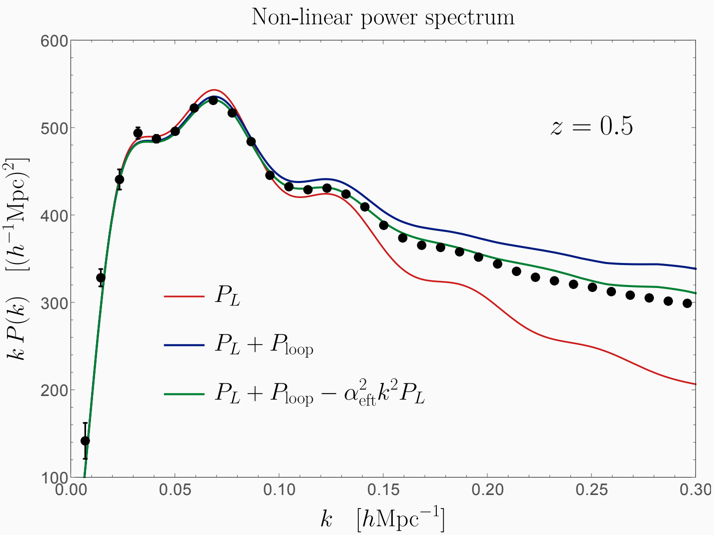
The objective of EFT is to cure the spurious high-k effects on statistics due to non modeled small scale physics, out of the reach of PT.

Further, dark matter evolution is dictated by the Boltzmann equation, and its simplified description with momentum conservation and Euler equation breaks down by nonlinear collapse which makes different streams to converge, leading to non-zero velocity dispersion and higher distribution function momenta. Hence, the very concept of CDM as a coherent fluid at all scales with no velocity dispersion is theoretically inconsistent because of gravitational collapse, breaking down at shell-crossing at best, and very rapidly all the Boltzmann hierarchy is necessary to describe the dynamics; this a key concern of EFT.

For the real space power spectrum, the leading order EFT correction counterterm is given by

$$P_{ct}(k) = -c_s^2(t)(k/k_o)^2 P_L(k) \quad \Rightarrow \; P^{\text{EFT}}(k) = P_L(k) + P_{\text{1-loop}}(k) + P_{ct}(k),$$

with c_s the effective speed of sound of dark matter arising from fluid equations of a non-perfect fluid.


Non-linear power spectrum
$z = 0.5$
P_L
$P_L + P_{\rm loop}$
$P_L + P_{\rm loop} - \alpha_{\rm eft}^2 k^2 P_L$
$k\,P(k) \quad [(h^{-1}{\rm Mpc})^2]$
$k \quad [h{\rm Mpc}^{-1}]$
600
500
400
300
200
100
0.00
0.05
0.10
0.15
0.20
0.25
0.30

Numerical Integration

$k = p$ divergence

$$P_{22}(k) \ni \frac{1}{2} Q_3(k) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p})$$

$$\text{with} \qquad q_3(\boldsymbol{k}, \boldsymbol{p}) = \frac{(\boldsymbol{k} \cdot \boldsymbol{p})^2 \left(\boldsymbol{k} \cdot (\boldsymbol{k} - \boldsymbol{p})\right)^2}{p^4 |\boldsymbol{k} - \boldsymbol{p}|^4} P_L(|\boldsymbol{k} - \boldsymbol{p}|) P_L(p)$$

But $q_3(\boldsymbol{k}, \boldsymbol{p})$ has a divergence when the internal momentum is equal to the external momentum: $\boldsymbol{p} = \boldsymbol{k}$.

$$\begin{aligned}
Q_3(k) &= \int \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}) = \int\limits_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}) + \int\limits_{p>|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}) \\
&= \int\limits_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}) + \int\limits_{\tilde{p}<|\boldsymbol{k}-\tilde{\mathbf{p}}|} \frac{d^3\tilde{p}}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{k} - \tilde{\mathbf{p}}) \\
&= 2 \int\limits_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}).
\end{aligned}$$

In the second equality we have split the region of integration in two pieces separated by the $\boldsymbol{p} = \boldsymbol{k}$ divergence. In the second integral of the third equality we redefined the variable $\boldsymbol{p} = \boldsymbol{k} - \tilde{\mathbf{p}}$. In the last equality we use the symmetry $q_3(\boldsymbol{k}, \boldsymbol{p}) = q_3(\boldsymbol{k}, \boldsymbol{k} - \boldsymbol{p})$.

$$\int_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k},\boldsymbol{p})$$

$$p < |\boldsymbol{k}-\boldsymbol{p}| = (k^2+p^2-2kpx)^{1/2} \quad \Longrightarrow \quad x < \frac{k}{2p},$$

with $x = \hat{k}\cdot\hat{p}$. For scalar rotational invariant function $f(\boldsymbol{k},\boldsymbol{p}) = f(k,p,x)$

$$\int_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} f(k,p,x) = \int_0^\infty \frac{dp}{4\pi^2} p^2 \int_{-1}^{\mathrm{Min}[1,k/(2p)]} dx\, f(k,p,x)$$

Further define: $\qquad r = \dfrac{p}{k}$

we obtain

$$\int_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k},\boldsymbol{p}) = \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{\mathrm{Min}[1,1/(2r)]} dx\, f(k,r,x)$$

Integration region for Q functions

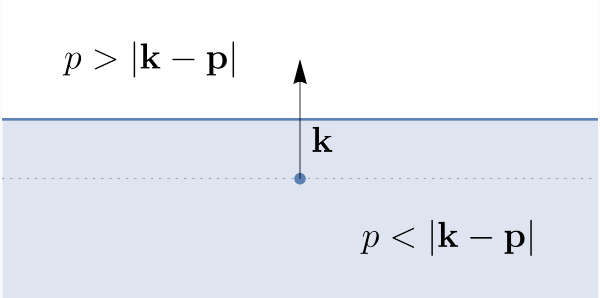
For a general function

$$F(k) = \int \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k}, \boldsymbol{p})$$

with symmetric kernel

$$f(\boldsymbol{k}, \boldsymbol{p}) = f(\boldsymbol{k}, \boldsymbol{k} - \boldsymbol{p})$$

$$F(k) = \int \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k}, \boldsymbol{p}) = 2 \int_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k}, \boldsymbol{p})$$



For numerical integration

$$x = \frac{\boldsymbol{k} \cdot \boldsymbol{p}}{kp}, \qquad r = p/k$$

$$Q_1(k) = \frac{k^3}{4\pi^2} \int_0^\infty dr P_L(kr) \int_{-1}^{1} dx \frac{r^2(1-x^2)^2}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx}),$$

$$Q_2(k) = \frac{k^3}{4\pi^2} \int_0^\infty dr P_L(kr) \int_{-1}^{1} dx \frac{rx(1-x^2)(1-rx)}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx})$$

$$Q_3(k) = \frac{k^3}{4\pi^2} \int_0^\infty dr P_L(kr) \int_{-1}^{1} dx \frac{x^2(1-rx)^2}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx})$$

$$R_1(k) = \frac{k^3}{4\pi^2} P_L(k) \int_0^\infty dr P_L(kr) \int_{-1}^{1} dx \frac{r^2(1-x^2)^2}{1+r^2-2rx}$$

$$R_2(k) = \frac{k^3}{4\pi^2} P_L(k) \int_0^\infty dr P_L(kr) \int_{-1}^{1} dx \frac{rx(1-rx)(1-x^2)}{1+r^2-2rx}$$

For numerical integration

$$x = \frac{\boldsymbol{k}\cdot\boldsymbol{p}}{kp}, \qquad r = p/k$$

$$Q_1(k) = 2\,\frac{k^3}{4\pi^2}\int_0^\infty dr\,P_L(kr)\int_{-1}^{\mathrm{Min}[1,\,1/2r]} dx\,\frac{r^2(1-x^2)^2}{(1+r^2-2rx)^2}P_L(k\sqrt{1+r^2-2rx}),$$

$$Q_2(k) = 2\,\frac{k^3}{4\pi^2}\int_0^\infty dr\,P_L(kr)\int_{-1}^{\mathrm{Min}[1,\,1/2r]} dx\,\frac{rx(1-x^2)(1-rx)}{(1+r^2-2rx)^2}P_L(k\sqrt{1+r^2-2rx})$$

$$Q_3(k) = 2\,\frac{k^3}{4\pi^2}\int_0^\infty dr\,P_L(kr)\int_{-1}^{\mathrm{Min}[1,\,1/2r]} dx\,\frac{x^2(1-rx)^2}{(1+r^2-2rx)^2}P_L(k\sqrt{1+r^2-2rx})$$

$$R_1(k) = \frac{k^3}{4\pi^2}P_L(k)\int_0^\infty dr\,P_L(kr)\int_{-1}^{1} dx\,\frac{r^2(1-x^2)^2}{1+r^2-2rx}$$

$$R_2(k) = \frac{k^3}{4\pi^2}P_L(k)\int_0^\infty dr\,P_L(kr)\int_{-1}^{1} dx\,\frac{rx(1-rx)(1-x^2)}{1+r^2-2rx}$$

$$P^{\mathrm{SPT}}_{\text{1-loop}}(k) = \underbrace{\frac{9}{98}Q_1(k)+\frac{3}{7}Q_2(k)+\frac{1}{2}Q_3(k)}_{P_{22}(k)}+\underbrace{\frac{10}{21}R_1(k)+\frac{6}{7}R_2(k)-\sigma_\Psi^2k^2P_L(k)}_{P_{13}(k)}$$

Radial integration

Trapezoidal rule

$$\int_{r_{\min}}^{r_{\max}} dr\, f(r) \approx \sum_{i=2}^{N_r} \frac{f(r_{i-1}) + f(r_i)}{2} \Delta r_i$$

where

$$r_i \in (r_1 = r_{\min}, r_2, \ldots, r_{r_N} = r_{\max})$$

$$\Delta r_i = r_i - r_{i-1}$$

- Higher precision than Riemann quadrature in the same number of steps

Angular integration: Gauss-Legendre quadrature

$$\int_{-1}^{1} dx\, f(x) \approx \sum_{i=1}^{N_x} w_i f(x_i)$$

- The quadrature nodes, x_i, are the roots of the N_x Legendre polynomial $\mathcal{P}_{N_x}(x)$.
- The quadrature weights, w_i, are given by

$$w_i = \frac{2}{(1-x_i^2)[\mathcal{P}'_{N_x}(x_i)]^2}$$

- Gauss-Legendre quadrature is exact for all polynomials up to degree $2N_x - 1$ over the interval $[-1, 1]$.

For arbitrary intervals:

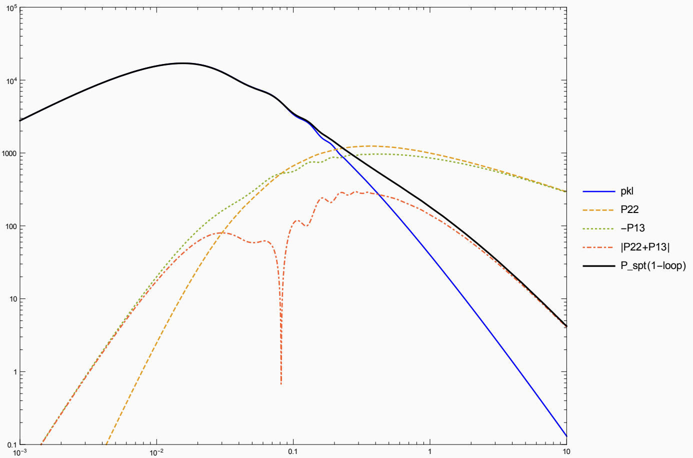
$$\int_a^b dx'\, f(x') = \int_{-1}^{1} dx\, \frac{b-a}{2} f(\frac{b-a}{2}x + \frac{a+b}{2})$$

substitute $\quad x_i \to \frac{b-a}{2}x_i + \frac{a+b}{2}, \qquad w_i \to \frac{b-a}{2}w_i$

Integrate $f(k, r, x)$.

Define arrays of external momenta (kT$[i]$), internal momenta (pT$[i]$) for trapezoidal rule, and nodes and weights (GLnodes$[h]$, GLweights$[h]$) for GL integration.

```
Do i = 1 to N_k                                   /* external momentum */
    k = kT[i];
    f_P = 0;  f_B = 0;  f_A = 0;
    Do j = 1 to N_p                               /* internal momentum integration */
        p = pT[j];  r = p/k;
        Do h = 1 to N_x                           /* angular integration */
            x = GLnodes[h];  w = GLweights[h];
            f_B = f_B + w · f(k, r, x);
        end do
        deltar = (pT[j]− pT[j − 1])/k;
        f_P = f_P + 0.5 · (f_B + f_A) · deltar;
        f_A = f_B;  f_B = 0;
    end do
    print (k, f_P)
end do
```



10^5
10^4
1000
100
10
1
0.1
10^-3
10^-2
0.1
1
10
pkl
P22
−P13
|P22+P13|
P_spt(1−loop)

Galaxy Bias

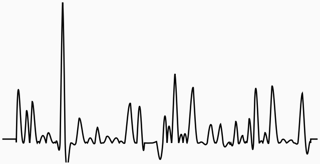
ON THE SPATIAL CORRELATIONS OF ABELL CLUSTERS

NICK KAISER
Institute for Theoretical Physics, University of California, Santa Barbara; and Department of Astronomy, University of California, Berkeley
Received 1984 April 2; accepted 1984 June 8

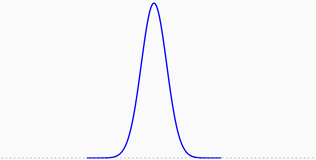
Effective field theory for biased tracers

- We observe biased tracers of the underlying dark matter distribution
- Their properties depend on baryonic and non-linear effects that are out of the reach of PT.
- Theories of bias are EFT.

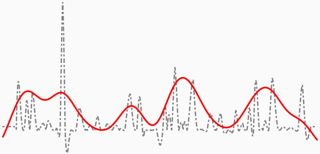
$\delta(\boldsymbol{x})$



$W_R(\boldsymbol{x})$



$$\delta_R(\boldsymbol{x}) = \int d^3x' W_R(\boldsymbol{x} - \boldsymbol{x}')\delta(\boldsymbol{x}')$$



Bias expansion

Consider tracers X with density number $n(\boldsymbol{x})$. The overdensity is

$$\delta_X(\boldsymbol{x}) = \frac{n_X(\boldsymbol{x}) - \bar{n}_X}{\bar{n}_X}, \qquad \langle \delta_X(\boldsymbol{x}) \rangle = 0$$

Introduce a bias function

$$1 + \delta_X(\boldsymbol{x}) = F_{\boldsymbol{x}}[\delta_R; \boldsymbol{x}]$$

and *local bias* parameters

$$c_n = F^{(n)}(0)$$

Expanding about $\delta_R = 0$

$$1 + \delta_X(\mathbf{r}) = c_0 + c_1 \delta_R(\mathbf{r}) + \frac{1}{2} c_2 \delta_R^2(\mathbf{r}) + \frac{1}{6} c_3 \delta_R^3(\mathbf{r}) + \cdots$$

Bias expansion

At large scales $\delta_R(\mathbf{r}) \ll 1$ and we can approximate the expansion as

$$1 + \delta_X(\mathbf{r}) \simeq c_0 + c_1 \delta_R(\mathbf{r})$$

but $\langle \delta_R(\mathbf{r}) \rangle = 0$, since $\langle \delta(\mathbf{r}) \rangle = 0$. Then

$$1 + \delta_X(\mathbf{r}) \simeq 1 + c_1 \delta_R(\mathbf{r})$$

Call $c_1 = b_1$ and one obtains the tracers fluctuation

$$\delta_X(\boldsymbol{x}) = b_1 \delta(\boldsymbol{x})$$

and the power spectrum for tracer X at large scales becomes

$$P_X(k) = \langle \delta_X(\boldsymbol{k}) \delta_X(\boldsymbol{k}') \rangle' = b_1^2 \langle \delta(\boldsymbol{k}) \delta(\boldsymbol{k}') \rangle' = b_1^2 P(k)$$

where $P(k)$ is the matter power spectrum.

Unidad V & VII, clase 8

5 de noviembre de 2021

Up to linear overdensity fields ($\delta_R = \delta_R^{(1)}$), and third order in bias expansion:

$$
\begin{aligned}
1 + \xi_X(r) &= \langle (1+\delta_X(\mathbf{r}_1))(1+\delta_X(\mathbf{r}_2)) \rangle \\
&= \langle (c_0 + c_1 \delta_R(\mathbf{r}_1) + \frac{1}{2} c_2 \delta_R^2(\mathbf{r}_1) + \frac{1}{6} c_3 \delta_R^3(\mathbf{r}_1) + \cdots) \\
&\qquad (c_0 + c_1 \delta_R(\mathbf{r}_2) + \frac{1}{2} c_2 \delta_R^2(\mathbf{r}_2) + \frac{1}{6} c_3 \delta_R^3(\mathbf{r}_2) + \cdots) \rangle \\
&= c_0^2 + c_0 c_2 \langle \delta^2 \rangle + c_1^2 \langle \delta_1 \delta_2 \rangle + \frac{1}{3} c_1 c_3 \langle \delta_1^3 \delta_2 \rangle + \frac{1}{4} c_2^2 \langle \delta_1^2 \delta_2^2 \rangle + \cdots \\
&= (c_0 + \frac{1}{2} c_2 \sigma_R^2 + \cdots)^2 + (c_1 + \frac{1}{2} c_3 \sigma_R^2 + \cdots)^2 \xi_R(r) + \frac{1}{2} (c_2 + \cdots)^2 \xi_R^2(r)
\end{aligned}
$$

zero-lag correlator:

$$
\sigma_R^2 = \langle \delta^2 \rangle = \langle \delta_R(\mathbf{r}_1) \delta_R(\mathbf{r}_1) \rangle = \int \frac{d^3k}{(2\pi)^3} \left[\tilde{W}_R(k) \right]^2 P_L(k) \approx \int_0^{1/R} \frac{dk}{2\pi^2} k^2 P_L(k)
$$

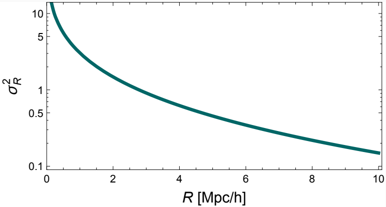
$$
\xi_R(r) = \langle \delta_R(\mathbf{r}_2) \delta_R(\mathbf{r}_1) \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{r}\cdot\boldsymbol{k}} \left[\tilde{W}_R(k) \right]^2 P_L(k).
$$

That is $\sigma_R^2 = \xi_R(0)$.

We used

$$
\begin{aligned}
\langle \delta_1^3 \delta_2 \rangle &= \langle \delta_1 \delta_1 \delta_1 \delta_2 \rangle = 3 \langle \delta_1 \delta_1 \rangle \langle \delta_1 \delta_2 \rangle = 3 \sigma_R^2 \xi_R \\
\langle \delta_1^2 \delta_2^2 \rangle &= \langle \delta_1 \delta_1 \delta_2 \delta_2 \rangle = \langle \delta_1 \delta_1 \rangle \langle \delta_2 \delta_2 \rangle + 2 \langle \delta_1 \delta_2 \rangle \langle \delta_1 \delta_2 \rangle = \sigma_R^4 + 2 \xi_R^2
\end{aligned}
$$

Statistics depend on the cutoff scale R even for very large scales $r \gg R$



Here we used a top-hat potential:

Config. Space:

$$W_R(r) = \frac{3}{4\pi R^3} \quad \text{if} \quad r \leq R, \qquad W_R(r) = 0 \quad \text{if} \quad r > R$$

Fourier Space:

$$\tilde{W}_R(k) = \frac{3}{(kR)^3}\left[\sin(kR) - kR\cos(kR)\right]$$

Renormalization

- Reorganization of bias parameters

$$b_0 = c_0 + \frac{1}{2} c_2 \sigma_R^2 + \cdots = 1 \qquad b_1 = c_1 + \frac{1}{2} c_3 \sigma_R^2 + \cdots$$

$$b_2 = c_2 + \cdots \qquad b_3 = c_3 + \cdots$$

- Correlation function:

$$\begin{aligned} 1 + \xi_X(r) &= (c_0 + \frac{1}{2} c_2 \sigma_R^2 + \cdots)^2 + (c_1 + \frac{1}{2} c_3 \sigma_R^2 + \cdots)^2 \xi_R(r) \\ &\quad + \frac{1}{2} (c_2 + \cdots)^2 \xi_R^2(r) + \cdots \\ &= 1 + b_1^2 \xi_R(r) + \frac{1}{2} b_2^2 \xi_R^2(r) + \cdots \end{aligned}$$

Linear bias

To linear order (at large scales) the galaxy density fluctuation is

$$\delta_g(\boldsymbol{x}) = b_1 \delta(\boldsymbol{x})$$

and the power spectrum becomes

$$P_g(k) = b_1^2 P_L(k)$$

Formal theory of bias

PHYSICAL REVIEW D **74,** 103512 (2006)

Clustering of dark matter tracers: Renormalizing the bias parameters

Patrick McDonald*
Canadian Institute for Theoretical Astrophysics, University of Toronto, Toronto, ON M5S 3H8, Canada
(Received 15 September 2006; published 10 November 2006)

Clustering of dark matter tracers: generalizing bias for the coming era of precision LSS

Patrick McDonald*
Canadian Institute for Theoretical Astrophysics, University of Toronto, Toronto, ON M5S 3H8, Canada

Arabindo Roy†
Department of Astronomy and Astrophysics, University of Toronto, Toronto, ON M5S 3H8, Canada

[JCAP 08 (2009) 020]

Generalizing bias

- Beyond Taylor expansion in density fields: $\delta_X = \sum_n \frac{c_n}{n!} \delta^n$
- Construct bias operators depending on velocity v_i and gravitational Φ fields.
- Equivalence principle implies biasing depends only on $\nabla_i v_j$ and $\nabla_i \nabla_j \Phi$.

Complete bias expansion

How the gravitational potential affect the evolution of tracers?

- A homogeneous shift in the gravitational field $\Phi \to \Phi + \Phi_0$ is not observable. Suggesting that biasing does not depend on Φ.
- A homogeneous shift in the gravitational force $\nabla^i \Phi \to \nabla^i \Phi + C^i$ can be removed by a change in coordinates. Suggesting that biasing does not depend on $\nabla^i \Phi$.
- Biasing depends on $\nabla_i \nabla_j \Phi$.

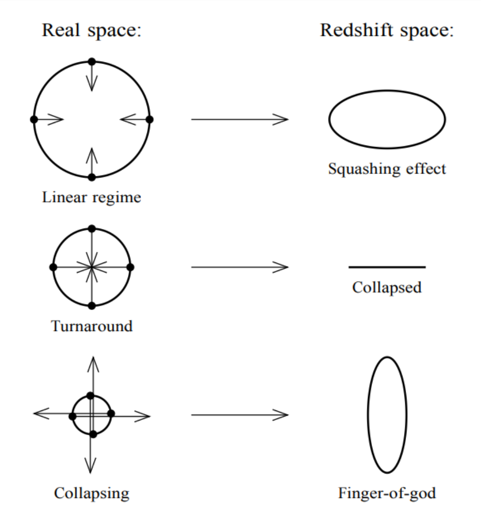
$$
\begin{aligned}
1 + \delta_X(\boldsymbol{x}, t) &= F_{\boldsymbol{x}}[\delta, \nabla_i \nabla_j \Phi, \nabla_i v_j; \boldsymbol{x}] = \sum_{\mathcal{O}} c_{\mathcal{O}}(t) \mathcal{O}(\boldsymbol{x}, t) \\
&= 1 + b_1 \delta + \frac{b_2}{2}(\delta^2 - \langle \delta^2 \rangle) + \frac{b_s}{2}(s^2 - \langle s^2 \rangle) + \cdots + \text{stochastic terms}
\end{aligned}
$$

with the stress tensor $\qquad s_{ij} = \left(k_i k_j - \frac{1}{3} \delta_{ij} k^2 \right) \overbrace{k^{-2} \delta}^{\propto \Phi}$

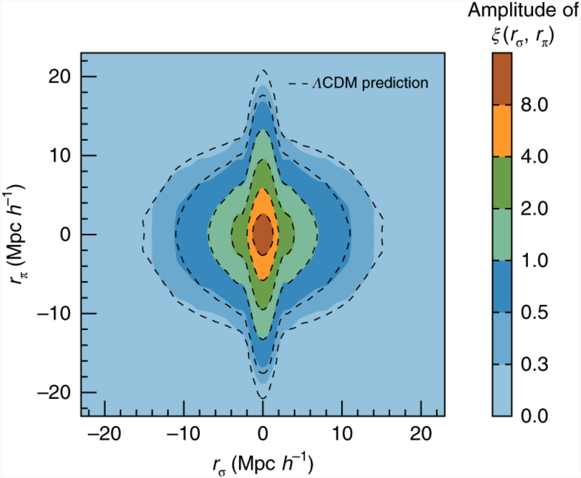
and tidal bias operator $\qquad s^2 = s_{ij} s^{ij}.$

Redshift space distortions

Effect of peculiar velocities

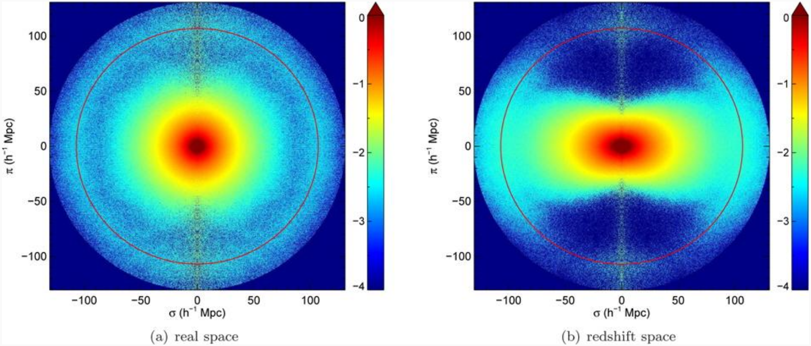


[From Hamilton astro-ph/9708102]



Amplitude of $\xi(r_\sigma, r_\pi)$
ΛCDM prediction
8.0
4.0
2.0
1.0
0.5
0.3
0.0
20
10
0
−10
−20
r_π (Mpc h^{-1})
r_σ (Mpc h^{-1})

Redshift space distortions



$$\delta_s(\boldsymbol{k}) = (1 + f\mu^2)\delta(\boldsymbol{k}) \qquad \text{with} \qquad f(t) = \frac{d\ln D_+(t)}{d\ln a(t)}$$

and μ the cosine of the angle between the line-of-sight and the wave-vector $\boldsymbol{k}$.

Redshift-space coordinates

We map objects in the sky by their angular coordinates $\hat{\boldsymbol{x}}$ and apparent positions $\boldsymbol{s}$ inferred by their redshifts z.

- The redshift to an object at a distance $d = ax$ (with $\boldsymbol{x}$ the comoving position) fixed to the Hubble flow is $z = Hd = aHx$.
- However, objects have peculiar velocities v

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{d\tau} = a\dot{\boldsymbol{x}},$$

  with $d\tau = a^{-1}dt$ the conformal time. Which induces a longitudinal (non-relativistic) Doppler effect along the line-of-sight direction $\hat{\boldsymbol{x}}$.

This two effects give the total redshift

$$z = aHx + \boldsymbol{v} \cdot \hat{\boldsymbol{x}}.$$

That is, an object located at a true position $\boldsymbol{x}$, is observed at an apparent position $\boldsymbol{s}$

$$\boldsymbol{s} = \boldsymbol{x} + \hat{\boldsymbol{x}} \frac{\boldsymbol{v} \cdot \hat{\boldsymbol{x}}}{aH},$$

due to the Doppler effect induced by its peculiar velocity.

Distant observer approximation

We shall assume that the objects of interest are located at a large distance to the observer relative to the angular size of the sample. In this case we can use $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}$ with $\hat{\boldsymbol{n}}$ the (fixed) direction to the sample.

In this case we can write

$$\boldsymbol{s} = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}),$$

with

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{\boldsymbol{v}(\boldsymbol{x}) \cdot \hat{\boldsymbol{n}}}{aH} \hat{\boldsymbol{n}}.$$

Notice that $\boldsymbol{v} \cdot \hat{\boldsymbol{n}}$ is usually written as $v_{\parallel}$.

- See, e.g., Castorina & White [arXiv:1709.09730] for beyond the plane parallel approximation.

Redshift-space density fields

We start with the coordinate transformation between real and redshift coordinates

$$\boldsymbol{s} = \boldsymbol{x} + \boldsymbol{u}$$

with "velocity" $\boldsymbol{u} = \hat{\boldsymbol{n}} \frac{\boldsymbol{v} \cdot \hat{\boldsymbol{n}}}{aH}$, with $\boldsymbol{v}$ the peculiar velocity.

We use matter (or tracers) conservation $\big(1 + \delta_s(\boldsymbol{s})\big) d^3 s = \big(1 + \delta(\boldsymbol{x})\big) d^3 x$:

$$\int \delta_{\mathrm{D}}(\boldsymbol{s} - \boldsymbol{x} - \boldsymbol{u}) \big(1 + \delta_s(\boldsymbol{s})\big) d^3 s = \int \delta_{\mathrm{D}}(\boldsymbol{s} - \boldsymbol{x} - \boldsymbol{u}) \big(1 + \delta(\boldsymbol{x})\big) d^3 x$$

$$1 + \delta_s(\boldsymbol{s}) = \int d^3 x \int \frac{d^3 k'}{(2\pi)^3} e^{i\boldsymbol{k}' \cdot (\boldsymbol{s} - \boldsymbol{x} - \boldsymbol{u})} \big(1 + \delta(\boldsymbol{x})\big)$$

$$1 + \delta_s(\boldsymbol{s}) = \int \frac{d^3 k'}{(2\pi)^3} e^{i\boldsymbol{k}' \cdot \boldsymbol{s}} \int d^3 x \big(1 + \delta(\boldsymbol{x})\big) e^{-i\boldsymbol{k}' \cdot (\boldsymbol{x} + \boldsymbol{u})}$$

Fourier tranforming (integrating against $\int d^3 s \, e^{-i\boldsymbol{k} \cdot \boldsymbol{s}}$), we obtain the overdensity field in redshift space

$$(2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k}) + \delta_s(\boldsymbol{k}) = \int d^3 x \big[1 + \delta(\boldsymbol{x})\big] e^{-i\boldsymbol{k} \cdot (\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}))}$$

Power spectrum

$$
\begin{aligned}
&\langle[(2\pi)^3\delta_{\rm D}(\boldsymbol{k})+\delta_s(\boldsymbol{k})][(2\pi)^3\delta_{\rm D}(\boldsymbol{k}')+\delta_s(\boldsymbol{k}')]\rangle\\
&=\langle\int d^3x_1\left[1+\delta(\boldsymbol{x}_1)\right]e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}_1+\boldsymbol{u}(\boldsymbol{x}_1))}\int d^3x_2\left[1+\delta(\boldsymbol{x}_2)\right]e^{-i\boldsymbol{k}'\cdot(\boldsymbol{x}_2+\boldsymbol{u}(\boldsymbol{x}_2))}\rangle\\
&=\int d^3x_1d^3x_2\langle\big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)e^{-i\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{x}_1)-i\boldsymbol{k}'\cdot\boldsymbol{u}(\boldsymbol{x}_2)}\rangle e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_1-i\boldsymbol{k}'\cdot\boldsymbol{x}_2}.
\end{aligned}
$$

Redefine $\boldsymbol{x}_2\rightarrow\boldsymbol{x}=\boldsymbol{x}_2-\boldsymbol{x}_1$, hence

$$
\begin{aligned}
&\langle[(2\pi)^3\delta_{\rm D}(\boldsymbol{k})+\delta_s(\boldsymbol{k})][(2\pi)^3\delta_{\rm D}(\boldsymbol{k}')+\delta_s(\boldsymbol{k}')]\rangle\\
&=\int d^3x_1d^3x\langle\big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}+\boldsymbol{x}_1)\big)e^{-i\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{x}_1)-i\boldsymbol{k}'\cdot\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{x}_1)}\rangle e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}_1}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}
\end{aligned}
$$

But the correlator $\langle\cdots\rangle$ does not depend on $\boldsymbol{x}_1$ since by homogeneity we can substract $\boldsymbol{x}_1$ from all its arguments without altering it.

Performing the integral d^3x_1 we obtain a Dirac delta function $(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')$. Hence

$$
\begin{aligned}
&\langle[(2\pi)^3\delta_{\rm D}(\boldsymbol{k})+\delta_s(\boldsymbol{k})][(2\pi)^3\delta_{\rm D}(\boldsymbol{k}')+\delta_s(\boldsymbol{k}')]\rangle\\
&=(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')\int d^3xe^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\langle\big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{u}}\rangle
\end{aligned}
$$

with $\Delta\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x}_2)-\boldsymbol{u}(\boldsymbol{x}_1)$ and $\boldsymbol{x}=\boldsymbol{x}_2-\boldsymbol{x}_1$.

Power spectrum

$$
\begin{aligned}
&\langle [(2\pi)^3 \delta_{\rm D}(\boldsymbol{k}) + \delta_s(\boldsymbol{k})][(2\pi)^3 \delta_{\rm D}(\boldsymbol{k}') + \delta_s(\boldsymbol{k}')] \rangle \\
&\qquad = (2\pi)^3 \delta_{\rm D}(\boldsymbol{k}) (2\pi)^3 \delta_{\rm D}(\boldsymbol{k}') + \langle \delta_s(\boldsymbol{k}) \delta_s(\boldsymbol{k}') \rangle
\end{aligned}
$$

We also have

$$\delta_{\rm D}(\boldsymbol{k}) \delta_{\rm D}(\boldsymbol{k}') = \delta_{\rm D}(\boldsymbol{k}) \delta_{\rm D}(\boldsymbol{k} + \boldsymbol{k}')$$

and

$$\langle \delta_s(\boldsymbol{k}) \delta_s(\boldsymbol{k}') \rangle = (2\pi)^3 \delta_{\rm D}(\boldsymbol{k} + \boldsymbol{k}') P_s(\boldsymbol{k}).$$

The *Redshift-Space Power Spectrum* is

$$(2\pi)^3 \delta_{\rm D}(\boldsymbol{k}) + P_s(\boldsymbol{k}) = \int d^3x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \big(1 + \delta(\boldsymbol{x}_1)\big) \big(1 + \delta(\boldsymbol{x}_2)\big) e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{u}} \rangle$$

with $\Delta\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}_2) - \boldsymbol{u}(\boldsymbol{x}_1)$ and $\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$.

Unidad V & VII, clase 9

8 de noviembre de 2021

Expansion of velocity density-weighted moments

For $\boldsymbol{k} \neq 0$ we can omit the Dirac function, and

$$P_s(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \left\langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big) e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{u}} \right\rangle$$

Expanding in Taylor series the exponential inside the correlator

$$e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{u}} = e^{-ik_i\Delta u_i} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} k_{i_1}\cdots k_{i_n} \Delta u_{i_1}\cdots\Delta u_{i_n}$$

The power spectrum in the moment expansion approach becomes

$$\boxed{\begin{aligned} &(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}) + P_s(\boldsymbol{k}) = \sum_{\mathrm{m}=0}^{\infty} \frac{(-i)^{\mathrm{m}}}{\mathrm{m}!} k_{i_1}\ldots k_{i_{\mathrm{m}}} \tilde{\Xi}^{\mathrm{m}}_{i_1\cdots i_{\mathrm{m}}}(\boldsymbol{k}), \\ &\qquad\qquad \text{with} \\ &\tilde{\Xi}^{\mathrm{m}}_{i_1\cdots i_{\mathrm{m}}}(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)\Delta u_{i_1}\cdots\Delta u_{i_{\mathrm{m}}}\rangle \end{aligned}}$$

Notice at linear order only $\mathrm{m} = 0, 1, 2$ are different from zero. e.g. $\mathrm{m} = 3$ has at least 3 fields.

- We broke isotropy by introducing the line-of-sight direction $\hat{\boldsymbol{n}}$. But we still have azimuthal symmetry about $\hat{\boldsymbol{n}}$.
- Hence, the power spectrum only depends on the angle between the wave-vector and the line-of-sight

$$P_s(\boldsymbol{k}) = P_s(k, \mu) \qquad \text{with} \qquad \mu = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}$$

- It is useful then to decompose the matter power spectrum in Legendre multipoles

$$P_\ell(k) = \frac{2\ell + 1}{2} \int_{-1}^{1} d\mu \, P_s(k, \mu) \mathcal{L}_\ell(\mu)$$

  with $\mathcal{L}_\ell$ the Legendre polynomial of degree ℓ.
- Because the symmetry $\mu \to -\mu$, only even multipoles are different from zero.
- At linear order only the monopole ($\ell = 0$), quadrupole ($\ell = 2$) and hexadecapole ($\ell = 4$) survive.

Momentum Expansion approach (linear order - Kaiser)

To linear order we can only have two fields on the correlations. So, the sum runs over $\mathrm{m} = 0, 1, 2$

$$P_s(k,\mu) = \tilde{\Xi}^{\mathrm{m}=0}(\boldsymbol{k}) - ik_i \tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) - \frac{1}{2} k_i k_j \tilde{\Xi}_{ij}^{\mathrm{m}=2}(\boldsymbol{k}),$$

with moments

$$\tilde{\Xi}^{\mathrm{m}}_{i_1\cdots i_{\mathrm{m}}}(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, \Xi^{\mathrm{m}}_{i_1\cdots i_{\mathrm{m}}}(\boldsymbol{x}) \quad :$$

$$\Xi^{\mathrm{m}=0}(\boldsymbol{x}) = \langle \big(1+\delta_1\big)\big(1+\delta_2\big)\rangle,$$

$$\Xi_i^{\mathrm{m}=1}(\boldsymbol{x}) = \langle \big(1+\delta_1\big)\big(1+\delta_2\big)\Delta u_i\rangle \stackrel{!}{=} \langle \Delta u_i(\delta_1+\delta_2)\rangle,$$

$$\Xi_{ij}^{\mathrm{m}=2}(\boldsymbol{x}) = \langle \big(1+\delta_1\big)\big(1+\delta_2\big)\Delta u_i \Delta u_j\rangle \stackrel{!}{=} \langle \Delta u_i \Delta u_j\rangle,$$

where the last equality (!) are valid to linear order only, and

.

Momentum Expansion approach (linear order)

- The zero order moment is

$$\Xi^{\mathrm{m}=0}(\boldsymbol{x}) = \langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)\rangle = 1+\xi(x),$$

which is the correlation function. In Fourier space

$$\tilde{\Xi}^{\mathrm{m}=0}(\boldsymbol{k}) = \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\big(1+\xi(x)\big) = (2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}) + P_{\delta\delta}(k)$$

the real-space (density-density) power spectrum $P_{\delta\delta}(k) = \langle\delta(\boldsymbol{k})\delta(\boldsymbol{k}')\rangle'$.

Since we are dealing with linear fields we can identify the δ-δ PS with

$$\begin{aligned} P_{\delta\delta}(k) &= P_L(k) \qquad \text{(for matter)}, \\ P_{\delta\delta}(k) &= b_1^2 P_L(k) \qquad \text{(for tracers)}. \end{aligned}$$

But we will keep the notation $P_{\delta\delta}(k)$ for the moment.

Momentum Expansion approach (linear order)

- The first velocity moment to linear order is

$$\Xi_i^{\mathrm{m}=1}(\boldsymbol{x}) = \langle \Delta u_i(\delta(\boldsymbol{x}_1) + \delta(\boldsymbol{x}_2))\rangle = 2\langle \Delta u_i \delta(\boldsymbol{x}_1)\rangle,$$

  In Fourier space

$$\tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) = 2\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \Delta u_i \delta(\boldsymbol{x}_1)\rangle$$

To go beyond, we notice that $\theta(\boldsymbol{x}) = -\nabla \cdot \boldsymbol{v}/(aHf)$ and $\boldsymbol{u}(\boldsymbol{x}) = \hat{\boldsymbol{n}}(\boldsymbol{v}\cdot\hat{\boldsymbol{n}})/(aH)$ imply

$$\boldsymbol{u}(\boldsymbol{k}) = if\hat{\boldsymbol{n}}\frac{\boldsymbol{k}\cdot\hat{\boldsymbol{n}}}{k^2}\theta(\boldsymbol{k}).$$

We can write

$$\Delta u_i \equiv u_i(\boldsymbol{x}_2) - u_i(\boldsymbol{x}_1) = \int \frac{d^3p}{(2\pi)^3} \left(e^{i\boldsymbol{p}\cdot\boldsymbol{x}_2} - e^{i\boldsymbol{p}\cdot\boldsymbol{x}_1}\right) \left(if \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2} \hat{n}_i\right) \theta(\boldsymbol{p}).$$

Hence

$$\begin{aligned}
&\langle \delta(\boldsymbol{x}_1) \Delta u_i \rangle \\
&= \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1} \left(e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_2} - e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_1}\right) \left(if \frac{\boldsymbol{k}_2\cdot\hat{\boldsymbol{n}}}{k_2^2} \hat{n}_i\right) \langle \delta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)\rangle \\
&= \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1} \left(e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_2} - e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_1}\right) \left(if \frac{\boldsymbol{k}_2\cdot\hat{\boldsymbol{n}}}{\boldsymbol{k}_2^2} \hat{n}_i\right) (2\pi)^3 \delta_{\rm D}(\boldsymbol{k}_1 + \boldsymbol{k}_2) P_{\delta\theta}(k_1) \\
&= -if\hat{n}_i \int \frac{d^3k_1}{(2\pi)^3} \left(e^{-i\boldsymbol{k}_1\cdot\boldsymbol{x}} - 1\right) \frac{\boldsymbol{k}_1\cdot\hat{\boldsymbol{n}}}{k_1^2} P_{\delta\theta}(k_1) = if\hat{n}_i \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2} P_{\delta\theta}(p),
\end{aligned}$$

In the last but one equality, the term containing the "1" vanishes because

$$\int \frac{d^3p}{(2\pi)^3} \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2} P_{\delta\theta}(p) = \int \frac{dp}{2\pi^2} p^2 \frac{P_{\delta\theta}(p)}{p} \hat{n}_i \frac{1}{4\pi} \int d\Omega_{\hat{\boldsymbol{p}}} \hat{p}^i = 0,$$

because the angular integral of $\hat{p}^i$ is zero. (In general, the solid angle integral of an odd number of $\hat{p}^{i_1} \cdots \hat{p}^{i_n}$ vanishes.)

$$
\begin{aligned}
\tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) &= 2\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \Delta u_i \delta(\boldsymbol{x}_1)\rangle \\
&= 2\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} (if\hat{n}_i) \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2} P_{\delta\theta}(p) \\
&= 2if\hat{n}_i \int \frac{d^3p}{(2\pi)^3} (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{p}-\boldsymbol{k}) \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2} P_{\delta\theta}(p) \\
&= 2if\hat{n}_i \frac{\boldsymbol{k}\cdot\hat{\boldsymbol{n}}}{k^2} P_{\delta\theta}(k)
\end{aligned}
$$

Hence, in Fourier space, and (we remark) to linear order

$$
\tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) = 2if\hat{n}_i \frac{\mu}{k} P_{\delta\theta}(k),
$$

and

$$
P_s(\boldsymbol{k}) \ni -ik_i \tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) = 2f\mu^2 P_{\delta\theta}(k).
$$

Analogoulsy, we compute $\Xi_{ij}^{m=2}(\boldsymbol{x}) = \langle(1+\delta_1)(1+\delta_2)\Delta u_i \Delta u_j\rangle \overset{!}{=} \langle \Delta u_i \Delta u_j\rangle$:

$$\begin{aligned}
\langle \Delta u_i \Delta u_j \rangle &= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\left(e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_2} - e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\right)\left(e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_2} - e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_1}\right) \\
&\qquad \times \left(if\frac{\boldsymbol{k}_1\cdot\hat{\boldsymbol{n}}}{k_1^2}\hat{n}_i\right)\left(if\frac{\boldsymbol{k}_2\cdot\hat{\boldsymbol{n}}}{k_2^2}\hat{n}_i\right)\langle\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)\rangle \\
&= 2f^2\sigma_v^2\hat{n}_i\hat{n}_j - 2f^2\hat{n}_i\hat{n}_j\int\frac{d^3p}{(2\pi)^3}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\frac{(\boldsymbol{p}\cdot\hat{\boldsymbol{n}})^2}{p^4}P_{\theta\theta}(p)
\end{aligned}$$

with the velocity variance

$$\sigma_v^2 = \frac{1}{6\pi^2}\int_0^\infty dp P_{\theta\theta}(p).$$

The second moment, to linear order, is

$$\tilde{\Xi}_{ij}^{m=2}(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{p}}\langle\Delta u_i\Delta u_j\rangle = -2f^2\hat{n}_i\hat{n}_j\frac{\mu^2}{k^2}P_{\theta\theta}(k),$$

up to a Dirac delta function. Hence

$$P_s(\boldsymbol{k}) \ni -\frac{1}{2}k_ik_j\tilde{\Xi}_{ij}^{\mathrm{m}=2}(\boldsymbol{k}) = f^2\mu^4P_{\theta\theta}(k).$$

Kaiser power spectrum

Summing up the three contributions

$$P_s(k,\mu) = \tilde{\Xi}^{\mathrm{m}=0}(\boldsymbol{k}) - ik_i\tilde{\Xi}_i^{\mathrm{m}=1,ud}(\boldsymbol{k}) - \frac{1}{2}k_ik_j\tilde{\Xi}_{ij}^{\mathrm{m}=2,dd}(\boldsymbol{k}),$$

we arrive to

$$P_s(k,\mu) = P_{\delta\delta}(k) + f\mu^2 P_{\delta\theta}(k) + f^2\mu^4 P_{\theta\theta}(k)$$

To linear order $\delta_m^{(1)} = \theta_m^{(1)}$.

Considering linear biased tracers: $\delta^{(1)} = b_1\delta_m^{(1)}$, and $\theta^{(1)} = \theta_m^{(1)}$.

Hence, for tracers

$$P_{\delta\delta} = b_1^2 P_L(k), \qquad P_{\delta\theta} = b_1 P_L(k), \qquad P_{\theta\theta} = P_L(k)$$

We obtain the *Kaiser power spectrum*

$$\boxed{P_s^K(k,\mu) = (1+\beta\mu^2)^2 b_1^2 P_L(k) \qquad \text{with} \quad \beta = \frac{f}{b_1}}$$

The factor $(1+\beta\mu^2)^2$ is called the *Kaiser boost*.

Multipoles of the Kaiser power spectrum

Using

$$P_\ell^K(k) = \frac{2\ell+1}{2}\int_{-1}^{1} P_s^K(k,\mu)\mathcal{L}_\ell(\mu),$$

with $\mathcal{L}_0(\mu) = 1$, $\mathcal{L}_2(\mu) = \frac{3}{2}(\mu^2 - \frac{1}{3})$ and $\mathcal{L}_4(\mu) = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$ we obtain

$$\begin{aligned}
P_0^K(k) &= \left(1 + \frac{2}{3}\beta + \frac{1}{5}\beta^2\right) b_1^2 P_L(k) \qquad \text{(monopole)},\\
P_2^K(k) &= \left(\frac{4}{3}\beta + \frac{4}{7}\beta^2\right) b_1^2 P_L(k) \qquad \text{(quadrupole)},\\
P_4^K(k) &= \frac{8}{35}\beta^2 b_1^2 P_L(k) \qquad \text{(hexadecapole)},
\end{aligned}$$

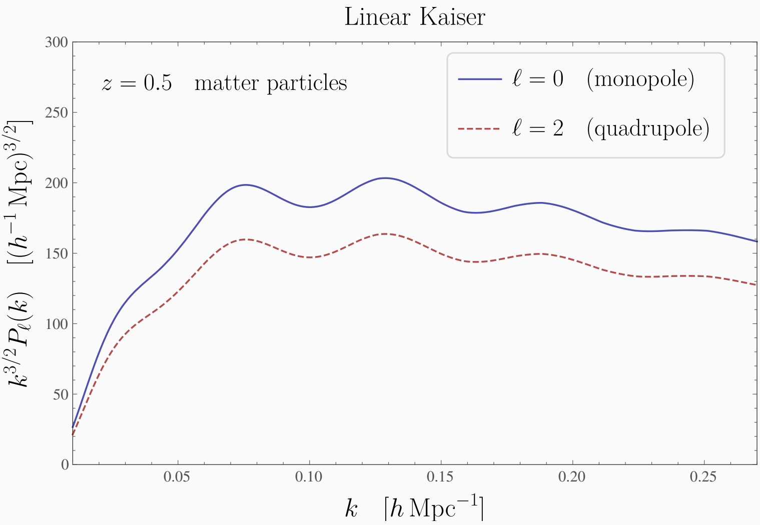
higher degree moments are zero.

Remind $\beta = f/b_1$ with

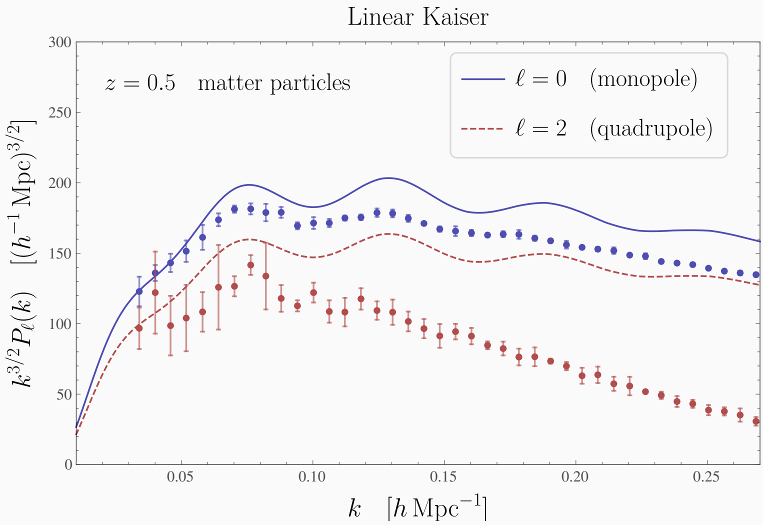
$$f = \frac{d\log D_+(t)}{d\log a(t)},$$

the growth rate.

Kaiser power spectrum

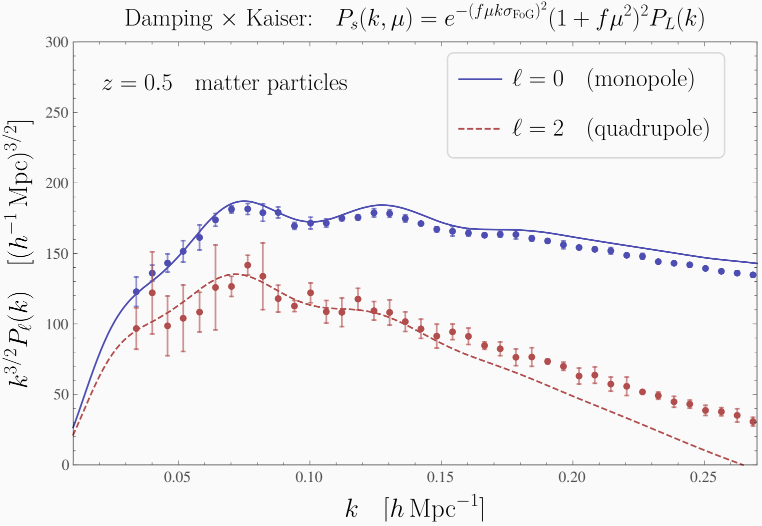


Kaiser power spectrum



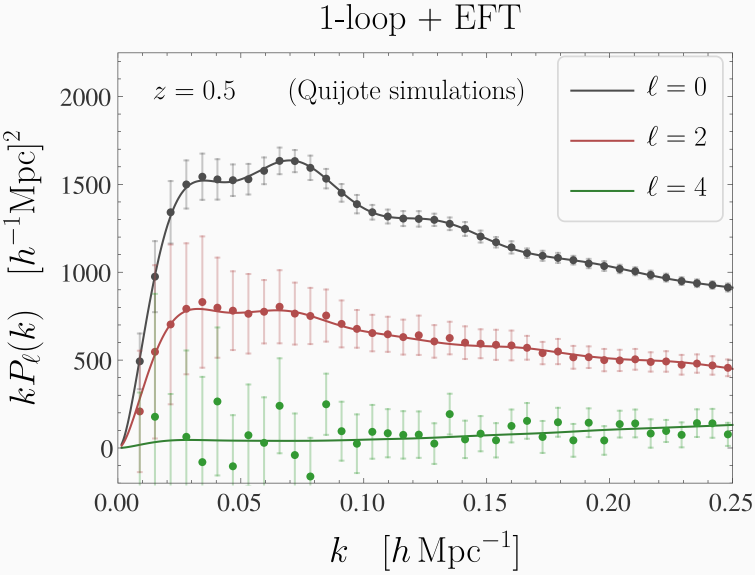
The main problem is that the fingers-of-god are a completely non-linear effect. Fingers-of-god lessen the observed clustering along the line-of-sight ($\mu \neq 0$). Hence, a simple prescription is to add a damping factor along the line-of-sight.

$$P_s(k,\mu) = \underbrace{e^{-(k\mu f \sigma_{\mathrm{FoG}})^2}}_{\text{Fingers of God}} \times \underbrace{\left(1 + f\mu^2\right)^2 P_L(k)}_{\text{Kaiser effect}}$$



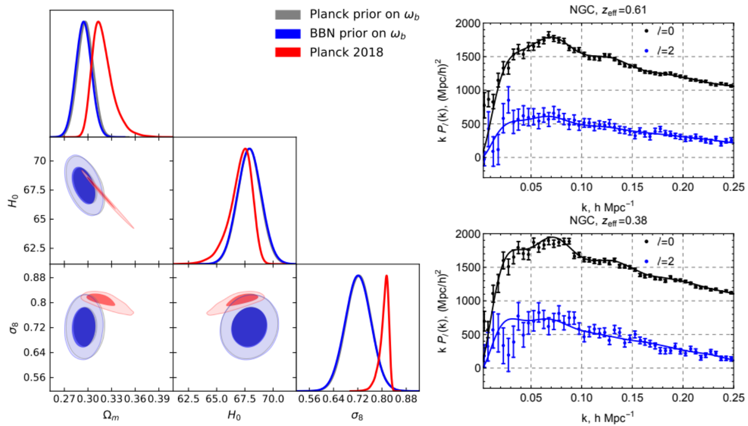
Damping × Kaiser: $P_s(k,\mu) = e^{-(f\mu k \sigma_{\rm FoG})^2}(1+f\mu^2)^2 P_L(k)$
$z = 0.5$ matter particles
$\ell = 0$ (monopole)
$\ell = 2$ (quadrupole)
$k^{3/2} P_\ell(k) \quad [(h^{-1}\,\mathrm{Mpc})^{3/2}]$
$k \quad [h\,\mathrm{Mpc}^{-1}]$
0
50
100
150
200
250
300
0.05
0.10
0.15
0.20
0.25

State of the art theory vs sims.



Cosmological parameters from the BOSS galaxy power spectrum

Mikhail M. Ivanov,[a,b] Marko Simonović[c] and Matias Zaldarriaga[d]



arxiv:1909.05277

Unidad V & VII, clase 10

10 de noviembre de 2021

Angular power spectra and correlations

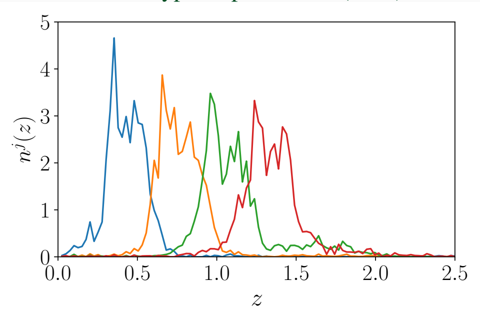
Photometric redshift

Not always we have access to precise redshift measurements of astronomical objects. This is the case of, e.g, photometric surveys. However we have high quality images in several wave-length bands, from which we can infer approximated redshifts called *photo-z*.

More precisely, we can infer a distribution of number of galaxies

$$n_g(z) = \frac{dN_g}{dz}$$

Subaru hyper suprime-cam (HSC)



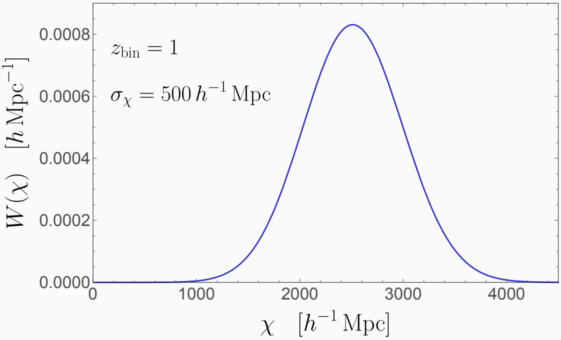
Instead of the redshift z, it may be useful to use the comoving radial distance $\chi(z)$

$$\chi(a) = \int_a^1 \frac{da'}{a'^2 H(a')}, \qquad \chi(z) = \int_0^z \frac{dz'}{H(z')}$$

And the distribution of galaxies $W(\chi) = \frac{1}{N_g} \frac{dN_g}{d\chi}$

N_g the number of galaxies, and W is normalized to unity: $\int_0^\infty W(\chi) d\chi = 1$.

$$W(\chi) \propto \chi^2 e^{-(\chi - \chi_{\rm bin})/2\sigma_\chi^2}$$



coordinates

A position of an event in space-time can be written as $\mathbf{p} = (\boldsymbol{x}, \eta)$.

The conformal time at χ is $\eta(\chi) = \eta_0 - \chi$. And $\boldsymbol{x} = \hat{\boldsymbol{\theta}}\chi$ with $\hat{\boldsymbol{\theta}}$ is the angular direction. (Dodelson & Schmidt textbook uses $\hat{\boldsymbol{n}}$ in chapter 11 for the angular position. But it uses $\boldsymbol{\theta}$ in chapter 13.)

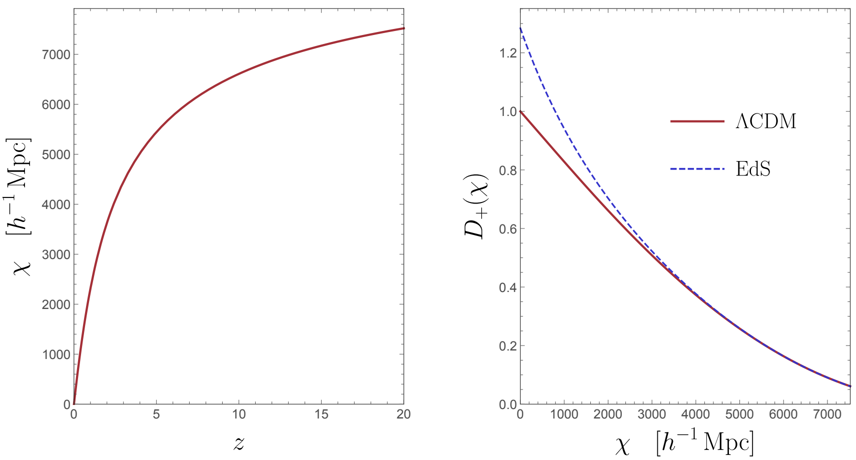
Hence we can write the position vector as $\mathbf{p} = (\hat{\boldsymbol{\theta}}\chi, \eta_0 - \chi)$. Or simpler:

$$\mathbf{p} = (\hat{\boldsymbol{\theta}}\chi, \chi)$$

This reflects the fact that we observe objects in our past light-cone and for an event hence $(\boldsymbol{x}, t)$ can be reduced to 3 numbers. in our case two numbers to determine the direction on the sky $\hat{\boldsymbol{\theta}}$, and one that give us both the radial position and the “time” of occurrence by χ.

It is important to keep in mind that χ, unlike $\hat{\boldsymbol{\theta}}$, is a cosmology dependent quantity.

Linear Growth function



projected density

Instead of measuring the 3-dim density field of galaxy, we measure its projection on the sky,

$$\Delta_g(\hat{\boldsymbol{\theta}}) = \int_0^\infty d\chi\, W(\chi)\delta_g(\hat{\boldsymbol{\theta}}\chi, \chi).$$

Now, we transform the 3D fluctuation to Fourier space:

$$\begin{aligned}
\Delta_g(\hat{\boldsymbol{\theta}}) =& \int_0^\infty d\chi\, W(\chi) \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\hat{\boldsymbol{\theta}}\chi}\delta_g(k,\chi) \\
=& 4\pi \int \frac{d^3k}{(2\pi)^3} \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell Y_{\ell m}(\hat{\boldsymbol{\theta}})Y^*_{\ell m}(\hat{\boldsymbol{k}}) \int_0^\infty d\chi\, W(\chi) j_\ell(k\chi)\delta_g(k,\chi)
\end{aligned}$$

where we used the plane wave expansion

$$e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 4\pi\sum_{\ell=0}^{\infty} i^\ell(2\ell+1)j_\ell(kx)\mathcal{P}_\ell(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{x}}) = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(kx) Y_{\ell m}(\hat{\boldsymbol{x}})Y^*_{\ell m}(\hat{\boldsymbol{k}})$$

Harmonic decomposition

We can rearrange

$$\Delta_g(\hat{\boldsymbol{\theta}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Delta_{g,\ell m} Y_{\ell m}(\hat{\boldsymbol{\theta}})$$

with

$$\Delta_{g,\ell m} = 4\pi \int \frac{d^3k}{(2\pi)^3} i^\ell Y^*_{\ell m}(\hat{\boldsymbol{k}}) \int_0^\infty d\chi\, W(\chi) j_\ell(k\chi) \delta_g(k,\chi).$$

In exact analogy with the CMB anisotropies (the $a_{\ell m}$), the angular power spectrum of galaxy counts on the sky is then proportional to the expectation value of $|\Delta_{g,\ell m}|^2$.

Angular power spectrum of galaxies

Let us evaluate the angular power spectrum

$$\langle \Delta_{g,\ell m} \Delta^*_{g,\ell' m'} \rangle = (4\pi)^2 i^{\ell-\ell'} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} Y^*_{\ell m}(\hat{\boldsymbol{k}}) Y_{\ell' m'}(\hat{\boldsymbol{k}}') \\ \times \int_0^\infty d\chi\, W(\chi) j_\ell(k\chi) \int_0^\infty d\chi'\, W(\chi') j_{\ell'}(k'\chi') \, \langle \delta_g(\boldsymbol{k}, \chi) \delta^*_g(\boldsymbol{k}', \chi') \rangle$$

The ensemble average of overdensities gives Dirac delta function $\delta_{\mathrm{D}}(\boldsymbol{k} - \boldsymbol{k}')$ (notice $\delta^*_g(\boldsymbol{k}') = \delta_g(-\boldsymbol{k})$) which sets $\boldsymbol{k} = \boldsymbol{k}'$. Hence we can use the orhonormality of spherical harmonics,

$$\int d\Omega \, Y^*_{\ell m}(\hat{\boldsymbol{n}}) Y_{\ell' m'}(\hat{\boldsymbol{n}}) = \delta_{\ell \ell'} \delta_{m m'},$$

to reduce the above expression.

We obtain

$$\langle \Delta_{g,\ell m} \Delta^*_{g,\ell' m'} \rangle = C_g(\ell) \delta_{\ell\ell'} \delta_{mm'}$$

with

$$C_g(\ell) = \frac{2}{\pi} \int_0^\infty d\chi\, W(\chi) \int_0^\infty d\chi'\, W(\chi') \int dk\, k^2 j_\ell(k\chi) j_\ell(k\chi') P_g(\boldsymbol{k}; \chi, \chi')$$

Notice:

- To compute the angular power spectrum we need to integrate the two overdensities at all "times" χ. And then the galaxy standard power spectrum is evaluated at two different times.
- The power spectrum evaluated at two different times $P_g(\boldsymbol{k}; \chi, \chi')$ is anisotropic because it depend on the value of two overdensities evaluated at two different times.

  Indeed, using the obvious transverse symmetry $\boldsymbol{k} = (k_{\|}, \boldsymbol{k}_\perp^{\mathrm{2D}})$ or $\boldsymbol{k} = (k\mu_k, \boldsymbol{k}_\perp^{\mathrm{2D}})$, where μ_k is the cosine angle of $\boldsymbol{k}$ and the line of sight.

Limber approximation

Consider the integral representation of the Dirac delta function:

$$\int dk\, k^2 j_\ell(k\chi) j_\ell(k\chi') = \frac{\pi}{2\chi^2}\delta_{\mathrm{D}}(\chi - \chi')$$

We have almost the above integral, but with $P(k)$ in the integrand. But for large ℓ one obtains

$$\int dk\, k^2 j_\ell(k\chi) j_\ell(k\chi') f(k) = \frac{\pi}{2\chi^2}\delta_{\mathrm{D}}(\chi - \chi') f(\ell/\chi)$$

which can be obtained by substituting

$$j_\ell(x) \approx \sqrt{\frac{\pi}{2\ell}}\,\delta_{\mathrm{D}}(\ell + \frac{1}{2} - x) \qquad \text{for large } \ell$$

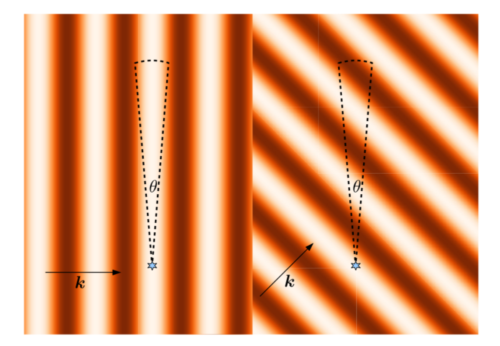
One obtains the angular power spectrum under Limber approximation

$$C_g(\ell) = \int_0^\infty \frac{d\chi}{\chi^2} W^2(\chi) P_g\left(k = \frac{\ell + 1/2}{\chi}, \mu_k = 0; \chi\right)$$

valid for large ℓ. Typically for $\ell > 20$.

We see that $\chi = \chi'$ in the Limber approximation, hence the power spectrum only involves equal-time densities. It also means that the k modes involved do not have a line-of-sight component, since that would mean different distances of different points along the perturbation, i.e. $\chi \neq \chi'$. So, $\boldsymbol{k}$ has to be transverse to the line of sight: $\mu_k = 0$.

Dodelson & Schmidt figure 11.8



Focusing on small scales corresponds to looking at small angles, $\theta \sim 1/\ell$. *Right panel:* Modes with longitudinal wavenumber $\mu_k k > \chi^{-1}$ (or $\mu_k > \theta$) do not give rise to angular correlations because of cancelations along the line of sight. *Left panel:* Only modes with $\mu_k k \lesssim \chi^{-1}$ (or $\mu_k < \theta$) lead to angular correlations. This corresponds to setting $\chi = \chi'$.

Angular galaxy power spectrum

$$C_{\rm g}(\ell) = \int d\chi \frac{W^2(\chi)}{\chi^2} P_{\rm g}\big(k = (\ell + 1/2)/\chi, \chi\big)$$