Manipultation of density-weighted velocity moments in the moment expansion approach to RSD modeling

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These notes seem to be correct, though a double check is worthy.

1 Manipulating momenta

The m-th density weighted velocity field moment of the generating function is an m-rank tensor defined as

$$\Xi^{\mathrm{m}}_{i_1\cdots i_{\mathrm{m}}}(\mathbf{x}) \equiv i^{\mathrm{m}} \frac{\partial^{\mathrm{m}}}{\partial J_{i_1}\cdots\partial J_{i_{\mathrm{m}}}}\big[1+\mathcal{M}(\mathbf{J},\mathbf{x})\big]\Big|_{\mathbf{J}=0} = \langle\big(1+\delta_1\big)\big(1+\delta_2\big)\Delta u_{i_1}\cdots\Delta u_{i_{\mathrm{m}}}\rangle, \tag{1.1}$$

with $\delta_1 = \delta(\mathbf{x}_1)$ and $\delta_2 = \delta(\mathbf{x}_2)$. The moment m piece of the power spectrum is

$$P^{\mathrm{m}}(k,\mu) \equiv \frac{(-i)^m}{m!} k_{i_1}\cdots k_{i_m} \tilde{\Xi}^{(m)}_{i_1\cdots i_m}(\mathbf{k}) = \sum_{n=0}^{m} \mu^{2n} f_0^m I_n^m(k), \tag{1.2}$$

with

$$\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}, \tag{1.3}$$

and f_0 will be defined shortly. The second equality in eq. (1.2) is an ansatz that works at least at 1-loop. But it should work at any order, I think. The redshift-space power spectrum is

$$P_s(k,\mu) = \sum_{m=0}^{\infty} P^{\mathrm{m}}(k,\mu). \tag{1.4}$$

To linear order we can truncate the sum at $m=2$. To 1-loop, at $m=4$.

For a rotational scalar function $S(\mathbf{k},\mathbf{p}) = S(k,p,x)$, with $x \equiv \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}$, we use the relation

$$\int \frac{d^3p}{(2\pi)^3} (\hat{\mathbf{p}}\cdot\hat{\mathbf{n}})^n S(\mathbf{k},\mathbf{p}) = \sum_{m=0}^{n} (\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})^m \int \frac{d^3p}{(2\pi)^3} G_{nm}(x) S(k,p,x), \tag{1.5}$$

where

$$\begin{aligned} G_{nm}(x) = \sum_{\ell=0}^{n} & \frac{(1+(-1)^{\ell+n})(2\ell+1)}{2(1+\ell+n)} \binom{\ell}{m}\binom{2\ell}{\ell}\binom{\frac{\ell+m-1}{2}}{\ell} \\ & \times {}_3F_2\Big(\frac{1-\ell}{2}, -\frac{\ell}{2}, \frac{1}{2}(-1-\ell-n); \frac{1}{2}-\ell, \frac{1}{2}(1-\ell-n); 1\Big)\mathcal{P}_\ell(x) \end{aligned} \tag{1.6}$$

where ${}_3F_2(\mathbf{a};\mathbf{b};z)$ is the hypergeometric function of the kind ($p=3$, $q=2$) evaluated at $z=1$, and $\mathcal{P}_\ell(x)$ is the Legendre polynomial of degree ℓ. This formula is the key to write the power spectrum as a sum of terms $\mu^{2n} f_0^m I_n^m(k)$.

We notice $G_{nn} = \mathcal{P}_n(x)$, and $G_{nm} = 0$ for $n < m$ or if $n+m$ is an odd integer. With this formula one obtains all the $G_{nm}(x)$ functions listed in arxiv.org/abs/1006.0699 (from now on, TNS paper), with the exception of $G_{55}(x) = \mathcal{P}_5(x)$ that in TNS paper has a typo. Note also that the indices are inverted $G_{nm}(x)^{\mathrm{Here}} = G_{mn}(x)^{\mathrm{TNS\ paper}}$.

From eq. (1.5) we can write

$$\int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^n S(\mathbf{k},\mathbf{p}) = \sum_{m=0}^{n} \mu^m \int \frac{d^3p}{(2\pi)^3} p^n G_{nm}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}) S(\mathbf{k},\mathbf{p})$$
$$= \frac{k^3}{4\pi^2} \sum_{m=0}^{n} \mu^m \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^n G_{nm}(x) S(k,r,x), \tag{1.7}$$

with $x = \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}$ and $r = p/k$.

The SPT kernels are general, not necessary EdS, so we will use also a $G_1(k)$ kernel, relating $\theta^{(1)}(\mathbf{k}) = G_1(k)\delta^{(1)}(\mathbf{k})$. This new k-dependence appears in theories beyond ΛCDM, for example if massive neutrinos are considered, or in modified gravity. Indeed, in such theories the linear growth function D_+, is no longer only time dependent, but it has also a dependence on the scale k.

We will use the notation

$$f(k,t) = \frac{d\log D_+(k,t)}{d\log a(t)} \tag{1.8}$$

for the growth rate, and

$$f_0 \equiv f(k=0,t). \tag{1.9}$$

In ΛCDM one obtains $f(k,t) = f(t) = f_0$.

Hence, through the notes, if you are interested only in ΛCDM, just replace $f_0 \to f$ and $G_1(k) = 1$.

In more general theories $G_1(k) = f(k)/f_0$.

WARNING: In these notes we define

$$D(k,\mu) = B(k,\mu) + C(k,\mu), \tag{1.10}$$

instead than $D(k,\mu) = B(k,\mu) + C(k,\mu) - (k\mu f_0\sigma_v)^2 P_s^K(k,\mu)$, as we do in other accompanying notes.

2 zeroth moment

$$\Xi^0(\mathbf{x}) = \langle \big(1+\delta_1\big)\big(1+\delta_2\big)\rangle = 1 + \langle\delta_1\delta_2\rangle \tag{2.1}$$

Hence

$$\tilde{\Xi}^{(0)}(\mathbf{k}) = P_{\delta\delta}(k), \tag{2.2}$$

and

$$P_s^0(\mathbf{k}) = I_0^0(k) = P_{\delta\delta}(k). \tag{2.3}$$

3 First moment

$$\Xi_i^1(\mathbf{x}) = \langle \big(1+\delta_1\big)\big(1+\delta_2\big)\Delta u_i\rangle$$
$$= \langle \Delta u_i(\delta_1+\delta_2)\rangle + \langle \Delta u_i \delta_1\delta_2\rangle \equiv \Xi_i^{1,ud}(\mathbf{x}) + \Xi_i^{1,udd}(\mathbf{x}) \tag{3.1}$$

3.1 ud

$$\tilde{\Xi}_i^{1,ud}(\mathbf{k}) = 2if_0\hat{n}_i\frac{\mu}{k}P_{\delta\theta}(k), \qquad -ik_i\tilde{\Xi}_i^{1,ud}(\mathbf{k}) = 2f_0\mu^2P_{\delta\theta}(k). \tag{3.2}$$

Hence

$$I_1^{1,ud}(k) = 2P_{\delta\theta}(k), \tag{3.3}$$

and

$$P^{1,ud}(k,\mu) = \mu^2 f_0 I_1^{1,ud}(k). \tag{3.4}$$

3.2 udd

$$\begin{aligned} P^{1,udd}(\mathbf{k}) = -ik_i\tilde{\Xi}_i^{1,udd} &= k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\big[B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k}-\mathbf{p},\mathbf{k})\big] \\ &= 2k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \end{aligned} \tag{3.5}$$

The last line can be done with any bispectrum function in the following. Even with B_σ in TNS paper. Don't know why the authors of TNS didn't realize of that.

Defining

$$S(\mathbf{k},\mathbf{p}) = \frac{2k}{p^2}B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \tag{3.6}$$

we have

$$P^{1,udd}(\mathbf{k}) = \mu f_0 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})S(\mathbf{k},\mathbf{p}) \tag{3.7}$$

We symmetrize the kernel of the integral

$$\begin{aligned} P^{1,udd}(\mathbf{k}) &= \frac{1}{2}\mu f_0 \int \frac{d^3p}{(2\pi)^3}\Big[(\mathbf{p}\cdot\hat{\mathbf{n}})S(\mathbf{k},\mathbf{p}) + (\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}S(\mathbf{k},\mathbf{k}-\mathbf{p})\Big] \\ &= \frac{1}{2}\mu f_0 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})\big[S(\mathbf{k},\mathbf{p}) - S(\mathbf{k},\mathbf{k}-\mathbf{p})\big] \\ &\quad + \frac{1}{2}k\mu^2 f_0 \int \frac{d^3p}{(2\pi)^3}S(\mathbf{k},\mathbf{k}-\mathbf{p}) \end{aligned} \tag{3.8}$$

Using r, x:

$$P^{1,udd}(\mathbf{k}) = \mu^2 f_0 I^{1,udd}(k) \tag{3.9}$$

$$I^{1,udd}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, \mathcal{I}^{1,udd}(\mathbf{k},\mathbf{p}) \tag{3.10}$$

with ($G_{11} = x$)

$$\mathcal{I}^{1,udd}(\mathbf{k},\mathbf{p}) = \frac{1}{2}kr^3x\big[S(\mathbf{k},\mathbf{p}) - S(\mathbf{k},\mathbf{k}-\mathbf{p})\big] + \frac{1}{2}kr^2S(\mathbf{k},\mathbf{k}-\mathbf{p}) \tag{3.11}$$

or

$$\mathcal{I}^{1,udd}(\mathbf{k},\mathbf{p}) = \frac{1}{2}kr^3xS(\mathbf{k},\mathbf{p}) + \frac{1}{2}kr^2(1-rx)S(\mathbf{k},\mathbf{k}-\mathbf{p}) \tag{3.12}$$

Developing a little bit more

$$
\begin{aligned}
\mathcal{I}^{1,udd}(\mathbf{k},\mathbf{p}) &= \frac{1}{2}kr^3x\frac{2k}{p^2}B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) + \frac{1}{2}kr^2(1-rx)\frac{2k}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\delta\delta}(\mathbf{k}-\mathbf{p},-\mathbf{k},\mathbf{p}) \\
&= \frac{k^2r^3x}{p^2}B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) + \frac{k^2r^2(1-rx)}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\delta\delta}(\mathbf{k}-\mathbf{p},-\mathbf{k},\mathbf{p}) \\
&= rxB_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) + \frac{r^2(1-rx)}{1+r^2-2rx}B_{\theta\delta\delta}(\mathbf{k}-\mathbf{p},-\mathbf{k},\mathbf{p}) \qquad (3.13)
\end{aligned}
$$

Now,

$$
\begin{aligned}
B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) &= 2G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})P_L(k)P_L(|\mathbf{k}-\mathbf{p}|) \\
&\quad + 2F_2(\mathbf{p},\mathbf{k}-\mathbf{p})G_1(p)P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) \\
&\quad + 2F_2(\mathbf{p},-\mathbf{k})G_1(p)P_L(p)P_L(k), \qquad (3.14) \\
B_{\theta\delta\delta}(\mathbf{k}-\mathbf{p},-\mathbf{k},\mathbf{p}) &= 2G_2(-\mathbf{k},\mathbf{p})P_L(k)P_L(p) \\
&\quad + 2F_2(\mathbf{k}-\mathbf{p},\mathbf{p})G_1(|\mathbf{k}-\mathbf{p}|)P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) \\
&\quad + 2F_2(\mathbf{k}-\mathbf{p},-\mathbf{k})G_1(|\mathbf{k}-\mathbf{p}|)P_L(|\mathbf{k}-\mathbf{p}|)P_L(k) \qquad (3.15)
\end{aligned}
$$

Arranging terms depending on the arguments of $P_L \times P_L$:

$$
\begin{aligned}
&\mathcal{I}^{1,udd}(\mathbf{k},\mathbf{p}) \\
&= 2\left[G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})rx + F_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(|\mathbf{k}-\mathbf{p}|)\frac{r^2(1-rx)}{1+r^2-2rx}\right]P_L(k)P_L(|\mathbf{k}-\mathbf{p}|) \\
&+ 2F_2(\mathbf{p},\mathbf{k}-\mathbf{p})\left[G_1(p)rx + G_1(|\mathbf{k}-\mathbf{p}|)\frac{r^2(1-rx)}{1+r^2-2rx}\right]P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) \\
&+ 2\left[F_2(-\mathbf{k},\mathbf{p})G_1(p)rx + G_2(-\mathbf{k},-\mathbf{p})\frac{r^2(1-rx)}{1+r^2-2rx}\right]P_L(p)P_L(k) \qquad (3.16)
\end{aligned}
$$

Now, we want to write this as in TNS paper:

$$
\begin{aligned}
I^{1,udd}(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\left\{A_{11}(\mathbf{k},\mathbf{p})P_L(k) + \tilde{A}_{11}(\mathbf{k},\mathbf{p})P_L(kr)\right\}\frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2} \\
&+ \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, a_{11}(\mathbf{k},\mathbf{p})P_L(k)P_L(kr) \qquad (3.17)
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{A_{11}(\mathbf{k},\mathbf{p})}{(1+r^2-2rx)^2} &= 2\left[G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})rx + F_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(|\mathbf{k}-\mathbf{p}|)\frac{r^2(1-rx)}{1+r^2-2rx}\right] \qquad (3.18) \\
\frac{\tilde{A}_{11}(\mathbf{k},\mathbf{p})}{(1+r^2-2rx)^2} &= 2F_2(\mathbf{p},\mathbf{k}-\mathbf{p})\left[G_1(p)rx + G_1(|\mathbf{k}-\mathbf{p}|)\frac{r^2(1-rx)}{1+r^2-2rx}\right] \qquad (3.19) \\
a_{11}(\mathbf{k},\mathbf{p}) &= 2\left[F_2(-\mathbf{k},\mathbf{p})G_1(p)rx + G_2(-\mathbf{k},-\mathbf{p})\frac{r^2(1-rx)}{1+r^2-2rx}\right] \qquad (3.20)
\end{aligned}
$$

For EdS kernels the dependence of A_{ij} on k vanishes. We have for EdS:

$$A_{11}^{\rm EdS}(r,x) = -\frac{r^3}{7}\left[x + 6x^3 + r^2 x(-3+10x^2) + r(-3+x^2-12x^4)\right], \tag{3.21}$$

$$\tilde{A}_{11}^{\rm EdS}(r,x) = \frac{1}{7}(x+r-2rx^2)(3r+7x-10rx^2), \tag{3.22}$$

$$a_{11}^{\rm EdS}(r,x) = \frac{-7x^2 + r^3 x(-3+10x^2) + 3rx(1+6x^2) + r^2(6-19x^2-8x^4)}{7(1+r^2-2rx)} \tag{3.23}$$

Functions $A_{11}^{\rm EdS}$ and $\tilde{A}_{11}^{\rm EdS}$ coincide with those reported in TNS paper. Since the power spectra accompaining a_{11} do not depend on x, the angular integral can be perfoemed analytically, if I do that I get $\int_{-1}^{1} dx a_{11}^{\rm EdS}(r,x) = a_{11}^{\rm TNS\ paper}(r,x)$, which is the expected result.

The main objective of symmetrize is to evaluate the integral $\int \frac{d^3p}{(2\pi)^3}$ over, not the whole 3D p-space, but only over $p < |\mathbf{k}-\mathbf{p}|$. Meaning that for a function $f(\mathbf{k},\mathbf{p})$ which is symmetrical against the substitution $\mathbf{p} \to \mathbf{k}-\mathbf{p}$ the following holds

$$\int \frac{d^3p}{(2\pi)^3} f(\mathbf{k},\mathbf{p}) = 2\int_{p<|\mathbf{k}-\mathbf{p}|} \frac{d^3p}{(2\pi)^3} f(\mathbf{k},\mathbf{p}) \tag{3.24}$$

or

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3} f(\mathbf{k},\mathbf{p}) &= \frac{k^2}{4\pi^2}\int_0^\infty dr \int_{-1}^{1} dx\, r^2 f(k,r,x) \\ &= 2\times \frac{k^2}{4\pi^2}\int_0^\infty dr \int_{-1}^{\mathrm{Min}[1,1/(2r)]} dx\, r^2 f(k,r,x).\end{aligned} \tag{3.25}$$

4 2nd moment

4.1 uu

$$\begin{aligned}P^{2,uu} &= \frac{i^2}{2} k_i k_j \tilde{\Xi}_{ij}^{2,uu}(\mathbf{k}) = -\frac{1}{2}k_i k_j \int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}} \langle \Delta u_i \Delta u_j\rangle \\ &= \mu^4 f_0^2 P_{\theta\theta}(k),\end{aligned} \tag{4.1}$$

then

$$I_2^{2,uu}(k) = P_{\theta\theta}(k), \tag{4.2}$$

and $I_0^{2,uu}(k) = I_1^{2,uu}(k) = 0$.

4.2 uud

$$P^{2,uud} = \frac{i^2}{2} k_i k_j \tilde{\Xi}_{ij}^{2,uud}(\mathbf{k}) \tag{4.3}$$

$$\begin{aligned}P^{2,uud}(\mathbf{k}) &= 2k\mu f_0^2 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2} B_{\theta\delta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \\ &\quad + 2k\mu^3 f_0^2 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} B_{\theta\theta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) = P_A^{\rm m=2} + P_B^{\rm m=2}\end{aligned} \tag{4.4}$$

Symmetrizing

$$P^{2,uud}(\mathbf{k}) = k\mu f_0^2 \int \frac{d^3p}{(2\pi)^3}\Big[\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2}B^A_{\theta\delta\theta} + \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}\frac{[\mathbf{p}\cdot\hat{\mathbf{n}}]^2}{p^2}B^B_{\theta\delta\theta}\Big]$$
$$+ k\mu^3 f_0^2 \int \frac{d^3p}{(2\pi)^3}\Big[\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}B^A_{\theta\theta\delta} + \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}B^B_{\theta\theta\delta}\Big] \quad (4.5)$$

where $B^A_{\theta\delta\theta} = B_{\theta\delta\theta}(\mathbf{p}, -\mathbf{k}, \mathbf{k}-\mathbf{p})$ and $B^B_{\theta\delta\theta} = B_{\theta\delta\theta}(\mathbf{k}-\mathbf{p}, -\mathbf{k}, \mathbf{p})$ Now

$$[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2 = (k\mu)^2 - 2k\mu\mathbf{p}\cdot\hat{\mathbf{n}} + (\mathbf{p}\cdot\hat{\mathbf{n}})^2 \quad (4.6)$$

Then

$$P^{2,uud}(\mathbf{k}) = k\mu f_0^2 \int \frac{d^3p}{(2\pi)^3}\Bigg\{\big[(\mathbf{p}\cdot\hat{\mathbf{n}})^3 - 2k\mu(\mathbf{p}\cdot\hat{\mathbf{n}})^2 + (k\mu)^2(\mathbf{p}\cdot\hat{\mathbf{n}})\big]\frac{B^A_{\theta\delta\theta}}{p^2|\mathbf{k}-\mathbf{p}|^2}$$
$$+ \big[k\mu(\mathbf{p}\cdot\hat{\mathbf{n}})^2 - (\mathbf{p}\cdot\hat{\mathbf{n}})^3\big]\frac{B^B_{\theta\delta\theta}}{p^2|\mathbf{k}-\mathbf{p}|^2}\Bigg\}$$
$$+ k\mu^3 f_0^2 \int \frac{d^3p}{(2\pi)^3}\Big[(\mathbf{p}\cdot\hat{\mathbf{n}})\frac{B^A_{\theta\theta\delta}}{p^2} + \big[k\mu - \mathbf{p}\cdot\hat{\mathbf{n}}\big]\frac{B^B_{\theta\theta\delta}}{|\mathbf{k}-\mathbf{p}|^2}\Big] \quad (4.7)$$

$$P^{2,uud} = \mu f_0^2 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) + \mu^2 f_0^2 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p})$$
$$+ \mu^3 f_0^2 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}}) S_1(\mathbf{k},\mathbf{p}) + \mu^4 f_0^2 \int \frac{d^3p}{(2\pi)^3} S_0(\mathbf{k},\mathbf{p}) \quad (4.8)$$

with

$$S_3(\mathbf{k},\mathbf{p}) = \frac{k}{p^2|\mathbf{k}-\mathbf{p}|^2}\big[B_{\theta\delta\theta}(\mathbf{p}, -\mathbf{k}, \mathbf{k}-\mathbf{p}) - B_{\theta\delta\theta}(\mathbf{k}-\mathbf{p}, -\mathbf{k}, \mathbf{p})\big] \quad (4.9)$$
$$= 0? \quad (4.10)$$
$$S_2(\mathbf{k},\mathbf{p}) = -\frac{k^2}{p^2|\mathbf{k}-\mathbf{p}|^2}\big[2B_{\theta\delta\theta}(\mathbf{p}, -\mathbf{k}, \mathbf{k}-\mathbf{p}) - B_{\theta\delta\theta}(\mathbf{k}-\mathbf{p}, -\mathbf{k}, \mathbf{p})\big] \quad (4.11)$$
$$= -\frac{k^2}{p^2|\mathbf{k}-\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p}, -\mathbf{k}, \mathbf{k}-\mathbf{p})? \quad (4.12)$$
$$S_1(\mathbf{k},\mathbf{p}) = \frac{k^3}{p^2|\mathbf{k}-\mathbf{p}|^2}\Big[B_{\theta\delta\theta}(\mathbf{p}, -\mathbf{k}, \mathbf{k}-\mathbf{p}) + \frac{|\mathbf{k}-\mathbf{p}|^2}{k^2}B_{\theta\theta\delta}(\mathbf{p}, -\mathbf{k}, \mathbf{k}-\mathbf{p})$$
$$- \frac{p^2}{k^2}B_{\theta\theta\delta}(\mathbf{k}-\mathbf{p}, -\mathbf{k}, \mathbf{p})\Big] \quad (4.13)$$
$$S_0(\mathbf{k},\mathbf{p}) = \frac{k^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{k}-\mathbf{p}, -\mathbf{k}, \mathbf{p}) \quad (4.14)$$

We will write $S_1(\mathbf{k},\mathbf{p}) = S_1(k,r,x)$

$$
\begin{aligned}
\int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\sum_{m=0}^{3}\mu^m \int_0^\infty dr\, r^2 \int_{-1}^{1} dx\,(kr)^3 G_{3m}(x) S_3(k,r,x) \\
\int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\sum_{m=0}^{2}\mu^m \int_0^\infty dr\, r^2 \int_{-1}^{1} dx\,(kr)^2 G_{2m}(x) S_2(k,r,x) \\
\int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}}) S_1(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\sum_{m=0}^{1}\mu^m \int_0^\infty dr\, r^2 \int_{-1}^{1} dx\,(kr) G_{1m}(x) S_1(k,r,x) \\
\int \frac{d^3p}{(2\pi)^3} S_0(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\sum_{m=0}^{1}\mu^m \int_0^\infty dr\, r^2 \int_{-1}^{1} dx\, S_1(k,r,x)
\end{aligned}
\tag{4.15}
$$

Given that

$$
\begin{aligned}
G_{11}(x) &= x, \\
G_{22}(x) &= \frac{1}{2}(3x^2-1) \qquad G_{20}(x) = \frac{1}{2}(1-x^2) \\
G_{33}(x) &= \frac{1}{2}x(5x^2-3) \qquad G_{31}(x) = \frac{3}{2}x(1-x^2)
\end{aligned}
\tag{4.16}
$$

and the rest vanish.

$$
\begin{aligned}
\int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) &= \mu\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr) G_{31}(x) S_3(k,r,x) \\
&\quad + \mu^3\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^3 G_{33}(x) S_3(k,r,x) \\
\int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) &= \mu^0\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^0 G_{20}(x) S_2(k,r,x) \\
&\quad + \mu^2\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^2 G_{22}(x) S_2(k,r,x) \\
\int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}}) S_1(\mathbf{k},\mathbf{p}) &= \mu\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr) G_{11}(x) S_1(k,r,x)
\end{aligned}
\tag{4.17}
$$

Then

$$\mu \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) = \mu^2 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^3 G_{31}(x) S_3(k,r,x) + \mu^4 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^3 G_{33}(x) S_3(k,r,x)$$

$$\mu^2 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) = \mu^2 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^2 G_{20}(x) S_2(k,r,x) + \mu^4 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^2 G_{22}(x) S_2(k,r,x)$$

$$\mu^3 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}}) S_1(\mathbf{k},\mathbf{p}) = \mu^4 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr) G_{11}(x) S_1(k,r,x) \tag{4.18}$$

$$\mu^4 \int \frac{d^3p}{(2\pi)^3} S_0(\mathbf{k},\mathbf{p}) = \mu^4 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx S_0(k,r,x) \tag{4.19}$$

Then,

$$P^{2,uud} = \mu^2 f_0^2 I_1^{2,uud}(k) + \mu^4 f_0^2 I_2^{2,uud}(k) \tag{4.20}$$

with

$$I_1^{2,uud}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx \Big[(kr)^3 G_{31}(x) S_3(k,r,x) + (kr)^2 G_{20}(x) S_2(k,r,x)\Big] \tag{4.21}$$

$$I_2^{2,uud}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx \Big[(kr)^3 G_{33}(x) S_3(k,r,x) + (kr)^2 G_{22}(x) S_2(k,r,x) + (kr) G_{11}(x) S_1(k,r,x) + S_0(k,r,x)\Big] \tag{4.22}$$

Now

$$B_{\theta\delta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) = 2G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(|\mathbf{k}-\mathbf{p}|)P_L(k)P_L(|\mathbf{k}-\mathbf{p}|) + 2F_2(\mathbf{p},\mathbf{k}-\mathbf{p})G_1(p)G_1(|\mathbf{k}-\mathbf{p}|)P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) + 2G_2(\mathbf{p},-\mathbf{k})G_1(p)P_L(p)P_L(k), \tag{4.23}$$

$$B_{\theta\delta\theta}(\mathbf{k}-\mathbf{p},-\mathbf{k},\mathbf{p}) = B_{\theta\delta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}), \tag{4.24}$$

$$B_{\theta\theta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) = 2G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(k)P_L(k)P_L(|\mathbf{k}-\mathbf{p}|) + 2G_2(\mathbf{p},\mathbf{k}-\mathbf{p})G_1(p)P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) + 2F_2(\mathbf{p},-\mathbf{k})G_1(p)G_1(k)P_L(p)P_L(k), \tag{4.25}$$

$$B_{\theta\theta\delta}(\mathbf{k}-\mathbf{p},-\mathbf{k},\mathbf{p}) = 2G_2(-\mathbf{k},\mathbf{p})G_1(k)P_L(k)P_L(p) + 2G_2(\mathbf{k}-\mathbf{p},\mathbf{p})G_1(|\mathbf{k}-\mathbf{p}|)P_L(p)P_L(|\mathbf{k}-\mathbf{p}|) + 2F_2(\mathbf{k}-\mathbf{p},-\mathbf{k})G_1(|\mathbf{k}-\mathbf{p}|)G_1(k)P_L(|\mathbf{k}-\mathbf{p}|)P_L(k), \tag{4.26}$$

4.2.1 Connection to TNS

$$I_1^{2,uud}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, k^2 r^4 G_{20}(x) S_2(k,r,x) \tag{4.27}$$

which can be written as TNS as

$$\begin{aligned} I_1^{2,uud}(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\left\{A_{12}(\mathbf{k},\mathbf{p})P_L(k) + \tilde{A}_{12}(\mathbf{k},\mathbf{p})P_L(kr)\right\}\frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2} \\ &+ \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, a_{12}(\mathbf{k},\mathbf{p})P_L(k)P_L(kr) \end{aligned} \tag{4.28}$$

with

$$\frac{A_{12}(\mathbf{k},\mathbf{p})}{(1+r^2-2rx)^2} = -\frac{r^2(1-x^2)}{1+r^2-2rx}G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(|\mathbf{k}-\mathbf{p}|) \tag{4.29}$$

$$\frac{\tilde{A}_{12}(\mathbf{k},\mathbf{p})}{(1+r^2-2rx)^2} = -\frac{r^2(1-x^2)}{1+r^2-2rx}F_2(\mathbf{p},\mathbf{k}-\mathbf{p})G_1(p)G_1(|\mathbf{k}-\mathbf{p}|) \tag{4.30}$$

$$a_{12}(\mathbf{k},\mathbf{p}) = -\frac{r^2(1-x^2)}{1+r^2-2rx}G_2(\mathbf{p},-\mathbf{k})G_1(p) \tag{4.31}$$

For EdS kernels the dependence of A_{ij} on k vanishes. We have for EdS:

$$A_{12}^{\mathrm{EdS}}(r,x) = \frac{r^4}{14}(x^2-1)(-1+7rx-6x^2), \tag{4.32}$$

$$\tilde{A}_{12}^{\mathrm{EdS}}(r,x) = \frac{r}{14}(x^2-1)(3r+7x-10rx^2), \tag{4.33}$$

$$a_{12}^{\mathrm{EdS}}(r,x) = \frac{r(x^2-1)[6r-7(1+r^2)x+8rx^2]}{14(1+r^2-2rx)} \tag{4.34}$$

A_{12}^{EdS} and $\tilde{A}_{12}^{\mathrm{EdS}}$ and a_{12}^{EdS} (after x integration) coincide with those of TNS paper.

$$\begin{aligned} I_2^{2,uud}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx\Big[&(kr)^2 G_{22}(x)S_2(k,r,x) + (kr)G_{11}(x)S_1(k,r,x) \\ &+ S_0(k,r,x)\Big] \end{aligned} \tag{4.35}$$

which can be written as TNS as

$$\begin{aligned} I_2^{2,uud}(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\left\{A_{22}(\mathbf{k},\mathbf{p})P_L(k) + \tilde{A}_{22}(\mathbf{k},\mathbf{p})P_L(kr)\right\}\frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2} \\ &+ \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, a_{22}(\mathbf{k},\mathbf{p})P_L(k)P_L(kr) \end{aligned} \tag{4.36}$$

with

$$\frac{A_{22}(k,x,r)}{(1+r^2-2rx)^2} = 2r^2\left[\frac{G_{11}(x)-rG_{22}(x)}{r(1+r^2-2rx)}G_1(|\mathbf{k}-\mathbf{p}|)+\frac{G_{11}(x)}{r}G_1(k)\right]G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})$$
$$+2r^2\frac{1-rG_{11}(x)}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)G_1(k)F_2(-\mathbf{k},\mathbf{k}-\mathbf{p}), \tag{4.37}$$

$$\frac{\tilde{A}_{22}(k,x,r)}{(1+r^2-2rx)^2} = 2\left[\frac{1-rG_{11}(x)}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)+\frac{G_{11}(x)}{r}G_1(p)\right]G_2(\mathbf{p},\mathbf{k}-\mathbf{p})$$
$$+2r^2\frac{G_{11}(x)-rG_{22}(x)}{r(1+r^2-2rx)}G_1(|\mathbf{k}-\mathbf{p}|)G_1(p)F_2(\mathbf{p},\mathbf{k}-\mathbf{p}), \tag{4.38}$$

$$a_{22}(k,x,r) = 2r^2\left[\frac{G_{11}(x)-rG_{22}(x)}{r(1+r^2-2rx)}G_1(p)+\frac{1-rG_{11}(x)}{1+r^2-2rx}G_1(k)\right]G_2(-\mathbf{k},\mathbf{p})$$
$$+2r^2\frac{G_{11}(x)}{r}G_1(p)G_1(k)F_2(-\mathbf{k},\mathbf{p}). \tag{4.39}$$

or, using the values of G_{11} and G_{22},

$$\frac{A_{22}(k,x,r)}{(1+r^2-2rx)^2} = \left[\frac{r^2(1-3x^2)+2xr}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)+2xrG_1(k)\right]G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})$$
$$+\frac{2r^2(1-rx)}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)G_1(k)F_2(-\mathbf{k},\mathbf{k}-\mathbf{p}), \tag{4.40}$$

$$\frac{\tilde{A}_{22}(k,x,r)}{(1+r^2-2rx)^2} = \left[\frac{2r^2(1-rx)}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)+2xrG_1(p)\right]G_2(\mathbf{p},\mathbf{k}-\mathbf{p})$$
$$+\frac{r^2(1-3x^2)+2xr}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)G_1(p)F_2(\mathbf{p},\mathbf{k}-\mathbf{p}), \tag{4.41}$$

$$a_{22}(k,x,r) = \left[\frac{r^2(1-3x^2)+2xr}{1+r^2-2rx}G_1(p)+\frac{2r^2(1-rx)}{1+r^2-2rx}G_1(k)\right]G_2(-\mathbf{k},\mathbf{p})$$
$$+2xrG_1(p)G_1(k)F_2(-\mathbf{k},\mathbf{p}). \tag{4.42}$$

For EdS kernels

$$A_{22}(r,x) = \frac{r^3}{14}\left[r^2x(13-41x^2)-4(x+6x^3)+r(5+9x^2+42x^4)\right] \tag{4.43}$$
$$\tilde{A}_{22}(r,x) = \frac{1}{14}\left[28x^2+rx(25-81x^2)+r^2(1-27x^2+54x^4)\right] \tag{4.44}$$
$$a_{22}(r,x) = \frac{-28x^2+r^3x(-13+41x^2)+rx(11+73x^2)-2r^2(-9+31x^2+20x^4)}{14(1+r^2-2rx)}, \tag{4.45}$$

which are the same as in TNS paper (after angular integration od a_{22}).

4.3 uudd

$$P^{2,uudd}(\mathbf{k}) = -\frac{1}{2}k_ik_j\tilde{\Xi}^{2,uudd}_{ij}(\mathbf{k}) \tag{4.46}$$

$$P^{2,uudd}(\mathbf{k}) = f_0^2k^2\mu^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\Big[P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|)-P_{\delta\delta}(k)\Big]$$
$$+f_0^2k^2\mu^2\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) \tag{4.47}$$

which here I write

$$P^{2,uudd}(\mathbf{k}) = -k^2\mu^2 f_0^2 P_{\delta\delta}(k)\sigma_v^2 + P_B^{2,uudd}(\mathbf{k}) + P_C^{2,uudd}(\mathbf{k}), \tag{4.48}$$

(remember that to linear order $P_{\delta\delta}(k) = P_L(k)$), with

$$P_B^{2,uudd}(\mathbf{k}) = f_0^2 k^2\mu^2 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} P_{\delta\theta}(p) P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) \tag{4.49}$$

$$P_C^{2,uudd}(\mathbf{k}) = f_0^2 k^2\mu^2 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} P_{\theta\theta}(p) P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|) \tag{4.50}$$

4.3.1 P2uudd B

$P_B^{2,uudd}(\mathbf{k})$ is already symmetric against $\mathbf{p} \to \mathbf{k}-\mathbf{p}$

$$\begin{aligned}
P_B^{2,uudd}(\mathbf{k}) &= k^2\mu^2 f_0^2 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} P_{\delta\theta}(p) P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&= k^2\mu^2 f_0^2 \int \frac{d^3p}{(2\pi)^3} \left[(k\mu)(\mathbf{p}\cdot\hat{\mathbf{n}}) - (\mathbf{p}\cdot\hat{\mathbf{n}})^2\right] \frac{P_{\delta\theta}(p) P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)}{p^2|\mathbf{k}-\mathbf{p}|^2} \\
&= \mu^2 f_0^2 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) + \mu^3 f_0^2 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}}) S_1(\mathbf{k},\mathbf{p})
\end{aligned} \tag{4.51}$$

with

$$S_2(\mathbf{k},\mathbf{p}) = \frac{-k^2}{p^2|\mathbf{k}-\mathbf{p}|^2} P_{\delta\theta}(p) P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) \tag{4.52}$$

$$S_1(\mathbf{k},\mathbf{p}) = \frac{k^3}{p^2|\mathbf{k}-\mathbf{p}|^2} P_{\delta\theta}(p) P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) \tag{4.53}$$

We use

$$\begin{aligned}
\mu^2 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) &= \mu^2 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^2 G_{20}(x) S_2(k,r,x) \\
&\quad + \mu^4 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^2 G_{22}(x) S_2(k,r,x) \\
\mu^3 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}}) S_1(\mathbf{k},\mathbf{p}) &= \mu^4 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr) G_{11}(x) S_1(k,r,x)
\end{aligned} \tag{4.54}$$

Then,

$$P_B^{2,uudd} = \mu^2 f_0^2 I_{1\,B}^{2,uudd}(k) + \mu^4 f_0^2 I_{2\,B}^{2,uudd}(k) \tag{4.55}$$

with

$$I_{1\,B}^{2,uudd}(k) = \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^2 G_{20}(x) S_2(k,r,x) \tag{4.56}$$

$$I_{2\,B}^{2,uudd}(k) = \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx \Big[(kr)^2 G_{22}(x) S_2(k,r,x) + (kr) G_{11}(x) S_1(k,r,x)\Big] \tag{4.57}$$

Comparing to the cases $(n = 1, a = 1, b = 1)$ and $(n = 2, a = 1, b = 1)$ of TNS paper equation A4:

$$
\begin{aligned}
I_{1\,B}^{2,uudd}(k) &= \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^{1} dx\, B_{11}^1(r,x)\frac{P_{\delta\theta}(kr)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)}{1+r^2-2rx} \\
I_{2\,B}^{2,uudd}(k) &= \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^{1} dx\, B_{11}^2(r,x)\frac{P_{\delta\theta}(kr)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)}{1+r^2-2rx}
\end{aligned} \tag{4.58}
$$

with

$$
B_{11}^1(r,x) = \frac{r^2}{2}(x^2-1), \tag{4.59}
$$

$$
B_{11}^2(r,x) = \frac{r}{2}(r-3rx^2+2x). \tag{4.60}
$$

These are the reported B_{ab}^n in TNS paper for EdS.

4.3.2 P2uudd C

$$
P_C^{2,uudd}(\mathbf{k}) = f_0^2 k^2\mu^2 \int \frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|) \tag{4.61}
$$

$$
\begin{aligned}
&= \frac{1}{2}f_0^2k^2\mu^2\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2\frac{1}{p^4}P_{\theta\theta}(p)P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|) \\
&\quad + \frac{1}{2}f_0^2k^2\mu^2\int\frac{d^3p}{(2\pi)^3}\left[(k\mu)^2-2(k\mu)(\mathbf{p}\cdot\hat{\mathbf{n}})+(\mathbf{p}\cdot\hat{\mathbf{n}})^2\right]\frac{1}{|\mathbf{k}-\mathbf{p}|^4}P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\delta\delta}(p)
\end{aligned} \tag{4.62}
$$

$$
= f_0^2\mu^2\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2S_2(\mathbf{k},\mathbf{p}) + f_0^2\mu^3\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})S_1(\mathbf{k},\mathbf{p}) + f_0^2\mu^4\int\frac{d^3p}{(2\pi)^3}S_0(\mathbf{k},\mathbf{p}) \tag{4.63}
$$

in the second line we symmetrized, with

$$
S_2(\mathbf{k},\mathbf{p}) = \frac{1}{2k^2}\frac{P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\delta\delta}(p)}{|\mathbf{k}-\mathbf{p}|^4k^{-4}} + \frac{k^2}{2p^4}P_{\theta\theta}(p)P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|) \tag{4.64}
$$

$$
S_1(\mathbf{k},\mathbf{p}) = \frac{-1}{k}\frac{P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\delta\delta}(p)}{|\mathbf{k}-\mathbf{p}|^4k^{-4}} \tag{4.65}
$$

$$
S_0(\mathbf{k},\mathbf{p}) = \frac{1}{2}\frac{P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\delta\delta}(p)}{|\mathbf{k}-\mathbf{p}|^4k^{-4}} \tag{4.66}
$$

We use

$$
\begin{aligned}
\mu^2\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2S_2(\mathbf{k},\mathbf{p}) &= \mu^2\frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dx(kr)^2G_{20}(x)S_2(k,r,x) \\
&\quad + \mu^4\frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dx(kr)^2G_{22}(x)S_2(k,r,x) \\
\mu^3\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})S_1(\mathbf{k},\mathbf{p}) &= \mu^4\frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dx(kr)G_{11}(x)S_1(k,r,x) \\
\mu^4\int\frac{d^3p}{(2\pi)^3}S_0(\mathbf{k},\mathbf{p}) &= \mu^4\frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dxS_1(k,r,x)
\end{aligned} \tag{4.67}
$$

Then,

$$P_C^{2,uudd} = \mu^2 f_0^2 I_{1\,C}^{2,uudd}(k) + \mu^4 f_0^2 I_{2\,C}^{2,uudd}(k) \tag{4.68}$$

with

$$I_{1\,C}^{2,uudd}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^2 G_{20}(x) S_2(k,r,x) \tag{4.69}$$

$$\begin{aligned} I_{2\,C}^{2,uudd}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx \Big[&(kr)^2 G_{22}(x) S_2(k,r,x) \\ &+ (kr) G_{11}(x) S_1(k,r,x) + S_0(k,r,x)\Big] \end{aligned} \tag{4.70}$$

Comparing to the cases $(n=1, a=1, b=1)$ and $(n=2, a=1, b=1)$:

$$\begin{aligned} I_{1\,C}^{2,uudd}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^{1} dx\, \frac{1}{4}(1-x^2) \Bigg\{ &P_{\delta\delta}(k\sqrt{1+r^2-2rx}) P_{\theta\theta}(kr) \\ &+ r^4 \frac{P_{\delta\delta}(kr) P_{\theta\theta}(k\sqrt{1+r^2-2rx})}{(1+r^2-2rx)^2} \Bigg\}, \end{aligned} \tag{4.71}$$

$$\begin{aligned} I_{2\,C}^{2,uudd}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^{1} dx \Bigg\{ &\frac{1}{4}(3x^2-1) P_{\delta\delta}(k\sqrt{1+r^2-2rx}) P_{\theta\theta}(kr) \\ &+ \frac{r^2}{4}\big[2 - 4rx + r^2(3x^2-1)\big] \frac{P_{\delta\delta}(kr) P_{\theta\theta}(k\sqrt{1+r^2-2rx})}{(1+r^2-2rx)^2} \Bigg\} \end{aligned} \tag{4.72}$$

We note these expressions have no counterpart in B, because the product here is $P_{\delta\delta}P_{\theta\theta}$, while the products for functions B are $P_{\delta\theta}P_{\delta\theta}$, $P_{\delta\theta}P_{\theta\theta}$ and $P_{\theta\theta}P_{\theta\theta}$. Therefore, it does not exist C_{11}^1 or C_{11}^2.

5 Third moment

5.1 uuu

$$P^{3,uuu}(\mathbf{k}) = -\frac{i^3}{3!} k_i k_j k_k \tilde{\Xi}_{ijk}^{3,uuu}(\mathbf{k}) \tag{5.1}$$

or

$$P^{3,uuu}(\mathbf{k}) = 2k\mu^3 f_0^3 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p}, -\mathbf{k}, \mathbf{k}-\mathbf{p}). \tag{5.2}$$

We note that $B_{\theta\theta\theta}(\mathbf{p}, -\mathbf{k}, \mathbf{k}-\mathbf{p})/\big[p^2|\mathbf{k}-\mathbf{p}|^2\big]$ is symmetric under the change of coordinates $\mathbf{p} \to \mathbf{k}-\mathbf{p}$. Then, symmetrizing $(\mathbf{p}\cdot\hat{\mathbf{n}})[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2$:

$$(\mathbf{p}\cdot\hat{\mathbf{n}})\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2 \longrightarrow \frac{1}{2}(k\mu)^2(\mathbf{p}\cdot\hat{\mathbf{n}}) - \frac{1}{2}(k\mu)(\mathbf{p}\cdot\hat{\mathbf{n}})^2. \tag{5.3}$$

Then

$$P^{3,uuu}(\mathbf{k}) = \mu^4 f_0^3 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) + \mu^5 f_0^3 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}}) S_1(\mathbf{k},\mathbf{p}) \tag{5.4}$$

with

$$S_2(\mathbf{k},\mathbf{p}) = -\frac{k^2}{p^2|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})$$
$$S_1(\mathbf{k},\mathbf{p}) = \frac{k^3}{p^2|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \tag{5.5}$$

We use

$$\begin{aligned}\mu^4\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2S_2(\mathbf{k},\mathbf{p}) &= \mu^4\frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dx(kr)^2G_{20}(x)S_2(k,r,x)\\ &+\mu^6\frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dx(kr)^2G_{22}(x)S_2(k,r,x)\\ \mu^5\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})S_1(\mathbf{k},\mathbf{p}) &= \mu^6\frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dx(kr)G_{11}(x)S_1(k,r,x)\end{aligned} \tag{5.6}$$

Then,

$$P^{3,uuu} = \mu^4 f_0^3 I_2^{3,uuu}(k) + \mu^6 f_0^3 I_3^{3,uuu}(k) \tag{5.7}$$

with

$$I_2^{3,uuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dx(kr)^2G_{20}(x)S_2(k,r,x) \tag{5.8}$$
$$I_3^{3,uuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dx\Big[(kr)^2G_{22}(x)S_2(k,r,x)+(kr)G_{11}(x)S_1(k,r,x)\Big] \tag{5.9}$$

5.1.1 As in TNS

$$I_2^{3,uuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\,r^2\int_{-1}^1 dx(kr)^2G_{20}(x)S_2(k,r,x) \tag{5.10}$$

which can be written as TNS as

$$\begin{aligned}I_2^{3,uuu}(k) &= \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\left\{A_{23}(\mathbf{k},\mathbf{p})P_L(k)+\tilde{A}_{23}(\mathbf{k},\mathbf{p})P_L(kr)\right\}\frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2}\\ &+\frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\,a_{23}(\mathbf{k},\mathbf{p})P_L(k)P_L(kr)\end{aligned} \tag{5.11}$$

with

$$\frac{A_{23}(k,x,r)}{(1+r^2-2rx)^2} = -\frac{2r^2G_{20}(x)}{1+r^2-2rx}G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(k)G_1(|\mathbf{k}-\mathbf{p}|), \tag{5.12}$$
$$\frac{\tilde{A}_{23}(k,x,r)}{(1+r^2-2rx)^2} = -\frac{2r^2G_{20}(x)}{1+r^2-2rx}G_2(\mathbf{p},\mathbf{k}-\mathbf{p})G_1(p)G_1(|\mathbf{k}-\mathbf{p}|), \tag{5.13}$$
$$a_{23}(k,x,r) = -\frac{2r^2G_{20}(x)}{1+r^2-2rx}G_2(-\mathbf{k},\mathbf{p})G_1(k)G_1(p). \tag{5.14}$$

or, using the values of $G_{20} = (1-x^2)/2$,

$$\frac{A_{23}(k,x,r)}{(1+r^2-2rx)^2} = \frac{r^2(x^2-1)}{1+r^2-2rx} G_2(-\mathbf{k},\mathbf{k}-\mathbf{p}) G_1(k) G_1(|\mathbf{k}-\mathbf{p}|), \tag{5.15}$$

$$\frac{\tilde{A}_{23}(k,x,r)}{(1+r^2-2rx)^2} = \frac{r^2(x^2-1)}{1+r^2-2rx} G_2(\mathbf{p},\mathbf{k}-\mathbf{p}) G_1(p) G_1(|\mathbf{k}-\mathbf{p}|), \tag{5.16}$$

$$a_{23}(k,x,r) = \frac{r^2(x^2-1)}{1+r^2-2rx} G_2(-\mathbf{k},\mathbf{p}) G_1(k) G_1(p). \tag{5.17}$$

For EdS kernels

$$A_{23}(r,x) = A_{12}(r,x) \tag{5.18}$$

$$\tilde{A}_{23}(r,x) = \frac{r}{14}(1-x^2)(r-7x+6rx^2) \tag{5.19}$$

$$a_{23}(r,x) = a_{12}(r,x), \tag{5.20}$$

which are the values reported in TNS paper.

Now,

$$I_3^{3,uuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx \Big[(kr)^2 G_{22}(x) S_2(k,r,x) + (kr) G_{11}(x) S_1(k,r,x)\Big] \tag{5.21}$$

in TNS form:

$$I_3^{3,uuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx \left\{ A_{33}(\mathbf{k},\mathbf{p}) P_L(k) + \tilde{A}_{33}(\mathbf{k},\mathbf{p}) P_L(kr) \right\} \frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2}$$
$$+ \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, a_{33}(\mathbf{k},\mathbf{p}) P_L(k) P_L(kr) \tag{5.22}$$

Now,

$$r^2\big[(kr)^2 G_{22}(x) S_2(k,r,x) + (kr) G_{11}(x) S_1(k,r,x)\big] =$$
$$\frac{k^4 r^3\big(-rG_{22}(x) + G_{11}(x)\big)}{p^2|\mathbf{k}-\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) = \frac{1}{2}\frac{r^2(1-3x^2)+2rx}{1+r^2-2rx} B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \tag{5.23}$$

Hence

$$\frac{A_{33}(k,x,r)}{(1+r^2-2rx)^2} = \frac{r^2(1-3x^2)+2rx}{1+r^2-2rx} G_2(-\mathbf{k},\mathbf{k}-\mathbf{p}) G_1(k) G_1(|\mathbf{k}-\mathbf{p}|), \tag{5.24}$$

$$\frac{\tilde{A}_{33}(k,x,r)}{(1+r^2-2rx)^2} = \frac{r^2(1-3x^2)+2rx}{1+r^2-2rx} G_2(\mathbf{p},\mathbf{k}-\mathbf{p}) G_1(p) G_1(|\mathbf{k}-\mathbf{p}|), \tag{5.25}$$

$$a_{33}(k,x,r) = \frac{r^2(1-3x^2)+2rx}{1+r^2-2rx} G_2(-\mathbf{k},\mathbf{p}) G_1(k) G_1(p). \tag{5.26}$$

In EdS

$$A_{33}^{\mathrm{EdS}}(x,r) = \frac{r^3}{14}(1-7rx+6x^2)[-2x+r(-1+3x^2)], \tag{5.27}$$

$$\tilde{A}_{33}^{\mathrm{EdS}}(x,r) = \frac{1}{14}(r-7x+6rx^2)(-2x-r+3rx^2), \tag{5.28}$$

$$a_{33}^{\mathrm{EdS}}(x,r) = \frac{[7x+r(-6+7rx-8x^2)][-2x+r(-1+3x^2)]}{14(1+r^2-2rx)}. \tag{5.29}$$

which are the values reported in TNS paper.

5.2 uuud

$$P^{3,uuud}(\mathbf{k}) = \frac{i}{6}k_ik_jk_k\tilde{\Xi}_{ijk}^{3,uuud}(\mathbf{k}) \tag{5.30}$$

$$P^{3,uuud}(\mathbf{k}) = 2(k\mu f_0)^3\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\left[\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)-\frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2}P_{\delta\theta}(k)\right] \tag{5.31}$$

We split it as

$$P^{3,uuud}(\mathbf{k}) = -2\mu^4f_0^3k^2\sigma_v^2P_{\delta\theta}(k)+P_D^{2,uuud}(\mathbf{k}) \tag{5.32}$$

with

$$\begin{aligned}
P_D^{3,uuud}(\mathbf{k}) &= 2(k\mu f_0)^3\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\\
&= (k\mu f_0)^3\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\\
&\quad+(k\mu f_0)^3\int\frac{d^3p}{(2\pi)^3}\frac{\left[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\right]^2}{|\mathbf{k}-\mathbf{p}|^4}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\delta\theta}(p)\\
&= \mu^3f_0^3\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^3S_3(\mathbf{k},\mathbf{p})+\mu^4f_0^3\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2S_2(\mathbf{k},\mathbf{p})\\
&\quad+\mu^5f_0^3\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})S_1(\mathbf{k},\mathbf{p})
\end{aligned} \tag{5.33}$$

with

$$S_3(\mathbf{k},\mathbf{p}) = \frac{k^3}{p^2|\mathbf{k}-\mathbf{p}|^2}\left[-\frac{P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)}{p^2}+\frac{P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\delta\theta}(p)}{|\mathbf{k}-\mathbf{p}|^2}\right] \tag{5.34}$$

$$S_2(\mathbf{k},\mathbf{p}) = \frac{k^4}{p^2|\mathbf{k}-\mathbf{p}|^2}\left[\frac{P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)}{p^2}-2\frac{P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\delta\theta}(p)}{|\mathbf{k}-\mathbf{p}|^2}\right] \tag{5.35}$$

$$S_1(\mathbf{k},\mathbf{p}) = \frac{k^5}{p^2|\mathbf{k}-\mathbf{p}|^4}P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\delta\theta}(p) \tag{5.36}$$

We use

$$\mu^3 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) = \mu^4 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^3 G_{31}(x) S_3(k,r,x)$$
$$+ \mu^6 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^3 G_{33}(x) S_3(k,r,x)$$
$$\mu^4 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) = \mu^4 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^2 G_{20}(x) S_2(k,r,x)$$
$$+ \mu^6 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^2 G_{22}(x) S_2(k,r,x)$$
$$\mu^5 \int \frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}}) S_1(\mathbf{k},\mathbf{p}) = \mu^6 \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr) G_{11}(x) S_1(k,r,x) \tag{5.37}$$

Then

$$P_D^{3,uuud} = \mu^4 f_0^3 I_{2\,D}^{3,uuud}(k) + \mu^6 f_0^3 I_{3\,D}^{3,uuud}(k) \tag{5.38}$$

with

$$I_{2\,D}^{3,uuud}(k) = \int \frac{d^3p}{(2\pi)^3}\Big[p^3 G_{31}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}) S_3(\mathbf{k},\mathbf{p}) + p^2 G_{20}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}) S_2(\mathbf{k},\mathbf{p})\Big] \tag{5.39}$$
$$I_{3\,D}^{3,uuud}(k) = \int \frac{d^3p}{(2\pi)^3}\Big[p^3 G_{33}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}) S_3(\mathbf{k},\mathbf{p}) + p^2 G_{22}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}) S_2(\mathbf{k},\mathbf{p})$$
$$+ p\, G_{11}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}) S_1(\mathbf{k},\mathbf{p})\Big] \tag{5.40}$$

or

$$I_{2\,D}^{3,uuud}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx \Big[(kr)^3 G_{31}(x) S_3(k,r,x) + (kr)^2 G_{20}(x) S_2(k,r,x)\Big] \tag{5.41}$$
$$I_{3\,D}^{3,uuud}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx \Big[(kr)^3 G_{33}(x) S_3(k,r,x) + (kr)^2 G_{22}(x) S_2(k,r,x)$$
$$+ (kr) G_{11}(x) S_1(k,r,x)\Big] \tag{5.42}$$

or

$$I_{2\,D}^{3,uuud}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, \mathcal{I}_{2\,D}^{3,uuud}(k,r,x) P_L(kr) P_L(k\sqrt{1+r^2-2rx}) \tag{5.43}$$
$$I_{3\,D}^{3,uuud}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, \mathcal{I}_{3\,D}^{3,uuud}(k,r,x) P_L(kr) P_L(k\sqrt{1+r^2-2rx}) \tag{5.44}$$

with

$$\mathcal{I}_{2\,D}^{3,uuud}(k,r,x) = \frac{G_1(p)G_1(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2}\Big[\frac{1}{2}(1+r^2-2rx)(1-3rx)(1-x^2) G_1(p)$$
$$- \frac{r^2}{2}(2-3rx)(1-x^2) G_1(|\mathbf{k}-\mathbf{p}|)\Big] \tag{5.45}$$
$$\mathcal{I}_{3\,D}^{3,uuud}(k,r,x) = \frac{G_1(p)G_1(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2}\Big[-\frac{1}{2}(1+r^2-2rx)(1-3rx-3x^2+5rx^3) G_1(p)$$
$$+ \frac{1}{2}(2r^2+2rx-3r^3x-6r^2x^2+5r^3x^3) G_1(|\mathbf{k}-\mathbf{p}|)\Big] \tag{5.46}$$

) In a TNS-like form

$$
\begin{aligned}
I^{3,uuud}_{2\,D}(k) = &-\frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, D^2_{21}(k,r,x)\frac{P_{\theta\theta}(k\sqrt{1+r^2-2rx})P_{\delta\theta}(kr)}{(1+r^2-2rx)^2} \\
&-\frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, D^2_{12}(k,r,x)\frac{P_{\delta\theta}(k\sqrt{1+r^2-2rx})P_{\theta\theta}(kr)}{(1+r^2-2rx)} \qquad (5.47)
\end{aligned}
$$

$$
\begin{aligned}
I^{3,uuud}_{3\,D}(k) = &-\frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, D^3_{21}(k,r,x)\frac{P_{\theta\theta}(k\sqrt{1+r^2-2rx})P_{\delta\theta}(kr)}{(1+r^2-2rx)^2} \\
&-\frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, D^3_{12}(k,r,x)\frac{P_{\delta\theta}(k\sqrt{1+r^2-2rx})P_{\theta\theta}(kr)}{(1+r^2-2rx)} \qquad (5.48)
\end{aligned}
$$

with

$$D^2_{21}(k,r,x) = \frac{r^2}{2}(2-3rx)(1-x^2) \qquad (5.49)$$

$$D^2_{12}(k,r,x) = -\frac{1}{2}(1-3rx)(1-x^2) \qquad (5.50)$$

$$D^3_{21}(k,r,x) = -\frac{1}{2}(2r^2+2rx-3r^3x-6r^2x^2+5r^3x^3) \qquad (5.51)$$

$$D^3_{12}(k,r,x) = \frac{1}{2}(1-3rx-3x^2+5rx^3) \qquad (5.52)$$

5.2.1 Splitting in P2uuudD B and C

I will do for C function, and check that $B = D - C$, with B that given in TNS paper.

$$
\begin{aligned}
P^{3,uuud}_{D} &= P^{3,uuud}_{B} + P^{3,uuud}_{C} \\
P^{3,uuud}_{B} &= 2(k\mu)^2 f_0^3\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^3}{|\mathbf{k}-\mathbf{p}|^4}P_{\delta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\
P^{3,uuud}_{C} &= 2(k\mu)^2 f_0^3\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) \qquad (5.53)
\end{aligned}
$$

Hence, symmetrizing $P^{3,uuud}_{C}$ we have

$$
\begin{aligned}
P^{3,uuud}_{C} &= (k\mu)^2 f_0^3\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&\quad + (k\mu)^2 f_0^3\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^2}\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2}{|\mathbf{k}-\mathbf{p}|^4}P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\delta\theta}(p) \\
&= \mu^2 f_0^3\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^4 S_4(\mathbf{k},\mathbf{p}) + \mu^3 f_0^3\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) \\
&\quad + \mu^4 f_0^3\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}), \qquad (5.54)
\end{aligned}
$$

with

$$S_4(\mathbf{k},\mathbf{p}) = \frac{1}{p^2k^2}\frac{P_{\delta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{k^{-4}|\mathbf{k}-\mathbf{p}|^4} + \frac{1}{p^4}\frac{P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\theta\theta}(p)}{k^{-2}|\mathbf{k}-\mathbf{p}|^2} \tag{5.55}$$

$$S_3(\mathbf{k},\mathbf{p}) = -\frac{2}{p^2k}\frac{P_{\delta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{k^{-4}|\mathbf{k}-\mathbf{p}|^4} - \frac{2k}{p^4}\frac{P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\theta\theta}(p)}{k^{-2}|\mathbf{k}-\mathbf{p}|^2} \tag{5.56}$$

$$S_2(\mathbf{k},\mathbf{p}) = \frac{1}{p^2}\frac{P_{\delta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{k^{-4}|\mathbf{k}-\mathbf{p}|^4} + \frac{k^2}{p^4}\frac{P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)P_{\theta\theta}(p)}{k^{-2}|\mathbf{k}-\mathbf{p}|^2} \tag{5.57}$$

We use

$$\begin{aligned}\mu^2\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^4 S_4(\mathbf{k},\mathbf{p}) &= \mu^2\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx (kr)^4 G_{40}(x)S_4(k,r,x)\\ &\quad+\mu^4\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx (kr)^4 G_{42}(x)S_4(k,r,x)\\ &\quad+\mu^6\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx (kr)^4 G_{44}(x)S_4(k,r,x)\end{aligned} \tag{5.58}$$

$$\begin{aligned}\mu^3\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) &= \mu^4\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx (kr)^3 G_{31}(x)S_3(k,r,x)\\ &\quad+\mu^6\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx (kr)^3 G_{33}(x)S_3(k,r,x)\end{aligned} \tag{5.59}$$

$$\begin{aligned}\mu^4\int\frac{d^3p}{(2\pi)^3}(\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) &= \mu^4\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx (kr)^2 G_{20}(x)S_2(k,r,x)\\ &\quad+\mu^6\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx (kr)^2 G_{22}(x)S_2(k,r,x)\end{aligned} \tag{5.60}$$

Then

$$P_C^{3,uuud} = \mu^2 f_0^3 I_{1\,C}^{3,uuud}(k) + \mu^4 f_0^3 I_{2\,C}^{3,uuud}(k) + \mu^6 f_0^3 I_{3\,C}^{3,uuud}(k) \tag{5.61}$$

with

$$I_{1\,C}^{3,uuud}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx (kr)^4 G_{40}(x)S_4(k,r,x) \tag{5.62}$$

$$\begin{aligned}I_{2\,C}^{3,uuud}(k) &= \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx\big[(kr)^4 G_{42}(x)S_4(k,r,x) + (kr)^3 G_{31}(x)S_3(k,r,x)\\ &\qquad + (kr)^2 G_{20}(x)S_2(k,r,x)\big]\end{aligned} \tag{5.63}$$

$$\begin{aligned}I_{3\,C}^{3,uuud}(k) &= \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2\int_{-1}^1 dx\big[(kr)^4 G_{44}(x)S_4(k,r,x) + (kr)^3 G_{33}(x)S_3(k,r,x)\\ &\qquad + (kr)^2 G_{22}(x)S_2(k,r,x)\big]\end{aligned} \tag{5.64}$$

$$\tag{5.65}$$

Now the first term, $I_{1\,C}^{3,uuud}(k)$ is

$$\begin{aligned}I_{1\,C}^{3,uuud}(k) &= -\frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, C_{21}^1(k,r,x)\frac{P_{\theta\theta}(k\sqrt{1+r^2-2rx})P_{\delta\theta}(kr)}{(1+r^2-2rx)^2}\\ &\quad-\frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, C_{12}^1(k,r,x)\frac{P_{\delta\theta}(k\sqrt{1+r^2-2rx})P_{\theta\theta}(kr)}{(1+r^2-2rx)}\end{aligned} \tag{5.66}$$

with

$$C^1_{21}(r,x) = -\frac{3r^4}{8}(x^2-1)^2 \tag{5.67}$$

$$C^1_{12}(r,x) = -\frac{3r^2}{8}(x^2-1)^2 \tag{5.68}$$

We have $C^1_{21} = -B^{1\,\text{TNS paper}}_{21}$ and $C^1_{12} = -B^{1\,\text{TNS paper}}_{12}$, consistent with $D^1_{21} = D^1_{12} = 0$ since $P^{3,uud}$ only has powers μ^4 and μ^6 and not μ^2 terms.

Now the term $I^{3,uuud}_{2\,C}(k)$ is

$$\begin{aligned} I^{3,uuud}_{2\,C}(k) = &-\frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, C^2_{21}(k,r,x)\frac{P_{\theta\theta}(k\sqrt{1+r^2-2rx})P_{\delta\theta}(kr)}{(1+r^2-2rx)^2} \\ &-\frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, C^2_{12}(k,r,x)\frac{P_{\delta\theta}(k\sqrt{1+r^2-2rx})P_{\theta\theta}(kr)}{(1+r^2-2rx)} \end{aligned} \tag{5.69}$$

with

$$C^2_{21}(r,x) = -\frac{r^2}{4}(1-x^2)(2-3r^2-12rx+15r^2x^2) \tag{5.70}$$

$$C^2_{12}(r,x) = -\frac{1}{4}(1-x^2)(2-3r^2-12rx+15r^2x^2) \tag{5.71}$$

We actually obtain the TNS paper results for B:

$$B^2_{21} = D^2_{21} - C^2_{21} = \frac{3}{4}r^2(-1+x^2)(-2+r^2+6rx-5r^2x^2) \tag{5.72}$$

$$B^2_{12} = D^2_{12} - C^2_{12} = -\frac{3}{4}r(-1+x^2)(-r-2x+5rx^2) \tag{5.73}$$

Now the term $I^{3,uuud}_{3\,C}(k)$ is

$$\begin{aligned} I^{3,uuud}_{3\,C}(k) = &-\frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, C^3_{21}(k,r,x)\frac{P_{\theta\theta}(k\sqrt{1+r^2-2rx})P_{\delta\theta}(kr)}{(1+r^2-2rx)^2} \\ &-\frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, C^3_{12}(k,r,x)\frac{P_{\delta\theta}(k\sqrt{1+r^2-2rx})P_{\theta\theta}(kr)}{(1+r^2-2rx)} \end{aligned} \tag{5.74}$$

with

$$C^3_{21}(r,x) = -\frac{r^2}{8}(-4+3r^2+24rx+12x^2-30r^2x^2-40rx^3+35r^2x^4) \tag{5.75}$$

$$C^3_{12}(r,x) = -\frac{1}{8}(-4+3r^2+24rx+12x^2-30r^2x^2-40rx^3+35r^2x^4) \tag{5.76}$$

And we recover the TNS paper results:

$$\begin{aligned} B^3_{21}(r,x) = D^3_{21} - C^3_{21} = \frac{r}{8}\Big\{ &-8x + r\big[-12+36x^2+12rx(3-5x^2) \\ &+ r^2(3-30x^2+35x^4)\big]\Big\} \end{aligned} \tag{5.77}$$

$$B^3_{12}(r,x) = D^3_{12} - C^3_{12} = \frac{r}{8}\big[4x(3-5x^2)+r(3-30x^2+35x^4)\big] \tag{5.78}$$

6 Fourth moment

$$P^{4,uuuu}(\mathbf{k}) = \frac{1}{4!} k_i k_j k_k k_l \tilde{\Xi}^{4,uuuu}_{ijkl}(\mathbf{k}), \tag{6.1}$$

hence

$$\begin{aligned} P^{4,uuuu}(\mathbf{k}) &= -k^2\mu^6 f_0^4 \sigma_v^2 P_{\theta\theta}(k) \\ &\quad + \frac{1}{2}(k\mu f_0)^4 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}-\mathbf{p}|^4} P_{\theta\theta}(p) P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) && (6.2) \\ &= -k^2\mu^6 f_0^4 \sigma_v^2 P_{\theta\theta}(k) + P_D^{4,uuuu}(\mathbf{k}) && (6.3) \end{aligned}$$

with the integrand of $P_D^{4,uuuu}(\mathbf{k})$ already symmetic under the interchange $\mathbf{p} \to \mathbf{k}-\mathbf{p}$. Now

$$\begin{aligned} P_D^{4,uuuu}(\mathbf{k}) &= \frac{1}{2}(k\mu f_0)^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^2 \left[(k\mu)^2 - 2k\mu(\mathbf{p}\cdot\hat{\mathbf{n}}) + (\mathbf{p}\cdot\hat{\mathbf{n}})^2\right] \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^4 p^4} \\ &= \mu^4 f_0^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^4 S_4(\mathbf{k},\mathbf{p}) + \mu^5 f_0^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) \\ &\quad + \mu^6 f_0^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) && (6.4) \end{aligned}$$

with

$$\begin{aligned} S_4(\mathbf{k},\mathbf{p}) &= \frac{1}{2p^4} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^4 k^{-4}} && (6.5) \\ S_3(\mathbf{k},\mathbf{p}) &= \frac{-k}{p^4} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^4 k^{-4}} && (6.6) \\ S_2(\mathbf{k},\mathbf{p}) &= \frac{k^2}{2p^4} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^4 k^{-4}} && (6.7) \end{aligned}$$

We use

$$\begin{aligned} \mu^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^4 S_4(\mathbf{k},\mathbf{p}) &= \mu^4 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^4 G_{40}(x) S_4(k,r,x) \\ &\quad + \mu^6 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^4 G_{42}(x) S_4(k,r,x) \\ &\quad + \mu^8 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^4 G_{44}(x) S_4(k,r,x) && (6.8) \\ \mu^5 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) &= \mu^6 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^3 G_{31}(x) S_3(k,r,x) \\ &\quad + \mu^8 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^3 G_{33}(x) S_3(k,r,x) && (6.9) \\ \mu^6 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) &= \mu^6 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^2 G_{20}(x) S_2(k,r,x) \\ &\quad + \mu^8 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^2 G_{22}(x) S_2(k,r,x) && (6.10) \end{aligned}$$

Then

$$P_D^{4,uuuu} = \mu^4 f_0^4 I_{2\,D}^{4,uuuu}(k) + \mu^6 f_0^3 I_{3\,D}^{4,uuuu}(k) + \mu^8 f_0^3 I_{4\,D}^{4,uuuu}(k) \tag{6.11}$$

with

$$I_{2\,D}^{4,uuuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx (kr)^4 G_{40}(x) S_4(k,r,x) \tag{6.12}$$

$$\begin{aligned} I_{3\,D}^{4,uuuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx \big[&(kr)^4 G_{42}(x) S_4(k,r,x) + (kr)^3 G_{31}(x) S_3(k,r,x) \\ &+ (kr)^2 G_{20}(x) S_2(k,r,x)\big] \end{aligned} \tag{6.13}$$

$$\begin{aligned} I_{4\,D}^{4,uuuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^1 dx \big[&(kr)^4 G_{44}(x) S_4(k,r,x) + (kr)^3 G_{33}(x) S_3(k,r,x) \\ &+ (kr)^2 G_{22}(x) S_2(k,r,x)\big] \end{aligned} \tag{6.14}$$

$$\tag{6.15}$$

or in TNS form

$$I_{2\,D}^{4,uuuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, D_{22}^2(r,x) \frac{P_{\theta\theta}(k\sqrt{1+r^2-2rx}) P_{\theta\theta}(kr)}{(1+r^2-2rx)^2}, \tag{6.16}$$

$$I_{3\,D}^{4,uuuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, D_{22}^3(r,x) \frac{P_{\theta\theta}(k\sqrt{1+r^2-2rx}) P_{\theta\theta}(kr)}{(1+r^2-2rx)^2}, \tag{6.17}$$

$$I_{4\,D}^{4,uuuu}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, D_{22}^4(r,x) \frac{P_{\theta\theta}(k\sqrt{1+r^2-2rx}) P_{\theta\theta}(kr)}{(1+r^2-2rx)^2}, \tag{6.18}$$

with

$$D_{22}^2(r,x) = \frac{3r^2}{16}(1-x^2)^2 \tag{6.19}$$

$$D_{22}^3(r,x) = \frac{1}{8}(1-x^2)(2-3r^2-12rx+15r^2x^2) \tag{6.20}$$

$$D_{22}^4(r,x) = \frac{1}{16}\big[-4+12x^2+8rx(3-5x^2)+r^2(3-30x^2+35x^4)\big] \tag{6.21}$$

6.1 Splitting P4uuuuD in B and C

$$P_D^{4,uuuu}(\mathbf{k}) = P_B^{4,uuuu}(\mathbf{k}) + P_C^{4,uuuu}(\mathbf{k}) \tag{6.22}$$

$$P_B^{4,uuuu}(\mathbf{k}) = (k\mu)^2 f_0^4 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^3}{|\mathbf{k}-\mathbf{p}|^4} P_{\theta\theta}(p) P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \tag{6.23}$$

$$P_C^{4,uuuu}(\mathbf{k}) = (k\mu)^2 f_0^4 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^4}{|\mathbf{k}-\mathbf{p}|^4} P_{\theta\theta}(p) P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \tag{6.24}$$

We will compute C and then use B of TNS paper and check if we recover D.

$$
\begin{aligned}
P_C^{4,uuuu}(\mathbf{k}) &= (k\mu)^2 f_0^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^2 \big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^4 \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{p^4|\mathbf{k}-\mathbf{p}|^4} \\
&= \frac{1}{2}(k\mu)^2 f_0^4 \int \frac{d^3p}{(2\pi)^3} \Big\{ (\mathbf{p}\cdot\hat{\mathbf{n}})^2 \big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^4 + (\mathbf{p}\cdot\hat{\mathbf{n}})^4 \big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2 \Big\} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{p^4|\mathbf{k}-\mathbf{p}|^4} \\
&= \frac{1}{2}(k\mu)^2 f_0^4 \int \frac{d^3p}{(2\pi)^3} \Big\{ (k\mu)^4(\mathbf{p}\cdot\hat{\mathbf{n}})^2 - 4(k\mu)^3(\mathbf{p}\cdot\hat{\mathbf{n}})^3 + 7(k\mu)^2(\mathbf{p}\cdot\hat{\mathbf{n}})^4 \\
&\qquad - 6(k\mu)(\mathbf{p}\cdot\hat{\mathbf{n}})^5 + 2(\mathbf{p}\cdot\hat{\mathbf{n}})^6 \Big\} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{p^4|\mathbf{k}-\mathbf{p}|^4} \qquad (6.25) \\
&= \mu^2 f_0^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^6 S_6 + \mu^3 f_0^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^5 S_5 + \mu^4 f_0^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^4 S_4 \\
&\quad + \mu^5 f_0^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3 + \mu^6 f_0^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2 \qquad (6.26)
\end{aligned}
$$

with

$$S_6(\mathbf{k},\mathbf{p}) = \frac{1}{k^2p^4} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^4 k^{-4}}, \qquad (6.27)$$

$$S_5(\mathbf{k},\mathbf{p}) = \frac{-3}{kp^4} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^4 k^{-4}}, \qquad (6.28)$$

$$S_4(\mathbf{k},\mathbf{p}) = \frac{7}{2p^4} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^4 k^{-4}}, \qquad (6.29)$$

$$S_3(\mathbf{k},\mathbf{p}) = \frac{-2k}{p^4} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^4 k^{-4}}, \qquad (6.30)$$

$$S_2(\mathbf{k},\mathbf{p}) = \frac{k^2}{2p^4} \frac{P_{\theta\theta}(p)P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)}{|\mathbf{k}-\mathbf{p}|^4 k^{-4}}. \qquad (6.31)$$

We use

$$\mu^2 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^6 S_6(\mathbf{k},\mathbf{p}) = \mu^2 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^6 G_{60}(x) S_6(k,r,x)$$
$$+ \mu^4 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^5 G_{62}(x) S_6(k,r,x)$$
$$+ \mu^6 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^5 G_{64}(x) S_6(k,r,x)$$
$$+ \mu^8 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^5 G_{66}(x) S_6(k,r,x) \tag{6.32}$$

$$\mu^3 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^5 S_5(\mathbf{k},\mathbf{p}) = \mu^4 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^5 G_{51}(x) S_5(k,r,x)$$
$$+ \mu^6 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^5 G_{53}(x) S_5(k,r,x)$$
$$+ \mu^8 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^5 G_{55}(x) S_5(k,r,x) \tag{6.33}$$

$$\mu^4 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^4 S_4(\mathbf{k},\mathbf{p}) = \mu^4 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^4 G_{40}(x) S_4(k,r,x)$$
$$+ \mu^6 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^4 G_{42}(x) S_4(k,r,x)$$
$$+ \mu^8 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^4 G_{44}(x) S_4(k,r,x) \tag{6.34}$$

$$\mu^5 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^3 S_3(\mathbf{k},\mathbf{p}) = \mu^6 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^3 G_{31}(x) S_3(k,r,x)$$
$$+ \mu^8 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^3 G_{33}(x) S_3(k,r,x) \tag{6.35}$$

$$\mu^6 \int \frac{d^3p}{(2\pi)^3} (\mathbf{p}\cdot\hat{\mathbf{n}})^2 S_2(\mathbf{k},\mathbf{p}) = \mu^6 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^2 G_{20}(x) S_2(k,r,x)$$
$$+ \mu^8 \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^2 G_{22}(x) S_2(k,r,x) \tag{6.36}$$

Then

$$P_C^{4,uuuu} = \mu^2 f_0^4 I_{1\,C}^{4,uuuu}(k) + \mu^4 f_0^4 I_{2\,C}^{4,uuuu}(k) + \mu^6 f_0^3 I_{3\,C}^{4,uuuu}(k) + \mu^8 f_0^3 I_{4\,C}^{4,uuuu}(k) \tag{6.37}$$

with

$$I^{4,uuuu}_{1\,C}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx (kr)^6 G_{60}(x) S_6(k,r,x) \tag{6.38}$$

$$\begin{aligned} I^{4,uuuu}_{2\,C}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} \big[&(kr)^6 G_{62}(x) S_6(k,r,x) + (kr)^5 G_{51}(x) S_5(k,r,x) \\ &+ (kr)^4 G_{40}(x) S_4(k,r,x)\big] \end{aligned} \tag{6.39}$$

$$\begin{aligned} I^{4,uuuu}_{3\,C}(k) = &\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx \big[(kr)^6 G_{64}(x) S_6(k,r,x) + (kr)^5 G_{53}(x) S_5(k,r,x) \\ &+ (kr)^4 G_{42}(x) S_4(k,r,x) + (kr)^3 G_{31}(x) S_3(k,r,x) + (kr)^2 G_{20}(x) S_2(k,r,x)\big] \end{aligned} \tag{6.40}$$

$$\begin{aligned} I^{4,uuuu}_{4\,C}(k) = &\frac{k^3}{4\pi^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} dx \big[(kr)^6 G_{66}(x) S_6(k,r,x) + (kr)^5 G_{55}(x) S_5(k,r,x) \\ &+ (kr)^4 G_{44}(x) S_4(k,r,x) + (kr)^3 G_{33}(x) S_3(k,r,x) + (kr)^2 G_{22}(x) S_2(k,r,x)\big] \end{aligned} \tag{6.41}$$

or in TNS form

$$I^{4,uuuu}_{n\,C}(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^{1} dx C^n_{22}(r,x) \frac{P_{\theta\theta}(kr)P_{\theta\theta}(k\sqrt{1+r^2-2rx})}{(1+r^2-2rx)^2} \tag{6.42}$$

with

$$C^1_{22} = -\frac{5r^4}{16}(-1+x^2)^3 \tag{6.43}$$

$$C^2_{22} = \frac{3r^2}{16}(-1+x^2)^2(7-5r^2-30rx+35r^2x^2) \tag{6.44}$$

$$\begin{aligned} C^3_{22} = -\frac{1}{16}(-1+x^2)\Big[4+3r\Big\{&-16x+r\big[-14+70x^2+20rx(3-7x^2) \\ &+5r^2(1-14x^2+21x^4)\big]\Big\}\Big] \end{aligned} \tag{6.45}$$

$$\begin{aligned} C^4_{22} = \frac{1}{16}\Big[&-4+12x^2+16rx(3-5x^2)+7r^2(3-30x^2+35x^4) \\ &-6r^3x(15-70x^2+63x^4)+r^4\big\{-5+21x^2(5-15x^2+11x^4)\big\}\Big] \end{aligned} \tag{6.46}$$

We remember that $D^1_{22} = 0$ as it should since $P^{4,uuuu}$ does not have μ^2 term. This is consistent because in TNS paper $C^1_{22} = -B^{1\,\text{TNS paper}}_{22}$, and hence $D^1_{22} = C^1_{22} + B^1_{22} = 0$. Indeed, using the values already found for D:

$$B^1_{22} = \frac{5r^4}{16}(-1+x^2)^3 \tag{6.47}$$

$$B^2_{22} = -\frac{3r^2}{16}(-1+x^2)^2(6-5r^2-30rx+35r^2x^2) \tag{6.48}$$

$$B^3_{22} = \frac{3r}{16}(x^2-1)\Big[-8x+r\big\{-12+60x^2+20rx(3-7x^2)+5r^2(1-14x^2+21x^4)\big\}\Big] \tag{6.49}$$

$$\begin{aligned} B^4_{22} = \frac{r}{16}\Big[&8x(-3+5x^2)-6r(3-30x^2+35x^4)+6r^2x(15-70x^2+63x^4) \\ &+r^3\big\{(5-21x^2(5-15x^2+11x^4)\big\}\Big] \end{aligned} \tag{6.50}$$

We recover $B_{22}^{1,2,3,4} = B_{22}^{1,2,3,4\,\text{TNS paper}}$.

7 Summary

7.1 A functions

$$
\begin{aligned}
I^{1,udd}(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx \left\{ A_{11}(\mathbf{k},\mathbf{p})P_L(k) + \tilde{A}_{11}(\mathbf{k},\mathbf{p})P_L(kr)\right\} \frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2} \\
&+ \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, a_{11}(\mathbf{k},\mathbf{p})P_L(k)P_L(kr) \qquad (7.1)
\end{aligned}
$$

with

$$
\frac{A_{11}(\mathbf{k},\mathbf{p})}{(1+r^2-2rx)^2} = 2\left[G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})rx + F_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(|\mathbf{k}-\mathbf{p}|)\frac{r^2(1-rx)}{1+r^2-2rx}\right] \qquad (7.2)
$$

$$
\frac{\tilde{A}_{11}(\mathbf{k},\mathbf{p})}{(1+r^2-2rx)^2} = 2F_2(\mathbf{p},\mathbf{k}-\mathbf{p})\left[G_1(p)rx + G_1(|\mathbf{k}-\mathbf{p}|)\frac{r^2(1-rx)}{1+r^2-2rx}\right] \qquad (7.3)
$$

$$
a_{11}(\mathbf{k},\mathbf{p}) = 2\left[F_2(-\mathbf{k},\mathbf{p})G_1(p)rx + G_2(-\mathbf{k},-\mathbf{p})\frac{r^2(1-rx)}{1+r^2-2rx}\right] \qquad (7.4)
$$

$$
\begin{aligned}
I_1^{2,uud}(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx \left\{ A_{12}(\mathbf{k},\mathbf{p})P_L(k) + \tilde{A}_{12}(\mathbf{k},\mathbf{p})P_L(kr)\right\} \frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2} \\
&+ \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, a_{12}(\mathbf{k},\mathbf{p})P_L(k)P_L(kr) \qquad (7.5)
\end{aligned}
$$

with

$$
\frac{A_{12}(\mathbf{k},\mathbf{p})}{(1+r^2-2rx)^2} = -\frac{r^2(1-x^2)}{1+r^2-2rx}G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(|\mathbf{k}-\mathbf{p}|) \qquad (7.6)
$$

$$
\frac{\tilde{A}_{12}(\mathbf{k},\mathbf{p})}{(1+r^2-2rx)^2} = -\frac{r^2(1-x^2)}{1+r^2-2rx}F_2(\mathbf{p},\mathbf{k}-\mathbf{p})G_1(p)G_1(|\mathbf{k}-\mathbf{p}|) \qquad (7.7)
$$

$$
a_{12}(\mathbf{k},\mathbf{p}) = -\frac{r^2(1-x^2)}{1+r^2-2rx}G_2(\mathbf{p},-\mathbf{k})G_1(p) \qquad (7.8)
$$

$$
\begin{aligned}
I_2^{2,uud}(\mathbf{k},\mathbf{p}) &= \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx \left\{ A_{22}(\mathbf{k},\mathbf{p})P_L(k) + \tilde{A}_{22}(\mathbf{k},\mathbf{p})P_L(kr)\right\} \frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2} \\
&+ \frac{k^3}{4\pi^2}\int_0^\infty dr \int_{-1}^1 dx\, a_{22}(\mathbf{k},\mathbf{p})P_L(k)P_L(kr) \qquad (7.9)
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{A_{22}(k,x,r)}{(1+r^2-2rx)^2} &= \left[\frac{r^2(1-3x^2)+2xr}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)+2xrG_1(k)\right]G_2(-\mathbf{k},\mathbf{k}-\mathbf{p}) \\
&\quad + \frac{2r^2(1-rx)}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)G_1(k)F_2(-\mathbf{k},\mathbf{k}-\mathbf{p}), \qquad (7.10)\\
\frac{\tilde{A}_{22}(k,x,r)}{(1+r^2-2rx)^2} &= \left[\frac{2r^2(1-rx)}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)+2xrG_1(p)\right]G_2(\mathbf{p},\mathbf{k}-\mathbf{p}) \\
&\quad + \frac{r^2(1-3x^2)+2xr}{1+r^2-2rx}G_1(|\mathbf{k}-\mathbf{p}|)G_1(p)F_2(\mathbf{p},\mathbf{k}-\mathbf{p}), \qquad (7.11)\\
a_{22}(k,x,r) &= \left[\frac{r^2(1-3x^2)+2xr}{1+r^2-2rx}G_1(p)+\frac{2r^2(1-rx)}{1+r^2-2rx}G_1(k)\right]G_2(-\mathbf{k},\mathbf{p}) \\
&\quad + 2xrG_1(p)G_1(k)F_2(-\mathbf{k},\mathbf{p}). \qquad (7.12)
\end{aligned}
$$

$$
\begin{aligned}
I_2^{3,uuu}(k) &= \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\left\{A_{23}(\mathbf{k},\mathbf{p})P_L(k)+\tilde{A}_{23}(\mathbf{k},\mathbf{p})P_L(kr)\right\}\frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2} \\
&\quad + \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, a_{23}(\mathbf{k},\mathbf{p})P_L(k)P_L(kr) \qquad (7.13)
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{A_{23}(k,x,r)}{(1+r^2-2rx)^2} &= \frac{r^2(x^2-1)}{1+r^2-2rx}G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(k)G_1(|\mathbf{k}-\mathbf{p}|), \qquad (7.14)\\
\frac{\tilde{A}_{23}(k,x,r)}{(1+r^2-2rx)^2} &= \frac{r^2(x^2-1)}{1+r^2-2rx}G_2(\mathbf{p},\mathbf{k}-\mathbf{p})G_1(p)G_1(|\mathbf{k}-\mathbf{p}|), \qquad (7.15)\\
a_{23}(k,x,r) &= \frac{r^2(x^2-1)}{1+r^2-2rx}G_2(-\mathbf{k},\mathbf{p})G_1(k)G_1(p). \qquad (7.16)
\end{aligned}
$$

$$
\begin{aligned}
I_3^{3,uuu}(k) &= \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\left\{A_{33}(\mathbf{k},\mathbf{p})P_L(k)+\tilde{A}_{33}(\mathbf{k},\mathbf{p})P_L(kr)\right\}\frac{P_L(|\mathbf{k}-\mathbf{p}|)}{(1+r^2-2rx)^2} \\
&\quad + \frac{k^3}{4\pi^2}\int_0^\infty dr\int_{-1}^1 dx\, a_{33}(\mathbf{k},\mathbf{p})P_L(k)P_L(kr) \qquad (7.17)
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{A_{33}(k,x,r)}{(1+r^2-2rx)^2} &= \frac{r^2(1-3x^2)+2rx}{1+r^2-2rx}G_2(-\mathbf{k},\mathbf{k}-\mathbf{p})G_1(k)G_1(|\mathbf{k}-\mathbf{p}|), \qquad (7.18)\\
\frac{\tilde{A}_{33}(k,x,r)}{(1+r^2-2rx)^2} &= \frac{r^2(1-3x^2)+2rx}{1+r^2-2rx}G_2(\mathbf{p},\mathbf{k}-\mathbf{p})G_1(p)G_1(|\mathbf{k}-\mathbf{p}|), \qquad (7.19)\\
a_{33}(k,x,r) &= \frac{r^2(1-3x^2)+2rx}{1+r^2-2rx}G_2(-\mathbf{k},\mathbf{p})G_1(k)G_1(p). \qquad (7.20)
\end{aligned}
$$

A Bispectra

Now, we want expressions for the bispectrum

$$\begin{aligned} Q =&(2\pi)^3\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_{\theta\delta\delta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \langle\theta(\mathbf{k}_1)\delta(\mathbf{k}_2)\delta(\mathbf{k}_3)\rangle \\ &= \langle\theta^{(2)}(\mathbf{k}_1)\delta^{(1)}(\mathbf{k}_2)\delta^{(1)}(\mathbf{k}_3)\rangle + \langle\theta^{(1)}(\mathbf{k}_1)\delta^{(2)}(\mathbf{k}_2)\delta^{(1)}(\mathbf{k}_3)\rangle + \langle\theta^{(1)}(\mathbf{k}_1)\delta^{(1)}(\mathbf{k}_2)\delta^{(2)}(\mathbf{k}_3)\rangle \\ &\equiv Q_1+Q_2+Q_3 = Q_1(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)+Q_2(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)+Q_2(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2) \end{aligned} \tag{A.1}$$

$$\begin{aligned} Q_1 &= \langle\theta^{(2)}(\mathbf{k}_1)\delta^{(1)}(\mathbf{k}_2)\delta^{(1)}(\mathbf{k}_3)\rangle \\ &= \int\frac{d^3p_1d^3p_2}{(2\pi)^6}(2\pi)^3\delta_{\rm D}(\mathbf{k}_1-\mathbf{p}_1-\mathbf{p}_2)G_2(\mathbf{p}_1,\mathbf{p}_2)\langle\delta^{(1)}(\mathbf{p}_1)\delta^{(1)}(\mathbf{p}_2)\delta^{(1)}(\mathbf{k}_2)\delta^{(1)}(\mathbf{k}_3)\rangle \\ &= 2\int\frac{d^3p_1d^3p_2}{(2\pi)^6}(2\pi)^9\delta_{\rm D}(\mathbf{k}_1-\mathbf{p}_1-\mathbf{p}_2)\delta_{\rm D}(\mathbf{p}_1+\mathbf{k}_2)\delta_{\rm D}(\mathbf{p}_2+\mathbf{k}_3)G_2(\mathbf{p}_1,\mathbf{p}_2)P_L(p_1)P_L(p_2) \\ &= (2\pi)^3\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)2G_2(\mathbf{k}_2,\mathbf{k}_3)P_L(k_2)P_L(k_3) \end{aligned} \tag{A.2}$$

where we used $G_2(\mathbf{k}_2,\mathbf{k}_3)=G_2(\mathbf{k}_2,\mathbf{k}_3)$

$$\begin{aligned} Q_2 &= \langle\theta^{(1)}(\mathbf{k}_1)\delta^{(2)}(\mathbf{k}_2)\delta^{(1)}(\mathbf{k}_3)\rangle \\ &= \int\frac{d^3p_1d^3p_2}{(2\pi)^3}\delta_{\rm D}(\mathbf{k}_2-\mathbf{p}_1-\mathbf{p}_2)F_2(\mathbf{p}_1,\mathbf{p}_2)G_1(\mathbf{k}_1)\langle\delta^{(1)}(\mathbf{k}_1)\delta^{(1)}(\mathbf{p}_1)\delta^{(1)}(\mathbf{p}_2)\delta^{(1)}(\mathbf{k}_3)\rangle \\ &= 2\int\frac{d^3p_1d^3p_2}{(2\pi)^6}(2\pi)^9\delta_{\rm D}(\mathbf{k}_1-\mathbf{p}_1-\mathbf{p}_2)\delta_{\rm D}(\mathbf{p}_1+\mathbf{k}_1)\delta_{\rm D}(\mathbf{p}_2+\mathbf{k}_3) \\ &\quad\times F_2(\mathbf{p}_1,\mathbf{p}_2)G_1(\mathbf{k}_1)P_L(p_1)P_L(p_2) \\ &= (2\pi)^3\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)2F_2(\mathbf{k}_1,\mathbf{k}_3)G_2(k_1)P_L(k_1)P_L(k_3) \end{aligned} \tag{A.3}$$

and

$$Q_3 = (2\pi)^3\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)2F_2(\mathbf{k}_1,\mathbf{k}_2)G_2(k_1)P_L(k_1)P_L(k_2) \tag{A.4}$$

Hence the bispectrum is

$$\boxed{\begin{aligned} B_{\theta\delta\delta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = &\quad 2G_2(\mathbf{k}_2,\mathbf{k}_3)P_L(k_2)P_L(k_3) \\ &+2F_2(\mathbf{k}_1,\mathbf{k}_3)G_1(k_1)P_L(k_1)P_L(k_3) \\ &+2F_2(\mathbf{k}_1,\mathbf{k}_2)G_1(k_1)P_L(k_1)P_L(k_2) \end{aligned}} \tag{A.5}$$

The other bispectra involved are

$$\boxed{\begin{aligned} B_{\theta\theta\delta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = &\quad 2G_2(\mathbf{k}_2,\mathbf{k}_3)G_1(k_2)P_L(k_2)P_L(k_3) \\ &+2G_2(\mathbf{k}_1,\mathbf{k}_3)G_1(k_1)P_L(k_1)P_L(k_3) \\ &+2F_2(\mathbf{k}_1,\mathbf{k}_2)G_1(k_1)G_1(k_2)P_L(k_1)P_L(k_2), \end{aligned}} \tag{A.6}$$

$$\boxed{\begin{aligned} B_{\theta\delta\theta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = &\quad 2G_2(\mathbf{k}_2,\mathbf{k}_3)G_1(k_3)P_L(k_2)P_L(k_3) \\ &+2F_2(\mathbf{k}_1,\mathbf{k}_3)G_1(k_1)G_1(k_3)P_L(k_1)P_L(k_3) \\ &+2G_2(\mathbf{k}_1,\mathbf{k}_2)G_1(k_1)P_L(k_1)P_L(k_2), \end{aligned}} \tag{A.7}$$

and

$$\boxed{\begin{aligned} B_{\theta\theta\theta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = &\quad 2G_2(\mathbf{k}_2,\mathbf{k}_3)G_1(k_2)G_1(k_3)P_L(k_2)P_L(k_3) \\ &+ 2G_2(\mathbf{k}_1,\mathbf{k}_3)G_1(k_1)G_1(k_3)P_L(k_1)P_L(k_3) \\ &+ 2G_2(\mathbf{k}_1,\mathbf{k}_2)G_1(k_1)G_1(k_2)P_L(k_1)P_L(k_2), \end{aligned}} \tag{A.8}$$

B Evaluation of kernels

$$\begin{aligned} G_2(\mathbf{k}_1,\mathbf{k}_2) = &\frac{3}{14f_0}\mathcal{A}(f_1+f_2) + \frac{3\dot{\mathcal{A}}}{14f_0H} + \left(\frac{f_1+f_2}{2} - \frac{3}{14}\mathcal{B}(f_1+f_2) - \frac{3\dot{\mathcal{B}}}{14H}\right)\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{f_0k_1^2k_2^2} \\ &+ \frac{\mathbf{k}_1\cdot\mathbf{k}_2}{2f_0k_1k_2}\left(f_1\frac{k_2}{k_1} + f_2\frac{k_1}{k_2}\right) \end{aligned} \tag{B.1}$$

New evaluation:

$$\begin{aligned} G_2(-\mathbf{k},\mathbf{k}-\mathbf{p}) = &\frac{3}{14f_0}\mathcal{A}(f_\mathbf{k}+f_{\mathbf{k}-\mathbf{p}}) + \frac{3\mathcal{A}'}{14f_0} \\ &+ \left(\frac{f_\mathbf{k}+f_{\mathbf{k}-\mathbf{p}}}{2} - \frac{3}{14}\mathcal{B}(f_\mathbf{k}+f_{\mathbf{k}-\mathbf{p}}) - \frac{3\mathcal{B}'}{14}\right)\frac{(1-rx)^2}{f_0(1+r^2-2rx)} \\ &- \frac{1-rx}{2f_0\sqrt{1+r^2-2rx}}\left(f_\mathbf{k}\sqrt{1+r^2-2rx} + f_{\mathbf{k}-\mathbf{p}}/\sqrt{1+r^2-2rx}\right) \end{aligned} \tag{B.2}$$

Q evaluation:

$$\begin{aligned} F_2(\mathbf{p},\mathbf{k}-\mathbf{p}) = &\frac{1}{2} + \frac{3}{14}\mathcal{A} + \left(\frac{1}{2} - \frac{3}{14}\mathcal{B}\right)\frac{(x-r)^2}{(1+r^2-2rx)} \\ &+ \frac{x-r}{2\sqrt{1+r^2-2rx}}\left(\frac{\sqrt{1+r^2-2rx}}{r} + \frac{r}{\sqrt{1+r^2-2rx}}\right) \end{aligned} \tag{B.3}$$

$$\begin{aligned} G_2(\mathbf{p},\mathbf{k}-\mathbf{p}) = &\frac{3}{14f_0}\mathcal{A}(f_\mathbf{p}+f_{\mathbf{k}-\mathbf{p}}) + \frac{3\mathcal{A}'}{14f_0} \\ &+ \left(\frac{f_\mathbf{p}+f_{\mathbf{k}-\mathbf{p}}}{2} - \frac{3}{14}\mathcal{B}(f_\mathbf{p}+f_{\mathbf{k}-\mathbf{p}}) - \frac{3\mathcal{B}'}{14}\right)\frac{(x-r)^2}{f_0(1+r^2-2rx)} \\ &+ \frac{x-r}{2f_0\sqrt{1+r^2-2rx}}\left(f_\mathbf{p}\frac{\sqrt{1+r^2-2rx}}{r} + f_{\mathbf{k}-\mathbf{p}}\frac{r}{\sqrt{1+r^2-2rx}}\right) \end{aligned} \tag{B.4}$$

R evaluation:

$$\begin{aligned} G_2(\mathbf{k},-\mathbf{p}) = &\frac{3}{14f_0}\mathcal{A}(f_\mathbf{p}+f_\mathbf{k}) + \frac{3\mathcal{A}'}{14f_0} \\ &+ \left(\frac{f_\mathbf{p}+f_\mathbf{k}}{2} - \frac{3}{14}\mathcal{B}(f_\mathbf{p}+f_\mathbf{k}) - \frac{3\mathcal{B}'}{14}\right)\frac{x^2}{f_0} \\ &- \frac{x}{2f_0}\left(f_\mathbf{p}/r + f_\mathbf{k}r\right) \end{aligned} \tag{B.5}$$