Temas Selectos de Cosmología – Parte 2
Teoría de perturbaciones no lineales para la formación de estructura cosmológica

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Curso PCF-UNAM

Teoría de perturbaciones no lineales para la formación de estructura cosmológica

1. Ecuación de Boltzmann y jerarquía de Boltzmann
2. Momentos de la función de distribución en espacio fase: descripción de partículas y descripción de campos.
3. Ecuaciones de fluctuaciones de materia: caso lineal y no lineal.
4. Teoría perturbativa euleriana (o estándar)
5. Espectro de potencias a 1-lazo
6. Teoría efectiva y resumados al infrarrojo.
7. Teoría de sesgo galáctico

8. Ajuste a datos sintéticos y observacionales.

9. Redshift Space Distortions?

clase 1

11 de septiembre de 2022

Cosmological principle

Background: The universe is homogeneous and isotropic

Perturbations: The universe is *statistically* homogeneous and isotropic

Fourier Transform conventions

$$f(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}), \qquad f(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{k})$$

Hence

$$(2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}') = \int d^3x\, e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}$$

For a real field $f(\boldsymbol{x})$

$$f(\boldsymbol{k}) = f^*(-\boldsymbol{k})$$

Statistical homogeneity

Translation: $\hat{T}_{\boldsymbol{y}} f(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{y})$

Fourier Space:

$$\hat{T}_{\boldsymbol{y}} f(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x} - \boldsymbol{y}) \overbrace{=}^{\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{y}} \int d^3x'\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}'} f(\boldsymbol{x}') e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}$$
$$= f(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}$$

Statistical translation invariance means $\langle f(\boldsymbol{k}) f(\boldsymbol{k}')\rangle = \langle \hat{T}_{\boldsymbol{y}} f(\boldsymbol{k})\, \hat{T}_{\boldsymbol{y}} f(\boldsymbol{k}')\rangle$

Then

$$\langle f(\boldsymbol{k}) f(\boldsymbol{k}')\rangle = \langle f(\boldsymbol{k}) f(\boldsymbol{k}')\rangle e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{y}} \qquad \text{(for any } \boldsymbol{y}\text{)}$$

Hence

$$\langle f(\boldsymbol{k}) f(\boldsymbol{k}')\rangle = (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k} + \boldsymbol{k}') P_f(\boldsymbol{k})$$

Statistical Isotropy

$$\text{Rotation:} \qquad \hat{R}f(\boldsymbol{x}) = f(R^{-1}\boldsymbol{x})$$

with R a rotation matrix ($\det R = 1$ & $R^{\mathrm{T}} = R^{-1}$)

Fourier Space:

$$\hat{R}f(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(R^{-1}\boldsymbol{x}) = \int d^3x\, e^{-i(R^{-1}\boldsymbol{k})\cdot(R^{-1}\boldsymbol{x})} f(R^{-1}\boldsymbol{x})$$

$$\overbrace{=}^{\boldsymbol{x}' = R^{-1}\boldsymbol{x}} \int d^3x'\, e^{-i(R^{-1}\boldsymbol{k})\cdot\boldsymbol{x}'} f(\boldsymbol{x}') = f(R^{-1}\boldsymbol{k})$$

Statistical rotational invariance means $\langle f(\boldsymbol{k}) f(\boldsymbol{k}')\rangle = \langle f(R^{-1}\boldsymbol{k})\, f(R^{-1}\boldsymbol{k}')\rangle$

That is,

$$P_f(\boldsymbol{k})\delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}') = P_f(R^{-1}\boldsymbol{k})\delta_{\mathrm{D}}(R^{-1}\boldsymbol{k}+R^{-1}\boldsymbol{k}') \qquad (\delta_{\mathrm{D}}(R^{-1}\boldsymbol{k}) = \det R\, \delta_{\mathrm{D}}(\boldsymbol{k}) = \delta_{\mathrm{D}}(\boldsymbol{k}))$$

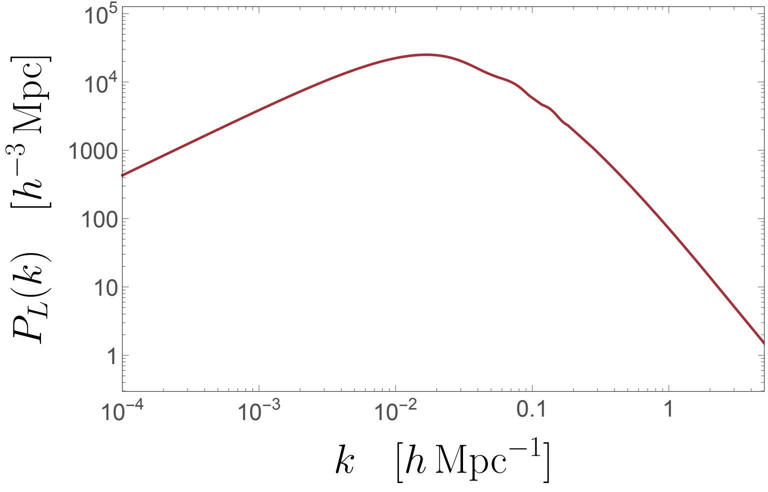
$$= P_f(R^{-1}\boldsymbol{k})\delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}') \qquad (\text{for any } R)$$

Hence, for statistically homogeneous and isotropic field f,

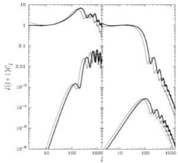
$$\boxed{\langle f(\boldsymbol{k}) f(\boldsymbol{k}')\rangle = (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}') P_f(k)}$$

Matter power spectrum

$$\langle \delta(\boldsymbol{k})\delta(\boldsymbol{k}')\rangle = (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}')P_L(k)$$



CAMB: https://camb.info/



Code for Anisotropies in the Microwave Background

by Antony Lewis and Anthony Challinor

Get help: [ ] Search Google™ Custom Search

Features:

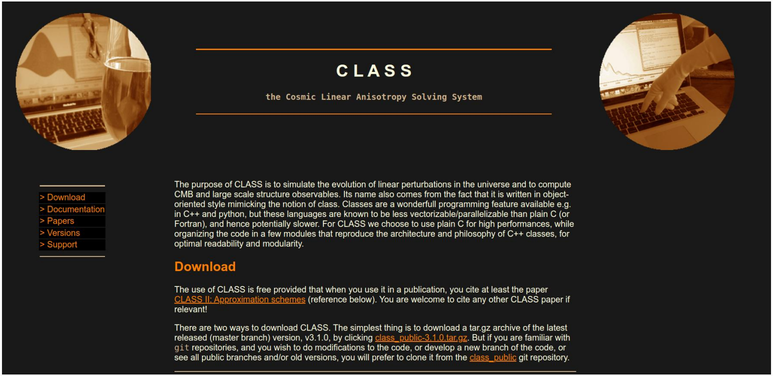
- Optimized Python and Fortran code
- Calculate CMB, lensing, source count and dark-age 21cm angular power spectra
- Matter transfer functions, power spectra, σ_8 and related quanties
- General background cosmology
- Support for closed, open and flat models
- Scalar, vector and tensor modes including polarization
- Fast computation to ~0.1% accuracy, with controllable accuracy level
- Object-oriented Python and easily-extensible modern Fortran 2008 classes
- Efficient support for massive neutrinos and arbitrary neutrino mass splittings
- Optional modelling of perturbed recombination and temperature perturbations
- Calculation of local primordial and CMB lensing bispectra (Fortran)

Download the source code and see:

- Python documentation
- Fortran documentation

See CosmoCoffee for support, and the BibTex file for references. There are also theory derivations and CAMB notes describing some conventions and approximations. The full set of linear equations

CLASS: lesgourg.github.io/class_public/class.html



CLASS

the Cosmic Linear Anisotropy Solving System

> Download
> Documentation
> Papers
> Versions
> Support

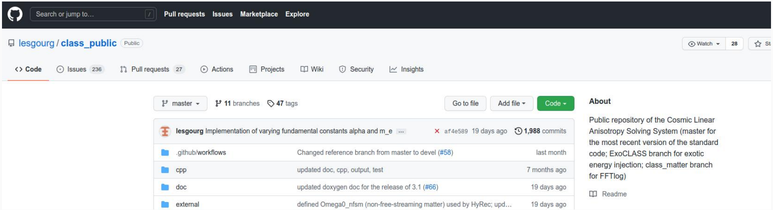
The purpose of CLASS is to simulate the evolution of linear perturbations in the universe and to compute CMB and large scale structure observables. Its name also comes from the fact that it is written in object-oriented style mimicking the notion of class. Classes are a wonderfull programming feature available e.g. in C++ and python, but these languages are known to be less vectorizable/parallelizable than plain C (or Fortran), and hence potentially slower. For CLASS we choose to use plain C for high performances, while organizing the code in a few modules that reproduce the architecture and philosophy of C++ classes, for optimal readability and modularity.

Download

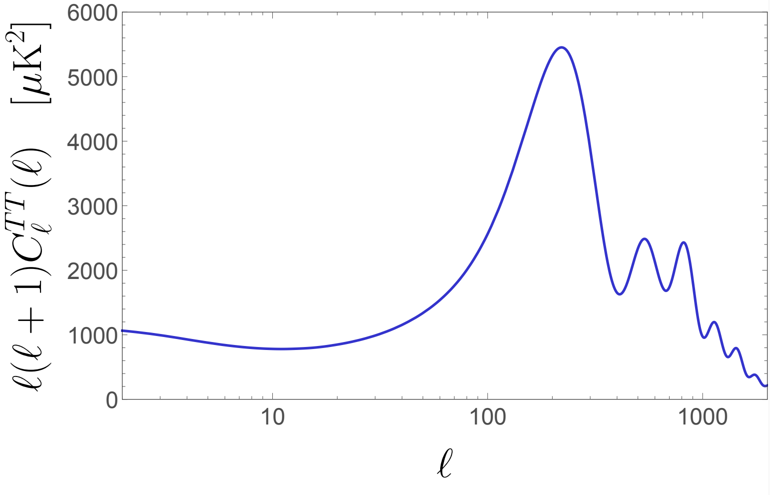
The use of CLASS is free provided that when you use it in a publication, you cite at least the paper CLASS II: Approximation schemes (reference below). You are welcome to cite any other CLASS paper if relevant!

There are two ways to download CLASS. The simplest thing is to download a tar.gz archive of the latest released (master branch) version, v3.1.0, by clicking class_public-3.1.0.tar.gz. But if you are familiar with `git` repositories, and you wish to do modifications to the code, or develop a new branch of the code, or see all public branches and/or old versions, you will prefer to clone it from the class_public git repository.

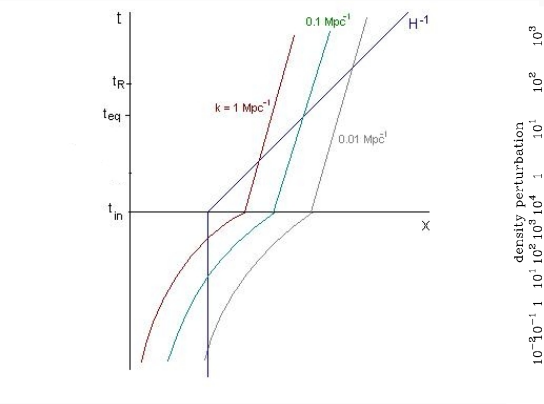
CLASS: github.com/lesgourg/class_public

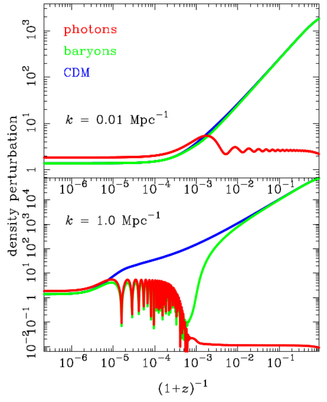


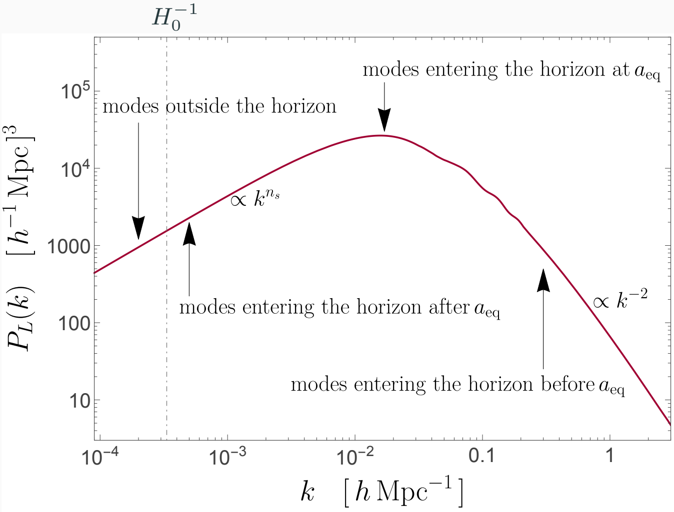
CMB angular TT power spectrum



Growth of perturbations





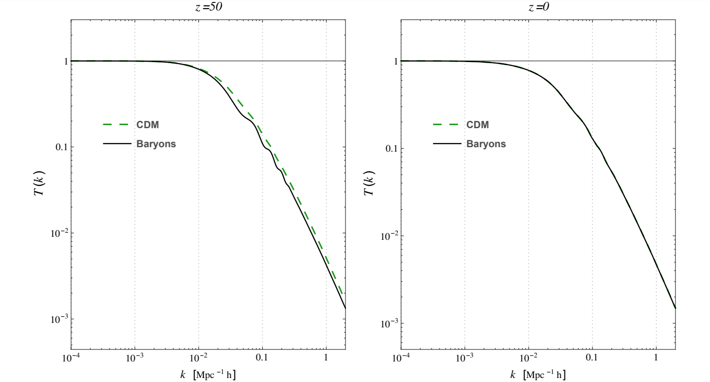


H_0^{-1}
modes entering the horizon at $a_{\rm eq}$
modes outside the horizon
$\propto k^{n_s}$
modes entering the horizon after $a_{\rm eq}$
$\propto k^{-2}$
modes entering the horizon before $a_{\rm eq}$
10^5
10^4
1000
100
10
10^{-4}
10^{-3}
10^{-2}
0.1
1
$P_L(k)\quad[h^{-1}\,\mathrm{Mpc}]^3$
$k\quad[h\,\mathrm{Mpc}^{-1}]$

Transfer function $T(k)$

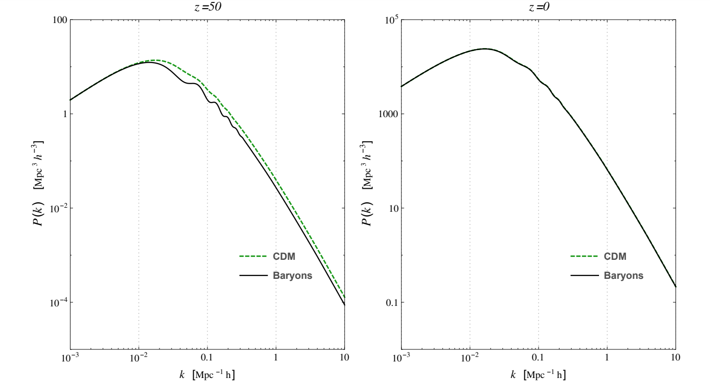
Warning: different definitions/conventions exist

$$P_L(k) = \underbrace{A_s \left(\frac{k}{k_0}\right)^{n_s}}_{\text{Primordial pk}} \times \overbrace{T^2(k)}^{\text{Transfer function}}$$



Power spectrum $P(k)$

$$P_L(k) = \underbrace{A_s\left(\frac{k}{k_0}\right)^{n_s}}_{\text{Primordial pk}} \times \overbrace{T^2(k)}^{\text{Transfer function}}$$



Drag epoch

The BAO scale is set by the radius of the *sound horizon at the drag epoch* $r_s^{\rm drag}$, that is at the time that baryons decouple from photons (or baryon optical depth is 1),

$$r_s^{\rm drag} = \int_0^{t_d} \frac{c_s(a)}{a(t)} dt = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)} dz$$

where z_d is the drag epoch redshift and the baryon-photon fluid speed of sound is

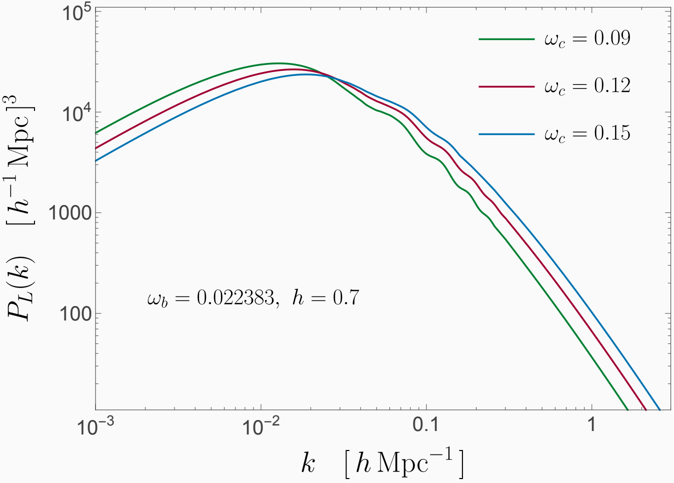
$$c_s(z) = \frac{1}{\sqrt{3\,(1+R(z))}}, \qquad R(z) = \frac{3}{4}\frac{\Omega_b}{\Omega_\gamma}\frac{1}{1+z}$$

Notice t_d is slightly larger than the CMB last scattering time t^* (defined as the time when the optical depth to Thompson scattering reaches unity). One can construct the *sound horizon at recombination* r_s^* by changing t_d by t^* in the above integral. $r_s^* \simeq 0.98\, r_s^{\rm drag}$.

Planck 2018: $r_d = (147.09 \pm 0.24)\,{\rm Mpc}$

$$r_d = \frac{55.154\, e^{-72.3(w_\nu+0.0006)^2}}{(w_c+w_b)^{0.25351} w_b^{0.12807}}\,{\rm Mpc}$$

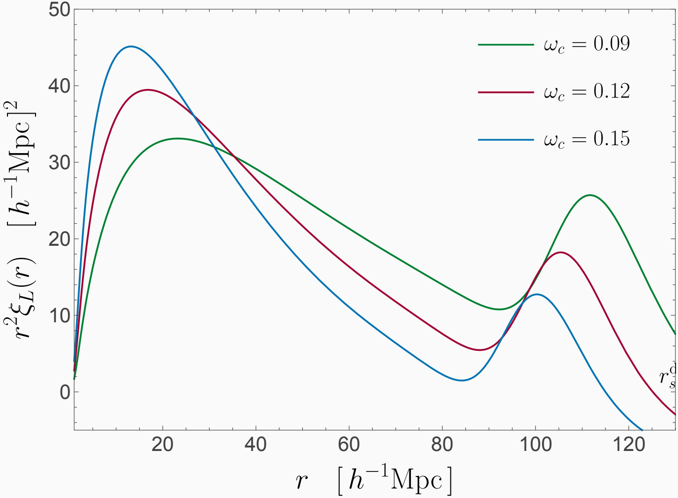
arxiv:1411.1074

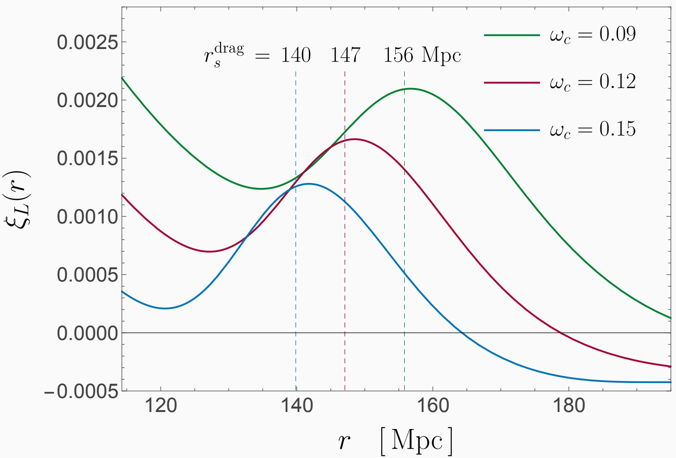


$\omega_c = 0.09$
$\omega_c = 0.12$
$\omega_c = 0.15$
$\omega_b = 0.022383, \ h = 0.7$
$P_L(k) \quad [h^{-1}\,\mathrm{Mpc}]^3$
$k \quad [h\,\mathrm{Mpc}^{-1}]$
10^5
10^4
1000
100
10^{-3}
10^{-2}
0.1
1

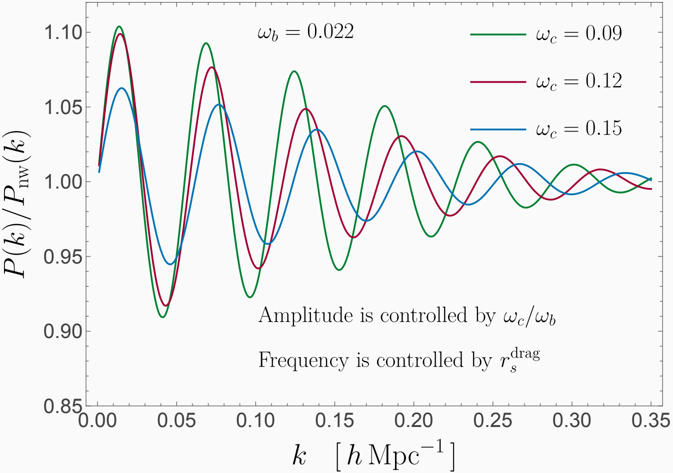
Correlation Function

$$\xi(r) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} P(k)$$





$\omega_c = 0.09$
$\omega_c = 0.12$
$\omega_c = 0.15$
$r_s^{\mathrm{drag}} = 140 \quad 147 \quad 156$ Mpc
$\xi_L(r)$
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
−0.0005
120
140
160
180
$r \quad [\,\mathrm{Mpc}\,]$



$\omega_b = 0.022$
$\omega_c = 0.09$
$\omega_c = 0.12$
$\omega_c = 0.15$
Amplitude is controlled by ω_c/ω_b
Frequency is controlled by $r_s^{\rm drag}$
$P(k)/P_{\rm nw}(k)$
$k \quad [\,h\,{\rm Mpc}^{-1}\,]$
1.10
1.05
1.00
0.95
0.90
0.85
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35

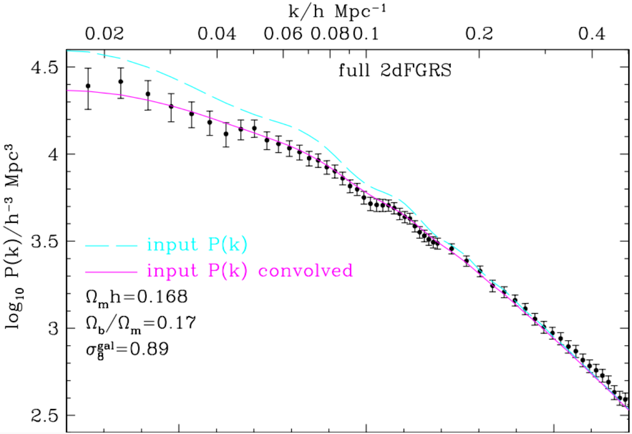
clase 2

20 de septiembre de 2022

Mon. Not. R. Astron. Soc. 362, 505–534 (2005) doi:10.1111/j.1365-2966.2005.09318.x

The 2dF Galaxy Redshift Survey: power-spectrum analysis of the final data set and cosmological implications

Shaun Cole,[1★] Will J. Percival,[2] John A. Peacock,[2] Peder Norberg,[3] Carlton M. Baugh,[1] Carlos S. Frenk,[1] Ivan Baldry,[4] Joss Bland-Hawthorn,[5] Terry Bridges,[6] Russell

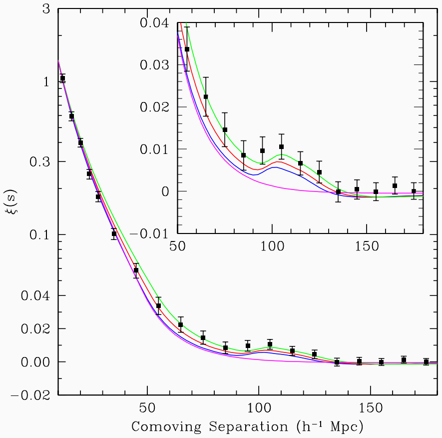


The (2-point) correlation function

SUBMITTED TO *The Astrophysical Journal* 12/31/2004
Preprint typeset using LaTeX style emulateapj v. 11/12/01

DETECTION OF THE BARYON ACOUSTIC PEAK IN THE LARGE-SCALE CORRELATION FUNCTION OF SDSS LUMINOUS RED GALAXIES

DANIEL J. EISENSTEIN[1,2], IDIT ZEHAVI[1], DAVID W. HOGG[3], ROMAN SCOCCIMARRO[3], MICHAEL R.



Correlation function

Average over a large volume V: $\langle X(\boldsymbol{x})\rangle = \int_V \frac{d\boldsymbol{x}}{V} X(\boldsymbol{x})$

We consider a point process $N(\boldsymbol{x})$ with number density $n(\boldsymbol{x})$ and mean number density $\bar{n} = \langle n(\boldsymbol{x})\rangle$. The number of particles N_1 inside a volume dV_1 (located at a position $\boldsymbol{x}_1$) is

$$N_1 = dV_1 n(\boldsymbol{x}_1) = dV_1 \bar{n}(1+\delta_1)$$

where $\delta_1 = \delta(\boldsymbol{x}_1)$ is the number overdensity in dV_1.

Over a very large volume V we have $N_T = \bar{n}V$.

Probability to find an object '1' inside a volume dV_1:

$$\mathcal{P}_1 = \frac{\langle N_1\rangle}{\langle N_T\rangle} = \frac{\bar{n}\langle 1+\delta(\boldsymbol{x}_1)\rangle dV_1}{\bar{n}V} = \frac{dV_1}{V}$$

Why $\langle \delta(\boldsymbol{x})\rangle = 0$?

Because $\quad \bar{n}(t) = \big\langle n(\boldsymbol{x},t)\big\rangle = \big\langle \bar{n}(t)(1+\delta(\boldsymbol{x},t))\big\rangle = \bar{n}(t)\big\langle 1+\delta(\boldsymbol{x},t)\big\rangle$

Correlation function

Probability to find an object '2' inside a volume dV_2 at postition $\boldsymbol{x}_2$, given that we have an object '1' inside a volume dV_1 located at a position $\boldsymbol{x}_1$

$$\mathcal{P}_{2|1} = \frac{\mathcal{P}_{1\cap 2}}{\mathcal{P}_1} = \frac{\langle n(\boldsymbol{x}_1) n(\boldsymbol{x}_2)\rangle}{N_T^2} \frac{1}{\mathcal{P}_1} = \frac{\bar{n}^2 \langle (1+\delta_1)(1+\delta_2)\rangle dV_1 dV_2}{(\bar{n}V)^2} \frac{V}{dV_1}$$
$$= \frac{\langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)\rangle dV_2}{V} = \big[1+\xi(\boldsymbol{x}_1, \boldsymbol{x}_2)\big] \frac{dV_2}{V}$$

where we defined the correlation function

$$\xi(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \langle \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\rangle$$

Statistical homogeneity and isotropy:

$$\xi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \xi(r), \qquad r = |\boldsymbol{x}_2 - \boldsymbol{x}_1|$$

That is, the correlation function gives the excess probability (compared to the expected from a uniform distribution) to find a particle a distance r away from another one.

The correlation function is the Fourier Transform of the power spectrum

$$
\begin{aligned}
\xi(r) = \langle \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\rangle &= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1 + i\boldsymbol{k}_2\cdot\boldsymbol{x}_2}\langle \delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}_1 + i\boldsymbol{k}_2\cdot\boldsymbol{x}_2}(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}_2)P(k) \\
&= \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}P(k)
\end{aligned}
$$

Now, defining $\mu = \hat{k}\cdot\hat{x}$,

$$
\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}P(k) = \frac{2\pi}{(2\pi)^3}\int_0^\infty dk\, k^2 P(k)\int_{-1}^{1} d\mu\, e^{ikr\mu}
$$

Then,

$$
\xi(r) = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 P(k) j_0(kr)
$$

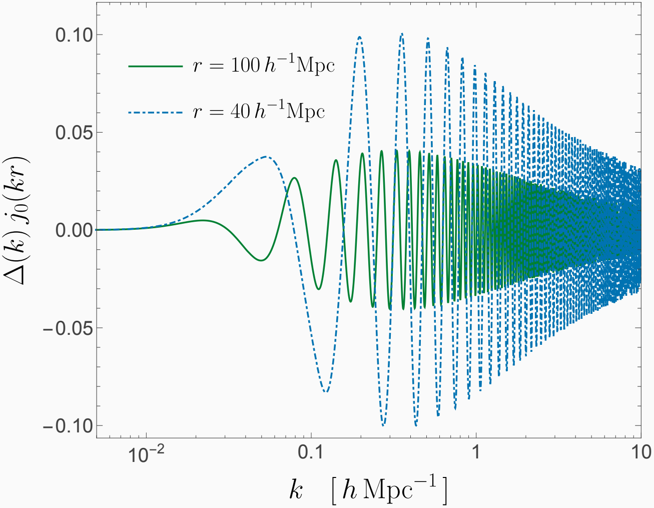
with

$$
j_0(x) = \frac{\sin x}{x}
$$

the spherical Bessel function of degree 0.

Numerical Issues

$$\xi(r) = \int_0^\infty \frac{dk}{2\pi^2} k^2 P_L(k)\, j_0(kr) = \int_0^\infty d(\log k)\, \Delta(k)\, j_0(kr)$$



Numerical Issues

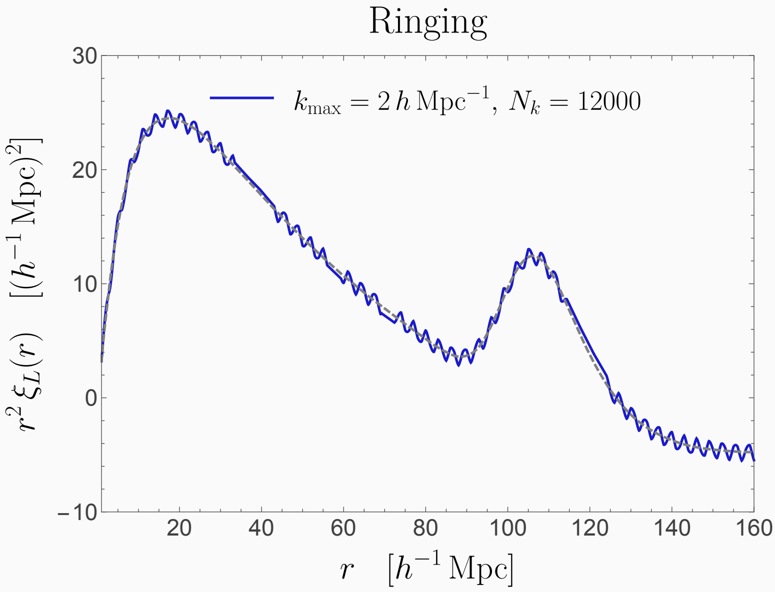
$$\xi(r) = \frac{1}{2\pi^2} \int_0^\infty dk\, k^2 P(k) j_0(kr)$$
$$= \frac{1}{2\pi^2} \int_{k_{\min}}^{k_{\max}} dk\, k^2 P(k) j_0(kr)$$

$$\xi(r) = \sum_{i=1}^{N_k} \frac{k_i^3}{2\pi^2} P(k_i) j_0(k_i r) \Delta(\log k_i)$$

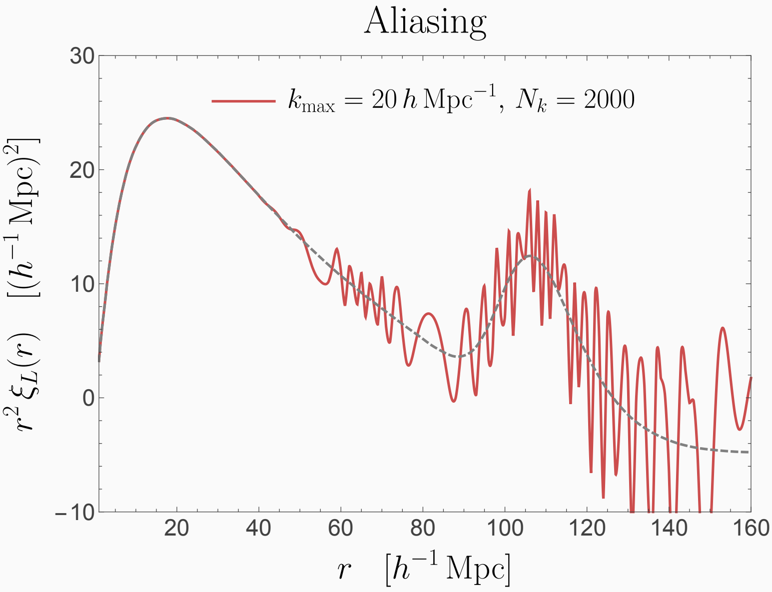
with

$$i = 1,\, 2, \cdots, N_k$$
$$k_i \in \{k_1 = k_{\min},\, k_2, \cdots, k_{N_k} = k_{\max}\}$$

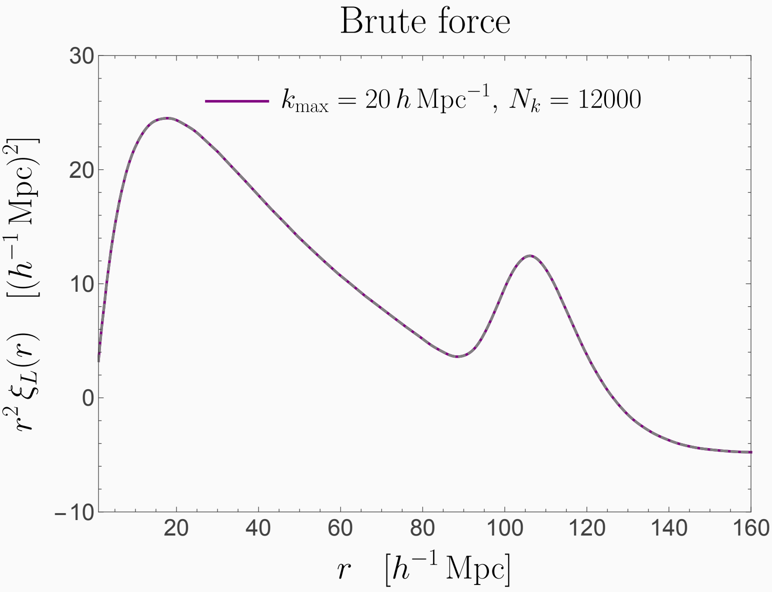
Ringing: cutting off high frequencies



Aliasing: poor sampling

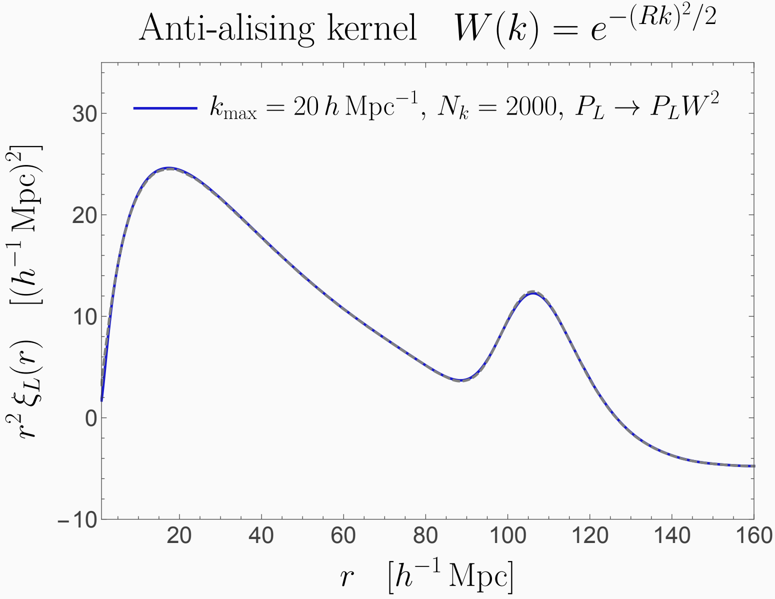


Brute force



Brute force
$k_{\max} = 20\, h\,\mathrm{Mpc}^{-1},\ N_k = 12000$
30
20
10
0
−10
20
40
60
80
100
120
140
160
$r^2\,\xi_L(r) \quad [(h^{-1}\,\mathrm{Mpc})^2]$
$r \quad [h^{-1}\,\mathrm{Mpc}]$

Damping (with $R = 1\,h^{-1}\,\mathrm{Mpc}$)



clase 3

22 de septiembre de 2022

Vlasov Equation

From particles to fluids

Euler-Lagrangian equation

- The Lagrangian of a particle at position

$$\boldsymbol{r} = a(t)\boldsymbol{x}$$

under the gravitational potential ϕ_{N} is given by

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2}m(\dot{a}\boldsymbol{x} + a\dot{\boldsymbol{x}})^2 - m\phi_{\mathrm{N}}(\boldsymbol{x}, t),$$

where $\boldsymbol{x}$ is the comoving position of the particle and $a(t)$ is the scale factor.

- Adding a total derivative dg/dt to the Lagrangian, with $g = -ma\dot{a}\boldsymbol{x}^2/2$, we rewrite the Lagrangian as

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2}ma^2\dot{\boldsymbol{x}}^2 - m\Phi(\boldsymbol{x}, t),$$

where

$$\Phi = \phi_{\mathrm{N}} + \frac{1}{2}a\ddot{a}\boldsymbol{x}^2.$$

Poisson equation

$$\nabla_{\boldsymbol{r}}^2 \phi_N = 4\pi G\rho = 4\pi G\bar{\rho}(1+\delta)$$

where

$$\nabla_{\boldsymbol{r}} = \frac{\partial}{\partial \boldsymbol{r}} = \frac{1}{a}\frac{\partial}{\partial \boldsymbol{x}} = \frac{1}{a}\nabla_{\boldsymbol{x}} \equiv \frac{1}{a}\nabla$$

but $\phi_{\mathrm{N}} = \Phi - a\ddot{a}\boldsymbol{x}^2/2$, hence

$$4\pi G\bar{\rho}(1+\delta) = \frac{1}{a^2}\nabla^2\left(\Phi - \frac{1}{2}a\ddot{a}\boldsymbol{x}^2\right) = \frac{1}{a^2}\nabla^2\Phi - 3\frac{\ddot{a}}{a}$$

But Friedmann equations give

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho}$$

Then

$$\frac{1}{a^2}\nabla^2\Phi(\boldsymbol{x},t) = 4\pi G\bar{\rho}(t)\delta(\boldsymbol{x},t)$$

is the Poisson equation for the perturbed field.

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2} m a^2 \dot{\boldsymbol{x}}^2 - m\Phi(\boldsymbol{x}, t),$$

- The conjugate momentum $\boldsymbol{p}$ to $\boldsymbol{x}$ is then

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = m a^2 \dot{\boldsymbol{x}} = m a \boldsymbol{u}$$

  where $\boldsymbol{u} = a\dot{\boldsymbol{x}} = d\boldsymbol{x}/d\tau$ is the peculiar velocity, i.e. the velocity of the particle with respect to the Hubble flow. The total velocity is $\boldsymbol{v}_T = a\boldsymbol{x}H + \boldsymbol{u}$.

- The equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} - \frac{\partial L}{\partial \boldsymbol{x}} = 0$$

  becomes

$$\frac{d\boldsymbol{p}}{dt} = -m\nabla\Phi,$$

  with $\nabla = \partial/\partial\boldsymbol{x}$.

Vlasov equation

The collisonless Boltzmann equation (or Vlasov equation) dictates the evolution of the phase-space particle distribution function $f(\boldsymbol{x}, \boldsymbol{p}, t)$

$$\begin{aligned}\frac{df}{dt}(\boldsymbol{x}, \boldsymbol{p}, t) &= \frac{\partial f}{\partial t} + \frac{\partial x^i}{\partial t}\frac{\partial f}{\partial x^i} + \frac{\partial p^i}{\partial t}\frac{\partial f}{\partial p^i} \\ &= \frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{ma^2}\cdot\frac{\partial f}{\partial \boldsymbol{x}} - m\nabla\Phi\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0.\end{aligned}$$

It could be convenient to use the conformal time τ instead of cosmic time t, recall that $ad\tau = dt$, getting

$$\frac{df}{d\tau}(\boldsymbol{x}, \boldsymbol{p}, \tau) = \frac{\partial f}{\partial \tau} + \frac{\boldsymbol{p}}{ma}\cdot\frac{\partial f}{\partial \boldsymbol{x}} - ma\nabla\Phi\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0$$

Moments of the distribution function

The comoving number density of particles is

$$n_C(\boldsymbol{x},\tau) = \int d^3p f(\boldsymbol{x},\boldsymbol{p},\tau).$$

The number density is related as $n = n_C/a^3$.

Assuming all dark matter particles have the same mass m, we use $\rho = mn$ to get

$$\rho(\boldsymbol{x},\tau) = \frac{m}{a^3}\int d^3p f(\boldsymbol{x},\boldsymbol{p},\tau).$$

In the following we will use the momentum average $\langle(\cdots)\rangle_p$ over the ensemble of matter particles. For a tensor $\mathbf{A}$

$$\langle\mathbf{A}\rangle_p = \frac{\int d^3p\mathbf{A}f}{\int d^3pf}, \quad \text{such that} \quad \rho\langle\mathbf{A}\rangle_p = \frac{m}{a^3}\int d^3p\mathbf{A}f$$

The mean velocity of particles is computed by taking $\langle p^i\rangle_p$ and the use of $\boldsymbol{p} = ma\boldsymbol{u}$

$$\rho v^i(\boldsymbol{x},\tau) \equiv \rho\langle u^i\rangle_p = \frac{1}{am}\rho\langle p^i\rangle_p = \frac{1}{a^4}\int d^3p\, p^i f(\boldsymbol{x},\boldsymbol{p},\tau).$$

We define the velocity dispersion tensor as the square of the difference between the peculiar velocities and the mean velocity of the particles averaged over the ensemble,

$$\sigma^{ij} = \langle (v^i - u^i)(v^j - u^j) \rangle_p .$$

Therefore, the second moment of the distribution function leads to

$$\rho \langle u^i u^j \rangle_p = \frac{\rho \langle p^i p^j \rangle_p}{m^2 a^2} = \frac{1}{m a^5} \int d^3p \, p^i p^j f = \rho (v^i v^j + \sigma^{ij}).$$

Higher rank tensors can be constructed, for example

$$\sigma^{ijk} \equiv \langle \Delta u^i \Delta u^j \Delta u^k \rangle_p = -\langle u^i u^j u^k \rangle_p + v^{\{i} \sigma^{jk\}} + v^i v^j v^k$$

where $\Delta \boldsymbol{u} = \boldsymbol{v} - \boldsymbol{u}$, and $T^{\{\cdots\}}$ indicate sum over cyclic permutations of indices. Note that

$$\rho \langle u^i u^j u^k \rangle_p = \frac{1}{m^2 a^6} \int d^3p \, p^i p^j p^k f.$$

Moments of the Boltzmann equation

We write the Boltzmann equation as

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 = 0,$$

with

$$\mathcal{B}_1 = \frac{\partial f}{\partial \tau}, \qquad \mathcal{B}_2 = \frac{p^i}{ma}\frac{\partial f}{\partial x_i}, \qquad \mathcal{B}_3 = -ma\frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial p^i},$$

and take moments:

$$\int d^3p\, p^{i_1} p^{i_2} \cdots p^{i_N} \mathcal{B} = 0$$

to obtain *the Boltzmann hierarchy*

0-moment

The zero moment of the Boltzmann equation is $\frac{m}{a^3}\int d^3p\, \mathcal{B} = 0$

- $\mathcal{B}_1$:

$$\frac{m}{a^3}\int d^3p\,\mathcal{B}_1 = \frac{m}{a^3}\int d^3p\,\frac{\partial f}{\partial \tau} = \frac{m}{a^3}\frac{\partial}{\partial \tau}\int d^3p\, f = \frac{m}{a^3}\frac{\partial}{\partial \tau}\left(\frac{a^3}{m}\rho\right)$$
$$= \partial_\tau \rho + 3\mathcal{H}\rho$$

with $\mathcal{H} = \partial_\tau a/a = aH$ the conformal Hubble rate

- $\mathcal{B}_2$:

$$\frac{m}{a^3}\int d^3p\,\mathcal{B}_2 = \frac{m}{a^3}\int d^3p\,\frac{p^i}{ma}\frac{\partial f}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\frac{1}{a^4}\int d^3p\, p^i f\right)$$
$$= \partial_i(\rho v^i)$$

Note the second equality comes from the fact that comoving position $\boldsymbol{x}$ and its conjugate momentum p are independent variables in phase-space.

0-moment

- $\mathcal{B}_3$:

$$\frac{m}{a^3}\int d^3p\,\mathcal{B}_3 = \frac{m}{a^3}\int d^3p\left(-ma\frac{\partial f}{\partial p^i}\frac{\partial \Phi}{\partial x_i}\right) = -\frac{m^2}{a^2}\frac{\partial \Phi}{\partial x_i}\int d^3p\,\frac{\partial f}{\partial p^i} = 0$$

  The second equality holds because Φ does not depend on p^i. In the last term we assume f is zero at infinity momentum.

From the above equations we have the evolution of the zero order moment of the Boltzmann equation

$$\partial_\tau\rho + 3\mathcal{H}\rho + \partial_i(\rho v^i) = 0,$$

which is called *the continuity equation*.

1-moment

We make

$$\frac{1}{a^4}\int d^3p\,p\,p^i\mathcal{B} = 0$$

- $\mathcal{B}_1$:

$$\begin{aligned}\frac{1}{a^4}\int d^3p\,p\,p^i\mathcal{B}_1 &= \frac{1}{a^4}\int d^3p\,p\,p^i\frac{\partial f}{\partial\tau} = \frac{1}{a^4}\frac{\partial}{\partial\tau}\int d^3p\,p\,p^i f = \frac{1}{a^4}\frac{\partial}{\partial\tau}(a^4\rho v^i)\\ &= \frac{\partial}{\partial\tau}(\rho v^i) + 4\mathcal{H}\rho v^i,\end{aligned}$$

where the second equality follows because p^i and τ are independent variables.

- $\mathcal{B}_2$:

$$\begin{aligned}\frac{1}{a^4}\int d^3p\,p\,p^i\mathcal{B}_2 &= \frac{1}{a^4}\int d^3p\,p\,p^i\frac{p^j}{ma}\frac{\partial f}{\partial x^j}\\ &= \frac{\partial}{\partial x^j}\frac{1}{ma^5}\int d^3p\,p\,p^i p^j f = \partial_j\left[\rho(v^i v^j + \sigma^{ij})\right]\end{aligned}$$

1-moment

- $\mathcal{B}_3$:

$$\frac{1}{a^4}\int d^3p\,p^i\mathcal{B}_3 = \frac{1}{a^4}\int d^3p\,p^i\left(-ma\frac{\partial\Phi}{\partial x_j}\frac{\partial f}{\partial p^j}\right) = -\frac{m}{a^3}\frac{\partial\Phi}{\partial x_j}\int d^3p\,p^i\frac{\partial f}{\partial p^j}$$
$$= -\frac{m}{a^3}\frac{\partial\Phi}{\partial x_j}\int d^3p\left[\frac{\partial}{\partial p^j}(p^i f) - f\frac{\partial p^i}{\partial p^j}\right] = \frac{m}{a^3}\frac{\partial\Phi}{\partial x_j}\int d^3p\,f\delta^i_j$$
$$= \frac{m}{a^3}\partial^i\Phi\frac{\rho a^3}{m} = \rho\partial^i\Phi$$

Summing up $a^{-4}\int d^3p\,(p^i\mathcal{B}_1 + p^i\mathcal{B}_2 + p^i\mathcal{B}_3) = 0$, we have

$$\partial_\tau(\rho v^i) + 4\mathcal{H}\rho v^i + \partial_j[\rho(v^iv^j + \sigma^{ij})] + \rho\partial^i\Phi = 0.$$

With the use of the continuity equation we arrive to the more usual form of the *Euler equation*

$$\partial_\tau v^i + \mathcal{H}v^i + v^j\partial_j v^i + \frac{1}{\rho}\partial_j(\rho\sigma^{ij}) + \partial^i\Phi = 0.$$

2-moment

From

$$\frac{1}{ma^5}\int d^3p\, p^i p^j \mathcal{B} = 0$$

A long computation gives

$$\partial_\tau \sigma^{ij} + 2\mathcal{H}\sigma^{ij} + v^k \partial_k \sigma^{ij} + \sigma^{ik}\partial_k v^j + \sigma^{jk}\partial_k v^i = \frac{1}{\rho}\partial_k(\rho\sigma^{ijk}),$$

the velocity dispersion tensor (VDT) *equation*.

Hydrodynamical equations

We rewrite the continuity, Euler and VDT equations

$$\partial_\tau \rho + 3\mathcal{H}\rho + \partial_i(\rho v^i) = 0$$

$$\partial_\tau v^i + \mathcal{H} v^i + v^j \partial_j v^i + \partial^i \Phi = -\frac{1}{\rho}\partial_j(\rho\sigma^{ij})$$

$$\partial_\tau \sigma^{ij} + 2\mathcal{H}\sigma^{ij} + v^k\partial_k\sigma^{ij} + \sigma^{ik}\partial_k v^j + \sigma^{jk}\partial_k v^i = \frac{1}{\rho}\partial_k(\rho\sigma^{ijk}).$$

Using $\rho(\boldsymbol{x},t) = \bar{\rho}(t)(1+\delta(\boldsymbol{x},t))$, and $\partial_\tau\bar{\rho} + 3\mathcal{H}\bar{\rho} = 0$, we get the useful form

$$\partial_\tau \delta(\boldsymbol{x},\tau) + \partial_i((1+\delta)v^i) = 0,$$

$$\partial_\tau v^i(\boldsymbol{x},\tau) + \mathcal{H} v^i + v^j \partial_j v^i + \partial^i \Phi = -\frac{1}{(1+\delta)}\partial_j((1+\delta)\sigma^{ij})$$

$$\partial_\tau \sigma^{ij}(\boldsymbol{x},\tau) + 2\mathcal{H}\sigma^{ij} + v^k\partial_k\sigma^{ij}, + \sigma^{ik}\partial_k v^j + \sigma^{jk}\partial_k v^i = \frac{1}{(1+\delta)}\partial_k((1+\delta)\sigma^{ijk}),$$

$$\cdots = \cdots$$

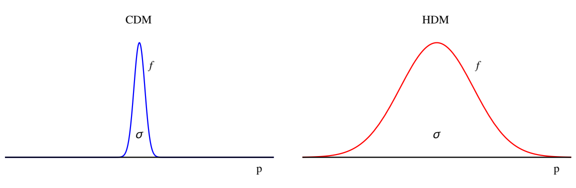
These equations are supplemented by the Poisson equation

$$\frac{1}{a^2}\nabla^2\Phi = 4\pi G\bar{\rho}_m\delta = \frac{3}{2}\Omega_m(a)H^2\delta$$

where we used the Friedmann equation in the second equality.

Cold Dark Matter

- CDM is a pressureless fluid (as we already have assumed). Then it is described by the full (infinite) Boltzmann hierarchy.
- Notice $\sigma = \langle (\boldsymbol{v} - \boldsymbol{u})^2 \rangle_p$ quantifies the failure of the particles to follow a *single stream*.



Velocity dispersions produce suppressions in clustering below the free-streaming scale: $\lambda_{\text{FS}} \sim \sqrt{\sigma_0}/\mathcal{H}$. We have structures down to few tens of kiloparsecs. Hence, σ should be very small.

- CDM is a fluid with no velocity dispersions $\sigma = 0$.

Hydrodynamical equations

Using cosmic time t

$$\dot{\delta}(\boldsymbol{x},t) + \frac{1}{a}\partial_i v^i = -\frac{1}{a}\partial_i(\delta v^i),$$
$$\dot{v}^i(\boldsymbol{x},t) + Hv^i + \frac{1}{a}\partial^i\Phi = -\frac{1}{a}v^j\partial_j v^i.$$

with the Poisson equation

$$\frac{1}{a^2}\nabla^2\Phi(\boldsymbol{x},t) = 4\pi G\bar{\rho}_m\delta = \frac{3}{2}\Omega_m(a)H^2\delta$$

Notice $\Omega_m(t)H^2 = \Omega_m^0 H_0^2 a^{-3}$.

- At linear order

$$\dot{\delta}(\boldsymbol{x},t) + \frac{1}{a}\partial_i v^i = 0,$$
$$\dot{v}^i(\boldsymbol{x},t) + Hv^i + \frac{1}{a}\partial^i\Phi = 0$$

Velocity field is longitudinal

Take the rotational of the linear Euler equation

$$\partial_t \nabla \times \boldsymbol{v} + H \nabla \times \boldsymbol{v} + \frac{1}{a} \underbrace{\nabla \times \nabla \Phi}_{=0} = 0$$

hence $\boldsymbol{w} = \nabla \times \boldsymbol{v} \propto 1/a$. If we keep the VDT into the hydrodynamical equations, a term $\nabla \times (\nabla \cdot \boldsymbol{\sigma})$ becomes a source to Euler equation allowing the vorticity to grow, even though its contribution is still small.

On large scales we can safely characterize the velocity by its divergence

$$\theta(\boldsymbol{x}, t) \equiv -\frac{\partial_i v^i}{aHf}$$

where for convenience we introduced a function $f(t)$, for the moment arbitrary.

Warning: There are several notations for $\theta \propto \nabla \cdot \mathbf{v}$ in the literature.

Linear Standard Perturbation Theory

Linear theory

Taking the divergence of Euler's linear equation, and using $\partial_i v^i = -aHf\theta$,

$$H^{-1}\frac{\partial\delta}{\partial t}(\boldsymbol{x},t) - f\theta = 0,$$

$$H^{-1}\frac{\partial(f\theta)}{\partial t}(\boldsymbol{x},t) + \left(2+\frac{\dot{H}}{H^2}\right)f\theta - \frac{3}{2}\Omega_m(a)\delta = 0$$

Combining these two equations we have (now in Fourier space)

$$\ddot{\delta}(\boldsymbol{k},t) + 2H\dot{\delta} - \frac{3}{2}\Omega_m(a)H^2\delta = 0.$$

This equation does not depend on $\boldsymbol{k}$, hence the solution can be separated $\delta(\boldsymbol{k},t) = D_+(t)A(\boldsymbol{k}) + D_-(t)B(\boldsymbol{k})$ with

$$\left(\frac{d^2}{dt^2} + 2H\frac{d}{dt} - \frac{3}{2}\Omega_m(t)H^2\right)D(t) = 0$$

The fastest growing solution is called the *linear growth function* D_+, then

$$\delta^{(1)}(\boldsymbol{k}, t) = D_+(t)\delta^{(1)}(\boldsymbol{k}, t_0)$$

where one normalize $D_+(t_0) = 1$. Typically one chooses t_0 to be the present time. The other solution is $D_- \propto H$.

Now, we choose f to be the *(logarithmic) growth factor*,

$$f(t) \equiv \frac{d \log D_+(t)}{d \log a(t)}.$$

Hence

$$\theta^{(1)}(\boldsymbol{k}, t) = \delta^{(1)}(\boldsymbol{k}, t)$$

Another popular notation defines $\theta = \nabla \cdot \boldsymbol{v}$, for which $\theta^{(1)} = -aHf\delta^{(1)}$

clase 4

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Solving for the linear growth equation

Define $\eta = \ln(a)$. Then

$$\frac{d}{dt} = H\frac{d}{d\eta}, \qquad \frac{d^2}{dt^2} = H^2\frac{d^2}{d\eta^2} + H\frac{dH}{d\eta}\frac{d}{d\eta}.$$

The equation for the growth function becomes

$$\left(\frac{d^2}{dt^2} + 2H\frac{d}{dt} - \frac{3}{2}\Omega_m(t)H^2\right)D(t) = 0$$

becomes

$$\left[\frac{d^2}{d\eta^2} + \left(2 + \frac{1}{H}\frac{dH}{d\eta}\right)\frac{d}{dt} - \frac{3}{2}\Omega_m(\eta)\right]D(\eta) = 0$$

Considering only matter and cosmological constant $\Omega_\Lambda = 1 - \Omega_m$, we have

$$\frac{1}{H}\frac{dH}{d\eta} = -\frac{3}{2}\Omega_m(\eta) = -\frac{3}{2\left(1 + \frac{1-\Omega_m}{\Omega_m}\exp(3\eta)\right)}.$$

Solving for the linear growth function

$$\left[\frac{d^2}{d\eta^2} + \left(2 - \frac{3}{2}\frac{1}{1+\frac{1-\Omega_m}{\Omega_m}\exp(3\eta)}\right)\frac{d}{d\eta} - \frac{3}{2}\frac{1}{1+\frac{1-\Omega_m}{\Omega_m}\exp(3\eta)}\right] D(\eta) = 0.$$

The solutions do not depend on H_0. Note $\eta(a = a_0 = 1) = 0$.

At early times $\eta \to -\infty$

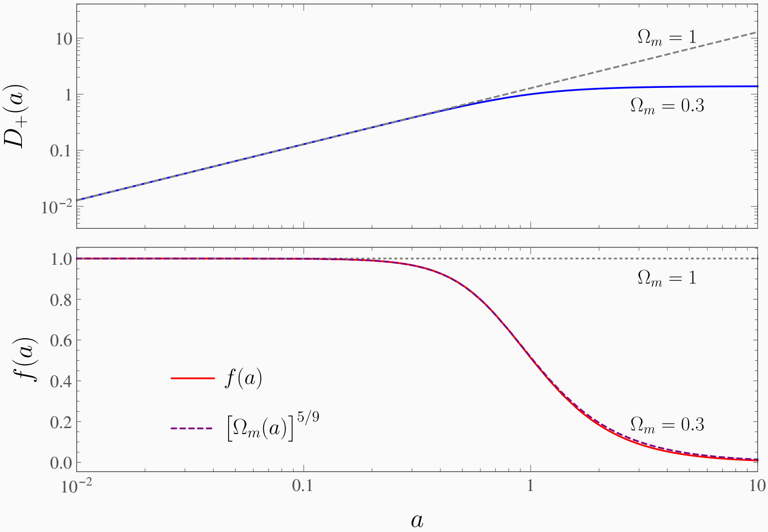
$$\left[\frac{d^2}{d\eta^2} + \frac{1}{2}\frac{d}{d\eta} - \frac{3}{2}\right] D(\eta) = 0.$$

Solutions: $\quad D_+(\eta) \propto e^{\eta} = a, \quad D_-(\eta) \propto e^{-\frac{3}{2}\eta} = a^{-3/2}$

Einstein-de Sitter solutions:

$$D_+(a) \propto a, \qquad f(a) = 1.$$

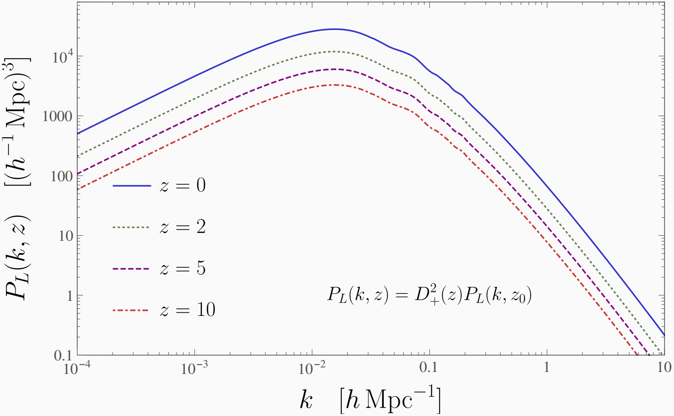
Linear growth function and growth rate

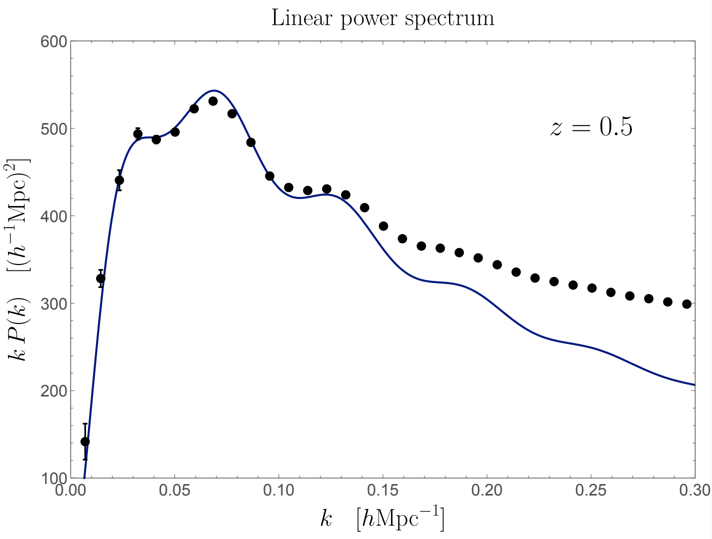


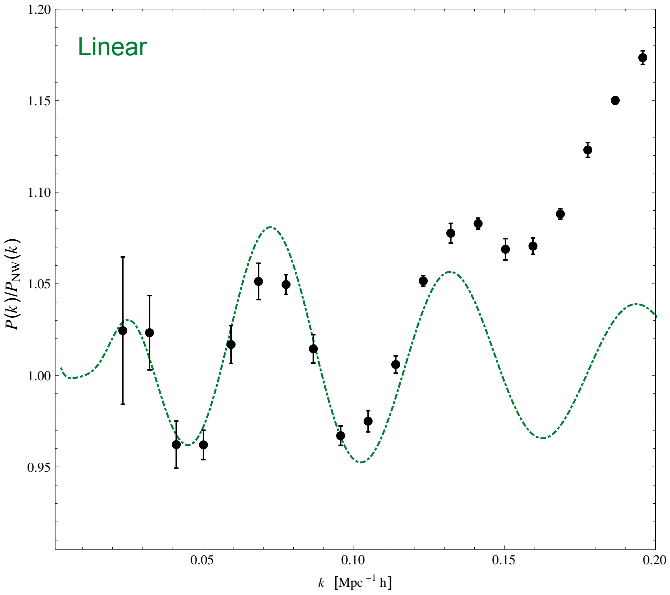
Linear power spectrum

$$\langle\delta^{(1)}(\boldsymbol{k},t)\delta^{(1)}(\boldsymbol{k}',t)\rangle = D_+^2(t)\langle\delta^{(1)}(\boldsymbol{k},t_0)\delta^{(1)}(\boldsymbol{k}',t_0)\rangle$$

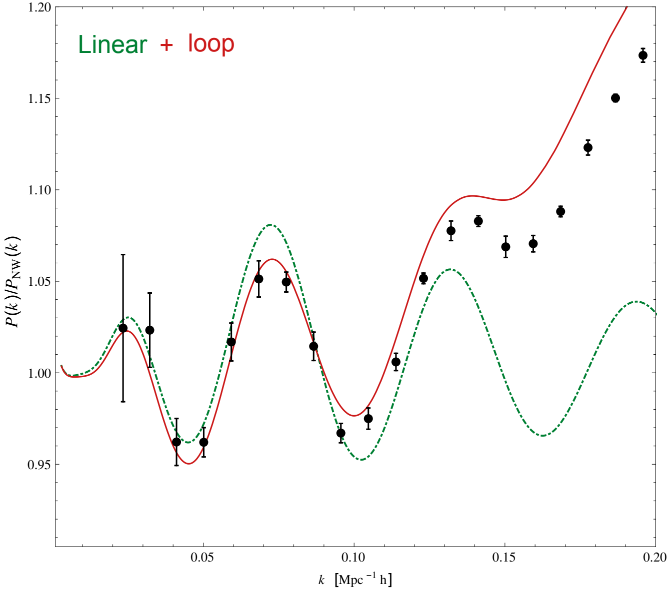
$$\Longrightarrow \qquad P_L(k,t) = D_+^2(t)P_L(k,t_0)$$

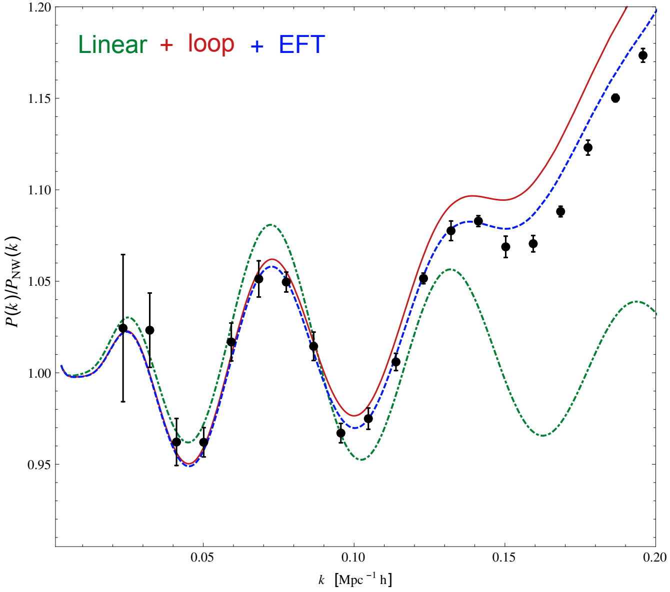



Linear power spectrum
$k\,P(k)\quad[(h^{-1}\mathrm{Mpc})^2]$
$k\quad[h\mathrm{Mpc}^{-1}]$
$z = 0.5$
600
500
400
300
200
100
0.00
0.05
0.10
0.15
0.20
0.25
0.30



Linear
1.20
1.15
1.10
1.05
1.00
0.95
0.05
0.10
0.15
0.20
$P(k)/P_{\rm NW}(k)$
k [Mpc^{-1} h]


Linear + loop
1.20
1.15
1.10
1.05
1.00
0.95
0.05
0.10
0.15
0.20
$P(k)/P_{\mathrm{NW}}(k)$
k [Mpc^{-1} h]


Linear + loop + EFT
1.20
1.15
1.10
1.05
1.00
0.95
0.05
0.10
0.15
0.20
$P(k)/P_{\mathrm{NW}}(k)$
k [Mpc^{-1} h]

Non-linearities

Standard Perturbation Theory

Fourier Transform conventions

$$f(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}), \qquad f(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{k})$$

Hence

$$(2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k}) = \int d^3x\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\mathcal{FT}\big[\,\partial_i f(\boldsymbol{x})\,\big](\boldsymbol{k}) = ik_i\,\mathcal{FT}\big[\,f(\boldsymbol{x})\,\big](\boldsymbol{k}) = ik_i f(\boldsymbol{k})$$

$$\partial_i \;\longrightarrow\; ik_i, \qquad \nabla^2 \;\longrightarrow\; -k^2$$

Hydrodynamical equations

$$\frac{\partial}{\partial t}\delta(\boldsymbol{x},t) + \frac{1}{a}\partial_i v^i = -\frac{1}{a}\partial_i(v^i\delta),$$

$$\frac{\partial}{\partial t}(\partial_i v^i)(\boldsymbol{x},t) + H\partial_i v^i + \frac{1}{a}\nabla^2\Phi = -\frac{1}{a}\partial_i(v^j\partial_j v^i).$$

with the Poisson equation $\frac{1}{a^2}\nabla^2\Phi(\boldsymbol{x},t) = 4\pi G\bar{\rho}_m\delta = \frac{3}{2}\Omega_m(a)H^2(a)\delta$

In Fourier Space

$$\frac{\partial}{\partial t}\delta(\boldsymbol{k},t) - Hf\theta(\boldsymbol{k},t) = \mathcal{FT}\left[-\frac{1}{a}\partial_i(v^i\delta)\right](\boldsymbol{k},t),$$

$$-\frac{\partial}{\partial t}(aHf\theta(\boldsymbol{k},t)) - aH^2f\theta(\boldsymbol{k},t) - a\frac{k^2}{a^2}\Phi(\boldsymbol{k},t) = \mathcal{FT}\left[-\frac{1}{a}\partial_i(v^j\partial_j v^i)\right](\boldsymbol{k},t)$$

Continuity equation NL:

$$\mathcal{FT}\left[-\frac{1}{a}\partial_i(v^i(\boldsymbol{x})\delta(\boldsymbol{x}))\right](\boldsymbol{k}) = -\frac{1}{a}\int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\partial}{\partial x^i}\left(v^i(\boldsymbol{x})\delta(\boldsymbol{x})\right)$$
$$= -\frac{1}{a}\int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\partial}{\partial x^i}\left[\int\frac{d^3k_1}{(2\pi)^3}\, e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}}v^i(\boldsymbol{k}_1)\int\frac{d^3k_2}{(2\pi)^3}e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}}\delta(\boldsymbol{k}_2)\right]$$
$$= -\frac{1}{a}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}i(k_1^i+k_2^i)v^i(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\int d^3x\, e^{-i(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}}$$
$$= -\frac{1}{a}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)i(k_1^i+k_2^i)v_i(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)$$

Using

$$\theta(\boldsymbol{x}) = -\frac{\partial_i v^i}{aHf} \qquad \Rightarrow \qquad \theta(\boldsymbol{k}) = -\frac{ik_iv^i(\boldsymbol{k})}{aHf} \qquad \Rightarrow \qquad v^i(\boldsymbol{k}) = i\frac{k^i}{k^2}aHf\theta(\boldsymbol{k})$$

We obtain

$$\mathcal{FT}\left[-\frac{1}{a}\partial_i(v^i(\boldsymbol{x})\delta(\boldsymbol{x}))\right](\boldsymbol{k}) = Hf\int_{\boldsymbol{k}_{12}=\boldsymbol{k}}\left(1+\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1^2}\right)\theta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)$$

Where we use the shorthand notation

$$\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} = \int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12}), \qquad \boldsymbol{k}_{12} = \boldsymbol{k}_1+\boldsymbol{k}_2$$

$$\int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} = \int\frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{1\cdots n}), \qquad \boldsymbol{k}_{1\cdots n} = \boldsymbol{k}_1+\cdots+\boldsymbol{k}_n$$

Euler's equation non-linear term:

$$
\begin{aligned}
&\mathcal{FT}\left[-\frac{1}{a}\partial_i\left(v^j(\boldsymbol{x})\partial_j v^i(\boldsymbol{x})\right)\right](\boldsymbol{k}) \\
&= -\frac{1}{a} i k_i \mathcal{FT}\left[v^j(\boldsymbol{x})\partial_j v^i(\boldsymbol{x})\right](\boldsymbol{k}) \qquad \text{(the FT of a multiplication is the convolution of the FTs of the factors)} \\
&= -\frac{i k_i}{a}\int_{\boldsymbol{k}_{12}=\boldsymbol{k}}\left[\frac{i k_1^j a H f \theta(\boldsymbol{k}_1)}{k_1^2} i k_2^j \frac{i k_2^i a H f \theta(\boldsymbol{k}_2)}{k_2^2}\right] \\
&= -aH^2 f^2 \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{k_1^2 k_2^2}\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2) \\
&= \frac{1}{2}\left[(1\to 1, 2\to 2) + (1\to 2, 2\to 1)\right] \qquad \text{(symmetrizing indices)} \\
&= -aH^2 f^2 \frac{1}{2}\left[\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{k_1^2 k_2^2}\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2) + \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)}{k_2^2 k_1^2}\theta(\boldsymbol{k}_2)\theta(\boldsymbol{k}_1)\right] \\
&= -aH^2 f^2 \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \frac{\boldsymbol{k}\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{2k_1^2 k_2^2}\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2) \\
&= -aH^2 f^2 \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \frac{k_{12}^2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{2k_1^2 k_2^2}\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)
\end{aligned}
$$

Hydrodynamical equations - Fourier space

$$H^{-1}\frac{\delta(\boldsymbol{k},t)}{\partial t} - f\theta = f\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2),$$

$$H^{-1}\frac{\partial(f\theta)}{\partial t}(\boldsymbol{k},t) + \left(2+\frac{\dot{H}}{H^2}\right)f\theta - \frac{3}{2}\Omega_m(a)\delta = f^2\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \beta(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)$$

with

$$\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + \frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1^2}, \qquad \beta(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{k_{12}^2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{2k_1^2k_2^2}.$$

Remind the definition

$$\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\delta_{\mathrm{D}}(\boldsymbol{k}_1+\boldsymbol{k}_2)$$

Hydrodynamical equations - Perturbation Theory

Expand the fields

$$\delta = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \cdots, \qquad \theta = \theta^{(1)} + \theta^{(2)} + \theta^{(3)} + \cdots.$$

Solve order by order:

$$H^{-1}\frac{\partial \delta^{(n)}(\boldsymbol{k},t)}{\partial t} - f\theta^{(n)} = \sum_{m=1}^{n-1} f \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta^{(m)}(\boldsymbol{k}_1)\delta^{(n-m)}(\boldsymbol{k}_2),$$

$$H^{-1}\frac{\partial (f\theta^{(n)}(\boldsymbol{k},t))}{\partial t} + \left(2 + \frac{\dot{H}}{H^2}\right) f\theta^{(n)} - \frac{3}{2}\Omega_m(a)\delta^{(n)} = \sum_{m=1}^{n-1} f^2 \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \beta(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta^{(m)}(\boldsymbol{k}_1)\theta^{(n-m)}(\boldsymbol{k}_2).$$

Propose solutions

$$\delta^{(n)}(\boldsymbol{k},t) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t)\cdots\delta_L(\boldsymbol{k}_n,t),$$

$$\theta^{(n)}(\boldsymbol{k},t) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t)\cdots\delta_L(\boldsymbol{k}_n,t).$$

At order n, each field is a (weighted) convolution of n linear density fields $\delta_L \equiv \delta^{(1)}$

$$F_1(\boldsymbol{k}) = 1, \qquad G_1(\boldsymbol{k}) = 1.$$

$$\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \alpha(\boldsymbol{k}_1, \boldsymbol{k}_2)\theta^{(m)}(\boldsymbol{k}_1)\delta^{(n-m)}(\boldsymbol{k}_2)$$
$$= \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \alpha(\boldsymbol{k}_{1\ldots m}, \boldsymbol{k}_{m+1\cdots n})G_m(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m)F_{n-m}(\boldsymbol{k}_{m+1}, \ldots \boldsymbol{k}_n)\delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n).$$

Tarea

Notation:

$$\alpha_{\bar{m},\bar{n}} = \alpha(\boldsymbol{k}_{1\ldots m}, \boldsymbol{k}_{m+1\cdots n}),$$
$$G_m(\bar{m}) = G_m(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m), \qquad F_{n-m}(\bar{n}) = F_{n-m}(\boldsymbol{k}_{m+1}, \ldots \boldsymbol{k}_n).$$

We obtain

$$\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \alpha(\boldsymbol{k}_1, \boldsymbol{k}_2)\theta^{(m)}(\boldsymbol{k}_1)\delta^{(n-m)}(\boldsymbol{k}_2) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \alpha_{\bar{m},\bar{n}}G_m(\bar{m})F_{n-m}(\bar{n})\delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n)$$
$$\int_{\boldsymbol{k}_{12}=\boldsymbol{k}} \beta(\boldsymbol{k}_1, \boldsymbol{k}_2)\theta^{(m)}(\boldsymbol{k}_1)\theta^{(n-m)}(\boldsymbol{k}_2) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \beta_{\bar{m},\bar{n}}G_m(\bar{m})G_{n-m}(\bar{n})\delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n)$$

clase 5

29 de septiembre de 2022

I Continuity Equation:

$$H^{-1}\frac{\partial\delta^{(n)}(\boldsymbol{k},t)}{\partial t} - f\theta^{(n)} = \sum_{m=1}^{n-1} f \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \alpha_{\bar{m},\bar{n}} G_m(\bar{m}) F_{n-m}(\bar{n}) \delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n),$$

II Euler Equation:

$$H^{-1}\frac{\partial(f\theta^{(n)}(\boldsymbol{k},t))}{\partial t} + \left(2+\frac{\dot{H}}{H^2}\right) f\theta^{(n)} - \frac{3}{2}\Omega_m(a)\delta^{(n)} =$$

$$\sum_{m=1}^{n-1} f^2 \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \beta_{\bar{m},\bar{n}} G_m(\bar{m}) G_{n-m}(\bar{n}) \delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n).$$

Continuity Equation

$$H^{-1}\frac{\partial\delta^{(n)}(\boldsymbol{k},t)}{\partial t} - f\theta^{(n)} = \sum_{m=1}^{n-1} f \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \alpha_{\bar{m},\bar{n}} G_m(\bar{m}) F_{n-m}(\bar{n}) \delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n),$$

The time derivative of the n-th density fluctuation is (remind in our ansatz F_n is time independent)

$$\dot{\delta}^{(n)}(\boldsymbol{k},t) = n \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\dot{\delta}^{(1)}(\boldsymbol{k}_1,t)\cdots\delta^{(1)}(\boldsymbol{k}_n,t),$$

$$\text{Since} \quad \dot{\delta}^{(1)} \underbrace{=}_{\text{linear cont. eq.}} fH\theta^{(1)} \underbrace{=}_{\text{linear solution.}} fH\delta^{(1)},$$

then, LHS of Continuity Equation above becomes:

$$\int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} \left[nfF_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) - fG_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\right]\delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n).$$

Then, we obtain

$$nF_n - G_n = \sum_{m=1}^{n-1} \alpha_{\bar{m},\bar{n}} G_m(\bar{m}) F_{n-m}(\bar{n}).$$

Euler Equation

$$H^{-1}\frac{\partial(f\theta^{(n)}(\boldsymbol{k},t))}{\partial t}+\left(2+\frac{\dot{H}}{H^2}\right)f\theta^{(n)}-\frac{3}{2}\Omega_m(a)\delta^{(n)}=$$

$$\sum_{m=1}^{n-1}f^2\int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}}\beta_{\bar{m},\bar{n}}G_m(\bar{m})G_{n-m}(\bar{n})\delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n).$$

Using $\dot{f}=\frac{3}{2}\Omega_m(a)H^2-H\left(2+\frac{\dot{H}}{H^2}\right)f-Hf^2$, we have

$$\frac{1}{fH}\dot{\theta}^{(n)}(\boldsymbol{k},t)-\theta+\frac{3}{2}\frac{\Omega_m}{f^2}(\theta-\delta)=$$

$$\sum_{m=1}^{n-1}\int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}}\beta_{\bar{m},\bar{n}}G_m(\bar{m})G_{n-m}(\bar{n})\delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n).$$

which finally gives

$$(n-1)\,G_n+\frac{3}{2}\frac{\Omega_m(a)}{f^2}\left(G_n-F_n\right)=\sum_{m=1}^{n-1}\beta_{\bar{m},\bar{n}}G_m(\bar{m})G_{n-m}(\bar{n})$$

Kernels EdS

For $\Omega_m(t) = f^2(t)$

$$nF_n - G_n = \sum_{m=1}^{n-1} \alpha_{\bar{m}\bar{n}} G_m(\bar{m}) F_{n-m}(\bar{n})$$

$$\frac{1}{2}(2n+1)\, G_n - \frac{3}{2} F_n = \sum_{m=1}^{n-1} \beta_{\bar{m}\bar{n}} G_m(\bar{m}) G_{n-m}(\bar{n})$$

Solving for F_n and G_n:

$$F_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) = \sum_{m=1}^{n-1} \frac{G_m(\bar{m})}{(2n+3)(n-1)} \Big[(2n+1)\alpha_{\bar{m}\bar{n}} F_{n-m}(\bar{n}) + 2\beta_{\bar{m}\bar{n}} G_{n-m}(\bar{n})\Big]$$

$$G_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) = \sum_{m=1}^{n-1} \frac{G_m(\bar{m})}{(2n+3)(n-1)} \Big[3\alpha_{\bar{m}\bar{n}} F_{n-m}(\bar{n}) + 2n\beta_{\bar{m}\bar{n}} G_{n-m}(\bar{n})\Big]$$

n-th order fluctuations

$$\delta^{(n)}(\boldsymbol{k},t) = D_+^n(t) \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t_0)\cdots\delta_L(\boldsymbol{k}_n,t_0),$$

$$\theta^{(n)}(\boldsymbol{k},t) = D_+^n(t) \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t_0)\cdots\delta_L(\boldsymbol{k}_n,t_0).$$

with $F_1(\boldsymbol{k}) = G_1(\boldsymbol{k}) = 1$, and

$$F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) = \sum_{m=1}^{n-1} \frac{G_m(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_m)}{(2n+3)(n-1)} \Big[(2n+1)\alpha(\boldsymbol{k}_{1\ldots m},\boldsymbol{k}_{m+1\cdots n})F_{n-m}(\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_n)$$
$$+ 2\beta(\boldsymbol{k}_{1\ldots m},\boldsymbol{k}_{m+1\cdots n})F_{n-m}(\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_n)\Big]$$

$$G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) = \sum_{m=1}^{n-1} \frac{G_m(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_m)}{(2n+3)(n-1)} \Big[3\alpha(\boldsymbol{k}_{1\ldots m},\boldsymbol{k}_{m+1\cdots n})F_{n-m}(\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_n)$$
$$+ 2n\beta(\boldsymbol{k}_{1\ldots m},\boldsymbol{k}_{m+1\cdots n})F_{n-m}(\boldsymbol{k}_{m+1},\ldots,\boldsymbol{k}_n)\Big]$$

with $\quad \alpha(\boldsymbol{p}_1,\boldsymbol{p}_2) = 1 + \dfrac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{p_1^2}, \qquad \beta(\boldsymbol{p}_1,\boldsymbol{p}_2) = \dfrac{p_{12}^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{2p_1^2p_2^2}$

Kernels F_2

Using $G_1 = F_1 = 1$

$$F_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{5}{7}\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2) + \frac{2}{7}\beta(\boldsymbol{k}_1, \boldsymbol{k}_2)$$
$$G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{3}{7}\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2) + \frac{4}{7}\beta(\boldsymbol{k}_1, \boldsymbol{k}_2)$$

Notice that $\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2)$ is not symmetric, but

$$\theta^{(2)}(\boldsymbol{k}) = \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} G_2(\boldsymbol{k}_1, \boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2) = \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} G_2^s(\boldsymbol{k}_1, \boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)$$

with $\quad G_2^s(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{1}{2}\big(G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) + G_2(\boldsymbol{k}_2, \boldsymbol{k}_1)\big)$

Symmetric F_2 and G_2 kernels

$$F_2^s(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{5}{14}\big(\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2) + \alpha(\boldsymbol{k}_2, \boldsymbol{k}_1)\big) + \frac{2}{7}\beta(\boldsymbol{k}_1, \boldsymbol{k}_2)$$
$$G_2^s(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{3}{14}\big(\alpha(\boldsymbol{k}_1, \boldsymbol{k}_2) + \alpha(\boldsymbol{k}_2, \boldsymbol{k}_1)\big) + \frac{4}{7}\beta(\boldsymbol{k}_1, \boldsymbol{k}_2)$$

Kernels F_2 and G_2

Developing:

$$F_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{5}{7} + \frac{2}{7}\frac{(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)^2}{k_1^2 k_2^2} + \frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2}{2k_1 k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)$$

$$G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{3}{7} + \frac{4}{7}\frac{(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)^2}{k_1^2 k_2^2} + \frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2}{2k_1 k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)$$

Notice:

- $F_2, G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = F_2, G_2(k_1, k_2, x) = F_2, G_2(k_1/k_2, x)$ with $x = \hat{k}_1 \cdot \hat{k}_2$.
- $F_2, G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = 0$ for $\boldsymbol{k}_2 = -\boldsymbol{k}_1$
  This means that at very large scales $\delta^{(2)}(\boldsymbol{k} \to 0) \to 0$ and $\theta^{(2)}(\boldsymbol{k} \to 0) \to 0$.
- $F_2, G_2(\boldsymbol{p}, \boldsymbol{k} - \boldsymbol{p}) \propto k^2/p^2$ for $k \ll p$.
- $F_2, G_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = F_2, G_2(\boldsymbol{k}_2, \boldsymbol{k}_1) = F_2, G_2(-\boldsymbol{k}_1, -\boldsymbol{k}_2)$

$$F_2(\boldsymbol{k},\boldsymbol{p}) = \frac{17}{21} + \frac{2}{7}\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\left(\hat{p}_i\hat{p}_j - \frac{1}{3}\delta_{ij}\right) + \frac{1}{2}\left(p_i\frac{k_i}{k^2} + k_i\frac{p_i}{p^2}\right)$$

$$\delta^{(2)}(\boldsymbol{k}) = \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)$$

- $$\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\delta(\boldsymbol{k}) \xrightarrow{\mathcal{FT}} \frac{2}{3a^2\Omega_m(a)H^2}\left(\partial_i\partial_j\Phi(\boldsymbol{x}) - \frac{1}{3}\delta_{ij}\nabla^2\Phi(\boldsymbol{x})\right)$$

- $$p_i\frac{k_i}{k^2}\delta^{(1)}(\boldsymbol{p})\delta^{(1)}(\boldsymbol{k}) \longrightarrow \left(\partial_i\delta^{(1)}(\boldsymbol{x})\right)\left(\frac{\partial_i}{\nabla^2}\delta^{(1)}(\boldsymbol{x})\right) = \Psi^i(\boldsymbol{x})\partial_i\delta^{(1)}(\boldsymbol{x})$$

where we used the *Lagrangian displacement at linear order* $\Psi_i(x) = \frac{\partial_i}{\nabla^2}\delta^{(1)}(\boldsymbol{x}) \propto \partial_i\Phi(\boldsymbol{x})$

In configuration space

$$\delta^{(2)}(\boldsymbol{x}) = \frac{17}{21}\left[\delta^{(1)}(\boldsymbol{x})\right]^2 + \Psi^i(\boldsymbol{x})\partial_i\delta^{(1)}(\boldsymbol{x}) + \frac{8}{63a^4\Omega_m(a)^2H^4}\left(\partial_i\partial_j\Phi(\boldsymbol{x}) - \frac{1}{2}\delta_{ij}\nabla^2\Phi(\boldsymbol{x})\right)^2$$

Notice $\quad \Psi^i(\boldsymbol{x})\partial_i\delta^{(1)}(\boldsymbol{x}) \simeq \delta^{(1)}(\boldsymbol{x}+\Psi) - \delta^{(1)}(\boldsymbol{x})$

F_3

$$F_3(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \frac{1}{18}\left[7\alpha(\boldsymbol{k}_1, \boldsymbol{k}_{23})F_2(\boldsymbol{k}_2, \boldsymbol{k}_3) + 2\beta(\boldsymbol{k}_1, \boldsymbol{k}_{23})G_2(\boldsymbol{k}_2, \boldsymbol{k}_3)\right] + \frac{G_2(\boldsymbol{k}_1, \boldsymbol{k}_2)}{18}\left[7\alpha(\boldsymbol{k}_{12}, \boldsymbol{k}_3) + 2\beta(\boldsymbol{k}_{12}, \boldsymbol{k}_3)\right]$$

$$F_3^s(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \frac{1}{3!}\left[F_3(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) + \text{ permutations}\right]$$

For fixed p: $\quad F_3^s(\boldsymbol{k}, \boldsymbol{p}, -\boldsymbol{p}) \propto \frac{k^2}{p^2}, \quad$ as $k \to 0$

EdS approximation

$$\delta^{(n)}(\boldsymbol{k},t) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t)\cdots\delta_L(\boldsymbol{k}_n,t),$$

$$\theta^{(n)}(\boldsymbol{k},t) = \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t)\cdots\delta_L(\boldsymbol{k}_n,t).$$

Using $\delta_L(k,t) = D_+(t)\delta_L(k,t_0)$

$$\delta^{(n)}(\boldsymbol{k},t) = D_+^n(t) \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t_0)\cdots\delta_L(\boldsymbol{k}_n,t_0),$$

$$\theta^{(n)}(\boldsymbol{k},t) = D_+^n(t) \int_{\boldsymbol{k}_{1\cdots n}=\boldsymbol{k}} G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_L(\boldsymbol{k}_1,t_0)\cdots\delta_L(\boldsymbol{k}_n,t_0).$$

This is called the Einstein-de Sitter approximation. It uses EdS kernels, but ΛCDM linear growth functions $D_+(t)$.

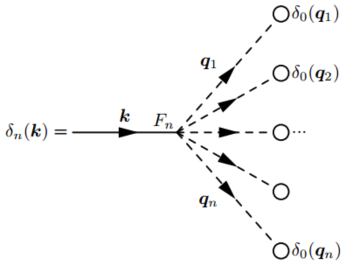
Perturbation Theories

Expansion of fields: $\phi = \lambda\phi^{(1)} + \lambda^2\phi^{(2)} + \lambda^3\phi^{(3)} + \cdots$

$$\phi^{(n)}_{\boldsymbol{k}} \sim [\phi^{(1)}_{\boldsymbol{k}_1} * \phi^{(1)}_{\boldsymbol{k}_2} * \cdots * \phi^{(1)}_{\boldsymbol{k}_n}]_{\sum \boldsymbol{k}_i = \boldsymbol{k}}$$

Linear modes interact to form higher order modes

$$\delta^{(n)}(\boldsymbol{k}) = \int \left(\prod_{m=1}^{n} \frac{d^3 q_m}{(2\pi)^3} \right) (2\pi)^3 \delta_{\rm D}(\boldsymbol{k} - \boldsymbol{q}_{1\cdots n}) F_n(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \delta_L(\boldsymbol{q}_1) \cdots \delta_L(\boldsymbol{q}_n)$$



Gaussian

- A random field is *Gaussian* if it is drawn from a Gaussian distribution function.

$$\langle f(\delta)\rangle = \int_{-\infty}^{\infty} d\delta \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\delta^2}{2\sigma^2}\right] f(\delta), \qquad \sigma^2 = \langle \delta^2 \rangle$$

- Initial conditidions set by Inflation are highly Gaussian

For Gaussian fields X with zero mean $\langle X\rangle = 0$.

$$\langle X(\boldsymbol{x}_1)\cdots X(\boldsymbol{x}_n)\rangle = \sum_{\text{products}} \prod_{\text{pairs } i\neq j} \langle X(\boldsymbol{x}_i) X(\boldsymbol{x}_j)\rangle,$$

which is called *Wick's Theorem*, or *Isserlis' Theorem*. If n is odd, the above correlator vanishes.

$$\begin{aligned}\langle X(\boldsymbol{x}_1)X(\boldsymbol{x}_2)X(\boldsymbol{x}_3)X(\boldsymbol{x}_4)\rangle &= \langle X(\boldsymbol{x}_1)X(\boldsymbol{x}_2)\rangle\langle X(\boldsymbol{x}_3)X(\boldsymbol{x}_4)\rangle \\ &+ \langle X(\boldsymbol{x}_1)X(\boldsymbol{x}_3)\rangle\langle X(\boldsymbol{x}_2)X(\boldsymbol{x}_4)\rangle + \langle X(\boldsymbol{x}_1)X(\boldsymbol{x}_4)\rangle\langle X(\boldsymbol{x}_2)X(\boldsymbol{x}_3)\rangle\end{aligned}$$

Notice $\langle X_1 \cdots X_n\rangle$ has $(n-1)!!$ terms.

Power spectrum

$$(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')P(k) = \langle\delta(\boldsymbol{k})\delta(\boldsymbol{k}')\rangle$$

- Assuming Gaussian linear order fields $\delta^{(s)}$'s

$$\begin{aligned}\langle\delta(\boldsymbol{k})\delta(\boldsymbol{k}')\rangle &= \langle(\delta^{(1)}(\boldsymbol{k})+\delta^{(2)}(\boldsymbol{k})+\delta^{(3)}(\boldsymbol{k})+\cdots)(\delta^{(1)}(\boldsymbol{k}')+\delta^{(2)}(\boldsymbol{k}')+\delta^{(3)}(\boldsymbol{k}')+\cdots)\rangle \\ &= (P_L(k)+2P^{(13)}(k)+P^{(22)}(k)+\cdots)(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}') \\ &= (2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')P^{\rm SPT}_{\text{1-loop}}(k)+\cdots\end{aligned}$$

where $$(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')P^{(nm)}(\boldsymbol{k}) = \langle\delta^{(n)}(\boldsymbol{k})\delta^{(m)}(\boldsymbol{k}')\rangle$$

- $P^{(12)} = \langle\delta^{(1)}\delta^{(2)}\rangle' \sim \int\langle\delta^{(1)}\delta^{(1)}\delta^{(1)}\rangle' = 0,$

No-PNG: The *bispectrum* is non-linear

$$B(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) \sim \langle\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(2)}(\boldsymbol{k}_3)\rangle' + \text{cyclic permutations}$$

clase 6

4 de octubre de 2022

P_{13}

$$P_{13}(k) \equiv P^{(13)}(k) + P^{(31)}(k) = 2P^{(13)}(k)$$

$$\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(3)}(\boldsymbol{k}')\right\rangle = \left\langle \delta^{(1)}(\boldsymbol{k}) \int_{\boldsymbol{k}_{123}=\boldsymbol{k}'} F_3(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle$$

$$= \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9}(2\pi)^3\delta_{\rm D}(\boldsymbol{k}' - \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)F_3(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\langle\delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\rangle$$

$$\begin{aligned}\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle =& \left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\right\rangle\left\langle \delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle \\ &+\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_2)\right\rangle\left\langle \delta^{(1)}(\boldsymbol{k}_3)\delta^{(1)}(\boldsymbol{k}_1)\right\rangle \\ &+\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_3)\right\rangle\left\langle \delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\right\rangle.\end{aligned}$$

But, inside the integral we can symmetrize $(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$: $F_3 \to F_3^s$ and

$$\begin{aligned}\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle &\to 3\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}_1)\right\rangle\left\langle \delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle \\ &= 3(2\pi)^6\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}_1)\delta_{\rm D}(\boldsymbol{k}_2+\boldsymbol{k}_3)P_L(k)P_L(k_2)\end{aligned}$$

P_{13}

$$\left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(3)}(\boldsymbol{k}')\right\rangle = 3(2\pi)^3 \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^3} \delta_{\rm D}(\boldsymbol{k}' - \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\delta_{\rm D}(\boldsymbol{k} + \boldsymbol{k}_1)\delta_{\rm D}(\boldsymbol{k}_2 + \boldsymbol{k}_3)$$
$$\times F_3^s(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) P_L(k) P_L(k_2)$$
$$= 3(2\pi)^3 \int \frac{d^3k_2}{(2\pi)^3} \delta_{\rm D}(\boldsymbol{k}' + \boldsymbol{k}) F_3^s(-\boldsymbol{k}, \boldsymbol{k}_2, -\boldsymbol{k}_2) P_L(k) P_L(k_2)$$
$$= (2\pi)^3 \delta_{\rm D}(\boldsymbol{k}' + \boldsymbol{k}) 3 P_L(k) \int \frac{d^3k_2}{(2\pi)^3} F_3^s(-\boldsymbol{k}, \boldsymbol{k}_2, -\boldsymbol{k}_2) P_L(k_2)$$
$$= (2\pi)^3 \delta_{\rm D}(\boldsymbol{k} + \boldsymbol{k}') 3 P_L(k) \int \frac{d^3p}{(2\pi)^3} F_3^s(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) P_L(p),$$

in the last equality we use $F_3^s(-\boldsymbol{k}, \boldsymbol{k}_2, -\boldsymbol{k}_2) = F_3^s(\boldsymbol{k}, -\boldsymbol{k}_2, \boldsymbol{k}_2)$ and define $\boldsymbol{p} = \boldsymbol{k}_2$.

Then

$$(2\pi)^3 \delta_{\rm D}(\boldsymbol{k} + \boldsymbol{k}') P_{13}(k) \equiv \left\langle \delta^{(1)}(\boldsymbol{k})\delta^{(3)}(\boldsymbol{k}')\right\rangle + \left\langle \delta^{(3)}(\boldsymbol{k})\delta^{(1)}(\boldsymbol{k}')\right\rangle$$

implies

$$\boxed{P_{13}(k) = 6 P_L(k) \int \frac{d^3p}{(2\pi)^3} F_3^s(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) P_L(p)}$$

P_{22}

$$(2\pi)^3 \delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')P_{22}(k) = \langle \delta^{(2)}(\boldsymbol{k})\delta^{(2)}(\boldsymbol{k}')\rangle$$

$$\left\langle \delta^{(2)}(\boldsymbol{k})\delta^{(2)}(\boldsymbol{k}')\right\rangle = \left\langle \int_{\boldsymbol{k}_{12}=\boldsymbol{k}} F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2) \int_{\boldsymbol{k}_{34}=\boldsymbol{k}'} F_2(\boldsymbol{k}_3,\boldsymbol{k}_4)\delta^{(1)}(\boldsymbol{k}_3)\delta^{(1)}(\boldsymbol{k}_4)\right\rangle$$

$$= \int \frac{d^3k_1 d^3k_2 d^3k_3 d^3k_4}{(2\pi)^6} \delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta_{\rm D}(\boldsymbol{k}'-\boldsymbol{k}_3-\boldsymbol{k}_4)F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)F_2(\boldsymbol{k}_3,\boldsymbol{k}_4)\langle\delta_1\delta_2\delta_3\delta_4\rangle$$

with $\delta_m \equiv \delta^{(1)}(\boldsymbol{k}_m, t)$.

$$\langle\delta_1\delta_2\delta_3\delta_4\rangle = \langle\delta_1\delta_2\rangle\langle\delta_3\delta_4\rangle + \langle\delta_1\delta_3\rangle\langle\delta_2\delta_4\rangle + \langle\delta_1\delta_4\rangle\langle\delta_2\delta_3\rangle$$

- The first correlator does not contribute since it yields $\langle\delta_1\delta_2\rangle\langle\delta_3\delta_4\rangle \ni \delta_{\rm D}(\boldsymbol{k}_1+\boldsymbol{k}_2)$; hence, $F_2(\boldsymbol{k}_1,\boldsymbol{k}_2) \to F_2(\boldsymbol{k}_1,-\boldsymbol{k}_1) = 0$.
- The integral is symmetric in k_3 and k_4, hence $\langle\delta_1\delta_3\rangle\langle\delta_2\delta_4\rangle + \langle\delta_1\delta_4\rangle\langle\delta_2\delta_3\rangle = 2\langle\delta_1\delta_3\rangle\langle\delta_2\delta_4\rangle$

Then

$$\langle\delta_1\delta_2\delta_3\delta_4\rangle \to 2\langle\delta_1\delta_3\rangle\langle\delta_2\delta_4\rangle = 2(2\pi)^3\delta_{\rm D}(\boldsymbol{k}_1+\boldsymbol{k}_3)\delta_{\rm D}(\boldsymbol{k}_2+\boldsymbol{k}_4)P_L(k_1)P_L(k_2)$$

P_{22}

$$\left\langle \delta^{(2)}(\boldsymbol{k})\delta^{(2)}(\boldsymbol{k}')\right\rangle = 2(2\pi)^3 \int \frac{d^3k_1 d^3k_2 d^3k_3 d^3k_4}{(2\pi)^6} F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)F_2(\boldsymbol{k}_3,\boldsymbol{k}_4)P_L(k_1)P_L(k_2)$$
$$\times\ \delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta_{\rm D}(\boldsymbol{k}'-\boldsymbol{k}_3-\boldsymbol{k}_4)\delta_{\rm D}(\boldsymbol{k}_1+\boldsymbol{k}_3)\delta_{\rm D}(\boldsymbol{k}_2+\boldsymbol{k}_4)$$
$$= 2(2\pi)^3 \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)F_2(-\boldsymbol{k}_1,-\boldsymbol{k}_2)P_L(k_1)P_L(k_2)\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta_{\rm D}(\boldsymbol{k}'+\boldsymbol{k}_1+\boldsymbol{k}_2)$$
$$= 2(2\pi)^3 \int \frac{d^3k_1}{(2\pi)^6} F_2(\boldsymbol{k}_1,-\boldsymbol{k}'-\boldsymbol{k}_1)F_2(-\boldsymbol{k}_1,\boldsymbol{k}'+\boldsymbol{k}_1)P_L(k_1)P_L(|-\boldsymbol{k}'-\boldsymbol{k}_1|)\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')$$
$$= (2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')2\int \frac{d^3p}{(2\pi)^6}\left[F_2(\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p})\right]^2 P_L(p)P_L(|\boldsymbol{k}-\boldsymbol{p}|)$$

in the last equality we used $F_2(\boldsymbol{k}_1,-\boldsymbol{k}'-\boldsymbol{k}_1) = F_2(-\boldsymbol{k}_1,\boldsymbol{k}'+\boldsymbol{k}_1)$, used the Dirac delta to substitute $\boldsymbol{k}' = -\boldsymbol{k}$, and defined $\boldsymbol{p} = \boldsymbol{k}_1$.

Hence

$$\boxed{P_{22}(k) = 2\int \frac{d^3p}{(2\pi)^3}\left[F_2(\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p})\right]^2 P_L(p)P_L(|\boldsymbol{k}-\boldsymbol{p}|).}$$

SPT power spectrum to 1-loop

$$P^{\text{SPT}}(k) = P_L(k) + P_{\text{1-loop}}(k) + \cdots$$
$$P_{\text{1-loop}}(k) = P_{22}(k) + P_{13}(k)$$

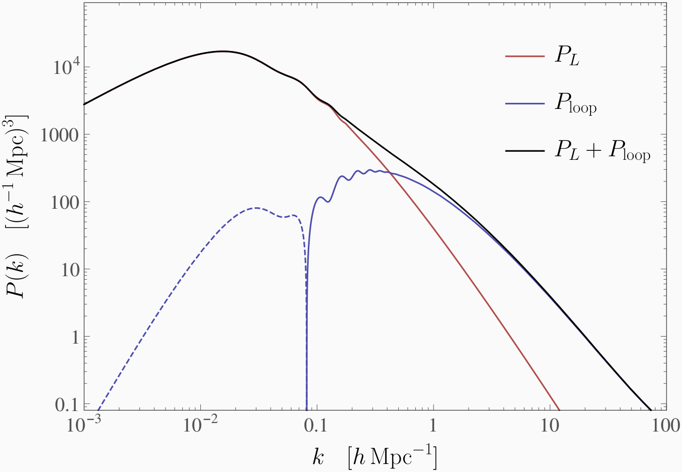
with
$$P_{22}(k) = 2\int \frac{d^3p}{(2\pi)^3}\left[F_2(\boldsymbol{p}, \boldsymbol{k}-\boldsymbol{p})\right]^2 P_L(p)P_L(|\boldsymbol{k}-\boldsymbol{p}|)$$
$$P_{13}(k) = 6P_L(k)\int \frac{d^3p}{(2\pi)^3} F_3^s(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p})P_L(p)$$

Since $P_L(k,t) \propto D_+^2(t)$, hence 1-loop corrections grow as $P_{\text{1-loop}}(k,t) = D_+^4(t)P_{\text{1-loop}}(k,t_0)$. At early times they are suppressed by linear growth.

We can write

$$P^{\text{SPT}}(k,t) = D_+^2(t)P_L(k,t_0) + D_+^4(t)\Big(P_{22}(k,t_0) + P_{13}(k,t_0)\Big) + \cdots$$

SPT power spectrum



$$
\begin{aligned}
2\big[F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})\big]^2 &= 2\left[\frac{5}{7}+\frac{2}{7}\frac{\big(\boldsymbol{p}\cdot(\boldsymbol{k}-\boldsymbol{p})\big)^2}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}+\frac{(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{p}}{2|\boldsymbol{k}-\boldsymbol{p}|p}\left(\frac{p}{|\boldsymbol{k}-\boldsymbol{p}|}+\frac{|\boldsymbol{k}-\boldsymbol{p}|}{p}\right)\right]^2 \\
&= \frac{9}{98}q_1(\boldsymbol{k},\boldsymbol{p})+\frac{3}{7}q_2(\boldsymbol{k},\boldsymbol{p})+\frac{1}{2}q_3(\boldsymbol{k},\boldsymbol{p})
\end{aligned}
$$

with

$$
\begin{aligned}
q_1(\boldsymbol{k},\boldsymbol{p}) &= \left[1-\frac{\big(\boldsymbol{p}\cdot(\boldsymbol{k}-\boldsymbol{p})\big)^2}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}\right]^2 \\
q_2(\boldsymbol{k},\boldsymbol{p}) &= \frac{(\boldsymbol{k}\cdot\boldsymbol{p})\,\big(\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{p})\big)}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}\left[1-\frac{\big(\boldsymbol{p}\cdot(\boldsymbol{k}-\boldsymbol{p})\big)^2}{p^2|\boldsymbol{k}-\boldsymbol{p}|^2}\right] \\
q_3(\boldsymbol{k},\boldsymbol{p}) &= \frac{(\boldsymbol{k}\cdot\boldsymbol{p})^2\,\big(\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{p})\big)^2}{p^4|\boldsymbol{k}-\boldsymbol{p}|^4}
\end{aligned}
$$

Notice $q_{1,2,3}(\boldsymbol{k},\boldsymbol{p})=q_{1,2,3}(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{p})$: A property inherited from the symmetric F_2.

$$6F_3(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) = \frac{10}{21} r_1(\boldsymbol{k}, \boldsymbol{p}) + \frac{6}{7} r_2(\boldsymbol{k}, \boldsymbol{p}) + r_3(\boldsymbol{k}, \boldsymbol{p})$$

with

$$r_1(\boldsymbol{k}, \boldsymbol{p}) = \frac{\left((\boldsymbol{k} \cdot \boldsymbol{p})\boldsymbol{k} - k^2 \boldsymbol{p}\right) \cdot (\boldsymbol{k} - \boldsymbol{p})}{p^2 |\boldsymbol{k} - \boldsymbol{p}|^2} \left[1 - \frac{\left(\boldsymbol{p} \cdot \boldsymbol{k}\right)^2}{p^2 k^2}\right]$$

$$r_2(\boldsymbol{k}, \boldsymbol{p}) = \frac{(\boldsymbol{k} \cdot \boldsymbol{p}) \left(\boldsymbol{k} \cdot (\boldsymbol{k} - \boldsymbol{p})\right)}{p^2 |\boldsymbol{k} - \boldsymbol{p}|^2} \left[1 - \frac{\left(\boldsymbol{p} \cdot \boldsymbol{k}\right)^2}{p^2 k^2}\right]$$

$$r_3(\boldsymbol{k}, \boldsymbol{p}) = -\frac{(\boldsymbol{k} \cdot \boldsymbol{p})^2}{p^4}$$

$$6F_3(\boldsymbol{k},-\boldsymbol{p},\boldsymbol{p}) = \frac{10}{21}r_1(\boldsymbol{k},\boldsymbol{p}) + \frac{6}{7}r_2(\boldsymbol{k},\boldsymbol{p}) - \frac{(\boldsymbol{k}\cdot\boldsymbol{p})^2}{p^4}$$

$$\begin{aligned} P_{13}(k) \ni P_L(k)\int \frac{d^3p}{(2\pi)^3}\left(-\frac{(\boldsymbol{k}\cdot\boldsymbol{p})^2}{p^4}\right)P_L(p) &= -k^2 P_L(k)\int \frac{dp}{4\pi^2}P_L(p)\int_{-1}^{1} dx\, x^2 \\ &= -\sigma_\Psi^2 k^2 P_L(k) \end{aligned}$$

where

$$\sigma_\Psi^2 = \frac{1}{6\pi^2}\int dp P_L(p)$$

is the variance of linear Lagrangian displacements.

$$P_{22}(k) = 2\int \frac{d^3p}{(2\pi)^3}\left[F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})\right]^2 P_L(|\boldsymbol{k}-\boldsymbol{p}|)P_L(p) = \frac{9}{98}Q_1(k) + \frac{3}{7}Q_2(k) + \frac{1}{2}Q_3(k)$$

$$P_{13}(k) = 6P_L(k)\int \frac{d^3p}{(2\pi)^3}F_3(\boldsymbol{k},-\boldsymbol{p},\boldsymbol{p})P_L(p) = \frac{10}{21}R_1(k) + \frac{6}{7}R_2(k) - \sigma_\Psi^2 k^2 P_L(k)$$

with

$$Q_1(k) = \int \frac{d^3p}{(2\pi)^3} q_1(\boldsymbol{k},\boldsymbol{p})P_L(|\boldsymbol{k}-\boldsymbol{p}|)P_L(p)$$

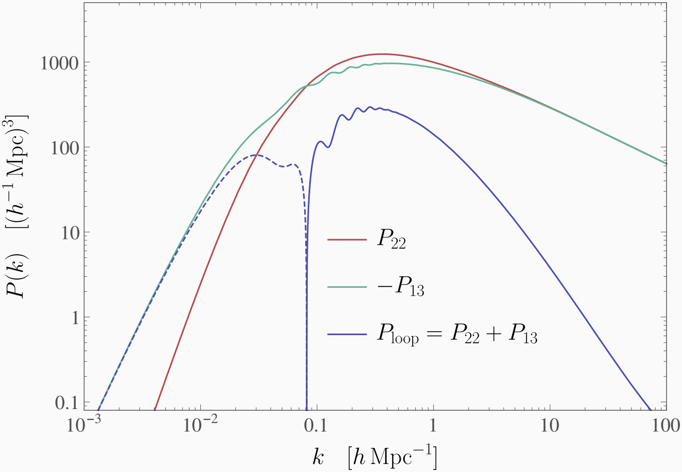
$$Q_2(k) = \int \frac{d^3p}{(2\pi)^3} q_2(\boldsymbol{k},\boldsymbol{p})P_L(|\boldsymbol{k}-\boldsymbol{p}|)P_L(p)$$

$$Q_3(k) = \int \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k},\boldsymbol{p})P_L(|\boldsymbol{k}-\boldsymbol{p}|)P_L(p)$$

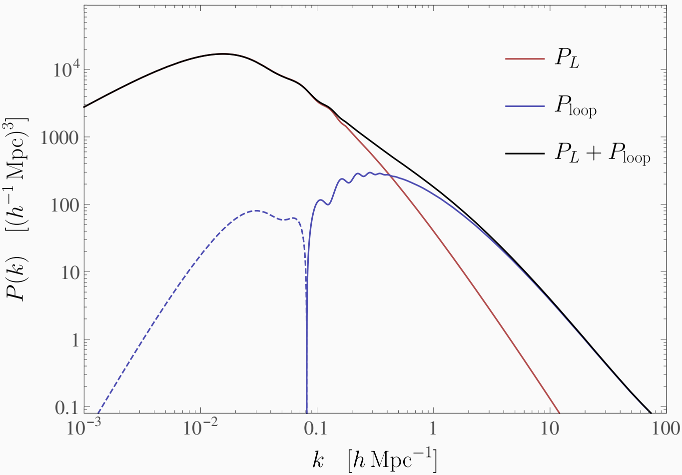
$$R_1(k) = P_L(k)\int \frac{d^3p}{(2\pi)^3} r_1(\boldsymbol{k},\boldsymbol{p})P_L(p)$$

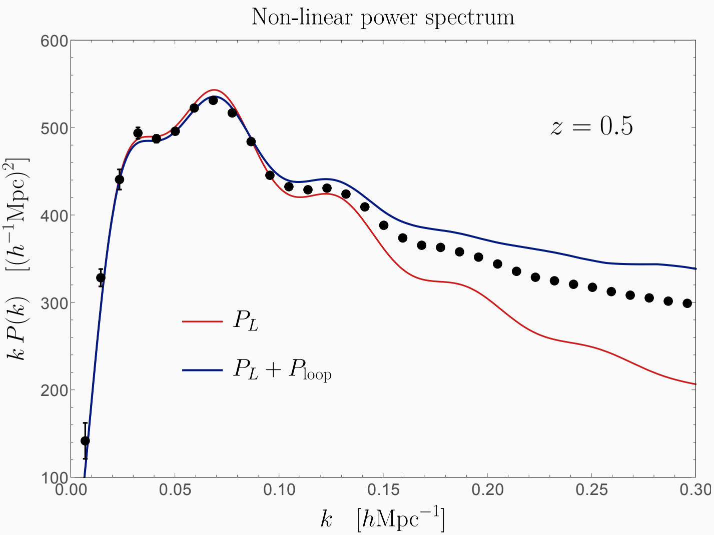
$$R_2(k) = P_L(k)\int \frac{d^3p}{(2\pi)^3} r_2(\boldsymbol{k},\boldsymbol{p})P_L(p)$$

Leading non-linear contributions



SPT power spectrum

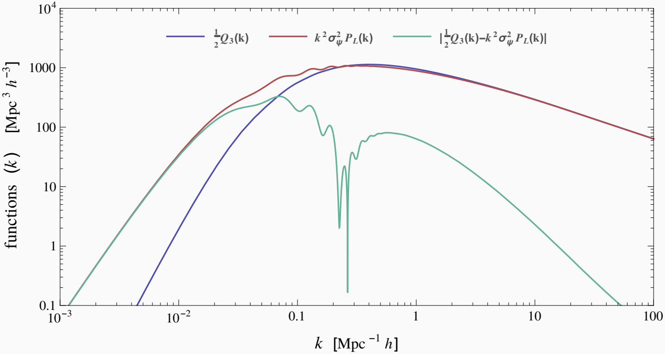




Non-linear power spectrum
$z = 0.5$
P_L
$P_L + P_{\rm loop}$
$k\,P(k) \quad [(h^{-1}\mathrm{Mpc})^2]$
$k \quad [h\mathrm{Mpc}^{-1}]$
600
500
400
300
200
100
0.00
0.05
0.10
0.15
0.20
0.25
0.30

Cancellations of small scales

$$\frac{1}{2}Q_3(k \to \infty) \to k^2 \sigma_\Psi^2 P_L(k)$$



Decoupling of scales

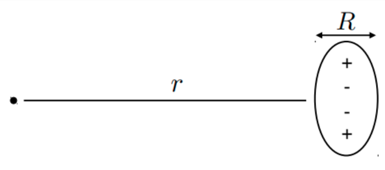


Figure from Blas++ 1408.2995

Assume outside a region R overdensities $\delta(\boldsymbol{x})$ vanish.

$$\delta(\boldsymbol{k},t) = \int d^3x\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\delta(\boldsymbol{x},t) = \int_R d^3x\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\delta(\boldsymbol{x},t)$$

Then, for $k \ll 1/R$

$$\delta(\boldsymbol{k},t) = \int_R d^3x\,\delta(\boldsymbol{x},t) + ik_i \int_R d^3x\, x^i \delta(\boldsymbol{x},t) - k_i k_j \int_R d^3x\, x^i x^j \delta(\boldsymbol{x},t)$$

- The first term vanishes because by definition

$$\int_R d^3x\,\delta(\boldsymbol{x},t) = \int d^3x\,\delta(\boldsymbol{x},t) = 0$$

$$\delta(\boldsymbol{k},t) = ik_i \int_R d^3x\, x^i \delta(\boldsymbol{x},t) + \mathcal{O}(k^2)$$

$$\begin{aligned}
\frac{d}{d\tau}\int_R d^3x\, x^i \delta(\boldsymbol{x},\tau) &= \int_R d^3x\, x^i \dot{\delta}(\boldsymbol{x},\tau) \\
&= \int_R d^3x \Big\{ -x^i \partial_j \Big[(1+\delta)v^j\Big] \Big\} \qquad \text{(using continuity eq.)} \\
&= \int_R d^3x \Big\{ -\partial_j \Big[(1+\delta)v^j x^i\Big] + (1+\delta)v^j \underbrace{\partial_j x^i}_{=\delta^i_j} \Big\} \\
&= -\hat{n}_j (1+\delta)v^j x^i \Big|_{\partial R} + \int_R d^3x\, (1+\delta)v^i \\
&= \frac{1}{\bar{\rho}} \int_R d^3x\, \rho(\boldsymbol{x},\tau) v^i = \frac{1}{\bar{\rho}} \times \text{Total momentum}
\end{aligned}$$

Conservation of momentum implies

$$\int_R d^3x\, x^i \delta(\boldsymbol{x},\tau) = C^i$$

But we can move to a reference frame where the total momentum is zero. Or it is zero after statistical averaging.

That is, short scale perturbations, over scales R affect the large scales overdensities as

$$\delta(\boldsymbol{k}) \sim k^2 \qquad k \ll 1/R$$

This is the *decoupling property of scales* in SPT.

It is the reason why, for $k \ll p$,

$$F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p}) \propto \frac{k^2}{p^2}, \qquad F_3(\boldsymbol{k},-\boldsymbol{p},\boldsymbol{p}) \propto \frac{k^2}{p^2}$$

And more generally

$$F_n(\boldsymbol{k}_1,\boldsymbol{k}_2,\dots,-\boldsymbol{p},\boldsymbol{p}) \propto \frac{k^2}{p^2} \qquad k = |\boldsymbol{k}_1+\boldsymbol{k}_2+\cdots|$$

For P_{22} (and Q functions)

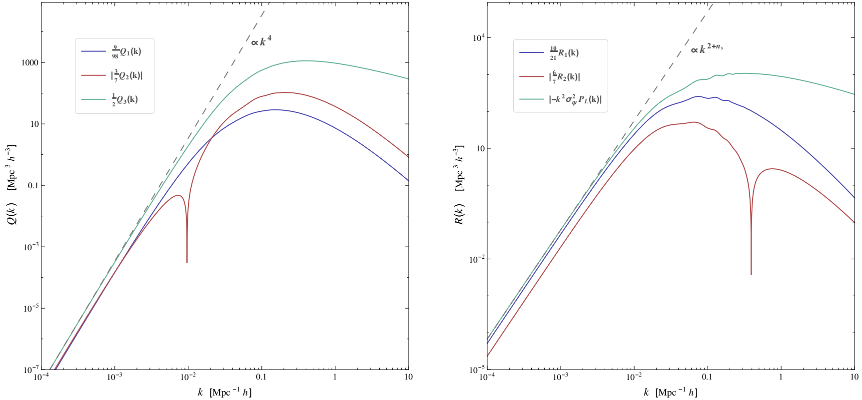
$$P_{22}(k) = \int \frac{d^3p}{(2\pi)^3}[F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})]^2 P_L(|\boldsymbol{k}-\boldsymbol{p}|)P_L(p) \propto k^4 \int \frac{d^3p}{(2\pi)^3}\frac{P_L^2(p)}{p^4}$$

For P_{13} (and R functions)

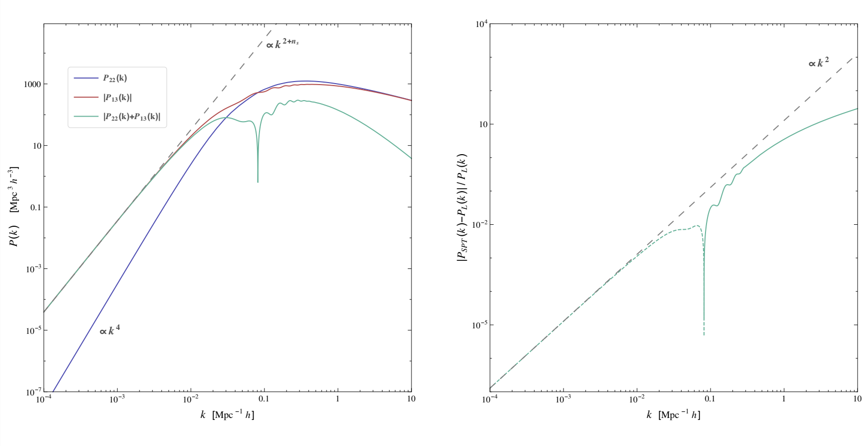
$$P_{13}(k) = 6P_L(k)\int \frac{d^3p}{(2\pi)^3} F_3(\boldsymbol{k},-\boldsymbol{p},\boldsymbol{p})P_L(p) \propto k^2 P_L(k)\int \frac{d^3p}{(2\pi)^3}\frac{P_L(p)}{p^2}$$

At large scales ($k \to 0$), loop corrections are dominated by $P_{13} \propto k^2 P_L(k) \propto k^{2+n_s}$, with $n_s \approx 1$ the primordial spectral index.

Q and R at large scales



Loop contributions at large scales



clase 7

6 de octubre de 2022

Numerical Integration

$\boldsymbol{k} = \boldsymbol{p}$ divergence

$$P_{22}(k) \ni \frac{1}{2} Q_3(k) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p})$$

$$\text{with} \qquad q_3(\boldsymbol{k}, \boldsymbol{p}) = \frac{(\boldsymbol{k} \cdot \boldsymbol{p})^2 \left(\boldsymbol{k} \cdot (\boldsymbol{k} - \boldsymbol{p})\right)^2}{p^4 |\boldsymbol{k} - \boldsymbol{p}|^4} P_L(|\boldsymbol{k} - \boldsymbol{p}|) P_L(p)$$

But $q_3(\boldsymbol{k}, \boldsymbol{p})$ has a divergence when the internal momentum is equal to the external momentum: $\boldsymbol{p} = \boldsymbol{k}$.

$$\begin{aligned} Q_3(k) = \int \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}) &= \int\limits_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}) + \int\limits_{p>|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}) \\ &= \int\limits_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}) + \int\limits_{\tilde{p}<|\boldsymbol{k}-\tilde{\mathbf{p}}|} \frac{d^3\tilde{p}}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{k} - \tilde{\mathbf{p}}) \\ &= 2 \int\limits_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} q_3(\boldsymbol{k}, \boldsymbol{p}). \end{aligned}$$

In the second equality we have split the region of integration in two pieces separated by the $\boldsymbol{p} = \boldsymbol{k}$ divergence. In the second integral of the third equality we redefined the variable $\boldsymbol{p} = \boldsymbol{k} - \tilde{\mathbf{p}}$. In the last equality we use the symmetry $q_3(\boldsymbol{k}, \boldsymbol{p}) = q_3(\boldsymbol{k}, \boldsymbol{k} - \boldsymbol{p})$.

$$\int_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k}, \boldsymbol{p})$$

$$p < |\boldsymbol{k} - \boldsymbol{p}| = (k^2 + p^2 - 2kpx)^{1/2} \quad \Longrightarrow \quad x < \frac{k}{2p},$$

with $x = \hat{k} \cdot \hat{p}$. For scalar rotational invariant function $f(\boldsymbol{k}, \boldsymbol{p}) = f(k, p, x)$

$$\int_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} f(k, p, x) = \int_0^\infty \frac{dp}{4\pi^2} p^2 \int_{-1}^{\mathrm{Min}[1, k/(2p)]} dx\, f(k, p, x)$$

Further define: $\qquad r = \dfrac{p}{k}$

we obtain

$$\int_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k}, \boldsymbol{p}) = \frac{k^3}{4\pi^2} \int_0^\infty dr\, r^2 \int_{-1}^{\mathrm{Min}[1, 1/(2r)]} dx\, f(k, r, x)$$

Integration region for Q functions

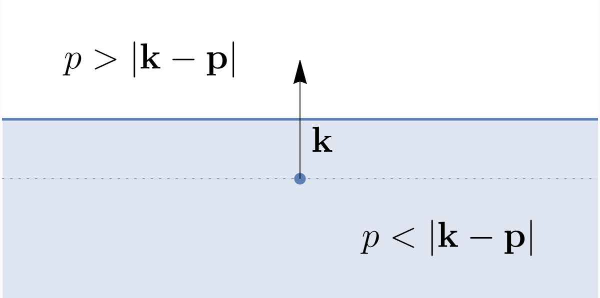
For a general function

$$F(k) = \int \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k}, \boldsymbol{p})$$

with symmetric kernel

$$f(\boldsymbol{k}, \boldsymbol{p}) = f(\boldsymbol{k}, \boldsymbol{k} - \boldsymbol{p})$$

$$F(k) = \int \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k}, \boldsymbol{p}) = 2 \int_{p<|\boldsymbol{k}-\boldsymbol{p}|} \frac{d^3p}{(2\pi)^3} f(\boldsymbol{k}, \boldsymbol{p})$$



For numerical integration

$$x = \frac{\boldsymbol{k} \cdot \boldsymbol{p}}{kp}, \qquad r = p/k$$

$$Q_1(k) = \frac{k^3}{4\pi^2} \int_0^\infty dr P_L(kr) \int_{-1}^1 dx \frac{r^2(1-x^2)^2}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx}),$$

$$Q_2(k) = \frac{k^3}{4\pi^2} \int_0^\infty dr P_L(kr) \int_{-1}^1 dx \frac{rx(1-x^2)(1-rx)}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx})$$

$$Q_3(k) = \frac{k^3}{4\pi^2} \int_0^\infty dr P_L(kr) \int_{-1}^1 dx \frac{x^2(1-rx)^2}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx})$$

$$R_1(k) = \frac{k^3}{4\pi^2} P_L(k) \int_0^\infty dr P_L(kr) \int_{-1}^1 dx \frac{r^2(1-x^2)^2}{1+r^2-2rx}$$

$$R_2(k) = \frac{k^3}{4\pi^2} P_L(k) \int_0^\infty dr P_L(kr) \int_{-1}^1 dx \frac{rx(1-rx)(1-x^2)}{1+r^2-2rx}$$

For numerical integration

$$x = \frac{\boldsymbol{k}\cdot\boldsymbol{p}}{kp}, \qquad r = p/k$$

$$Q_1(k) = 2\frac{k^3}{4\pi^2}\int_0^\infty dr P_L(kr)\int_{-1}^{\mathrm{Min}[1,1/2r]} dx \frac{r^2(1-x^2)^2}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx}),$$

$$Q_2(k) = 2\frac{k^3}{4\pi^2}\int_0^\infty dr P_L(kr)\int_{-1}^{\mathrm{Min}[1,1/2r]} dx \frac{rx(1-x^2)(1-rx)}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx})$$

$$Q_3(k) = 2\frac{k^3}{4\pi^2}\int_0^\infty dr P_L(kr)\int_{-1}^{\mathrm{Min}[1,1/2r]} dx \frac{x^2(1-rx)^2}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx})$$

$$R_1(k) = \frac{k^3}{4\pi^2} P_L(k)\int_0^\infty dr P_L(kr)\int_{-1}^{1} dx \frac{r^2(1-x^2)^2}{1+r^2-2rx}$$

$$R_2(k) = \frac{k^3}{4\pi^2} P_L(k)\int_0^\infty dr P_L(kr)\int_{-1}^{1} dx \frac{rx(1-rx)(1-x^2)}{1+r^2-2rx}$$

$$P^{\mathrm{SPT}}_{\text{1-loop}}(k) = \underbrace{\frac{9}{98}Q_1(k) + \frac{3}{7}Q_2(k) + \frac{1}{2}Q_3(k)}_{P_{22}(k)} + \underbrace{\frac{10}{21}R_1(k) + \frac{6}{7}R_2(k) - \sigma_\Psi^2 k^2 P_L(k)}_{P_{13}(k)}$$

Radial integration

Trapezoidal rule

$$\int_{r_{\min}}^{r_{\max}} dr\, f(r) \approx \sum_{i=2}^{N_r} \frac{f(r_{i-1}) + f(r_i)}{2} \Delta r_i$$

where

$$r_i \in (r_1 = r_{\min}, r_2, \ldots, r_{r_N} = r_{\max})$$

$$\Delta r_i = r_i - r_{i-1}$$

- Higher precision than Riemann quadrature in the same number of steps

Angular integration: Gauss-Legendre quadrature

$$\int_{-1}^{1} dx\, f(x) \approx \sum_{i=1}^{N_x} w_i f(x_i)$$

Such that, the approximation becomes exact for polynomials up to degree $2N_x - 1$ over the interval $[-1, 1]$

- The quadrature nodes, x_i, are the roots of the N_x Legendre polynomial $\mathcal{P}_{N_x}(x)$.
- The quadrature weights, w_i, are given by

$$w_i = \frac{2}{(1 - x_i^2)[\mathcal{P}'_{N_x}(x_i)]^2}$$

For arbitrary intervals:

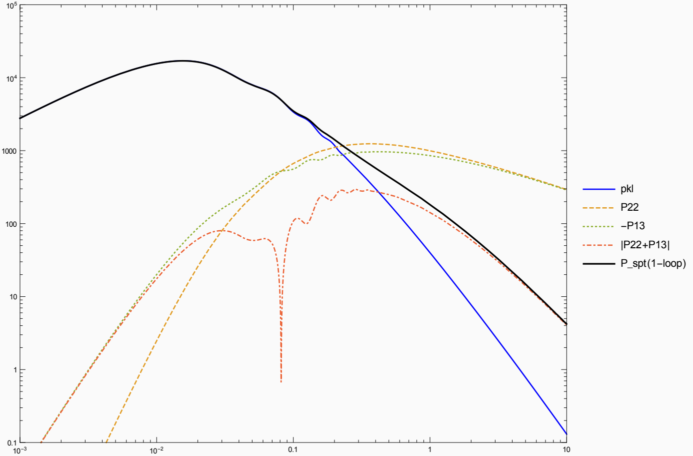
$$\int_a^b dx'\, f(x') = \int_{-1}^{1} dx\, \frac{b-a}{2} f(\frac{b-a}{2} x + \frac{a+b}{2})$$

substitute $\quad x_i \to \frac{b-a}{2} x_i + \frac{a+b}{2}, \qquad w_i \to \frac{b-a}{2} w_i$

Integrate $f(k, r, x)$.

Define arrays of external momenta (kT$[i]$), internal momenta (pT$[i]$) for trapezoidal rule, and nodes and weights (GLnodes$[h]$, GLweights$[h]$) for GL integration.

```
Do i = 1 to N_k                                   /* external momentum */
    k = kT[i];
    f_P = 0;  f_B = 0;  f_A = 0;
    Do j = 1 to N_p                               /* internal momentum integration */
        p = pT[j];  r = p/k;
        Do h = 1 to N_x                           /* angular integration */
            x = GLnodes[h];  w = GLweights[h];
            f_B = f_B + w · f(k, r, x);
        end do
        deltar = (pT[j]− pT[j − 1])/k;
        f_P = f_P + 0.5 · (f_B + f_A) · deltar;
        f_A = f_B;  f_B = 0;
    end do
    print (k, f_P)
end do
```



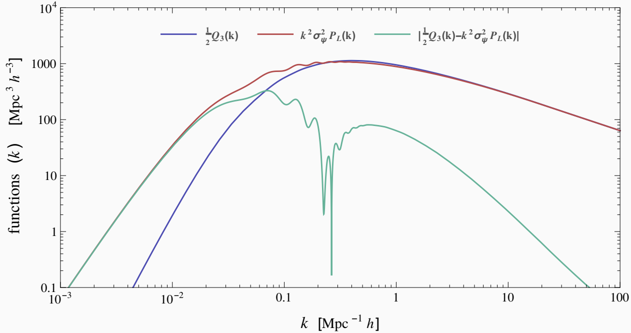
10^5
10^4
1000
100
10
1
0.1
10^{-3}
10^{-2}
0.1
1
10
pkl
P22
−P13
|P22+P13|
P_spt(1−loop)

Cancellations of small scales

$$Q_3(k) = 2\frac{k^3}{4\pi^2}\int_0^\infty dr P_L(kr)\int_{-1}^{\mathrm{Min}[1,\,1/2r]} dx \frac{x^2(1-rx)^2}{(1+r^2-2rx)^2} P_L(k\sqrt{1+r^2-2rx})$$

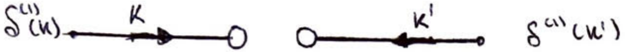
For $k \gg p$ we have $r \ll 1$:

$$\begin{aligned}\frac{1}{2}Q_3(k\to\infty) &\approx \frac{k^3}{4\pi^2}P_L(k)\int_0^\infty dr P_L(kr)\int_{-1}^{1} dx\, x^2 = k^2 P_L(k)\int \frac{dp}{6\pi^2}P_L(p)\\ &= k^2\sigma_\Psi^2 P_L(k)\end{aligned}$$

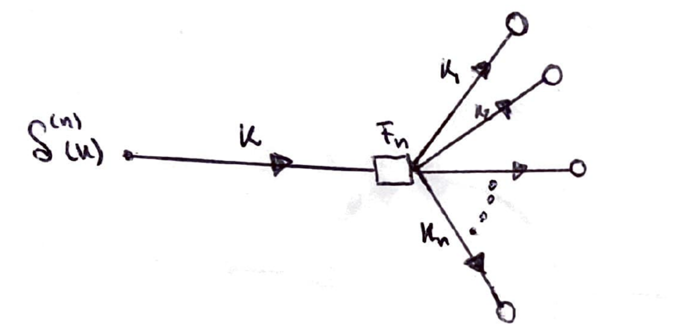


Diagrammatics

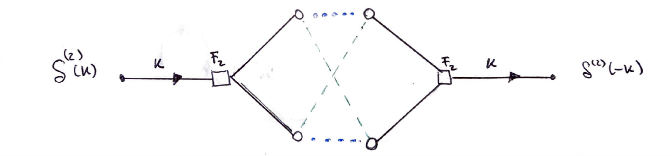
Linear pk

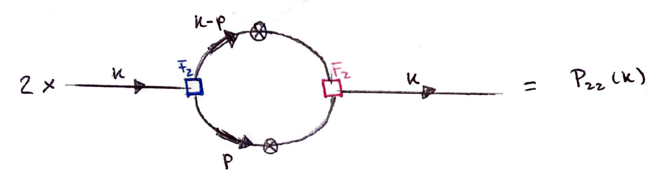


$$\equiv (2\pi)^3 \delta_D(k+k')\, P_L(k)$$

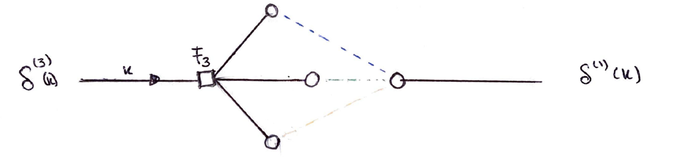


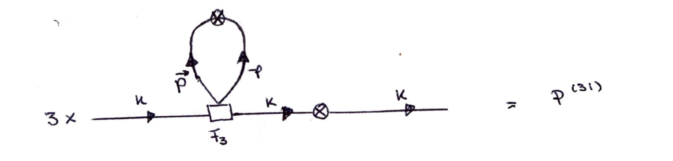
$$\delta^{(n)}(\boldsymbol{k}) = \int_{\boldsymbol{k}_{12\cdots n}=\boldsymbol{k}} F_n(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_n)\delta^{(1)}(\boldsymbol{k}_1)\cdots\delta^{(1)}(\boldsymbol{k}_n)$$





$$
\begin{aligned}
P_{22}(k) &= 2\int \frac{d^3p}{(2\pi)^3} F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p}) F_2(-\boldsymbol{k}+\boldsymbol{p},-\boldsymbol{p}) P_L(|\boldsymbol{k}-\boldsymbol{p}|) P_L(p) \\
&= 2\int \frac{d^3p}{(2\pi)^3} \left[F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})\right]^2 P_L(|\boldsymbol{k}-\boldsymbol{p}|) P_L(p)
\end{aligned}
$$

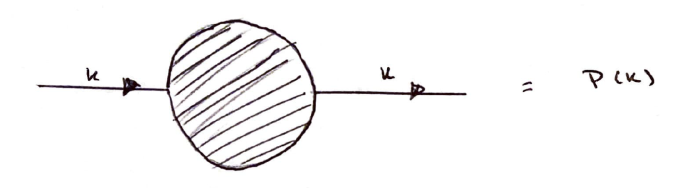




$$P^{(31)}(k) = 3P_L(k) \int \frac{d^3p}{(2\pi)^3} F_3(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) P_L(p)$$

$$P_{13}(k) = 2P^{(31)}(k) = 6P_L(k) \int \frac{d^3p}{(2\pi)^3} F_3(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) P_L(p)$$

P(k)



2-loops

$$P_{\text{2-loops}}(k) = 2\langle\delta^{(5)}\delta^{(1)}\rangle' + 2\langle\delta^{(4)}\delta^{(2)}\rangle' + \langle\delta^{(3)}\delta^{(3)}\rangle'$$

2-loops

$$P_{\text{2-loops}}(k) = 2\langle\delta^{(5)}\delta^{(1)}\rangle' + 2\langle\delta^{(4)}\delta^{(2)}\rangle' + \langle\delta^{(3)}\delta^{(3)}\rangle'$$

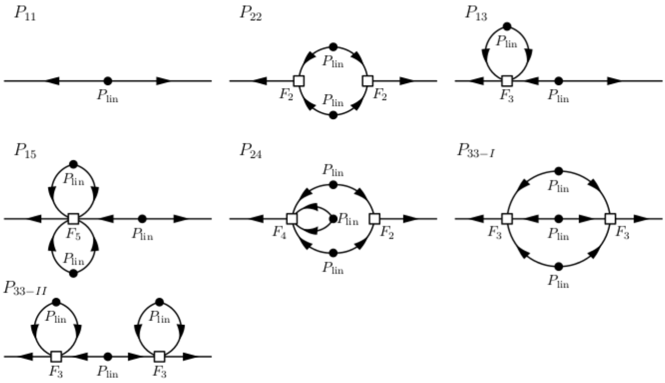
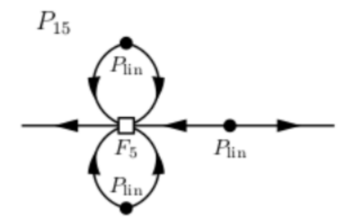


FIG. 1. Diagrams for the tree level, one- and two-loop expressions of the SPT power spectrum.

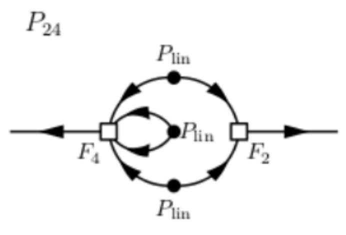
Image from [Baldauf, Mercolli & Zaldarriaga (2016)]

2-loops



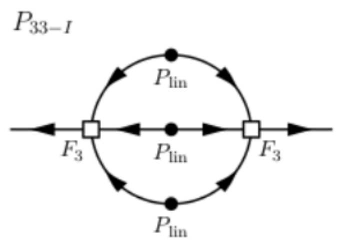
$$P^{(15)}(k) = 15 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} F_5(\boldsymbol{k}, \boldsymbol{p}, -\boldsymbol{p}, \boldsymbol{q}, -\boldsymbol{q}) P_L(k) P_L(q) P_L(p)$$

2-loops



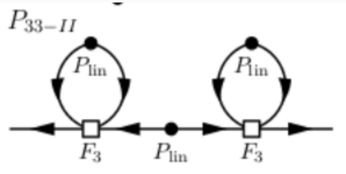
$$P^{(42)}(k) = 12 \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} F_4(\boldsymbol{q}, -\boldsymbol{q}, \boldsymbol{k}-\boldsymbol{p}, \boldsymbol{p}) F_2(\boldsymbol{k}-\boldsymbol{p}, \boldsymbol{p}) P_L(p) P_L(q) P_L(|\boldsymbol{k}-\boldsymbol{p}|)$$

2-loops



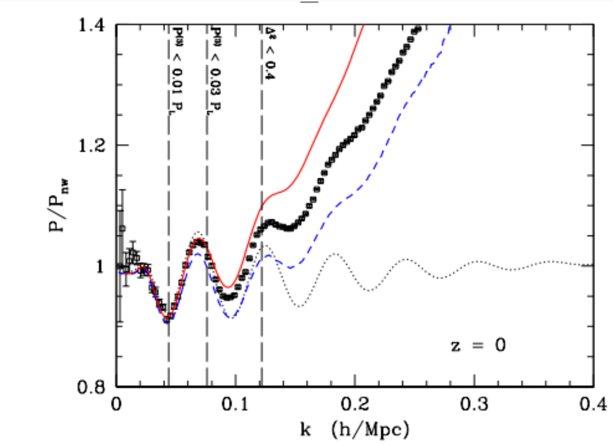
$$P^{(33)I}(k) = 6 \int \frac{d^3p d^3q}{(2\pi)^6} \left[F_3(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}, \boldsymbol{q}, \boldsymbol{p})\right]^2 P_L(p) P_L(q) P_L(|\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}|)$$

2-loops



$$P^{(33)II}(k) = 9 \int \frac{d^3p d^3q}{(2\pi)^6} F_3(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) F_3(\boldsymbol{k}, -\boldsymbol{q}, \boldsymbol{q}) P_L(p) P_L(q) P_L(k)$$

Power spectrum up to 2-loops



[Carlson, White & Padmanabhan (2009)]

clase 8

11 de octubre de 2022

Divergences and Effective Field Theory

UV divergences

Let the internal momentum goes as Λ and send it to infinity. *i.e.* for large p,

$$P_{13} \simeq k^2 P_L(k) \int^{\Lambda} dp P_L(p)$$
$$P_{22} \simeq k^4 \int^{\Lambda} dp \frac{P_L^2(p)}{p^2}$$

Assume scale invariant power spectrum $P_L(p) \propto p^n$, which is a good approximation for large p,

$$P_{13} \simeq k^2 P_L(k) \int^{\Lambda} dp\, p^n \propto \Lambda^{n+1}$$
$$P_{22} \simeq k^4 \int^{\Lambda} dp\, p^{2n-2} \propto \Lambda^{2n-1}$$

- P_{13} power spectrum contributions is UV-divergent for spectral index $n \geq -1$.
- P_{22} power spectrum contributions is UV-divergent for spectral index $n \geq 1/2$

Typically the power spectrum has $n \simeq -2$.

IR divergences

Let the internal momentum goes as ϵ and send it to zero

- P_{13} power spectrum contributions is IR-divergent for spectral index $n \leq -1$.
- P_{22} power spectrum contributions is IR-divergent for spectral index $n \leq -1$

Typically the power spectrum has $n \simeq -2$.

But, $\frac{1}{2}Q_3 \in P_{22}(k)$ and $-\sigma_\Psi^2 k^2 P_L(k) \in P_{13}(k)$ cancel out at leading order, bringing the IR-divergence of the full loop contribution $P_{22} + P_{13}$ appears only for $n \leq -3$

Effective Field Theory

So far, we have followed a standard PT approach. However, loop integrals are of the form $I(k) = \int_{\boldsymbol{p}} K(\boldsymbol{k}, \boldsymbol{p})$ and are computed over all internal momentum space, although $K(\boldsymbol{k}, \boldsymbol{p})$ does not hold at all scales, particularly for high internal momentum.

Though these kernels are typically suppressed for regions $p \gg k$, such that small scales do not affect considerably the $I(k)$ functions at moderate, quasilinear scales, they pose a fundamentally wrong UV behaviour —in particular P_{13}.

The Effective Field Theory for Large Scale Structure (EFT, Baumman et al, arxiv:1004.2488) formalism cuts-off the loop integrals, by directly smoothing the overdensity fields by an arbitrary scale, and introduces counterterms necessary to remove the cut-off dependence on the final expressions.

$$\bar{I}(k) = \int_{\boldsymbol{p}} K(\boldsymbol{k}, \boldsymbol{p}) \quad \longrightarrow \quad I_{reg}(k, \Lambda) = \int_{p<\Lambda} K(\boldsymbol{k}, \boldsymbol{p})$$
$$\longrightarrow \quad I(k) = I_{reg}(k, \Lambda) + I_{ct}(k, \Lambda)$$

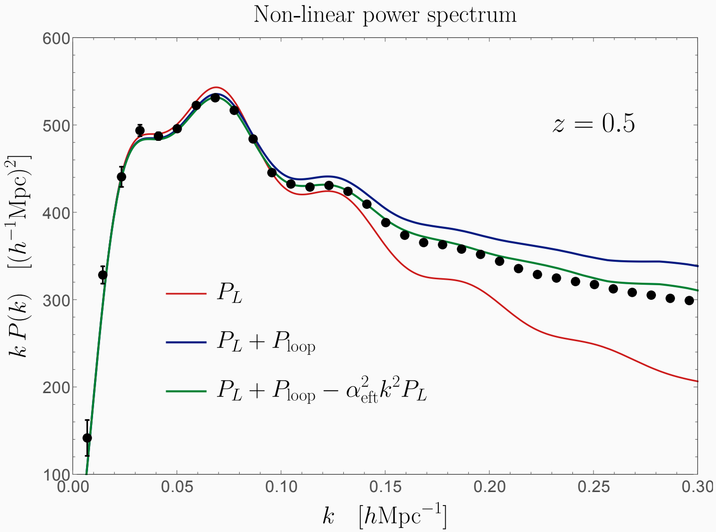
The objective of EFT is to cure the spurious high-k effects on statistics due to non modeled small scale physics, out of the reach of PT.

Further, dark matter evolution is dictated by the Boltzmann equation, and its simplified description with momentum conservation and Euler equation breaks down by nonlinear collapse which makes different streams to converge, leading to non-zero velocity dispersion and higher distribution function momenta. Hence, the very concept of CDM as a coherent fluid at all scales with no velocity dispersion is theoretically inconsistent because of gravitational collapse, breaking down at shell-crossing at best, and very rapidly all the Boltzmann hierarchy is necessary to describe the dynamics; this a key concern of EFT.

For the real space power spectrum, the leading order EFT correction counterterm is given by

$$P_{ct}(k) = -c_s^2(t)(k/k_o)^2 P_L(k) \quad \Rightarrow \; P^{\mathrm{EFT}}(k) = P_L(k) + P_{\text{1-loop}}(k) + P_{ct}(k),$$

with c_s the effective speed of sound of dark matter arising from fluid equations of a non-perfect fluid.



Non-linear power spectrum
$z = 0.5$
P_L
$P_L + P_{\rm loop}$
$P_L + P_{\rm loop} - \alpha_{\rm eft}^2 k^2 P_L$
$k\,P(k) \quad [(h^{-1}{\rm Mpc})^2]$
$k \quad [h{\rm Mpc}^{-1}]$
600
500
400
300
200
100
0.00
0.05
0.10
0.15
0.20
0.25
0.30

Galaxy Bias

ON THE SPATIAL CORRELATIONS OF ABELL CLUSTERS

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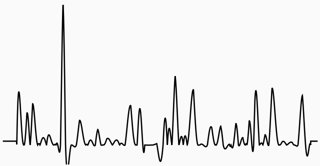
Received 1984 April 2; accepted 1984 June 8

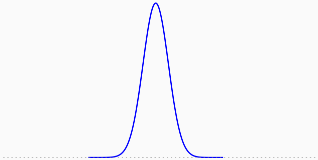
Effective field theory for biased tracers

- We observe biased tracers of the underlying dark matter distribution
- Their properties depend on baryonic and non-linear effects that are out of the reach of PT.
- Theories of bias are EFT.

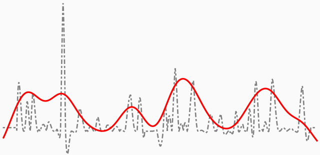
$\delta(\boldsymbol{x})$



$W_R(\boldsymbol{x})$



$$\delta_R(\boldsymbol{x}) = \int d^3x' W_R(\boldsymbol{x} - \boldsymbol{x}')\delta(\boldsymbol{x}')$$



Bias expansion

Consider tracers X with density number $n(\boldsymbol{x})$. The overdensity is

$$\delta_X(\boldsymbol{x}) = \frac{n_X(\boldsymbol{x}) - \bar{n}_X}{\bar{n}_X}, \qquad \langle \delta_X(\boldsymbol{x}) \rangle = 0$$

Introduce a bias function

$$1 + \delta_X(\boldsymbol{x}) = F_{\boldsymbol{x}}[\delta_R; \boldsymbol{x}]$$

and *local bias* parameters

$$c_n = F^{(n)}(0)$$

Expanding about $\delta_R = 0$

$$1 + \delta_X(\mathbf{r}) = c_0 + c_1 \delta_R(\mathbf{r}) + \frac{1}{2} c_2 \delta_R^2(\mathbf{r}) + \frac{1}{6} c_3 \delta_R^3(\mathbf{r}) + \cdots$$

Bias expansion

At large scales $\delta_R(\mathbf{r}) \ll 1$ and we can approximate the expansion as

$$1 + \delta_X(\mathbf{r}) \simeq c_0 + c_1 \delta_R(\mathbf{r})$$

but $\langle \delta_R(\mathbf{r}) \rangle = 0$, since $\langle \delta(\mathbf{r}) \rangle = 0$. Then

$$1 + \delta_X(\mathbf{r}) \simeq 1 + c_1 \delta_R(\mathbf{r})$$

Call $c_1 = b_1$ and one obtains the tracers fluctuation

$$\delta_X(\boldsymbol{x}) = b_1 \delta(\boldsymbol{x})$$

and the power spectrum for tracer X at large scales becomes

$$P_X(k) = \langle \delta_X(\boldsymbol{k}) \delta_X(\boldsymbol{k}') \rangle' = b_1^2 \langle \delta(\boldsymbol{k}) \delta(\boldsymbol{k}') \rangle' = b_1^2 P(k)$$

where $P(k)$ is the matter power spectrum.

Up to linear overdensity fields ($\delta_R = \delta_R^{(1)}$), and third order in bias expansion:

$$
\begin{aligned}
1 + \xi_X(r) &= \langle (1 + \delta_X(\mathbf{r}_1))(1 + \delta_X(\mathbf{r}_2)) \rangle \\
&= \langle (c_0 + c_1 \delta_R(\mathbf{r}_1) + \frac{1}{2} c_2 \delta_R^2(\mathbf{r}_1) + \frac{1}{6} c_3 \delta_R^3(\mathbf{r}_1) + \cdots) \\
&\qquad (c_0 + c_1 \delta_R(\mathbf{r}_2) + \frac{1}{2} c_2 \delta_R^2(\mathbf{r}_2) + \frac{1}{6} c_3 \delta_R^3(\mathbf{r}_2) + \cdots) \rangle \\
&= c_0^2 + c_0 c_2 \langle \delta^2 \rangle + c_1^2 \langle \delta_1 \delta_2 \rangle + \frac{1}{3} c_1 c_3 \langle \delta_1^3 \delta_2 \rangle + \frac{1}{4} c_2^2 \langle \delta_1^2 \delta_2^2 \rangle + \cdots \\
&= (c_0 + \frac{1}{2} c_2 \sigma_R^2 + \cdots)^2 + (c_1 + \frac{1}{2} c_3 \sigma_R^2 + \cdots)^2 \xi_R(r) + \frac{1}{2} (c_2 + \cdots)^2 \xi_R^2(r)
\end{aligned}
$$

zero-lag correlator:

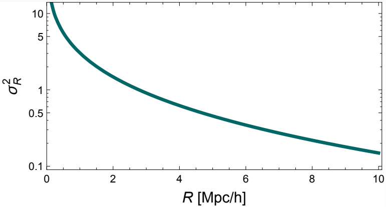
$$
\sigma_R^2 = \langle \delta^2 \rangle = \langle \delta_R(\mathbf{r}_1) \delta_R(\mathbf{r}_1) \rangle = \int \frac{d^3k}{(2\pi)^3} \left[\tilde{W}_R(k) \right]^2 P_L(k) \approx \int_0^{1/R} \frac{dk}{2\pi^2} k^2 P_L(k)
$$

$$
\xi_R(r) = \langle \delta_R(\mathbf{r}_2) \delta_R(\mathbf{r}_1) \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{r}\cdot\boldsymbol{k}} \left[\tilde{W}_R(k) \right]^2 P_L(k).
$$

That is $\sigma_R^2 = \xi_R(0)$.

We used
$$
\begin{aligned}
\langle \delta_1^3 \delta_2 \rangle &= \langle \delta_1 \delta_1 \delta_1 \delta_2 \rangle = 3 \langle \delta_1 \delta_1 \rangle \langle \delta_1 \delta_2 \rangle = 3 \sigma_R^2 \xi_R \\
\langle \delta_1^2 \delta_2^2 \rangle &= \langle \delta_1 \delta_1 \delta_2 \delta_2 \rangle = \langle \delta_1 \delta_1 \rangle \langle \delta_2 \delta_2 \rangle + 2 \langle \delta_1 \delta_2 \rangle \langle \delta_1 \delta_2 \rangle = \sigma_R^4 + 2 \xi_R^2
\end{aligned}
$$

Statistics depend on the cutoff scale R even for very large scales $r \gg R$



Here we used a top-hat potential:

Config. Space:

$$W_R(r) = \frac{3}{4\pi R^3} \quad \text{if} \quad r \leq R, \qquad W_R(r) = 0 \quad \text{if} \quad r > R$$

Fourier Space:

$$\tilde{W}_R(k) = \frac{3}{(kR)^3}\Big[\sin(kR) - kR\cos(kR)\Big]$$

Renormalization

- Reorganization of bias parameters

$$b_0 = c_0 + \frac{1}{2}c_2\sigma_R^2 + \cdots = 1 \qquad b_1 = c_1 + \frac{1}{2}c_3\sigma_R^2 + \cdots$$

$$b_2 = c_2 + \cdots \qquad b_3 = c_3 + \cdots$$

- Correlation function:

$$\begin{aligned} 1 + \xi_X(r) &= (c_0 + \frac{1}{2}c_2\sigma_R^2 + \cdots)^2 + (c_1 + \frac{1}{2}c_3\sigma_R^2 + \cdots)^2\xi_R(r) \\ &\quad + \frac{1}{2}(c_2 + \cdots)^2\xi_R^2(r) + \cdots \\ &= 1 + b_1^2\xi_R(r) + \frac{1}{2}b_2^2\xi_R^2(r) + \cdots \end{aligned}$$

Linear bias

To linear order (at large scales) the galaxy density fluctuation is

$$\delta_g(\boldsymbol{x}) = b_1 \delta(\boldsymbol{x})$$

and the power spectrum becomes

$$P_g(k) = b_1^2 P_L(k)$$

Formal theory of bias

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Clustering of dark matter tracers: Renormalizing the bias parameters

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Clustering of dark matter tracers: generalizing bias for the coming era of precision LSS

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Generalizing bias

- Beyond Taylor expansion in density fields: $\delta_X = \sum_n \frac{c_n}{n!}\delta^n$
- Construct bias operators depending on velocity v_i and gravitational Φ fields.
- Equivalence principle implies biasing depends only on $\nabla_i v_j$ and $\nabla_i \nabla_j \Phi$.

Complete bias expansion

How the gravitational potential affect the evolution of tracers?

- A homogeneous shift in the gravitational field $\Phi \to \Phi + \Phi_0$ is not observable. Suggesting that biasing does not depend on Φ.
- A homogeneous shift in the gravitational force $\nabla^i \Phi \to \nabla^i \Phi + C^i$ can be removed by a change in coordinates. Suggesting that biasing does not depend on $\nabla^i \Phi$.
- Biasing depends on $\nabla_i \nabla_j \Phi$.

$$
\begin{aligned}
1 + \delta_X(\boldsymbol{x}, t) &= F_{\boldsymbol{x}}[\delta, \nabla_i \nabla_j \Phi, \nabla_i v_j; \boldsymbol{x}] = \sum_{\mathcal{O}} c_{\mathcal{O}}(t) \mathcal{O}(\boldsymbol{x}, t) \\
&= 1 + b_1 \delta + \frac{b_2}{2}(\delta^2 - \langle \delta^2 \rangle) + \frac{b_s}{2}(s^2 - \langle s^2 \rangle) + \cdots + \text{stochastic terms}
\end{aligned}
$$

with the stress tensor $\quad s_{ij} = \left(k_i k_j - \dfrac{1}{3} \delta_{ij} k^2 \right) \overbrace{k^{-2} \delta}^{\propto \Phi}$

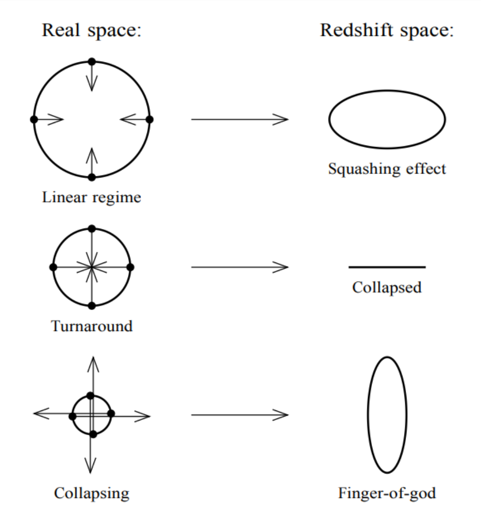
and tidal bias operator $\quad s^2 = s_{ij} s^{ij}.$

clase 9

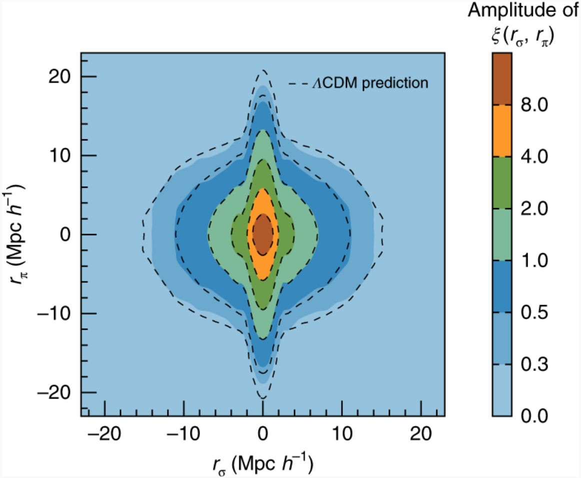
12 de octubre de 2022

Redshift space distortions

Effect of peculiar velocities

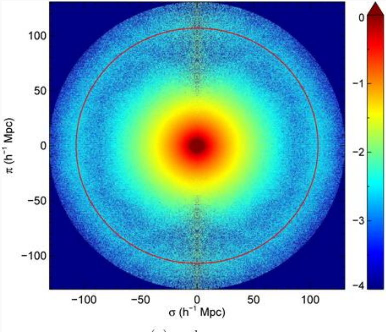


[From Hamilton astro-ph/9708102]

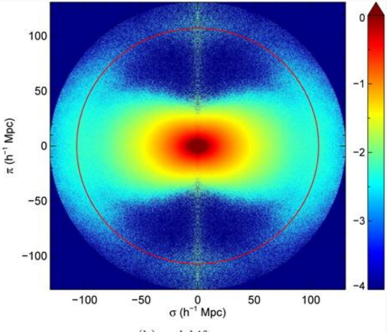


Amplitude of $\xi(r_\sigma, r_\pi)$
8.0
4.0
2.0
1.0
0.5
0.3
0.0
ΛCDM prediction
20
10
0
−10
−20
r_π (Mpc h^{-1})
r_σ (Mpc h^{-1})

Redshift space distortions



(a) real space



(b) redshift space

$$\delta_s(\boldsymbol{k}) = (1 + f\mu^2)\delta(\boldsymbol{k}) \qquad \text{with} \qquad f(t) = \frac{d\ln D_+(t)}{d\ln a(t)}$$

and μ the cosine of the angle between the line-of-sight and the wave-vector $\boldsymbol{k}$.

Redshift-space coordinates

We map objects in the sky by their angular coordinates $\hat{\boldsymbol{x}}$ and apparent positions $\boldsymbol{s}$ inferred by their redshifts z.

- The redshift to an object at a distance $d = ax$ (with $\boldsymbol{x}$ the comoving position) fixed to the Hubble flow is $z = Hd = aHx$.
- However, objects have peculiar velocities v

  $$\boldsymbol{v} = \frac{d\boldsymbol{x}}{d\tau} = a\dot{\boldsymbol{x}},$$

  with $d\tau = a^{-1}dt$ the conformal time. Which induces a longitudinal (non-relativistic) Doppler effect along the line-of-sight direction $\hat{\boldsymbol{x}}$.

This two effects give the total redshift

$$z = \underbrace{aHx}_{\text{Hubble flow}} + \underbrace{\boldsymbol{v} \cdot \hat{\boldsymbol{x}}}_{\text{peculiar velocity}} .$$

That is, an object located at a true position $\boldsymbol{x}$, is observed at an apparent position $\boldsymbol{s}$

$$\boldsymbol{s} = \boldsymbol{x} + \hat{\boldsymbol{x}} \frac{\boldsymbol{v} \cdot \hat{\boldsymbol{x}}}{aH},$$

due to the Doppler effect induced by its peculiar velocity.

Distant observer approximation

We shall assume that the objects of interest are located at a large distance to the observer relative to the angular size of the sample. In this case we can use $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}$ with $\hat{\boldsymbol{n}}$ the (fixed) direction to the sample.

In this case we can write

$$\boldsymbol{s} = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}),$$

with

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{\boldsymbol{v}(\boldsymbol{x}) \cdot \hat{\boldsymbol{n}}}{aH} \hat{\boldsymbol{n}}.$$

Notice that $\boldsymbol{v} \cdot \hat{\boldsymbol{n}}$ is usually written as $v_{\parallel}$.

- See, e.g., Castorina & White [arXiv:1709.09730] for beyond the plane parallel approximation.

Redshift-space density fields

We start with the coordinate transformation between real and redshift coordinates

$$\boldsymbol{s} = \boldsymbol{x} + \boldsymbol{u}$$

with "velocity" $\boldsymbol{u} = \hat{\boldsymbol{n}} \frac{\boldsymbol{v}\cdot\hat{\boldsymbol{n}}}{aH}$, with $\boldsymbol{v}$ the peculiar velocity.

We use matter (or tracers) conservation $\big(1+\delta_s(\boldsymbol{s})\big) d^3s = \big(1+\delta(\boldsymbol{x})\big) d^3x$:

$$\int \delta_{\rm D}(\boldsymbol{s}-\boldsymbol{x}-\boldsymbol{u})\big(1+\delta_s(\boldsymbol{s})\big) d^3s = \int \delta_{\rm D}(\boldsymbol{s}-\boldsymbol{x}-\boldsymbol{u})\big(1+\delta(\boldsymbol{x})\big) d^3x$$

$$1+\delta_s(\boldsymbol{s}) = \int d^3x \int \frac{d^3k'}{(2\pi)^3} e^{i\boldsymbol{k}'\cdot(\boldsymbol{s}-\boldsymbol{x}-\boldsymbol{u})} \big(1+\delta(\boldsymbol{x})\big)$$

$$1+\delta_s(\boldsymbol{s}) = \int \frac{d^3k'}{(2\pi)^3} e^{i\boldsymbol{k}'\cdot\boldsymbol{s}} \int d^3x \big(1+\delta(\boldsymbol{x})\big) e^{-i\boldsymbol{k}'\cdot(\boldsymbol{x}+\boldsymbol{u})}$$

Fourier tranforming (integrating against $\int d^3s\, e^{-i\boldsymbol{k}\cdot\boldsymbol{s}}$), we obtain the overdensity field in redshift space

$$(2\pi)^3\delta_{\rm D}(\boldsymbol{k}) + \delta_s(\boldsymbol{k}) = \int d^3x \big[1+\delta(\boldsymbol{x})\big] e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{u}(\boldsymbol{x}))}$$

NOTICE: the wavenumber here is dual to the redhisft position $\boldsymbol{s}$ and not to the real space postion $\boldsymbol{x}$.

Power spectrum

$$\left\langle [(2\pi)^3\delta_{\rm D}(\boldsymbol{k})+\delta_s(\boldsymbol{k})][(2\pi)^3\delta_{\rm D}(\boldsymbol{k}')+\delta_s(\boldsymbol{k}')]\right\rangle$$
$$=\left\langle \int d^3x_1\,[1+\delta(\boldsymbol{x}_1)]\,e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}_1+\boldsymbol{u}(\boldsymbol{x}_1))}\int d^3x_2\,[1+\delta(\boldsymbol{x}_2)]\,e^{-i\boldsymbol{k}'\cdot(\boldsymbol{x}_2+\boldsymbol{u}(\boldsymbol{x}_2))}\right\rangle$$
$$=\int d^3x_1 d^3x_2 e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_1-i\boldsymbol{k}'\cdot\boldsymbol{x}_2}\left\langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)e^{-i\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{x}_1)-i\boldsymbol{k}'\cdot\boldsymbol{u}(\boldsymbol{x}_2)}\right\rangle.$$

Redefine $\boldsymbol{x}_2\to\boldsymbol{x}=\boldsymbol{x}_2-\boldsymbol{x}_1$, hence

$$\left\langle [(2\pi)^3\delta_{\rm D}(\boldsymbol{k})+\delta_s(\boldsymbol{k})][(2\pi)^3\delta_{\rm D}(\boldsymbol{k}')+\delta_s(\boldsymbol{k}')]\right\rangle$$
$$=\int d^3x_1 d^3x e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}_1}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\left\langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}+\boldsymbol{x}_1)\big)e^{-i\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{x}_1)-i\boldsymbol{k}'\cdot\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{x}_1)}\right\rangle$$

But the correlator $\langle\cdots\rangle$ does not depend on $\boldsymbol{x}_1$ since by homogeneity we can substract $\boldsymbol{x}_1$ from all its arguments without altering it.

Performing the integral d^3x_1 we obtain a Dirac delta function $(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')$. Hence

$$\left\langle [(2\pi)^3\delta_{\rm D}(\boldsymbol{k})+\delta_s(\boldsymbol{k})][(2\pi)^3\delta_{\rm D}(\boldsymbol{k}')+\delta_s(\boldsymbol{k}')]\right\rangle$$
$$=(2\pi)^3\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\left\langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{u}}\right\rangle$$

with $\Delta\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x}_2)-\boldsymbol{u}(\boldsymbol{x}_1)$ and $\boldsymbol{x}=\boldsymbol{x}_2-\boldsymbol{x}_1$.

Power spectrum

$$\langle[(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k})+\delta_s(\boldsymbol{k})][(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}')+\delta_s(\boldsymbol{k}')]\rangle$$
$$=(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k})(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}')+\langle\delta_s(\boldsymbol{k})\delta_s(\boldsymbol{k}')\rangle$$

We also have

$$\delta_{\mathrm{D}}(\boldsymbol{k})\delta_{\mathrm{D}}(\boldsymbol{k}')=\delta_{\mathrm{D}}(\boldsymbol{k})\delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}')$$

and

$$\langle\delta_s(\boldsymbol{k})\delta_s(\boldsymbol{k}')\rangle=(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}')P_s(\boldsymbol{k}).$$

The *Redshift-Space Power Spectrum* is

$$(2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k})+P_s(\boldsymbol{k})=\int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\langle\big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{u}}\rangle$$

with $\Delta\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x}_2)-\boldsymbol{u}(\boldsymbol{x}_1)$ and $\boldsymbol{x}=\boldsymbol{x}_2-\boldsymbol{x}_1$.

Expansion of velocity density-weighted moments

For $\boldsymbol{k} \neq 0$ we can omit the Dirac function, and

$$P_s(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \left\langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big) e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{u}} \right\rangle$$

Expanding in Taylor series the exponential inside the correlator

$$e^{-i\boldsymbol{k}\cdot\Delta\boldsymbol{u}} = e^{-ik_i\Delta u_i} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} k_{i_1}\cdots k_{i_n}\Delta u_{i_1}\cdots\Delta u_{i_n}$$

The power spectrum in the moment expansion approach becomes

$$(2\pi)^3\delta_{\rm D}(\boldsymbol{k}) + P_s(\boldsymbol{k}) = \sum_{\rm m=0}^{\infty} \frac{(-i)^{\rm m}}{\rm m!} k_{i_1}\dots k_{i_{\rm m}} \tilde{\Xi}^{\rm m}_{i_1\cdots i_{\rm m}}(\boldsymbol{k}),$$

with

$$\tilde{\Xi}^{\rm m}_{i_1\cdots i_{\rm m}}(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)\Delta u_{i_1}\cdots\Delta u_{i_{\rm m}}\rangle$$

Notice at linear order only $\mathrm{m} = 0, 1, 2$ are different from zero. e.g. $\mathrm{m} = 3$ has at least 3 fields.

- We broke isotropy by introducing the line-of-sight direction $\hat{\boldsymbol{n}}$. But we still have azimuthal symmetry about $\hat{\boldsymbol{n}}$.
- Hence, the power spectrum only depends on the angle between the wave-vector and the line-of-sight

$$P_s(\boldsymbol{k}) = P_s(k, \mu) \qquad \text{with} \qquad \mu = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}$$

- It is useful then to decompose the matter power spectrum in Legendre multipoles

$$P_\ell(k) = \frac{2\ell + 1}{2} \int_{-1}^{1} d\mu \, P_s(k, \mu) \mathcal{L}_\ell(\mu)$$

with $\mathcal{L}_\ell$ the Legendre polynomial of degree ℓ.

- Because the symmetry $\mu \to -\mu$, only even multipoles are different from zero.
- At linear order only the monopole ($\ell = 0$), quadrupole ($\ell = 2$) and hexadecapole ($\ell = 4$) survive.

Momentum Expansion approach (linear order - Kaiser)

To linear order we can only have two fields on the correlations. So, the sum runs over $\mathrm{m} = 0, 1, 2$

$$P_s(k,\mu) = \tilde{\Xi}^{\mathrm{m}=0}(\boldsymbol{k}) - ik_i \tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) - \frac{1}{2} k_i k_j \tilde{\Xi}_{ij}^{\mathrm{m}=2}(\boldsymbol{k}),$$

with moments

$$\tilde{\Xi}^{\mathrm{m}}_{i_1 \cdots i_{\mathrm{m}}}(\boldsymbol{k}) = \int d^3x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \, \Xi^{\mathrm{m}}_{i_1 \cdots i_{\mathrm{m}}}(\boldsymbol{x}) \quad :$$

$$\Xi^{\mathrm{m}=0}(\boldsymbol{x}) = \langle \big(1+\delta_1\big)\big(1+\delta_2\big) \rangle,$$

$$\Xi_i^{\mathrm{m}=1}(\boldsymbol{x}) = \langle \big(1+\delta_1\big)\big(1+\delta_2\big) \Delta u_i \rangle \overset{!}{=} \langle \Delta u_i (\delta_1 + \delta_2) \rangle,$$

$$\Xi_{ij}^{\mathrm{m}=2}(\boldsymbol{x}) = \langle \big(1+\delta_1\big)\big(1+\delta_2\big) \Delta u_i \Delta u_j \rangle \overset{!}{=} \langle \Delta u_i \Delta u_j \rangle,$$

where the last equality (!) are valid to linear order only, and

.

Momentum Expansion approach (linear order)

- The zero order moment is

$$\Xi^{\mathrm{m}=0}(\boldsymbol{x}) = \langle \big(1+\delta(\boldsymbol{x}_1)\big)\big(1+\delta(\boldsymbol{x}_2)\big)\rangle = 1 + \xi(x),$$

which is the correlation function. In Fourier space

$$\tilde{\Xi}^{\mathrm{m}=0}(\boldsymbol{k}) = \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\big(1+\xi(x)\big) = (2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}) + P_{\delta\delta}(k)$$

the real-space (density-density) power spectrum $P_{\delta\delta}(k) = \langle\delta(\boldsymbol{k})\delta(\boldsymbol{k}')\rangle'$.

Since we are dealing with linear fields we can identify the δ-δ PS with

$$\begin{aligned} P_{\delta\delta}(k) &= P_L(k) \qquad \text{(for matter)},\\ P_{\delta\delta}(k) &= b_1^2 P_L(k) \qquad \text{(for tracers)}. \end{aligned}$$

But we will keep the notation $P_{\delta\delta}(k)$ for the moment.

Momentum Expansion approach (linear order)

- The first velocity moment to linear order is

$$\Xi_i^{\mathrm{m}=1}(\boldsymbol{x}) = \langle \Delta u_i(\delta(\boldsymbol{x}_1) + \delta(\boldsymbol{x}_2))\rangle = 2\langle \Delta u_i \delta(\boldsymbol{x}_1)\rangle,$$

  In Fourier space

$$\tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) = 2\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \Delta u_i \delta(\boldsymbol{x}_1)\rangle$$

To go beyond, we notice that $\theta(\boldsymbol{x}) = -\nabla\cdot\boldsymbol{v}/(aHf)$ and $\boldsymbol{u}(\boldsymbol{x}) = \hat{\boldsymbol{n}}(\boldsymbol{v}\cdot\hat{\boldsymbol{n}})/(aH)$ imply

$$\boldsymbol{u}(\boldsymbol{k}) = if\hat{\boldsymbol{n}}\frac{\boldsymbol{k}\cdot\hat{\boldsymbol{n}}}{k^2}\theta(\boldsymbol{k}).$$

We can write

$$\Delta u_i \equiv u_i(\boldsymbol{x}_2) - u_i(\boldsymbol{x}_1) = \int \frac{d^3p}{(2\pi)^3} \left(e^{i\boldsymbol{p}\cdot\boldsymbol{x}_2} - e^{i\boldsymbol{p}\cdot\boldsymbol{x}_1}\right) \left(if \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2}\hat{n}_i\right) \theta(\boldsymbol{p}).$$

Hence

$$\begin{aligned}
&\langle \delta(\boldsymbol{x}_1)\Delta u_i\rangle \\
&= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1} \left(e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_2} - e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_1}\right) \left(if \frac{\boldsymbol{k}_2\cdot\hat{\boldsymbol{n}}}{k_2^2}\hat{n}_i\right) \langle \delta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)\rangle \\
&= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1} \left(e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_2} - e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_1}\right) \left(if \frac{\boldsymbol{k}_2\cdot\hat{\boldsymbol{n}}}{\boldsymbol{k}_2^2}\hat{n}_i\right) (2\pi)^3\delta_{\mathrm{D}}(\boldsymbol{k}_1+\boldsymbol{k}_2) P_{\delta\theta}(k_1) \\
&= -if\hat{n}_i \int \frac{d^3k_1}{(2\pi)^3} \left(e^{-i\boldsymbol{k}_1\cdot\boldsymbol{x}} - 1\right) \frac{\boldsymbol{k}_1\cdot\hat{\boldsymbol{n}}}{k_1^2} P_{\delta\theta}(k_1) = if\hat{n}_i \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2} P_{\delta\theta}(p),
\end{aligned}$$

In the last but one equality, the term containing the “1” vanishes because

$$\int \frac{d^3p}{(2\pi)^3} \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2} P_{\delta\theta}(p) = \int \frac{dp}{2\pi^2} p^2 \frac{P_{\delta\theta}(p)}{p} \hat{n}_i \frac{1}{4\pi} \int d\Omega_{\hat{\boldsymbol{p}}} \hat{p}^i = 0,$$

because the angular integral of $\hat{p}^i$ is zero. (In general, the solid angle integral of an odd number of $\hat{p}^{i_1}\cdots\hat{p}^{i_n}$ vanishes.)

$$
\begin{aligned}
\tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) &= 2\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \Delta u_i \delta(\boldsymbol{x}_1)\rangle \\
&= 2\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} (if\hat{n}_i) \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2} P_{\delta\theta}(p) \\
&= 2if\hat{n}_i \int \frac{d^3p}{(2\pi)^3} (2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{p}-\boldsymbol{k}) \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{n}}}{p^2} P_{\delta\theta}(p) \\
&= 2if\hat{n}_i \frac{\boldsymbol{k}\cdot\hat{\boldsymbol{n}}}{k^2} P_{\delta\theta}(k)
\end{aligned}
$$

Hence, in Fourier space, and (we remark) to linear order

$$
\tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) = 2if\hat{n}_i \frac{\mu}{k} P_{\delta\theta}(k),
$$

and

$$
P_s(\boldsymbol{k}) \ni -ik_i \tilde{\Xi}_i^{\mathrm{m}=1}(\boldsymbol{k}) = 2f\mu^2 P_{\delta\theta}(k).
$$

Analogoulsy, we compute $\Xi_{ij}^{m=2}(\boldsymbol{x}) = \langle (1+\delta_1)(1+\delta_2)\Delta u_i \Delta u_j \rangle \overset{!}{=} \langle \Delta u_i \Delta u_j \rangle$:

$$\begin{aligned} \langle \Delta u_i \Delta u_j \rangle &= \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \left(e^{i\boldsymbol{k}_1 \cdot \boldsymbol{x}_2} - e^{i\boldsymbol{k}_1 \cdot \boldsymbol{x}_1}\right)\left(e^{i\boldsymbol{k}_2 \cdot \boldsymbol{x}_2} - e^{i\boldsymbol{k}_2 \cdot \boldsymbol{x}_1}\right) \\ &\quad \times \left(if \frac{\boldsymbol{k}_1 \cdot \hat{\boldsymbol{n}}}{k_1^2} \hat{n}_i\right)\left(if \frac{\boldsymbol{k}_2 \cdot \hat{\boldsymbol{n}}}{k_2^2} \hat{n}_i\right) \langle \theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)\rangle \\ &= 2f^2\sigma_v^2 \hat{n}_i \hat{n}_j - 2f^2 \hat{n}_i \hat{n}_j \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{(\boldsymbol{p}\cdot\hat{\boldsymbol{n}})^2}{p^4} P_{\theta\theta}(p) \end{aligned}$$

with the velocity variance

$$\sigma_v^2 = \frac{1}{6\pi^2} \int_0^\infty dp P_{\theta\theta}(p).$$

The second moment, to linear order, is

$$\tilde{\Xi}_{ij}^{m=2}(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{p}} \langle \Delta u_i \Delta u_j \rangle = -2f^2 \hat{n}_i \hat{n}_j \frac{\mu^2}{k^2} P_{\theta\theta}(k),$$

up to a Dirac delta function. Hence

$$P_s(\boldsymbol{k}) \ni -\frac{1}{2} k_i k_j \tilde{\Xi}_{ij}^{\mathrm{m}=2}(\boldsymbol{k}) = f^2 \mu^4 P_{\theta\theta}(k).$$

Kaiser power spectrum

Summing up the three contributions

$$P_s(k,\mu) = \tilde{\Xi}^{\mathrm{m}=0}(\boldsymbol{k}) - ik_i\tilde{\Xi}_i^{\mathrm{m}=1,ud}(\boldsymbol{k}) - \frac{1}{2}k_ik_j\tilde{\Xi}_{ij}^{\mathrm{m}=2,dd}(\boldsymbol{k}),$$

we arrive to

$$P_s(k,\mu) = P_{\delta\delta}(k) + f\mu^2 P_{\delta\theta}(k) + f^2\mu^4 P_{\theta\theta}(k)$$

To linear order $\delta_m^{(1)} = \theta_m^{(1)}$.

Considering linear biased tracers: $\delta^{(1)} = b_1\delta_m^{(1)}$, and $\theta^{(1)} = \theta_m^{(1)}$.

Hence, for tracers

$$P_{\delta\delta} = b_1^2 P_L(k), \qquad P_{\delta\theta} = b_1 P_L(k), \qquad P_{\theta\theta} = P_L(k)$$

We obtain the *Kaiser power spectrum*

$$\boxed{P_s^K(k,\mu) = (1+\beta\mu^2)^2 b_1^2 P_L(k) \qquad \text{with} \quad \beta = \frac{f}{b_1}}$$

The factor $(1+\beta\mu^2)^2$ is called the *Kaiser boost*.

Multipoles of the Kaiser power spectrum

Using

$$P_\ell^K(k) = \frac{2\ell+1}{2}\int_{-1}^{1} P_s^K(k,\mu)\mathcal{L}_\ell(\mu),$$

with $\mathcal{L}_0(\mu) = 1$, $\mathcal{L}_2(\mu) = \frac{3}{2}(\mu^2 - \frac{1}{3})$ and $\mathcal{L}_4(\mu) = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$ we obtain

$$\begin{aligned}
P_0^K(k) &= \left(1 + \frac{2}{3}\beta + \frac{1}{5}\beta^2\right) b_1^2 P_L(k) \qquad \text{(monopole)},\\
P_2^K(k) &= \left(\frac{4}{3}\beta + \frac{4}{7}\beta^2\right) b_1^2 P_L(k) \qquad \text{(quadrupole)},\\
P_4^K(k) &= \frac{8}{35}\beta^2 b_1^2 P_L(k) \qquad \text{(hexadecapole)},
\end{aligned}$$

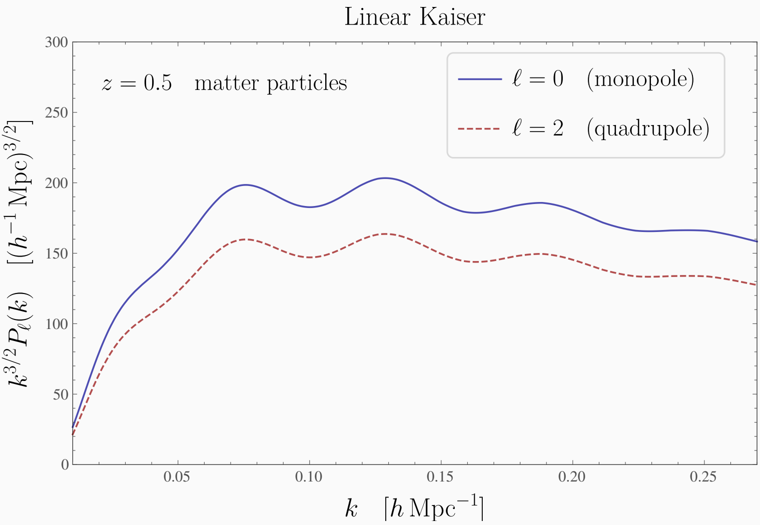
higher degree moments are zero.

Remind $\beta = f/b_1$ with

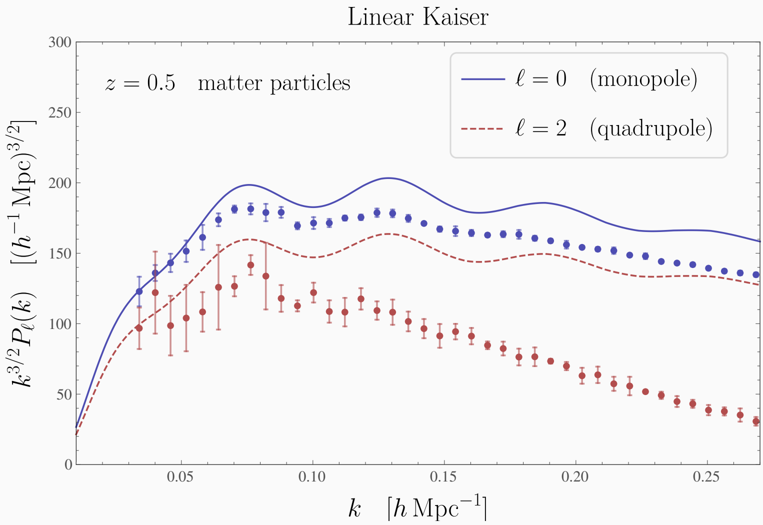
$$f = \frac{d\log D_+(t)}{d\log a(t)},$$

the growth rate.

Kaiser power spectrum

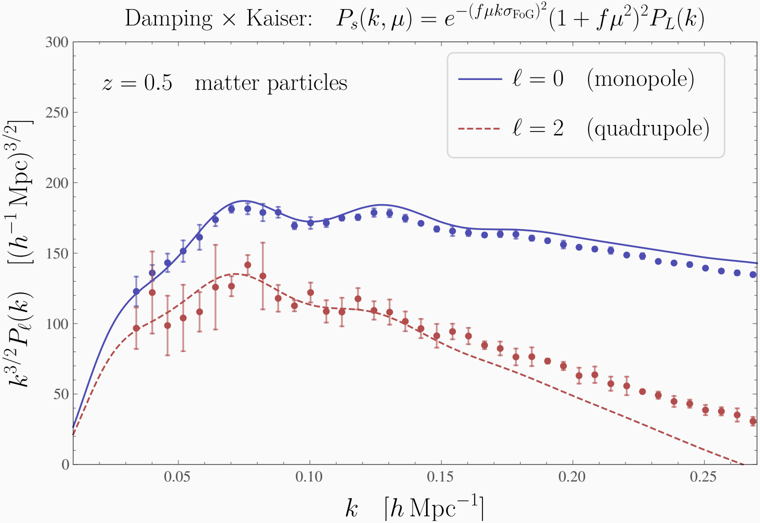


Kaiser power spectrum

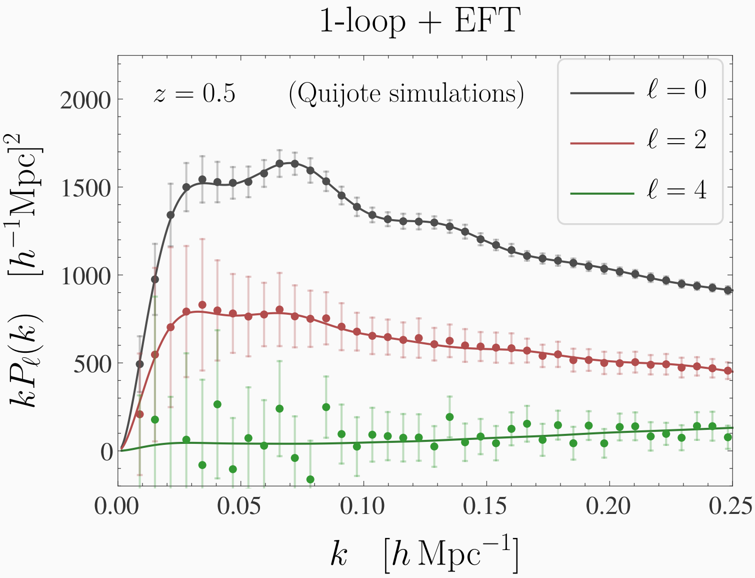


The main problem is that the fingers-of-god are a completely non-linear effect. Fingers-of-god lessen the observed clustering along the line-of-sight ($\mu \neq 0$). Hence, a simple prescription is to add a damping factor along the line-of-sight.

$$P_s(k,\mu) = \underbrace{e^{-(k\mu f \sigma_{\mathrm{FoG}})^2}}_{\text{Fingers of God}} \times \underbrace{\left(1 + f\mu^2\right)^2 P_L(k)}_{\text{Kaiser effect}}$$

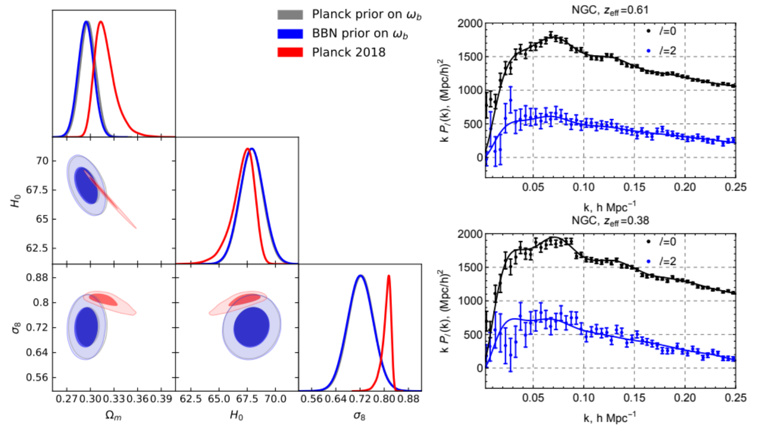

Damping × Kaiser: $P_s(k,\mu) = e^{-(f\mu k\sigma_{\rm FoG})^2}(1+f\mu^2)^2 P_L(k)$
$z = 0.5$ matter particles
$\ell = 0$ (monopole)
$\ell = 2$ (quadrupole)
$k^{3/2}P_\ell(k)$ $[(h^{-1}\,\mathrm{Mpc})^{3/2}]$
k $[h\,\mathrm{Mpc}^{-1}]$
0
50
100
150
200
250
300
0.05
0.10
0.15
0.20
0.25

State of the art theory vs sims.



Cosmological parameters from the BOSS galaxy power spectrum

Mikhail M. Ivanov,[a,b] Marko Simonović[c] and Matias Zaldarriaga[d]



arxiv:1909.05277

¡Gracias!

slides después del curso

$$P_s(k) = A_s \left(\frac{k}{k_p}\right)^{n_s - 1} \tag{1}$$

$$P_s(k) = A_{s'} \left(\frac{k}{k_{p'}}\right)^{n_s - 1} \quad \text{con} \quad A_{s'} \equiv \frac{A_s k_{p'}^{n_s-1}}{k_p^{n_s-1}} \tag{2}$$

$$A_{s'} = \frac{A_s k_{p'}^{n_s-1}}{k_p^{n_s-1}} \tag{3}$$