Notes on the momentum expansion approach to RSD modeling

Alejandro Aviles (avilescervantes@gmail.com)

These notes seem to be correct, though a double check is worthy. We will use some of the methods of [1, 2].

1 RSD effects in matter clustering

As we observe objects in the sky we map them through their angular position, $\hat{\mathbf{n}}$, and the radial position as inferred from their redshift. The latter is given by the Hubble flow and their peculiar velocity $\mathbf{v}$. Hence, an object located at a comoving coordinate $\mathbf{x}$ is observed to be at an apparent position $\mathbf{s}$, such that the mapping between real and redshift space positions is given by the (non-relativistic) Doppler effect,

$$\mathbf{s} = \mathbf{x} + \mathbf{u} \tag{1.1}$$

with "velocity" $\mathbf{u}$ defined as

$$\mathbf{u} \equiv \hat{\mathbf{n}} \frac{\mathbf{v} \cdot \hat{\mathbf{n}}}{aH}. \tag{1.2}$$

Conservation of number of objects is

$$\big(1 + \delta_s(\mathbf{s})\big) d^3 s = \big(1 + \delta(\mathbf{x})\big) d^3 x, \tag{1.3}$$

or

$$(2\pi)^3 \delta_{\rm D}(\mathbf{k}) + \delta_s(\mathbf{k}) = \int d^3 x \big(1 + \delta(\mathbf{x})\big) e^{-i\mathbf{k}\cdot(\mathbf{x}+\mathbf{u}(\mathbf{x}))}, \tag{1.4}$$

The redshift-space power spectrum, $P_s(\mathbf{k}) = \langle \delta_s(\mathbf{k}) \delta_s(\mathbf{k}') \rangle'$, becomes

$$(2\pi)^3 \delta_{\rm D}(\mathbf{k}) + P_s(\mathbf{k}) = \int d^3 x e^{-i\mathbf{k}\cdot\mathbf{x}} \Big[1 + \mathcal{M}(\mathbf{J} = \mathbf{k}, \mathbf{x})\Big] \tag{1.5}$$

with generating function

$$1 + \mathcal{M}(\mathbf{J}, \mathbf{x}) = \Big\langle \big(1 + \delta(\mathbf{x}_1)\big)\big(1 + \delta(\mathbf{x}_2)\big) e^{-i\mathbf{J}\cdot\Delta\mathbf{u}} \Big\rangle \tag{1.6}$$

and $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$ and $\Delta\mathbf{u} = \mathbf{u}(\mathbf{x}_2) - \mathbf{u}(\mathbf{x}_1)$.

Throughout we will use the logarithmic growth factor, $f_0(t)$,

$$f_0 \equiv \frac{d \log D_+(t)}{d \log a(t)}. \tag{1.7}$$

That is, we use the notation f_0 instead of the usual f.

In the following, we will use the distant observer approximation, where $\hat{\mathbf{n}}$ is a constant vector in the direction of the survey, instead of being equal to the position unit vector $\hat{\mathbf{x}}$.

2 Expantion in moments

The m-th density weighted velocity field moment of the generating function is an m-rank tensor defined as

$$\Xi^{\mathrm{m}}_{i_1\cdots i_{\mathrm{m}}}(\mathbf{x}) \equiv i^{\mathrm{m}} \frac{\partial^{\mathrm{m}}}{\partial J_{i_1}\cdots \partial J_{i_{\mathrm{m}}}} \big[1+\mathcal{M}(\mathbf{J},\mathbf{x})\big]\Big|_{\mathbf{J}=0} = \langle \big(1+\delta_1\big)\big(1+\delta_2\big)\Delta u_{i_1}\cdots \Delta u_{i_{\mathrm{m}}}\rangle, \tag{2.1}$$

with $\delta_1 = \delta(\mathbf{x}_1)$ and $\delta_2 = \delta(\mathbf{x}_2)$. The RSD power spectrum is then

$$(2\pi)^3\delta_{\mathrm{D}}(\mathbf{k}) + P_s(\mathbf{k}) = \sum_{n=0}^{\infty} \frac{(-i)^{\mathrm{m}}}{\mathrm{m}!} k_{i_1}\dots k_{i_{\mathrm{m}}} \tilde{\Xi}^{\mathrm{m}}_{i_1\cdots i_{\mathrm{m}}}(\mathbf{k}), \tag{2.2}$$

with

$$\tilde{\Xi}^{\mathrm{m}}_{i_1\cdots i_n}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \Xi^{\mathrm{m}}_{i_1\cdots i_{\mathrm{m}}}(\mathbf{x}) \tag{2.3}$$

That we write here as

$$P_s(\mathbf{k}) = \sum_{\mathrm{m}=0}^{\infty} P^{\mathrm{m}}(k,\mu) \tag{2.4}$$

We introduce the "velocity" θ as

$$\theta(\mathbf{x}) = -\frac{\nabla\cdot\mathbf{v}}{aHf_0} \quad\Longrightarrow\quad \theta(\mathbf{k}) = -\frac{i\mathbf{k}\cdot\mathbf{v}(\mathbf{k})}{aHf_0} \tag{2.5}$$

and

$$u_i(\mathbf{k}) = i f_0 \hat{n}_i \frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2}\theta(\mathbf{k}). \tag{2.6}$$

Only the terms $\mathrm{m}=0,1,2,3,4$ contribute to the PS in eq. (2.2) to 1-loop:

$$P_s(\mathbf{k}) = \tilde{\Xi}^0(\mathbf{k}) - ik_i\tilde{\Xi}^1_i(\mathbf{k}) - \frac{1}{2}k_ik_j\tilde{\Xi}^2_{ij}(\mathbf{k}) + \frac{i}{6}k_ik_jk_k\tilde{\Xi}^3_{ijk}(\mathbf{k}) + \frac{1}{24}k_ik_jk_kk_l\tilde{\Xi}^4_{ijkl}(\mathbf{k}). \tag{2.7}$$

In the following, we compute the moments $\tilde{\Xi}^{\mathrm{m}=0,1,2,3,4}(\mathbf{k})$.

3 Zero order moments

$$\Xi^0(\mathbf{x}) = \langle \big(1+\delta_1\big)\big(1+\delta_2\big)\rangle = 1 + \langle\delta_1\delta_2\rangle = 1+\xi(\mathbf{x}) \tag{3.1}$$

Hence

$$\boxed{\tilde{\Xi}^0(\mathbf{k}) = P_{\delta\delta}(k),} \tag{3.2}$$

up to a Dirac delta function at $\mathbf{k}=0$.

4 First order moments

$$\begin{aligned}\Xi^1_i(\mathbf{x}) &= \langle \big(1+\delta_1\big)\big(1+\delta_2\big)\Delta u_i\rangle = \langle \Delta u_i(\delta_1+\delta_2)\rangle + \langle \Delta u_i\delta_1\delta_2\rangle &(4.1)\\ &\equiv \Xi^{1,ud}_i(\mathbf{x}) + \Xi^{1,udd}_i(\mathbf{x}) &(4.2)\end{aligned}$$

Hence

$$\tilde{\Xi}^1_i(\mathbf{k}) = \tilde{\Xi}^{1,ud}_i(\mathbf{k}) + \tilde{\Xi}^{1,udd}_i(\mathbf{k}) \tag{4.3}$$

4.1 Moment (1,ud)

We start with

$$\begin{aligned}\langle\delta(\mathbf{x}_1)\Delta u_i\rangle &= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\left(e^{i\mathbf{k}_2\cdot\mathbf{x}_2}-e^{i\mathbf{k}_2\cdot\mathbf{x}_1}\right)\left(if_0\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}\hat{n}_i\right)\langle\delta(\mathbf{k}_1)\theta(\mathbf{k}_2)\rangle \\ &= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\left(e^{i\mathbf{k}_2\cdot\mathbf{x}_2}-e^{i\mathbf{k}_2\cdot\mathbf{x}_1}\right)\left(if_0\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}\hat{n}_i\right)(2\pi)^3\delta_{\mathrm{D}}(\mathbf{k}_1+\mathbf{k}_2)P_{\delta\theta}(k_1) \\ &= -if_0\hat{n}_i\int\frac{d^3k_1}{(2\pi)^3}\left(e^{-i\mathbf{k}_1\cdot\mathbf{x}}-1\right)\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}P_{\delta\theta}(k_1) = if_0\hat{n}_i\int\frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}P_{\delta\theta}(p) \end{aligned} \tag{4.4}$$

Hence

$$\tilde{\Xi}_i^{1,ud_1}(\mathbf{k}) = if_0\hat{n}_i\frac{\mu}{k}P_{\delta\theta}(k) \tag{4.5}$$

and

$$-ik_i\tilde{\Xi}_i^{1,ud_1}(\mathbf{k}) = f_0\mu^2P_{\delta\theta}(k). \tag{4.6}$$

Following the same steps as above we have $\tilde{\Xi}_i^{1,ud_2}(\mathbf{k}) = \tilde{\Xi}_i^{1,ud_1}(\mathbf{k})$. Hence

$$-ik_i\tilde{\Xi}_i^{1,ud}(\mathbf{k}) = 2f_0\mu^2P_{\delta\theta}(k) \tag{4.7}$$

4.2 Moment (1,udd)

$$\begin{aligned}\langle\Delta u_i\delta_1\delta_2\rangle &= \int\frac{d^3k_1d^3k_2d^3k_3}{(2\pi)^9}\left(e^{i\mathbf{k}_1\cdot\mathbf{x}_2}-e^{i\mathbf{k}_2\cdot\mathbf{x}_2}\right)e^{i\mathbf{k}_2\cdot\mathbf{x}_1}e^{i\mathbf{k}_3\cdot\mathbf{x}_2} \\ &\quad\left(if_0\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\hat{n}_i\right)(2\pi)^3\delta_{\mathrm{D}}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_{\theta\delta\delta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \\ &= i\hat{n}_if_0\int\frac{d^3k_1d^3k_2}{(2\pi)^6}e^{-i\mathbf{k}_2\cdot\mathbf{x}}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}B_{\theta\delta\delta}(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2) \\ &\quad - i\hat{n}_if_0\int\frac{d^3k_1d^3k_3}{(2\pi)^6}e^{i\mathbf{k}_3\cdot\mathbf{x}}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}B_{\theta\delta\delta}(\mathbf{k}_1,-\mathbf{k}_1-\mathbf{k}_3,\mathbf{k}_3),\end{aligned} \tag{4.8}$$

where $(2\pi)^3\delta_{\mathrm{D}}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_{\theta\delta\delta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \langle\theta(\mathbf{k}_1)\delta(\mathbf{k}_2)\delta(\mathbf{k}_3)\rangle$ is a cross-bispectrum. Hence

$$\begin{aligned}\tilde{\Xi}_i^{1,udd}(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\langle\Delta u_i\delta_1\delta_2\rangle \\ &= i\hat{n}_if_0\int\frac{d^3k_1d^3k_2}{2\pi}\delta_{\mathrm{D}}(\mathbf{k}+\mathbf{k}_2)\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}B_{\theta\delta\delta}(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2) \\ &\quad - i\hat{n}_if_0\int\frac{d^3k_1d^3k_3}{2\pi}\delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_3)\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}B_{\theta\delta\delta}(\mathbf{k}_1,-\mathbf{k}_1-\mathbf{k}_3,\mathbf{k}_3) \\ &= i\hat{n}_if_0\int\frac{d^3k_1}{2\pi}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\left[B_{\theta\delta\delta}(\mathbf{k}_1,-\mathbf{k},\mathbf{k}-\mathbf{k}_1)-B_{\theta\delta\delta}(\mathbf{k}_1,-\mathbf{k}-\mathbf{k}_1,\mathbf{k})\right]\end{aligned} \tag{4.9}$$

Hence

$$\tilde{\Xi}_i^{1,udd}(\mathbf{k}) = i\hat{n}_if_0\int\frac{d^3p}{2\pi}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\left[B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})-B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k}-\mathbf{p},\mathbf{k})\right] \tag{4.10}$$

and

$$-ik_i\tilde{\Xi}_i^{1,udd}(\mathbf{k}) = k\mu f_0\int\frac{d^3p}{2\pi}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\left[B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})-B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k}-\mathbf{p},\mathbf{k})\right] \tag{4.11}$$

4.3 Together 1st order moments

We obtain

$$\tilde{\Xi}^1_i(\mathbf{k}) = if_0\hat{n}_i\Big\{ 2\frac{\mu}{k}P_{\delta\theta}(k) + \int \frac{d^3p}{2\pi}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\big[B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k}-\mathbf{p},\mathbf{k})\big]\Big\} \quad (4.12)$$

and $P^1_s(\mathbf{k}) = -ik_i\tilde{\Xi}^1_i(\mathbf{k})$ with

$$\boxed{-ik_i\tilde{\Xi}^1_i(\mathbf{k}) = 2\mu^2 f_0 P_{\delta\theta}(k) + k\mu f_0\int \frac{d^3p}{2\pi}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\big[B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k}-\mathbf{p},\mathbf{k})\big]} \quad (4.13)$$

Note the symmetries $B_{\theta\delta\delta}(1,2,3) = B_{\theta\delta\delta}(1,3,2)$.

5 Second order moments

The second order moment is

$$\begin{aligned}\Xi^2_{ij}(\mathbf{x}) &= \langle\big(1+\delta_1\big)\big(1+\delta_2\big)\Delta u_i\Delta u_j\rangle \\ &= \langle\Delta u_i\Delta u_j\rangle + \langle\Delta u_i\Delta u_j(\delta_1+\delta_2)\rangle + \langle\Delta u_i\Delta u_j\delta_1\delta_2\rangle \quad (5.1)\\ &= \Xi^{2,uu}_{ij}(\mathbf{x}) + \Xi^{2,uud}_{ij}(\mathbf{x}) + \Xi^{2,uudd}_{ij}(\mathbf{x}), \quad (5.2)\end{aligned}$$

and Fourier transforms

$$\tilde{\Xi}^2_{ij}(\mathbf{k}) = \tilde{\Xi}^{2,uu}_{ij}(\mathbf{k}) + \tilde{\Xi}^{2,uud}_{ij}(\mathbf{k}) + \tilde{\Xi}^{2,uudd}_{ij}(\mathbf{k}). \quad (5.3)$$

5.1 Moment (2,uu)

We split

$$\langle\Delta u_i\Delta u_j\rangle = 2\langle u_i(\mathbf{0})u_j(\mathbf{0})\rangle - 2\langle u_i(\mathbf{x}_1)u_j(\mathbf{x}_2)\rangle \quad (5.4)$$

Now,

$$\begin{aligned}\langle u_i(\mathbf{x}_1)u_j(\mathbf{x}_2)\rangle &= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}e^{i\mathbf{k}_1\cdot\mathbf{x}_1}e^{i\mathbf{k}_2\cdot\mathbf{x}_2}\left(if_0\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\hat{n}_i\right)\left(if_0\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}\hat{n}_j\right)\langle\theta(\mathbf{k}_1)\theta(\mathbf{k}_2)\rangle \\ &= -f_0^2\hat{n}_i\hat{n}_j\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}e^{i\mathbf{k}_1\cdot\mathbf{x}_1}e^{i\mathbf{k}_2\cdot\mathbf{x}_2}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}\,\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_1^2k_2^2}(2\pi)^3\delta_{\mathrm{D}}(\mathbf{k}_1+\mathbf{k}_2)P_{\theta\theta}(k_1) \\ &= f_0^2\hat{n}_i\hat{n}_j\int \frac{d^3p}{(2\pi)^3}e^{-i\mathbf{p}\cdot\mathbf{x}}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p) \quad (5.5)\end{aligned}$$

Equivalently we have

$$\langle u_i(\mathbf{0})u_j(\mathbf{0})\rangle = f_0^2\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_l\int \frac{d^3p}{(2\pi)^3}\frac{p_kp_l}{p^4}P_{\theta\theta}(p) = f_0^2\hat{n}_i\hat{n}_j\sigma_v^2 \quad (5.6)$$

with the velocity dispersion

$$\sigma_v^2 = \frac{1}{6\pi^2}\int_0^\infty dpP_{\theta\theta}(p). \quad (5.7)$$

Hence

$$\langle \Delta u_i \Delta u_j \rangle = 2f_0^2 \hat{n}_i \hat{n}_j \int \frac{d^3p}{(2\pi)^3} \left(1 - e^{i\mathbf{p}\cdot\mathbf{x}}\right) \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} P_{\theta\theta}(p) \tag{5.8}$$

$$= 2f_0^2 \hat{n}_i \hat{n}_j \sigma_v^2 - 2f_0^2 \hat{n}_i \hat{n}_j \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} P_{\theta\theta}(p) \tag{5.9}$$

And then,

$$\tilde{\Xi}_{ij}^{2,uu}(\mathbf{k}) = 2f_0^2 \hat{n}_i \hat{n}_j \sigma_v^2 \delta_{\rm D}(\mathbf{k}) - 2f_0^2 \hat{n}_i \hat{n}_j \frac{\mu^2}{k^2} P_{\theta\theta}(\mathbf{k}) \tag{5.10}$$

and then, for $\mathbf{k} \neq 0$

$$-\frac{1}{2} k_i k_j \tilde{\Xi}_{ij}^{2,uu}(\mathbf{k}) = f_0^2 \mu^4 P_{\theta\theta}(\mathbf{k}) \tag{5.11}$$

5.2 Moment (2,uud)

$$\begin{aligned}
\langle \Delta u_i \Delta u_j (\delta_1 + \delta_2) \rangle &= \int \frac{d^9 k_{123}}{(2\pi)^9} \left(e^{i\mathbf{k}_1\cdot\mathbf{x}_2} - e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_2\cdot\mathbf{x}_2} - e^{i\mathbf{k}_2\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_3\cdot\mathbf{x}_1} + e^{i\mathbf{k}_3\cdot\mathbf{x}_2}\right) \\
&\quad \left(i f_0 \frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2} \hat{n}_i\right)\left(i f_0 \frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2} \hat{n}_j\right) (2\pi)^3 \delta_{\rm D}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_{\theta\theta\delta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\
&= -f_0^2 \hat{n}_i \hat{n}_j \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \left[2 + e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} + e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} - 2(e^{i\mathbf{k}_2\cdot\mathbf{x}} + e^{-i\mathbf{k}_2\cdot\mathbf{x}})\right] \\
&\quad \frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2} \frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2} B_{\theta\theta\delta}(\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_1 - \mathbf{k}_2) \\
&= -f_0^2 \hat{n}_i \hat{n}_j \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \left[2 + 2e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} - 2(e^{i\mathbf{k}_2\cdot\mathbf{x}} + e^{-i\mathbf{k}_2\cdot\mathbf{x}})\right] \\
&\quad \frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2} \frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2} B_{\theta\theta\delta}(\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_1 - \mathbf{k}_2)
\end{aligned} \tag{5.12}$$

where in the second equality we use that the interchange $\mathbf{k}_1 \leftrightarrow \mathbf{k}_2$ leaves the integral invariant, and in the third we make the interchange $(\mathbf{k}_1, \mathbf{k}_2) \to (-\mathbf{k}_1, -\mathbf{k}_2)$ and use the symmetry $B_{\theta\theta\delta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = B_{\theta\theta\delta}(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3)$.

Now,

$$\begin{aligned}
\tilde{\Xi}_{ij}^{2,uud} &\equiv \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \langle \Delta u_i \Delta u_j (\delta_1 + \delta_2) \rangle \\
&- 2f_0^2 \hat{n}_i \hat{n}_j \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \left[\delta_{\rm D}(\mathbf{k}) + \delta_{\rm D}(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) - \delta_{\rm D}(\mathbf{k} - \mathbf{k}_2) - \delta_{\rm D}(\mathbf{k} + \mathbf{k}_2)\right] \\
&\quad \frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2} \frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2} B_{\theta\theta\delta}(\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_1 - \mathbf{k}_2) \\
&= -2f_0^2 \hat{n}_i \hat{n}_j \Bigg\{ \int \frac{d^3k_1}{(2\pi)^3} \frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2} \frac{(\mathbf{k} - \mathbf{k}_1)\cdot\hat{\mathbf{n}}}{|\mathbf{k} - \mathbf{k}_1|^2} B_{\theta\theta\delta}(\mathbf{k}_1, \mathbf{k} - \mathbf{k}_1, -\mathbf{k}) \\
&\qquad - \int \frac{d^3k_1}{(2\pi)^3} \frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2} \frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2} B_{\theta\theta\delta}(\mathbf{k}_1, \mathbf{k}, -\mathbf{k} - \mathbf{k}_1) \\
&\qquad - \int \frac{d^3k_1}{(2\pi)^3} \frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2} \frac{(-\mathbf{k})\cdot\hat{\mathbf{n}}}{k^2} B_{\theta\theta\delta}(\mathbf{k}_1, -\mathbf{k}, \mathbf{k} - \mathbf{k}_1) \Bigg\}
\end{aligned} \tag{5.13}$$

Organizing some terms, basically renaming $\mathbf{k}_1 \to \mathbf{p}$ we have

$$\tilde{\Xi}^{2,uud}_{ij}(\mathbf{k}) = -2f_0\hat{n}_i\hat{n}_j \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k}-\mathbf{p},-\mathbf{k})$$
$$+ 2f_0\hat{n}_i\hat{n}_j \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2}\big[B_{\theta\theta\delta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p}) - B_{\theta\theta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})\big] \tag{5.14}$$

and

$$-\frac{1}{2}k_ik_j\tilde{\Xi}^{2,uud}_{ij}(\mathbf{k}) = (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k}-\mathbf{p},-\mathbf{k})$$
$$+ (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2}\big[B_{\theta\theta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - B_{\theta\theta\delta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\big] \tag{5.15}$$

Let us look at the first term in the last equation

$$\begin{aligned}
&(k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k}-\mathbf{p},-\mathbf{k})\\
&= k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}f_0\frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k}-\mathbf{p},-\mathbf{k})\\
&+ k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^2}f_0\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k}-\mathbf{p},-\mathbf{k})\\
&= k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}f_0\frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k}-\mathbf{p},-\mathbf{k})\\
&- k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}f_0\frac{[(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{k}+\mathbf{p},-\mathbf{p},-\mathbf{k})\\
&= k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0\frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k}-\mathbf{p},-\mathbf{k})\\
&\qquad - f_0\frac{[(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\theta\delta}(\mathbf{k}+\mathbf{p},-\mathbf{p},-\mathbf{k})\Bigg]\\
&= k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0\frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})\\
&\qquad - f_0\frac{[(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg]
\end{aligned} \tag{5.16}$$

In the first equality we substitue one of the powers in $(k\mu)^2$ by $k\mu = (\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}+\mathbf{p}\cdot\hat{\mathbf{n}}$. In the second integral of the second equality we have substituted $\mathbf{p}\to\mathbf{k}+\mathbf{p}$. Now, $B_{\theta\theta\delta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = B_{\theta\delta\theta}(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2) = B_{\theta\delta\theta}(\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_1)$ followed by $B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = B(-\mathbf{k}_1,-\mathbf{k}_2,-\mathbf{k}_3)$ is used in the fourth equality. Hence we can write

$$
\boxed{
\begin{aligned}
-\frac{1}{2}k_ik_j\tilde{\Xi}^{2,uud}_{ij}(\mathbf{k}) &= k\mu f_0\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0\frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})\\
&\quad - f_0\frac{[(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg]\\
&+(k\mu f_0)\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}B_{\theta\theta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - f_0\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg].
\end{aligned}
}
\tag{5.17}
$$

5.3 Moment (m=2,uudd)

$$
\begin{aligned}
\langle\Delta u_i\Delta u_j\delta_1\delta_2\rangle &= \int\frac{d^3k_1d^3k_2d^3k_3d^3k_4}{(2\pi)^9}\left(e^{i\mathbf{k}_1\cdot\mathbf{x}_2}-e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_2\cdot\mathbf{x}_2}-e^{i\mathbf{k}_2\cdot\mathbf{x}_1}\right)e^{i\mathbf{k}_3\cdot\mathbf{x}_1}e^{i\mathbf{k}_4\cdot\mathbf{x}_2}\\
&\quad\left(if_0\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\hat{n}_i\right)\left(if_0\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}\hat{n}_j\right)\langle\theta(\mathbf{k}_1)\theta(\mathbf{k}_2)\delta(\mathbf{k}_3)\delta(\mathbf{k}_4)\rangle
\end{aligned}
\tag{5.18}
$$

Note

$$
\begin{aligned}
\langle\theta_1\theta_2\delta_3\delta_4\rangle &= \langle\theta_1\theta_2\rangle\langle\delta_3\delta_4\rangle + 2\langle\theta_1\delta_3\rangle\langle\theta_2\delta_4\rangle\\
&= (2\pi)^6\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_2)\delta_{\rm D}(\mathbf{k}_3+\mathbf{k}_4)P_{\theta\theta}(k_1)P_{\delta\delta}(k_4)\\
&\quad + 2(2\pi)^6\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_3)\delta_{\rm D}(\mathbf{k}_2+\mathbf{k}_4)P_{\delta\theta}(k_1)P_{\delta\theta}(k_4) = 1+2
\end{aligned}
\tag{5.19}
$$

Hence, we split in two pieces

$$
\begin{aligned}
\langle\Delta u_i\Delta u_j\delta_1\delta_2\rangle_1 &= \int\frac{d^3k_1d^3k_2d^3k_3d^3k_4}{(2\pi)^9}\left(e^{i\mathbf{k}_1\cdot\mathbf{x}_2}-e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_2\cdot\mathbf{x}_2}-e^{i\mathbf{k}_2\cdot\mathbf{x}_1}\right)e^{i\mathbf{k}_3\cdot\mathbf{x}_1}e^{i\mathbf{k}_4\cdot\mathbf{x}_2}\\
&\quad\left(if_0\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\hat{n}_i\right)\left(if_0\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}\hat{n}_j\right)(2\pi)^6\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_2)\delta_{\rm D}(\mathbf{k}_3+\mathbf{k}_4)P_{\theta\theta}(k_1)P_{\delta\delta}(k_4)\\
&= 2f_0^2\hat{n}_i\hat{n}_j\int\frac{d^3k_1d^3k_4}{(2\pi)^6}\left(e^{i\mathbf{k}_4\cdot\mathbf{x}}-e^{i(\mathbf{k}_1+\mathbf{k}_4)\cdot\mathbf{x}}\right)\frac{(\mathbf{k}_1\cdot\hat{\mathbf{n}})^2}{k_1^4}P_{\theta\theta}(k_1)P_{\delta\delta}(k_4)
\end{aligned}
\tag{5.20}
$$

Hence

$$
\begin{aligned}
\Xi^{2,uudd_1}_{ij}(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\langle\Delta u_i\Delta u_j\delta_1\delta_2\rangle_1\\
&= 2f_0^2\hat{n}_i\hat{n}_j\int\frac{d^3k_1d^3k_4}{(2\pi)^6}\left(\delta_{\rm D}(\mathbf{k}-\mathbf{k}_4)-\delta_{\rm D}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_4)\right)\frac{(\mathbf{k}_1\cdot\hat{\mathbf{n}})^2}{k_1^4}P_{\theta\theta}(k_1)P_{\delta\delta}(k_4)\\
&= -2f_0^2\hat{n}_i\hat{n}_j\int\frac{d^3k_1}{(2\pi)^3}\frac{(\mathbf{k}_1\cdot\hat{\mathbf{n}})^2}{k_1^4}P_{\theta\theta}(k_1)\Big[P_{\delta\delta}(|\mathbf{k}-\mathbf{k}_1|)-P_{\delta\delta}(k)\Big]
\end{aligned}
\tag{5.21}
$$

Now, for the piece 2

$$\langle\Delta u_i\Delta u_j\delta_1\delta_2\rangle_2 = 2\int\frac{d^3k_1d^3k_2d^3k_3d^3k_4}{(2\pi)^9}\left(e^{i\mathbf{k}_1\cdot\mathbf{x}_2}-e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_2\cdot\mathbf{x}_2}-e^{i\mathbf{k}_2\cdot\mathbf{x}_1}\right)e^{i\mathbf{k}_3\cdot\mathbf{x}_1}e^{i\mathbf{k}_4\cdot\mathbf{x}_2}$$
$$\left(-if_0\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\hat{n}_i\right)\left(-if_0\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}\hat{n}_j\right)(2\pi)^6\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_3)\delta_{\rm D}(\mathbf{k}_2+\mathbf{k}_4)P_{\delta\theta}(k_1)P_{\delta\theta}(k_4)$$
$$= -2f_0^2\hat{n}_i\hat{n}_j\int\frac{d^3k_1d^3k_2}{(2\pi)^6}\left(e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}-2e^{i\mathbf{k}_1\cdot\mathbf{x}}+1\right)\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}P_{\delta\theta}(k_1)P_{\delta\theta}(k_2) \quad (5.22)$$

The second and third terms of the above equation are exactly zero because $\int d\Omega_{\hat{\mathbf{k}}}\hat{k}_i = 0$, hence

$$\langle\Delta u_i\Delta u_j\delta_1\delta_2\rangle_2 = -2f_0^2\hat{n}_i\hat{n}_j\int\frac{d^3k_1d^3k_2}{(2\pi)^6}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}P_{\delta\theta}(k_1)P_{\delta\theta}(k_2). \quad (5.23)$$

The Fourier space moment is

$$\tilde{\Xi}_{ij}^{2,uudd_2}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\langle\Delta u_i\Delta u_j\delta_1\delta_2\rangle_2$$
$$= -2f_0^2\hat{n}_i\hat{n}_j\int\frac{d^3k_1d^3k_2}{(2\pi)^6}\delta_{\rm D}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}P_{\delta\theta}(k_1)P_{\delta\theta}(k_2)$$
$$= -2f_0^2\hat{n}_i\hat{n}_j\int\frac{d^3k_1}{(2\pi)^3}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{(\mathbf{k}-\mathbf{k}_1)\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{k}_1|^2}P_{\delta\theta}(k_1)P_{\delta\theta}(|\mathbf{k}-\mathbf{k}_1|). \quad (5.24)$$

Hence

$$\tilde{\Xi}_{ij}^{2,uudd}(\mathbf{k}) = \tilde{\Xi}_{ij}^{2,uudd_1}(\mathbf{k}) + \tilde{\Xi}_{ij}^{2,uudd_2}(\mathbf{k})$$
$$= -2f_0^2\hat{n}_i\hat{n}_j\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\Big[P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|)-P_{\delta\delta}(k)\Big]$$
$$-2f_0^2\hat{n}_i\hat{n}_j\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|). \quad (5.25)$$

Hence

$$-\frac{1}{2}k_ik_j\tilde{\Xi}_{ij}^{2,uudd}(\mathbf{k}) = f_0^2k^2\mu^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\Big[P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|)-P_{\delta\delta}(k)\Big]$$
$$+f_0^2k^2\mu^2\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|). \quad (5.26)$$

5.4 2nd moments together

Writing together eqs. (5.11,5.17,5.26) we have

$$
\boxed{\begin{aligned}
&-\frac{1}{2}k_ik_j\tilde{\Xi}^2_{ij}(\mathbf{k}) = f_0^2\mu^4 P_{\theta\theta}(\mathbf{k})\\
&+k\mu f_0\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0\frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})\\
&\qquad\qquad -f_0\frac{[(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg]\\
&+(k\mu f_0)\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}B_{\theta\theta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})-f_0\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg]\\
&+(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\Big[P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|)-P_{\delta\delta}(k)\Big]\\
&+(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|).
\end{aligned}}
\tag{5.27}
$$

6 Third order moments

$$
\begin{aligned}
\Xi^3_{ijk}(\mathbf{x}) &= \langle\big(1+\delta_1\big)\big(1+\delta_2\big)\Delta u_i\Delta u_j\Delta u_k\rangle\\
&= \langle\Delta u_i\Delta u_j\Delta u_k\rangle + \langle(\delta_1+\delta_2)\Delta u_i\Delta u_j\Delta u_k\rangle + \langle\delta_1\delta_2\Delta u_i\Delta u_j\Delta u_k\rangle
\end{aligned}
\tag{6.1}
$$

The third term is $\mathcal{O}(P_L^3)$, so we do not consider it here. Hence,

$$
\tilde{\Xi}^3_{ijk}(\mathbf{k}) = \tilde{\Xi}^{3,uuu}_{ijk}(\mathbf{k}) + \tilde{\Xi}^{3,uuud}_{ijk}(\mathbf{k}).
\tag{6.2}
$$

Contrary to the previous moments, whose expressions are valid to arbitrary PT order, in the following we keep only terms up to $\mathcal{O}(P_L^2)$.

6.1 Moment (m=3,uuu)

$$\langle \Delta u_i \Delta u_j \Delta u_k \rangle = \int \frac{d^9 k_{123}}{(2\pi)^9} \left(e^{i\mathbf{k}_1\cdot\mathbf{x}_2} - e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_2\cdot\mathbf{x}_2} - e^{i\mathbf{k}_2\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_3\cdot\mathbf{x}_2} - e^{i\mathbf{k}_3\cdot\mathbf{x}_1}\right)$$
$$\left(if_0\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\hat{n}_i\right)\left(if_0\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}\hat{n}_j\right)\left(if_0\frac{\mathbf{k}_3\cdot\hat{\mathbf{n}}}{k_3^2}\hat{n}_k\right)(2\pi)^3\delta_{\mathrm{D}}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_{\theta\theta\theta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \tag{6.3}$$

Now, the exponential terms evaluated in $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3 = 0$ give

$$\begin{aligned}
&\left[\left(e^{i\mathbf{k}_1\cdot\mathbf{x}_2} - e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_2\cdot\mathbf{x}_2} - e^{i\mathbf{k}_2\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_3\cdot\mathbf{x}_2} - e^{i\mathbf{k}_3\cdot\mathbf{x}_1}\right)\right]_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0} \\
&= (e^{i\mathbf{k}_1\cdot\mathbf{x}} - e^{-i\mathbf{k}_1\cdot\mathbf{x}}) + (e^{i\mathbf{k}_2\cdot\mathbf{x}} - e^{-i\mathbf{k}_2\cdot\mathbf{x}}) - (e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} - e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}) \\
&= 2(e^{i\mathbf{k}_2\cdot\mathbf{x}} - e^{-i\mathbf{k}_2\cdot\mathbf{x}}) - (e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} - e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}})
\end{aligned} \tag{6.4}$$

where the second equality is valid inside the integral. Hence

$$\begin{aligned}
\langle \Delta u_i \Delta u_j \Delta u_k \rangle = -if_0^3\hat{n}_i\hat{n}_j\hat{n}_k \int \frac{d^3k_1 d^3k_2}{(2\pi)^6}&\left[2(e^{i\mathbf{k}_2\cdot\mathbf{x}} - e^{-i\mathbf{k}_2\cdot\mathbf{x}}) - (e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} - e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}})\right] \\
&\times \frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}\frac{(-\mathbf{k}_1-\mathbf{k}_2)\cdot\hat{\mathbf{n}}}{|\mathbf{k}_1+\mathbf{k}_2|^2}B_{\theta\theta\theta}(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2).
\end{aligned} \tag{6.5}$$

Hence

$$\begin{aligned}
&\tilde{\Xi}^{3,uuu}_{ijk}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\langle \Delta u_i \Delta u_j \Delta u_k \rangle = \\
&= if_0^3\hat{n}_i\hat{n}_j\hat{n}_k \int \frac{d^3k_1 d^3k_2}{(2\pi)^6}\Big[2\delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_2) - 2\delta_{\mathrm{D}}(\mathbf{k}+\mathbf{k}_2) \\
&\quad - \delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) + \delta_{\mathrm{D}}(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\Big]\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2}\frac{(\mathbf{k}_1+\mathbf{k}_2)\cdot\hat{\mathbf{n}}}{|\mathbf{k}_1+\mathbf{k}_2|^2}B_{\theta\theta\theta}(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2) \\
&= if_0^3\hat{n}_i\hat{n}_j\hat{n}_k\Bigg\{2\int\frac{d^3k_1}{(2\pi)^3}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2}\frac{(\mathbf{k}+\mathbf{k}_1)\cdot\hat{\mathbf{n}}}{|\mathbf{k}+\mathbf{k}_1|^2}B_{\theta\theta\theta}(\mathbf{k}_1,\mathbf{k},-\mathbf{k}-\mathbf{k}_1) \\
&\qquad - 2\int\frac{d^3k_1}{(2\pi)^3}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{(-\mathbf{k})\cdot\hat{\mathbf{n}}}{k^2}\frac{(-\mathbf{k}+\mathbf{k}_1)\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{k}_1|^2}B_{\theta\theta\theta}(\mathbf{k}_1,-\mathbf{k},\mathbf{k}-\mathbf{k}_1) \\
&\qquad - \int\frac{d^3k_1}{(2\pi)^3}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{(\mathbf{k}-\mathbf{k}_1)\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{k}_1|^2}\frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2}B_{\theta\theta\theta}(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1,-\mathbf{k}) \\
&\qquad + \int\frac{d^3k_1}{(2\pi)^3}\frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2}\frac{(-\mathbf{k}-\mathbf{k}_1)\cdot\hat{\mathbf{n}}}{|\mathbf{k}+\mathbf{k}_1|^2}\frac{(-\mathbf{k})\cdot\hat{\mathbf{n}}}{k^2}B_{\theta\theta\theta}(\mathbf{k}_1,-\mathbf{k}-\mathbf{k}_1,\mathbf{k})\Bigg\}
\end{aligned} \tag{6.6}$$

Hence, by making $\mathbf{k}_1 \to \mathbf{p}$

$$
\begin{aligned}
\tilde{\Xi}^{3,uuu}_{ijk}(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \langle \Delta u_i \Delta u_j \Delta u_k \rangle = \\
&= 2if_0^3 \hat{n}_i \hat{n}_j \hat{n}_k \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2} \\
&\quad \left[\frac{(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}+\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p}) - \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \right] \\
&+ if_0^3 \hat{n}_i \hat{n}_j \hat{n}_k \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2} \\
&\quad \left[\frac{(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}+\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k}-\mathbf{p},\mathbf{k}) - \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},\mathbf{k}-\mathbf{p},-\mathbf{k}) \right]
\end{aligned} \tag{6.7}
$$

Using now that $B_{\theta\theta\theta}$ is symmetric over all their arguments

$$
\begin{aligned}
\tilde{\Xi}^{3,uuu}_{ijk}(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \langle \Delta u_i \Delta u_j \Delta u_k \rangle = 3if_0^3 \hat{n}_i \hat{n}_j \hat{n}_k \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2} \\
&\times \left[\frac{(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}+\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p}) - \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \right],
\end{aligned} \tag{6.8}
$$

and

$$
\begin{aligned}
\frac{(-i)^3}{3!} k_i k_j k_k \tilde{\Xi}^{3,uuu}_{ijk}(\mathbf{k}) &= -\frac{1}{2}(k\mu f_0)^3 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2} \\
&\quad \left[\frac{(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}+\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p}) - \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \right]
\end{aligned} \tag{6.9}
$$

Rearranging terms we have

$$
\begin{aligned}
\frac{i}{6} k_i k_j k_k \tilde{\Xi}^{3,uuu}_{ijk}(\mathbf{k}) &= k\mu f_0 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \left[f_0^2 \frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}-\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \right. \\
&\left. - f_0^2 \frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2} \frac{((\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}+\mathbf{p}|^2} B_{\theta\theta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p}) \right]
\end{aligned} \tag{6.10}
$$

6.2 Moment (m=3,uuud)

$$
\begin{aligned}
\langle \Delta u_i \Delta u_j \Delta u_k (\delta_1 + \delta_2) \rangle &= \int \frac{d^3k_1 d^3k_2 d^3k_3 d^3k_4}{(2\pi)^{12}} \left(e^{i\mathbf{k}_1\cdot\mathbf{x}_2} - e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_2\cdot\mathbf{x}_2} - e^{i\mathbf{k}_2\cdot\mathbf{x}_1}\right) \\
&\times \left(e^{i\mathbf{k}_3\cdot\mathbf{x}_2} - e^{i\mathbf{k}_3\cdot\mathbf{x}_1}\right)\left(e^{i\mathbf{k}_4\cdot\mathbf{x}_2} + e^{i\mathbf{k}_4\cdot\mathbf{x}_1}\right) \left(if_0 \frac{\mathbf{k}_1\cdot\hat{\mathbf{n}}}{k_1^2} \hat{n}_i \right) \left(if_0 \frac{\mathbf{k}_2\cdot\hat{\mathbf{n}}}{k_2^2} \hat{n}_j \right) \\
&\times \left(if_0 \frac{\mathbf{k}_3\cdot\hat{\mathbf{n}}}{k_3^2} \hat{n}_k \right) \langle \theta(\mathbf{k}_1)\theta(\mathbf{k}_2)\theta(\mathbf{k}_3)\delta(\mathbf{k}_4) \rangle.
\end{aligned} \tag{6.11}
$$

Now,

$$\langle\theta_1\theta_2\theta_3\delta_4\rangle = 3\langle\theta_1\theta_2\rangle\langle\theta_3\delta_4\rangle = 3(2\pi)^6\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_2)\delta_{\rm D}(\mathbf{k}_3+\mathbf{k}_4)P_{\theta\theta}(k_1)P_{\delta\theta}(k_3), \tag{6.12}$$

where we used the symmetry $\mathbf{k}_1 \leftrightarrow \mathbf{k}_2 \leftrightarrow \mathbf{k}_3$. The exponential factors *exps* give

$$exps|_{\mathbf{k}_2=-\mathbf{k}_1,\mathbf{k}_4=-\mathbf{k}_3} = \left(2 - e^{i\mathbf{k}_1\cdot\mathbf{x}} - e^{-i\mathbf{k}_1\cdot\mathbf{x}}\right)\left(e^{i\mathbf{k}_3\cdot\mathbf{x}} - e^{-i\mathbf{k}_3\cdot\mathbf{x}}\right) \tag{6.13}$$

yielding

$$\begin{aligned}\langle\Delta u_i\Delta u_j\Delta u_k(\delta_1+\delta_2)\rangle = 3if_0^3\hat{n}_i\hat{n}_j\hat{n}_k\int\frac{d^3k_1d^3k_3}{(2\pi)^6}\left(2 - e^{i\mathbf{k}_1\cdot\mathbf{x}} - e^{-i\mathbf{k}_1\cdot\mathbf{x}}\right)\\ \left(e^{i\mathbf{k}_3\cdot\mathbf{x}} - e^{-i\mathbf{k}_3\cdot\mathbf{x}}\right)\frac{(\mathbf{k}_1\cdot\hat{\mathbf{n}})^2}{k_1^4}\frac{\mathbf{k}_3\cdot\hat{\mathbf{n}}}{k_3^2}P_{\theta\theta}(k_1)P_{\delta\theta}(k_3)\end{aligned} \tag{6.14}$$

The integral is even against $\mathbf{k}_1 \to -\mathbf{k}_1$ and odd against $\mathbf{k}_3 \to -\mathbf{k}_3$, then $\Xi^{3,uuud}_{ijk}(\mathbf{x}) = \langle\Delta u_i\Delta u_j\Delta u_k(\delta_1+\delta_2)\rangle$ is

$$\Xi^{3,uuud}_{ijk}(\mathbf{x}) = 12if_0^3\hat{n}_i\hat{n}_j\hat{n}_k\int\frac{d^3k_1d^3k_3}{(2\pi)^6}\left(e^{i\mathbf{k}_3\cdot\mathbf{x}} - e^{i(\mathbf{k}_1+\mathbf{k}_3)\cdot\mathbf{x}}\right)\frac{(\mathbf{k}_1\cdot\hat{\mathbf{n}})^2}{k_1^4}\frac{\mathbf{k}_3\cdot\hat{\mathbf{n}}}{k_3^2}P_{\theta\theta}(k_1)P_{\delta\theta}(k_3). \tag{6.15}$$

Now, the k-moment is

$$\begin{aligned}\tilde{\Xi}^{3,uuud}_{ijk}(\mathbf{k}) &= \int e^{-i\mathbf{k}\cdot\mathbf{x}}\Xi^{3,uuud}_{ijk}(\mathbf{x})\\ &= 12if_0^3\hat{n}_i\hat{n}_j\hat{n}_k\int\frac{d^3k_1d^3k_3}{(2\pi)^3}\left(\delta_{\rm D}(\mathbf{k}-\mathbf{k}_3) - \delta_{\rm D}(\mathbf{k}-\mathbf{k}_{13})\right)\frac{(\mathbf{k}_1\cdot\hat{\mathbf{n}})^2}{k_1^4}\frac{\mathbf{k}_3\cdot\hat{\mathbf{n}}}{k_3^2}P_{\theta\theta}(k_1)P_{\delta\theta}(k_3)\\ &= 12if_0^3\hat{n}_i\hat{n}_j\hat{n}_k\int\frac{d^3k_1}{(2\pi)^3}\frac{(\mathbf{k}_1\cdot\hat{\mathbf{n}})^2}{k_1^4}\frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2}P_{\theta\theta}(k_1)P_{\delta\theta}(k)\\ &\quad - 12if_0^3\hat{n}_i\hat{n}_j\hat{n}_k\int\frac{d^3k_1}{(2\pi)^3}\frac{(\mathbf{k}_1\cdot\hat{\mathbf{n}})^2}{k_1^4}\frac{(\mathbf{k}-\mathbf{k}_1)\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{k}_1|^2}P_{\theta\theta}(k_1)P_{\delta\theta}(|\mathbf{k}-\mathbf{k}_1|)\end{aligned} \tag{6.16}$$

Then,

$$\begin{aligned}\frac{i}{6}k_ik_jk_k\tilde{\Xi}^{3,uuud}_{ijk}(\mathbf{k}) = -2(k\mu f_0)^3\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{\mathbf{k}\cdot\hat{\mathbf{n}}}{k^2}P_{\theta\theta}(p)P_{\delta\theta}(k)\\ + 2(k\mu f_0)^3\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|).\end{aligned} \tag{6.17}$$

We will use

$$\sigma_v^2 = \int\frac{dp}{6\pi^2}P_{\theta\theta}(p), \tag{6.18}$$

for the first term. The second term is

$$
\begin{aligned}
&2(k\mu f_0)^2\big[f_0(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}+f_0\mathbf{p}\cdot\hat{\mathbf{n}}\big]\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\\
&=2(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\\
&\quad+2(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\theta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\\
&=(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\left[2f_0\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\right]\\
&\quad+2(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}P_{\delta\theta}(p)f_0\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^3}{|\mathbf{k}-\mathbf{p}|^4}P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)
\end{aligned}
\tag{6.19}
$$

where in the last equality we have make the substitution $\mathbf{p}\to\mathbf{k}-\mathbf{p}$.

Then

$$
\begin{aligned}
&\frac{i}{6}k_ik_jk_k\tilde{\Xi}^{3,uuud}_{ijk}(\mathbf{k})=-2k^2\mu^4f_0^3\sigma_v^2P_{\delta\theta}(\mathbf{k})\\
&\quad+(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\left[2f_0\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\right]\\
&\quad+2(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}P_{\delta\theta}(p)f_0\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^3}{|\mathbf{k}-\mathbf{p}|^4}P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|).
\end{aligned}
\tag{6.20}
$$

6.3 Together third order moments

$$
\boxed{
\begin{aligned}
&\frac{i}{6}k_ik_jk_k\tilde{\Xi}^{3}_{ijk}(\mathbf{k})=-2k^2\mu^4f_0^3\sigma_v^2P_{\delta\theta}(\mathbf{k})\\
&\quad+k\mu f_0\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\left[f_0^2\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})\right.\\
&\qquad\left.-f_0^2\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}\frac{((\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\theta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\right]\\
&\quad+(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\left[2f_0\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\right]\\
&\quad+2(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}P_{\delta\theta}(p)f_0\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^3}{|\mathbf{k}-\mathbf{p}|^4}P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)
\end{aligned}
}
\tag{6.21}
$$

7 Fourth order moments

$$
\begin{aligned}
&\Xi^{4,uuuu}_{ijkl}(\mathbf{x})=\langle\Delta u_i\Delta u_j\Delta u_k\Delta u_l\rangle\\
&=\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_lf_0^4\int\prod_{a=1}^{4}\left[\frac{d^3k_a}{(2\pi)^3}\left(e^{i\mathbf{k}_a\cdot\mathbf{x}_2}-e^{i\mathbf{k}_a\cdot\mathbf{x}_1}\right)\left(i\frac{\mathbf{k}_a\cdot\hat{\mathbf{n}}}{k_a^2}\right)\right]\langle\theta(\mathbf{k}_1)\theta(\mathbf{k}_2)\theta(\mathbf{k}_3)\theta(\mathbf{k}_4)\rangle
\end{aligned}
\tag{7.1}
$$

We have that all θ in the correlator are linear and by using the symmetries $\mathbf{k}_a \leftrightarrow \mathbf{k}_b$ inside the integral we have

$$\langle \Delta u_i \Delta u_j \Delta u_k \Delta u_l \rangle = 12 \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_l f_0^4 \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \big(1 + e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot \mathbf{x}} - 2 e^{i\mathbf{k}_1\cdot\mathbf{x}}\big) \\ \times \frac{(\mathbf{k}_1 \cdot \hat{\mathbf{n}})^2 (\mathbf{k}_2 \cdot \hat{\mathbf{n}})^2}{k_1^4 k_2^4} P_{\theta\theta}(k_1) P_{\theta\theta}(k_2) = 12 f_0^4 \sigma_v^4 \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_l \\ + 12 \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_l f_0^4 \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \big(e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot \mathbf{x}} - 2 e^{i\mathbf{k}_1\cdot\mathbf{x}}\big) \frac{(\mathbf{k}_1 \cdot \hat{\mathbf{n}})^2 (\mathbf{k}_2 \cdot \hat{\mathbf{n}})^2}{k_1^4 k_2^4} P_{\theta\theta}(k_1) P_{\theta\theta}(k_2) \tag{7.2}$$

To 1-loop $\Xi^4_{ijkl}(\mathbf{x}) = \langle \Delta u_i \Delta u_j \Delta u_k \Delta u_l \rangle$. Fourier transforming,

$$\tilde{\Xi}^{4,uuuu}_{ijkl}(\mathbf{k}) \equiv \int d^3 x e^{-i\mathbf{k}\cdot\mathbf{x}} \Xi^4_{ijkl}(\mathbf{x}) = 12 \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_l f_0^4 \sigma_v^4 \delta_{\rm D}(\mathbf{k}) \\ + 12 \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_l f_0^4 \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \big(-2\delta_{\rm D}(-\mathbf{k}+\mathbf{k}_1) + \delta_{\rm D}(-\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\big) \\ \times \frac{(\mathbf{k}_1 \cdot \hat{\mathbf{n}})^2 (\mathbf{k}_2 \cdot \hat{\mathbf{n}})^2}{k_1^4 k_2^4} P_{\theta\theta}(k_1) P_{\theta\theta}(k_2) \\ = -24 \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_l f_0^4 \int \frac{d^3 p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} \frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^4} P_{\theta\theta}(p) P_{\theta\theta}(k) \\ + 12 \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_l f_0^4 \int \frac{d^3 p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}-\mathbf{p}|^4} P_{\theta\theta}(p) P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \tag{7.3}$$

Hence

$$\frac{1}{4!} k_i k_j k_k k_l \tilde{\Xi}^{4,uuuu}_{ijkl}(\mathbf{k}) = -k^2 \mu^6 f_0^4 \sigma_v^2 P_{\theta\theta}(k) \\ + \frac{1}{2}(k\mu f_0)^4 \int \frac{d^3 p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}-\mathbf{p}|^4} P_{\theta\theta}(p) P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \tag{7.4}$$

Taking $(k\mu)^2 = \big((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big)^2 + 2((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})(\mathbf{p}\cdot\hat{\mathbf{n}}) + (\mathbf{p}\cdot\hat{\mathbf{n}})^2$ the second term is rewritten as

$$\frac{1}{2}(k\mu f_0)^4 \int \frac{d^3 p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}-\mathbf{p}|^4} P_{\theta\theta}(p) P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\ = \frac{1}{2}(k\mu f_0)^2 \int \frac{d^3 p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^4}{|\mathbf{k}-\mathbf{p}|^4} f_0 P_{\theta\theta}(p) f_0 P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\ + (k\mu f_0)^2 \int \frac{d^3 p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^3}{|\mathbf{k}-\mathbf{p}|^4} f_0 P_{\theta\theta}(p) f_0 P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\ + \frac{1}{2}(k\mu f_0)^2 \int \frac{d^3 p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^4}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}-\mathbf{p}|^4} f_0 P_{\theta\theta}(p) f_0 P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\ = (k\mu f_0)^2 \int \frac{d^3 p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^3}{|\mathbf{k}-\mathbf{p}|^4} f_0 P_{\theta\theta}(p) f_0 P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\ + (k\mu f_0)^2 \int \frac{d^3 p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^4}{|\mathbf{k}-\mathbf{p}|^4} f_0 P_{\theta\theta}(p) f_0 P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \tag{7.5}$$

Hence,

$$\boxed{\begin{aligned}&\frac{1}{4!}k_ik_jk_kk_l\tilde{\Xi}^{4,uuuu}_{ijkl}(\mathbf{k}) = -k^2\mu^6 f_0^4\sigma_v^2 P_{\theta\theta}(k)\\ &+(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^3}{|\mathbf{k}-\mathbf{p}|^4}f_0P_{\theta\theta}(p)f_0P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)\\ &+(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^4}{|\mathbf{k}-\mathbf{p}|^4}f_0P_{\theta\theta}(p)f_0P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)\end{aligned}} \tag{7.6}$$

with $\tilde{\Xi}^4_{ijkl}(\mathbf{k}) = \tilde{\Xi}^{4,uuuu}_{ijkl}(\mathbf{k})$ to 1-loop.

8 All moments

We rewrite all moments:

$$\tilde{\Xi}^0(\mathbf{k}) = P_{\delta\delta}(k), \tag{8.1}$$

$$ik_i\tilde{\Xi}^1_i(\mathbf{k}) = 2\mu f_0 P_{\delta\theta}(k) + k\mu f_0\int\frac{d^3p}{2\pi}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\big[B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k}-\mathbf{p},\mathbf{k})\big], \tag{8.2}$$

$$\begin{aligned}-\frac{1}{2}k_ik_j\tilde{\Xi}^2_{ij}(\mathbf{k}) &= f_0^2\mu^4 P_{\theta\theta}(\mathbf{k})\\ &+k\mu f_0\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0\frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})\\ &\qquad - f_0\frac{[(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg]\\ &+(k\mu f_0)\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}B_{\theta\theta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - f_0\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg]\\ &+(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\Big[P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|) - P_{\delta\delta}(k)\Big]\\ &+(k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|),\end{aligned} \tag{8.3}$$

$$
\begin{aligned}
-\frac{i}{6}k_ik_jk_k\tilde{\Xi}^3_{ijk}(\mathbf{k}) &= -2k^2\mu^4 f_0^3\sigma_v^2 P_{\delta\theta}(\mathbf{k}) \\
&+ k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0^2\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) \\
&\qquad - f_0^2\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}\frac{((\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\theta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg] \\
&+ (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\left[2f_0\frac{\left[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\right]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\right] \\
&+ (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}P_{\delta\theta}(p)f_0\frac{\left[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\right]^3}{|\mathbf{k}-\mathbf{p}|^4}P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&+ (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)f_0\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4}P_{\theta\theta}(p),
\end{aligned}
\tag{8.4}
$$

$$
\begin{aligned}
\frac{1}{4!}k_ik_jk_kk_l\Xi^4_{ijkl}(\mathbf{k}) &= -k^2\mu^6 f_0^4\sigma_v^2 P_{\theta\theta}(k) \\
&+ (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^3}{|\mathbf{k}-\mathbf{p}|^4}f_0P_{\theta\theta}(p)f_0P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&+ (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^4}{|\mathbf{k}-\mathbf{p}|^4}f_0P_{\theta\theta}(p)f_0P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|).
\end{aligned}
\tag{8.5}
$$

9 Ordering terms that contain bispectra

Terms with bispectrum are given in eqs. (4.11,5.17,6.10)

$$
\begin{aligned}
A(\mathbf{k}) \equiv -ik_i\Xi^{1,udd}_i - \frac{1}{2}k_ik_j\tilde{\Xi}^{2,uud}_{ij}(\mathbf{k}) + \frac{i}{3!}k_ik_jk_k\tilde{\Xi}^{3,uuu}_{ijk}(\mathbf{k}) &= \\
k\mu f_0 \int \frac{d^3p}{2\pi}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\left[B_{\theta\delta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - B_{\theta\delta\delta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\right]& \\
+ k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0\frac{[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})& \\
- f_0\frac{[(\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}}]^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\delta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg]& \\
+ k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\left[f_0\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}B_{\theta\theta\delta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - f_0\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}B_{\theta\theta\delta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\right]& \\
+ k\mu f_0 \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\Bigg[f_0^2\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}-\mathbf{p}|^2}B_{\theta\theta\theta}(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p})& \\
- f_0^2\frac{(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{k^2}\frac{((\mathbf{k}+\mathbf{p})\cdot\hat{\mathbf{n}})^2}{|\mathbf{k}+\mathbf{p}|^2}B_{\theta\theta\theta}(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\Bigg]&
\end{aligned}
\tag{9.1}
$$

Let us define

$$B_\sigma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \equiv B_{\theta\delta\delta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + f_0\frac{(\mathbf{k}_3\cdot\hat{\mathbf{n}})^2}{k_3^2}B_{\theta\delta\theta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$$
$$+ f_0\frac{(\mathbf{k}_2\cdot\hat{\mathbf{n}})^2}{k_2^2}B_{\theta\theta\delta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + f_0^2\frac{(\mathbf{k}_2\cdot\hat{\mathbf{n}})^2}{k_2^2}\frac{(\mathbf{k}_3\cdot\hat{\mathbf{n}})^2}{k_3^2}B_{\theta\theta\theta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \quad (9.2)$$

Then,

$$A(k,\mu) = k\mu f_0\int\frac{d^3p}{2\pi}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\big[B_\sigma(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}) - B_\sigma(\mathbf{p},\mathbf{k},-\mathbf{k}-\mathbf{p})\big]. \quad (9.3)$$

Now, note we can write eq. (9.2) as

$$B_\sigma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \left\langle\theta(\mathbf{k}_1)\left[\delta(\mathbf{k}_2) + f_0\frac{(\mathbf{k}_2\cdot\hat{\mathbf{n}})^2}{k_2^2}\theta(\mathbf{k}_2)\right]\left[\delta(\mathbf{k}_3) + f_0\frac{(\mathbf{k}_3\cdot\hat{\mathbf{n}})^2}{k_3^2}\theta(\mathbf{k}_3)\right]\right\rangle', \quad (9.4)$$

which shows the symmetry $B_\sigma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = B_\sigma(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2)$. Renaming $\mathbf{p}\to-\mathbf{p}$ in the second term of the integral in eq. (9.3), and using $B_\sigma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = B_\sigma(-\mathbf{k}_1,-\mathbf{k}_2,-\mathbf{k}_3)$ we obtain that the terms containing bispectra yields

$$A(k,\mu) = 2k\mu f_0\int\frac{d^3p}{2\pi}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}B_\sigma(\mathbf{p},-\mathbf{k},\mathbf{k}-\mathbf{p}). \quad (9.5)$$

This function was first introduced in [3] for the Taruya-Nishimichi-Saito (TNS) model.

10 Ordering All

We have extracted the bispectrum terms in the previous section. Now, then

$$\begin{aligned}
&P_s(\mathbf{k}) = P_{\delta\delta}(k) + 2\mu f_0 P_{\delta\theta}(k) + f_0^2\mu^4 P_{\theta\theta}(\mathbf{k}) + A(k,\mu)\\
[1]\quad &+ (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\Big[P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|) - P_{\delta\delta}(k)\Big]\\
[2]\quad &+ (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(p)P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\\
[3]\quad &+ 0\\
[4]\quad &- 2k^2\mu^4 f_0^3\sigma_v^2 P_{\delta\theta}(\mathbf{k})\\
[5]\quad &+ (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\left[2f_0\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\right]\\
[6]\quad &+ (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}P_{\delta\theta}(p)f_0\frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^3}{|\mathbf{k}-\mathbf{p}|^4}P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)\\
[7]\quad &+ (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)f_0\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4}P_{\theta\theta}(p)\\
[8]\quad &- k^2\mu^6 f_0^4\sigma_v^2 P_{\theta\theta}(k)\\
[9]\quad &+ (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^3}{|\mathbf{k}-\mathbf{p}|^4}f_0P_{\theta\theta}(p)f_0P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)\\
[10]\quad &+ (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^4}{|\mathbf{k}-\mathbf{p}|^4}f_0P_{\theta\theta}(p)f_0P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \qquad (10.1)
\end{aligned}$$

[1] and [2] come from Ξ^2, [4],[5],[6] and [7] from Ξ^3 and [8], [9] and [10] from Ξ^4. Line [3] is zero, it was another term but I realized that it was zero exactly. Later I will fix this.

Term [1] can be rewritten as

$$[1] = [1a] + [1b] = (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} P_{\theta\theta}(p) P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|) - (k\mu f_0)^2 P_{\delta\delta}(k)\sigma_v^2 \tag{10.2}$$

Putting together [1b] + [4]+[8] we have

$$\begin{aligned}
&P_s(\mathbf{k}) = \Big[1 - (k\mu f_0\sigma)^2\Big]\Big[P_{\delta\delta}(k) + 2\mu f_0 P_{\delta\theta}(k) + f_0^2\mu^4 P_{\theta\theta}(k)\Big] + A(k,\mu) \\
&[1a] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} P_{\theta\theta}(p) P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|) \\
&[2] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} P_{\delta\theta}(p) P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&[3] \quad + 0 \\
&[5] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} P_{\theta\theta}(p) \left[2 f_0 \frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^2}{|\mathbf{k}-\mathbf{p}|^2} P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\right] \\
&[6] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} P_{\delta\theta}(p) f_0 \frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^3}{|\mathbf{k}-\mathbf{p}|^4} P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&[7] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) f_0 \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4} P_{\theta\theta}(p) \\
&[9] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^3}{|\mathbf{k}-\mathbf{p}|^4} f_0 P_{\theta\theta}(p) f_0 P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&[10] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^4}{|\mathbf{k}-\mathbf{p}|^4} f_0 P_{\theta\theta}(p) f_0 P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)
\end{aligned} \tag{10.3}$$

Now, summing up [2], [6], [7] and [9]

$$\begin{aligned}
&[2] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} P_{\delta\theta}(p) P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&[6] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} P_{\delta\theta}(p) f_0 \frac{\big[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big]^3}{|\mathbf{k}-\mathbf{p}|^4} P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&[7] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) f_0 \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4} P_{\theta\theta}(p) \\
&[9] \quad + (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^3}{p^4} \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^3}{|\mathbf{k}-\mathbf{p}|^4} f_0 P_{\theta\theta}(p) f_0 P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|) \\
&= (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2} \left[P_{\delta\theta}(p) + f_0 \frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^2} P_{\theta\theta}(p)\right] \\
&\quad \frac{(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}}{|\mathbf{k}-\mathbf{p}|^2} \left[P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) + f_0 \frac{\big((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\big)^2}{|\mathbf{k}-\mathbf{p}|^2} P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)\right]
\end{aligned} \tag{10.4}$$

We define, as in TNS paper [3],

$$B(k,\mu) \equiv (k\mu f_0)^2 \int \frac{d^3p}{(2\pi)^3} F(\mathbf{p}) F(\mathbf{k}-\mathbf{p}) \tag{10.5}$$

with

$$F(\mathbf{p}) = \frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{p^2}\left[P_{\delta\theta}(p) + f_0\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^2}P_{\theta\theta}(p)\right] \tag{10.6}$$

we have

$$B(k,\mu) = [2] + [6] + [7] + [9]. \tag{10.7}$$

We obtain

$$\begin{aligned}
&P_s(\mathbf{k}) = \left[1 - (k\mu f_0\sigma)^2\right]\left[P_{\delta\delta}(k) + 2\mu f_0 P_{\delta\theta}(k) + f_0^2\mu^4 P_{\theta\theta}(k)\right] + A(k,\mu) + B(k,\mu)\\
&[1a] \quad + (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|)\\
&[3] \quad + 0\\
&[5] \quad + (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\left[2f_0\frac{\left[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\right]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|)\right]\\
&[10] \quad + (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}\frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^4}{|\mathbf{k}-\mathbf{p}|^4}f_0P_{\theta\theta}(p)f_0P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)
\end{aligned} \tag{10.8}$$

We have

$$\begin{aligned}
[1a] + [5] + [10] = {}& (k\mu f_0)^2\int\frac{d^3p}{(2\pi)^3}\frac{(\mathbf{p}\cdot\hat{\mathbf{n}})^2}{p^4}P_{\theta\theta}(p)\Bigg[P_{\delta\delta}(|\mathbf{k}-\mathbf{p}|)\\
&+ 2f_0\frac{\left[(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}}\right]^2}{|\mathbf{k}-\mathbf{p}|^2}P_{\delta\theta}(|\mathbf{k}-\mathbf{p}|) + \frac{((\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{n}})^4}{|\mathbf{k}-\mathbf{p}|^4}f_0^2P_{\theta\theta}(|\mathbf{k}-\mathbf{p}|)\Bigg]\\
&\equiv C(k,\mu)
\end{aligned} \tag{10.9}$$

Hence, the redshift-space power spectrum in the moment expansion approach is

$$\boxed{\begin{aligned}P_s(k,\mu) = {}&\left[1 - (k\mu f_0\sigma_v)^2\right]\left[P_{\delta\delta}(k) + 2\mu f_0 P_{\delta\theta}(k) + f_0^2\mu^4 P_{\theta\theta}(k)\right]\\ &+ A(k,\mu) + B(k,\mu) + C(k,\mu).\end{aligned}} \tag{10.10}$$

References

[1] R. Scoccimarro, *Redshift-space distortions, pairwise velocities and nonlinearities*, *Phys. Rev.* **D70** (2004) 083007, [astro-ph/0407214].

[2] Z. Vlah and M. White, *Exploring redshift-space distortions in large-scale structure*, *JCAP* **1903** (2019) 007, [1812.02775].

[3] A. Taruya, T. Nishimichi and S. Saito, *Baryon Acoustic Oscillations in 2D: Modeling Redshift-space Power Spectrum from Perturbation Theory*, *Phys. Rev.* **D82** (2010) 063522, [1006.0699].