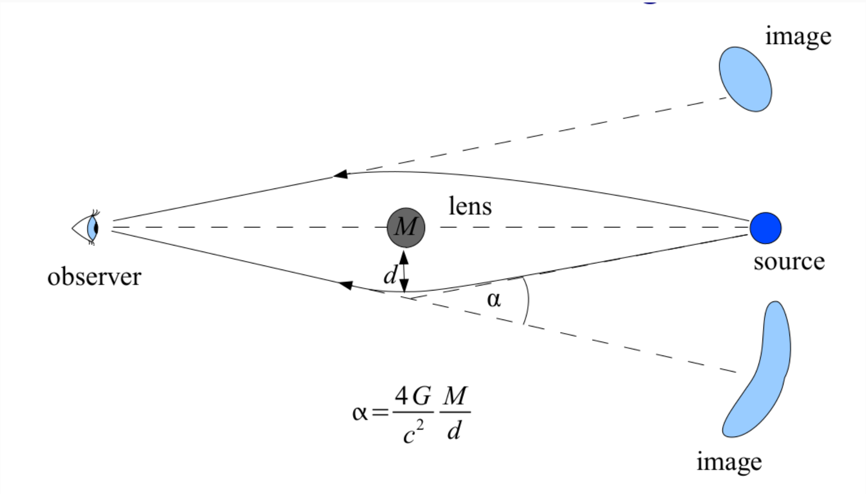
Cosmological weak lensing of galaxy sources

Alejandro Avilés (CONACyT/ININ)
avilescervantes@gmail.com

Curso de Primavera 2022 en “Weak Lensing: theory & estimators”

Curso IAC

1. Introduction and projected fields



Credit: Stefan Hilbert (MPA)

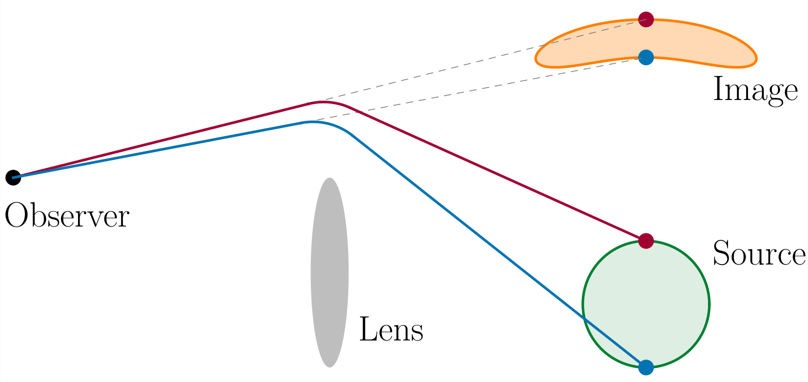


Image
Observer
Source
Lens

What this course is NOT about:

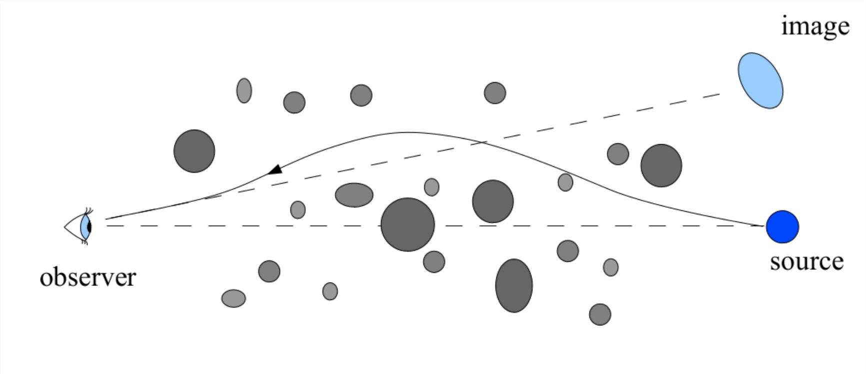
- Strong Lensing
- Time delays: e.g. H0LiCOW arxiv:1907.04869.
- Microlensing
- CMB lensing

This is not an introductory course to Cosmology. I will assume you know basic stuff ranging from background cosmology to transfer functions. I will assume you know what is a power spectrum and a correlation function (matter, 3-dim case) .

This is an introductory course to Weak Lensing

Gravitational lensing is an important tool because light paths respond directly to the gravitational potentials. That is, lensing probes the whole energy-momentum tensor, including, of course, its dark components.

- Dodelson & Schmidt, *Modern Cosmology*. Second Edition. Academic Press
- M. Kilbinger, *Cosmology with cosmic shear observations: a review*. Rep. Prog. Phys. 78 (2015) 086901. [arxiv:1411.0115].
- R. Mandelbaum, *Weak lensing for precision cosmology*. Ann. Rev. Astron. Astrophys. 56 (2018) 393. [arxiv:1710.03235]



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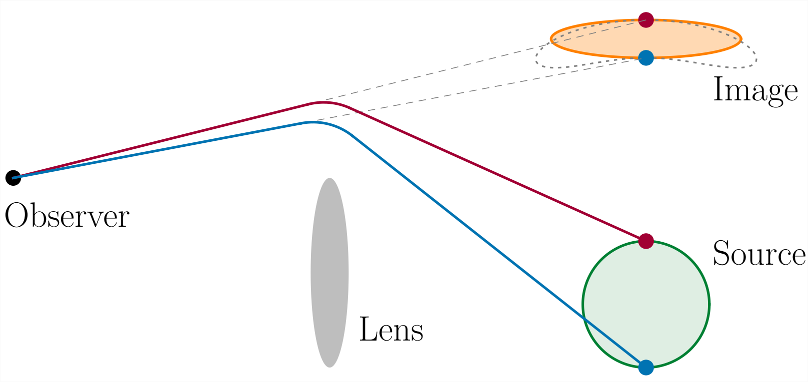
Image
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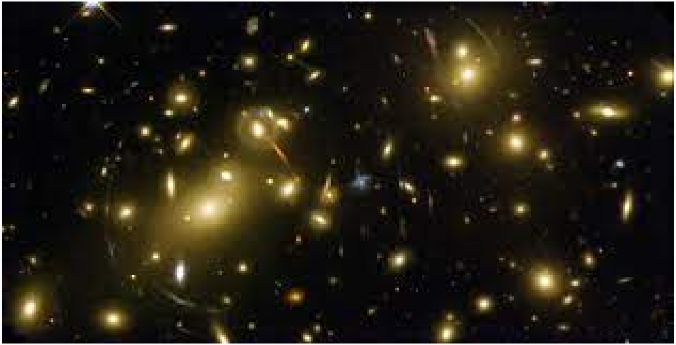
The statistical properties of the cosmic shear are directly linked to the statistical properties of density fluctuations (that is, to the total matter power spectrum). Hence, contrary to other Cosmological tests, WL is probing the dark matter itself



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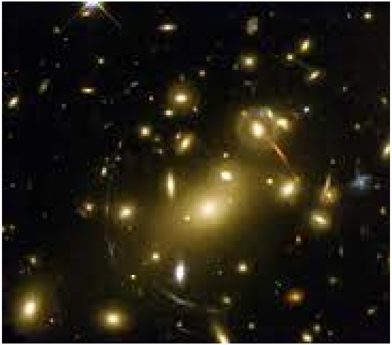
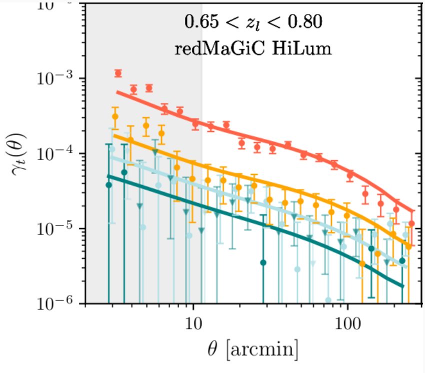
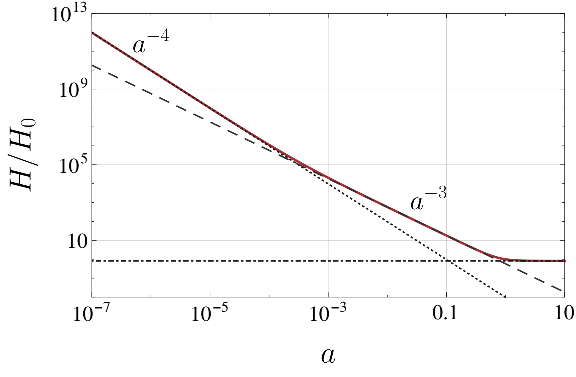




Background (homogeneous and isotropic) Cosmology

Cosmological Principle $\quad + \quad$ General Relativity

$$\Longrightarrow \qquad H^2 = \frac{8\pi G}{3}\sum \bar{\rho}_A + \frac{\Lambda}{3} - \frac{k}{a^2}, \qquad \dot{\bar{\rho}}_A = -3H(\bar{\rho}_A + \bar{P}_A)$$



Comoving radial distance χ

The starting point for the calculation of distances in cosmology is the comoving distance. Consider the comoving distance between a distant light source and us. In a small time interval dt, light travels a comoving *radial* distance

$$d\chi = \frac{dt}{a} \qquad \text{for} \quad d\chi = \frac{dr}{\sqrt{1-kr^2}}$$

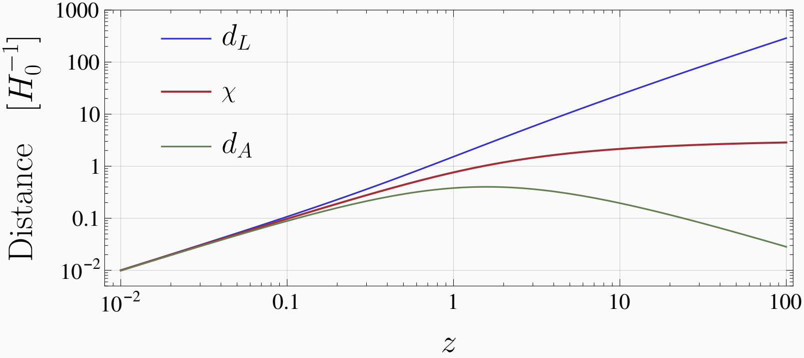
such the total comoving distance traveled by light emitted by a source at time t, when the scale factor was equal to $a = a(t)$, up to today is

$$\chi(t) = \int_t^{t_0} \frac{dt'}{a(t')} = \int_0^z \frac{dz'}{H(z')}$$

Conformal time: $$\eta(t) = \int_0^t \frac{dt'}{a(t')} = \eta_0 - \chi(t)$$

Summary of distances

$$\chi(z) = \frac{c}{H_0}\int_0^z \frac{dz'}{H(z')/H_0}, \quad d_L(z) = (1+z)\chi(z), \quad d_A(z) = \frac{1}{1+z}\chi(z)$$



At small redshift, the three distances coincide: $d = H_0^{-1} z$, (or $z = H_0 d$)

Longitudinal Doppler effect: $\lambda_0/\lambda_S = \sqrt{(1+v)/(1-v)} \approx 1 + v$, then $v = H_0 d$

Cosmological principle

Background: The universe is homogeneous and isotropic

$$X(\boldsymbol{x}, t) = X(\boldsymbol{x} + \boldsymbol{b}, t), \qquad X(\boldsymbol{x}, t) = X(\mathbf{R}\,\boldsymbol{x}, t)$$

+ perturbations: The universe is *statistically* homogeneous and isotropic

$$\begin{aligned}\langle X(\boldsymbol{x}_1, t) Y(\boldsymbol{x}_2, t) \cdots Z(\boldsymbol{x}_n, t)\rangle &= \langle X(\boldsymbol{x}_1 + \boldsymbol{b}) Y(\boldsymbol{x}_2 + \boldsymbol{b}) \cdots Z(\boldsymbol{x}_n + \boldsymbol{b})\rangle \\ &= \langle X(\mathbf{R}\,\boldsymbol{x}_1) Y(\mathbf{R}\,\boldsymbol{x}_2) \cdots Z(\mathbf{R}\,\boldsymbol{x}_n)\rangle\end{aligned}$$

Density fluctuations

$$\rho(\boldsymbol{x}, t) = \bar{\rho}(t)(1 + \delta(\boldsymbol{x}, t))$$

Since $\quad \bar{\rho}(t) = \langle \rho(\boldsymbol{x}, t) \rangle, \quad$ then $\quad \langle \delta(\boldsymbol{x}, t) \rangle = 0.$

2-point correlation function (2PCF): $\qquad \xi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \langle \delta(\boldsymbol{x}_1, t)\delta(\boldsymbol{x}_2, t) \rangle$

$$\xi(\boldsymbol{x}_1, \boldsymbol{x}_2) \quad \underset{\text{homogeneity}}{\longrightarrow} \quad \xi(\boldsymbol{x}_2 - \boldsymbol{x}_1) \quad \underset{\text{isotropy}}{\longrightarrow} \quad \xi(r = |\boldsymbol{x}_2 - \boldsymbol{x}_1|)$$

Fourier Transform conventions

$$f(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}), \qquad f(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{k})$$

Hence

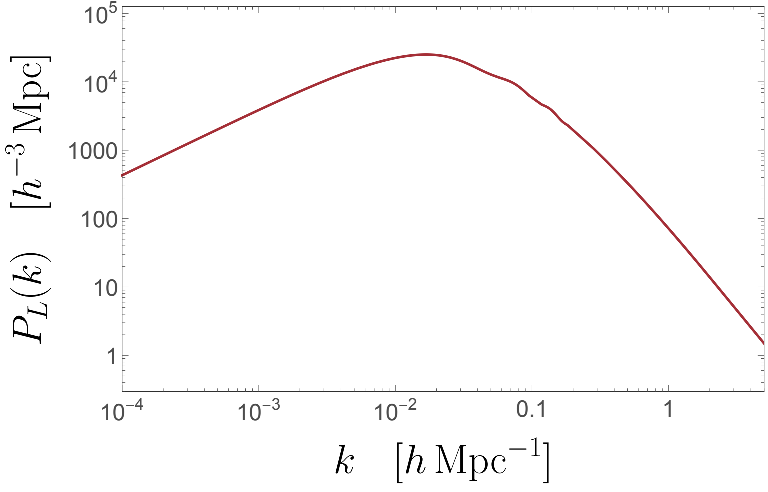
$$(2\pi)^3 \delta_{\mathrm{D}}(\boldsymbol{k}+\boldsymbol{k}') = \int d^3x\, e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}$$

For a real field $f(\boldsymbol{x})$

$$f(\boldsymbol{k}) = f^*(-\boldsymbol{k})$$

Matter power spectrum

$$\langle \delta(\boldsymbol{k})\delta(\boldsymbol{k}')\rangle = (2\pi)^3 \delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')P_L(k)$$

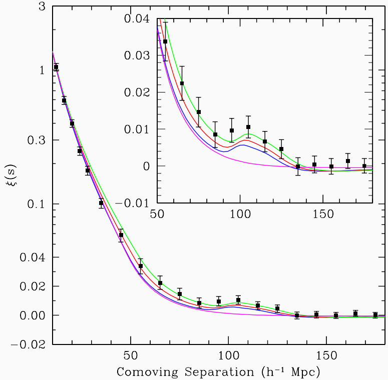


The (2-point) matter correlation function

SUBMITTED TO *The Astrophysical Journal* 12/31/2004
Preprint typeset using LaTeX style emulateapj v. 11/12/01

DETECTION OF THE BARYON ACOUSTIC PEAK IN THE LARGE-SCALE CORRELATION FUNCTION OF SDSS LUMINOUS RED GALAXIES

DANIEL J. EISENSTEIN[1,2], IDIT ZEHAVI[1], DAVID W. HOGG[3], ROMAN SCOCCIMARRO[3], MICHAEL R.



The correlation function is the Fourier Transform of the power spectrum

$$\xi(r) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} P(k)$$

Because $P(\boldsymbol{k}) = P(k)$, the angular integral can be performed analytically

$$\xi(r) = \frac{1}{2\pi^2} \int_0^\infty dk\, k^2 P(k) j_0(kr)$$

with

$$j_0(x) = \frac{\sin x}{x}$$

the spherical Bessel function of degree zero.

This is a Hankel transformation. Due to the oscillatory behaviour of j_0, is computational expensive to integrate it correctly.

Use FFTLog methods: Hamilton (2000), arxiv:9905191 [astro-ph].

Numerical Issues

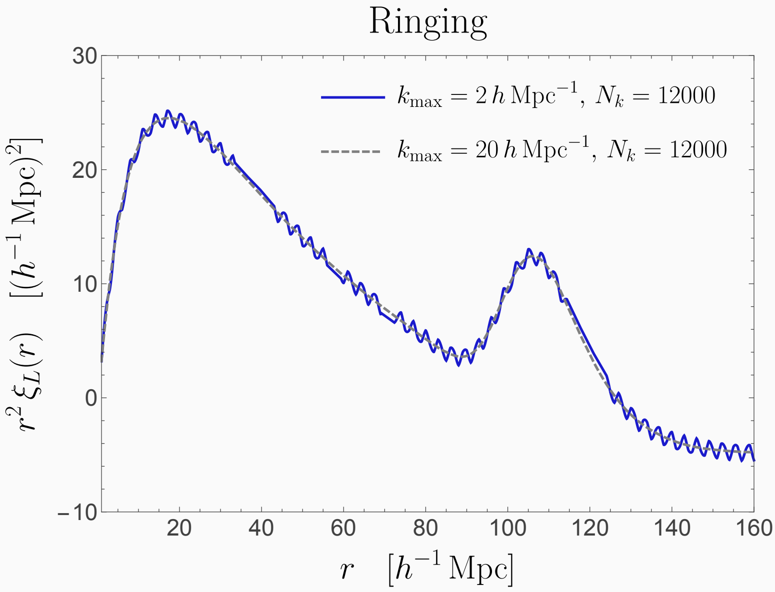
$$\xi(r) = \frac{1}{2\pi^2}\int_0^\infty dk\,k^2 P(k) j_0(kr)$$
$$= \frac{1}{2\pi^2}\int_{k_{\rm min}}^{k_{\rm max}} dk\,k^2 P(k) j_0(kr)$$

$$\xi(r) = \sum_{i=1}^{N_k} \frac{k_i^3}{2\pi^2} P(k_i) j_0(k_i r)\Delta(\log k_i)$$

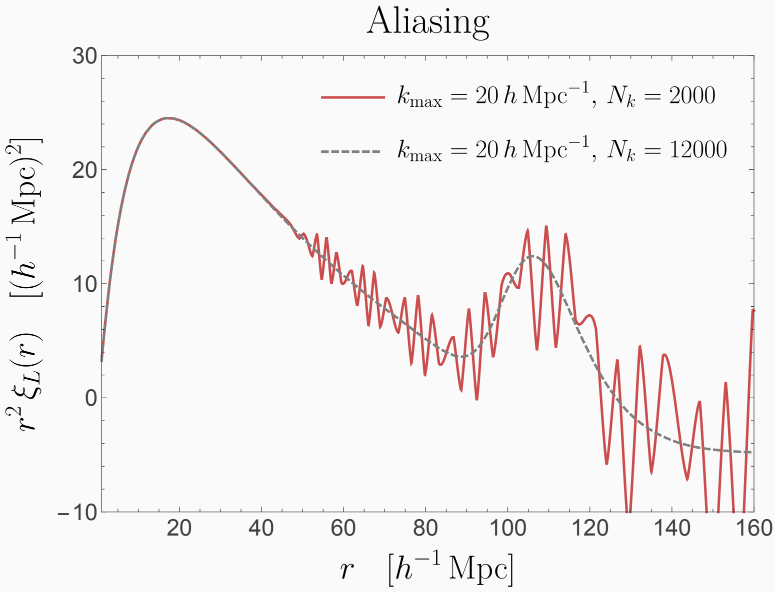
with

$$i = 1,\, 2, \cdots, N_k$$
$$k_i \in \{k_1 = k_{\rm min},\, k_2, \cdots, k_{N_k} = k_{\rm max}\}$$

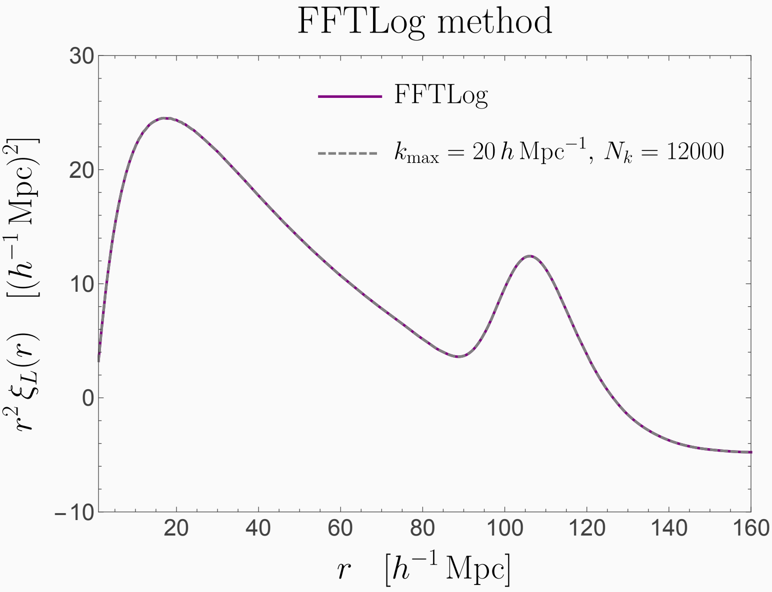
Ringing: cutting off high frequencies



Aliasing: poor sampling



FFTLog. Hamilton (2000)



Matter fluctuations linear evolution

Boltzmann eq. $\longrightarrow$ Fluid eqs. $\longrightarrow$ linearize

$$\Longrightarrow \quad \ddot{\delta}(\boldsymbol{k}, t) + 2H\dot{\delta} - \frac{3}{2}\Omega_m(a)H^2\delta = 0.$$

This equation does not depend on $\boldsymbol{k}$, hence the solution is separable

$$\delta(\boldsymbol{k}, t) = D_+(t)A(\boldsymbol{k}) + D_-(t)B(\boldsymbol{k})$$

with $D_\pm$ the solutions to the equation

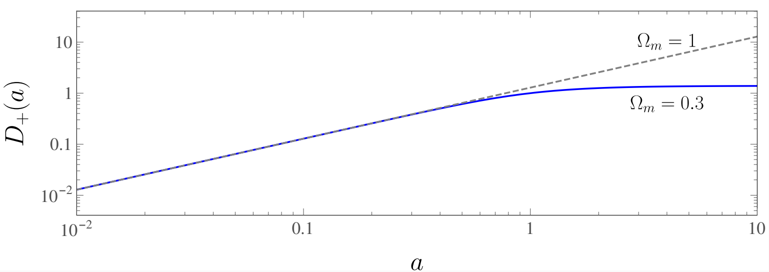
$$\left(\frac{d^2}{dt^2} + 2H\frac{d}{dt} - \frac{3}{2}\Omega_m(a)H^2\right)D(t) = 0$$

Linear growth function

The fastest growing solution is called the *linear growth function* D_+, then

$$\delta^{(1)}(\boldsymbol{k}, t) = D_+(t)\delta^{(1)}(\boldsymbol{k}, t_0)$$

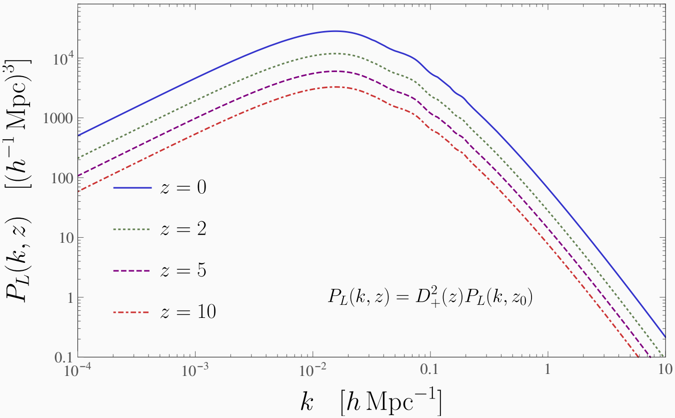
where one normalize $D_+(t_0) = 1$. Typically one chooses t_0 to be the present time. The other solution is $D_- \propto H$.

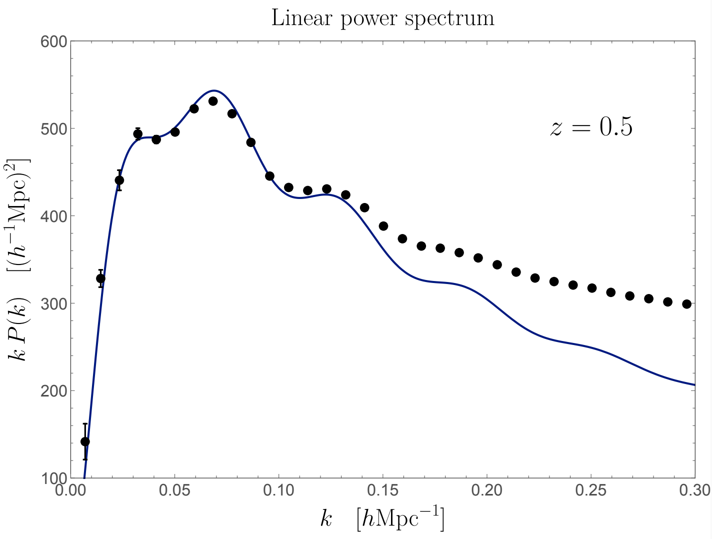


Linear power spectrum

$$\langle \delta^{(1)}(\boldsymbol{k},t)\delta^{(1)}(\boldsymbol{k}',t)\rangle = D_+^2(t)\langle \delta^{(1)}(\boldsymbol{k},t_0)\delta^{(1)}(\boldsymbol{k}',t_0)\rangle$$

$$\Longrightarrow \qquad P_L(k,t) = D_+^2(t)P_L(k,t_0)$$




Linear power spectrum
$z = 0.5$
$k\,P(k) \quad [(h^{-1}\mathrm{Mpc})^2]$
$k \quad [h\mathrm{Mpc}^{-1}]$
600
500
400
300
200
100
0.00
0.05
0.10
0.15
0.20
0.25
0.30

Perturbation theory

$$P^{\text{SPT/EFT}}(k) = (1 - \alpha_{\text{EFT}} k^2) P_L(k) + P_{\text{1-loop}}(k) + \cdots$$

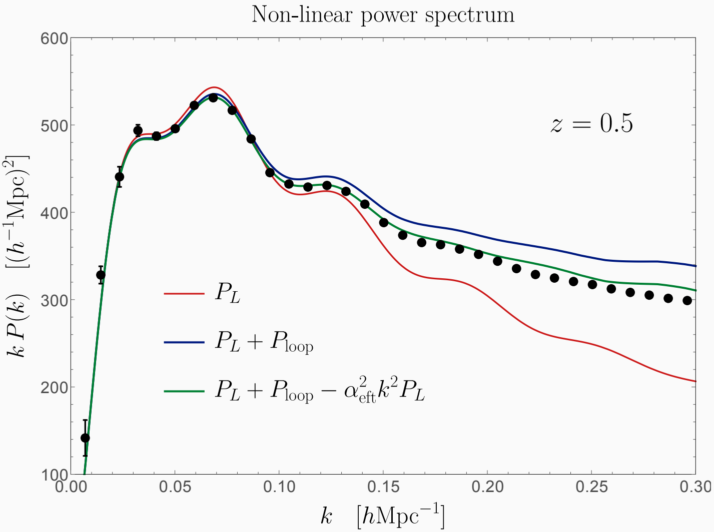
$$P_{\text{1-loop}}(k) = P_{22}(k) + P_{13}(k)$$

with

$$P_{22}(k) = \langle \delta^{(2)}(\boldsymbol{k}) \delta^{(2)}(\boldsymbol{k}') \rangle' = 2 \int \frac{d^3 p}{(2\pi)^3} \left[F_2(\boldsymbol{p}, \boldsymbol{k} - \boldsymbol{p}) \right]^2 P_L(p) P_L(|\boldsymbol{k} - \boldsymbol{p}|)$$

$$P_{13}(k) = 2 \langle \delta^{(1)}(\boldsymbol{k}) \delta^{(3)}(\boldsymbol{k}') \rangle' = 6 P_L(k) \int \frac{d^3 p}{(2\pi)^3} F_3(\boldsymbol{k}, -\boldsymbol{p}, \boldsymbol{p}) P_L(p)$$

$$P^{\text{EFT}}_{\text{1-loop}}(k, t) = \left[1 - \alpha_{\text{EFT}}(t) k^2 \right] D_+^2(t) P_L(k, t_0) + D_+^4(t) P_{\text{1-loop}}(k, t_0) + \mathcal{O}(D_+^6 P^3)$$



Non-linear power spectrum
$z = 0.5$
P_L
$P_L + P_{\rm loop}$
$P_L + P_{\rm loop} - \alpha_{\rm eft}^2 k^2 P_L$
$k\,P(k)\quad[(h^{-1}{\rm Mpc})^2]$
$k\quad[h{\rm Mpc}^{-1}]$
600
500
400
300
200
100
0.00
0.05
0.10
0.15
0.20
0.25
0.30

Bias

We observe galaxies in the sky. But galaxies do not exactly follow the matter distribution, they are biased tracers. To linear order (at large scales) the galaxy density fluctuation is

$$\delta_g(\boldsymbol{x}) = b_1 \delta(\boldsymbol{x}) \qquad \longrightarrow \qquad P_g(k) = b_1^2 P_L(k)$$

Formal theory of bias (McDonald & Roy [JCAP 08 (2009) 020])

$$\delta_g(\boldsymbol{x}) = \sum_{\mathcal{O}} b_{\mathcal{O}} \mathcal{O}(\boldsymbol{x})$$

where $\mathcal{O}$ are a set of local and non-local operators (functions) of the gravitational and velocity potentials. e.g.: $\delta, \delta^2, \cdots \in \mathcal{O}$

Angular power spectra and correlations

Photometric redshift

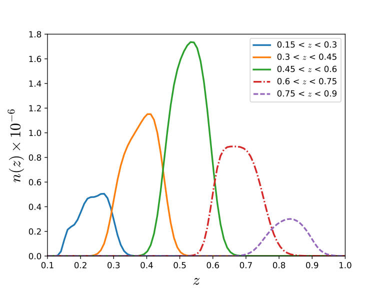
We do not always have access to accurate redshift measurements of astronomical objects. This is the case of, for example, photometric surveys. However, we do have high-quality images in several wavelength bands. These colors can be converted into rough estimates of redshift, so-called photometric redshifts, which can be used as proxies (with significant spread) for true redshifts.

More precisely, we can infer distributions of number of galaxies,

$$W_g(z) = \frac{1}{N_g}\frac{dN_g}{dz},$$

and depending on the features of each galaxy decide to which distribution it belongs.

DES 1 yr results (1708.01536)



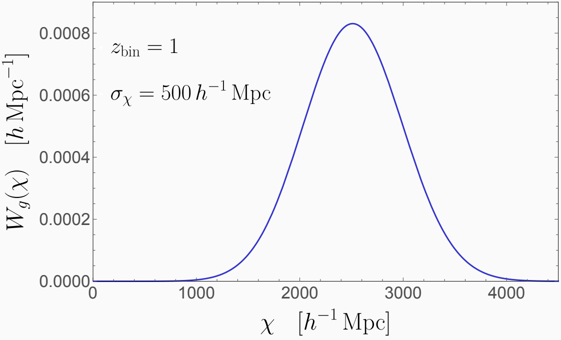
Instead of the redshift z, it is common to use the comoving radial distance $\chi(z)$

$$\chi(a) = \int_a^1 \frac{da'}{a'^2 H(a')}, \qquad \chi(z) = \int_0^z \frac{dz'}{H(z')}$$

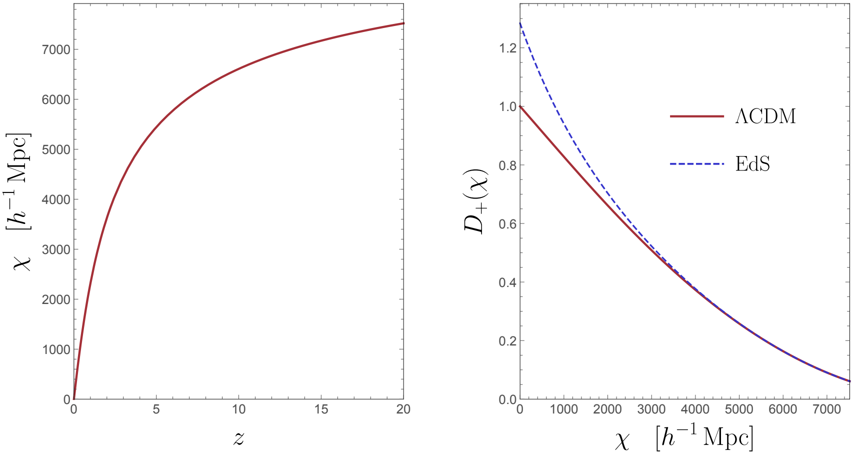
And the distribution of galaxies becomes $W_g(\chi) = W_g(z(\chi))\frac{dz}{d\chi}$

normalized to unity: $\int_0^\infty W_g(\chi) d\chi = 1$.

$$W_g(\chi) \propto \chi^2 e^{-(\chi-\chi_{\rm bin})/2\sigma_\chi^2}$$



Linear Growth function

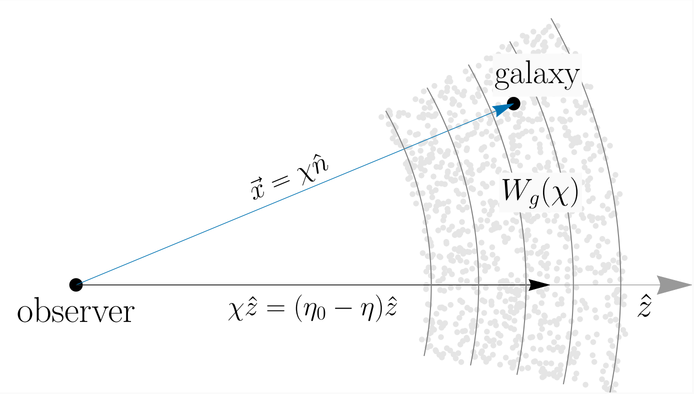


Coordinates

An event in space-time can be written as $\mathbf{p} = (\boldsymbol{x}, \eta)$, with $\boldsymbol{x}$ the comoving coordinates with the expansion and η the conformal time.

The conformal time at the comoving radial distance χ is $\eta(\chi) = \eta_0 - \chi$. And $\boldsymbol{x} = \hat{\boldsymbol{n}}\chi$ with $\hat{\boldsymbol{n}}$ the angular direction

Then, we can write $\mathbf{p} = (\hat{\boldsymbol{n}}\chi, \eta_0 - \chi)$: two numbers for $\hat{\boldsymbol{n}}$, and one for χ.



projected density

If we have no access to the redshift of individual galaxies, instead of measuring the 3-dim galaxy density field, we measure its 2-dim projection on the sky,

$$\Delta_g(\hat{\boldsymbol{n}}) = \int_0^\infty d\chi\, W(\chi)\delta_g(\hat{\boldsymbol{n}}\chi, \eta = \eta_0 - \chi).$$

Now, we transform the 3D fluctuation to (3D) Fourier space:

$$\begin{aligned}
\Delta_g(\hat{\boldsymbol{n}}) &= \int_0^\infty d\chi\, W_g(\chi) \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\hat{\boldsymbol{n}}\chi} \delta_g(k, \chi) \\
&= \int_0^\infty d\chi\, W_g(\chi) \int \frac{d^3k}{(2\pi)^3} \left\{ 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell j_\ell(k\chi) Y_{\ell m}(\hat{\boldsymbol{n}}) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \right\} \delta_g(k, \chi) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\boldsymbol{n}}) \left\{ 4\pi \int \frac{d^3k}{(2\pi)^3} i^\ell Y^*_{\ell m}(\hat{\boldsymbol{k}}) \int_0^\infty d\chi\, W_g(\chi) j_\ell(k\chi)\delta_g(k, \chi) \right\}
\end{aligned}$$

where we used the plane wave expansion

$$e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell j_\ell(kx) Y_{\ell m}(\hat{\boldsymbol{x}}) Y^*_{\ell m}(\hat{\boldsymbol{k}})$$

with j_ℓ the spherical Bessel functions and $Y_{\ell m}$ the spherical harmonics.

Decomposition in spherical harmonics

That is, we have expanded the projected overdensity in spherical harmonics

$$\Delta_g(\hat{\boldsymbol{n}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Delta_{g,\ell m} Y_{\ell m}(\hat{\boldsymbol{n}})$$

with

$$\Delta_{g,\ell m} = 4\pi i^{\ell} \int \frac{d^3k}{(2\pi)^3} Y^*_{\ell m}(\hat{\boldsymbol{k}}) \int_0^{\infty} d\chi \, W_g(\chi) j_\ell(k\chi) \delta_g(k,\chi).$$

The $\Delta_{g,\ell m}$ are the analogs to the $a_{\ell m}$ in the CMB anisotropies. The angular power spectrum of galaxy counts on the sky is then proportional to the expectation value of $|\Delta_{g,\ell m}|^2$.

Angular power spectrum of galaxies

Let us evaluate the angular power spectrum

$$\langle \Delta_{g,\ell m} \Delta^*_{g,\ell' m'} \rangle = (4\pi)^2 i^{\ell-\ell'} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} Y^*_{\ell m}(\hat{\boldsymbol{k}}) Y_{\ell' m'}(\hat{\boldsymbol{k}}') \times \int_0^\infty d\chi \, W_g(\chi) j_\ell(k\chi) \int_0^\infty d\chi' \, W_g(\chi') j_{\ell'}(k'\chi') \, \langle \delta_g(\boldsymbol{k},\chi) \delta^*_g(\boldsymbol{k}',\chi') \rangle$$

The ensemble average of overdensities yields a Dirac delta function $\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}')$, that we use to perform the d^3k' integral and sets $\boldsymbol{k}' = \boldsymbol{k}$. Hence, we can use the orhonormality of spherical harmonics,

$$\int d\Omega_{\boldsymbol{k}} \, Y^*_{\ell m}(\hat{\boldsymbol{k}}) Y_{\ell' m'}(\hat{\boldsymbol{k}}) = \delta_{\ell\ell'} \delta_{mm'},$$

to integrate the angular part of the d^3k integral.

We obtain

$$\langle \Delta_{g,\ell m} \Delta^*_{g,\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_g(\ell)$$

with

$$C_g(\ell) = \frac{2}{\pi} \int_0^\infty d\chi \, W_g(\chi) \int_0^\infty d\chi' \, W_g(\chi') \int_0^\infty dk \, k^2 j_\ell(k\chi) j_\ell(k\chi') P_g(\boldsymbol{k}; \chi, \chi')$$

Notice:

- To compute the angular power spectrum we need to integrate the two overdensities at all “times” χ. And then the galaxy standard power spectrum is evaluated at two different times.
- The power spectrum evaluated at two different times $P_g(\boldsymbol{k}; \chi, \chi')$ is anisotropic because it depends on the value of two overdensities evaluated at two different times.

  Indeed, using the obvious transverse symmetry $\boldsymbol{k} = (k_\parallel, \boldsymbol{k}_\perp^{\rm 2D})$ or $\boldsymbol{k} = (k\mu_k, \boldsymbol{k}_\perp^{\rm 2D})$, where μ_k is the cosine angle of $\boldsymbol{k}$ and the line of sight.

Limber approximation

Consider the integral representation of the Dirac delta function:

$$\int_0^\infty dk\, k^2 j_\ell(k\chi) j_\ell(k\chi') = \frac{\pi}{2\chi^2}\delta_{\rm D}(\chi - \chi')$$

We have almost the above integral, but with $P(k)$ in the integrand. But for large ℓ one gets

$$\int dk\, k^2 j_\ell(k\chi) j_\ell(k\chi') f(k) = \frac{\pi}{2\chi^2}\delta_{\rm D}(\chi - \chi') f\big(\frac{\ell + 1/2}{\chi}\big)$$

which can be obtained by using the approximation

$$j_\ell(x) \approx \sqrt{\frac{\pi}{2\ell}}\,\delta_{\rm D}(\ell + \frac{1}{2} - x) \qquad \text{for large } \ell$$

which is accurate inside the integral for large ℓ.

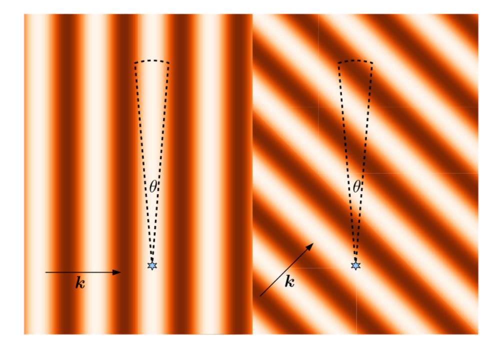
One obtains the angular power spectrum under the Limber approximation

$$C_g(\ell) = \int_0^\infty \frac{d\chi}{\chi^2} W_g^2(\chi) P_g\left(k = \frac{\ell + 1/2}{\chi}, \mu_k = 0; \chi\right)$$

valid for large ℓ. Typically for $\ell > 20$.

- We see that $\chi = \chi'$ in the Limber approximation, hence only the power spectrum at equal-time densities in the integral is important (at large ℓ).
- It means also that the k modes involved do not have a line-of-sight component, since that would mean different distances of different points along the perturbation, i.e. $\chi \neq \chi'$.
- So, $\boldsymbol{k}$ has to be transverse to the line of sight: $\mu_k = 0$. The longitudinal components cancel out!

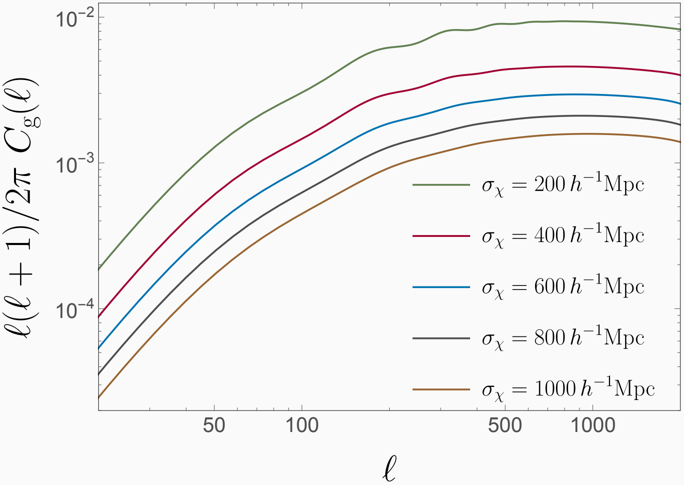
Dodelson & Schmidt figure 11.8



Focusing on small scales corresponds to looking at small angles, $\theta \sim 1/\ell$. *Right panel:* Modes with longitudinal wavenumber $\mu_k k > \chi^{-1}$ (or $\mu_k > \theta$) don't provide angular correlations because of cancelations along the line of sight. *Left panel:* Only modes with $\mu_k k \lesssim \chi^{-1}$ (or $\mu_k < \theta$) lead to angular correlations. Since χ is typically large, compared to k^{-1}, this corresponds to setting $\chi = \chi'$.

Angular galaxy power spectrum

$$C_{\rm g}(\ell) = \int d\chi \frac{W_g^2(\chi)}{\chi^2} P_{\rm g}\Big(\frac{\ell+1/2}{\chi}, \eta_0 - \chi\Big)$$

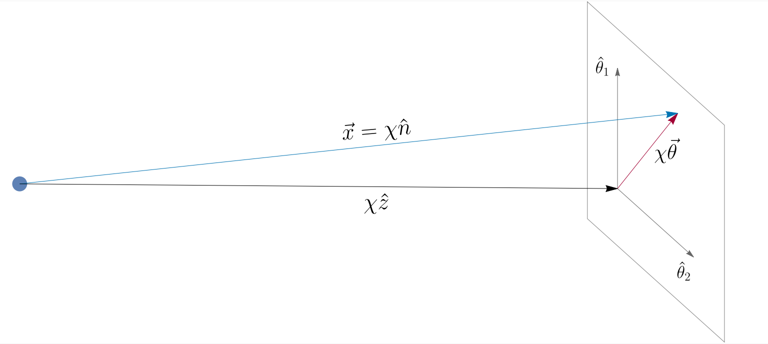


Coordinates in the plane parallel approximation

We choose a basis vector on the plane, $\{\hat{\theta}_1, \hat{\theta}_2\}$. Hence the 3-dim spatial position is given by

$$\boldsymbol{x} = \chi(z)(\theta_1, \theta_2, 1),$$

$(\hat{\theta}_1 \times \hat{\theta}_2 = \hat{z})$, while the position on the plane is $\boldsymbol{\theta} = (\theta_1, \theta_2)$

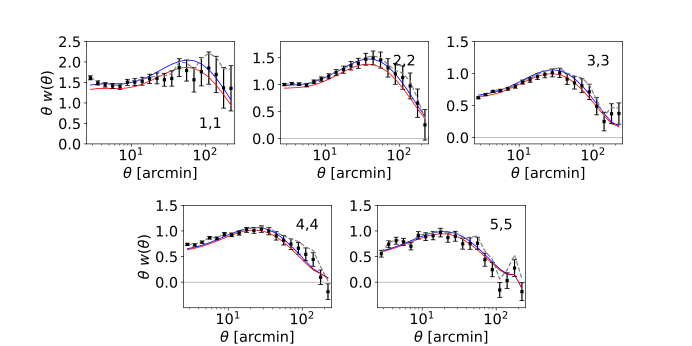


Under the plane parallel approximation $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}} + \hat{\boldsymbol{\theta}}$.

Angular correlation function

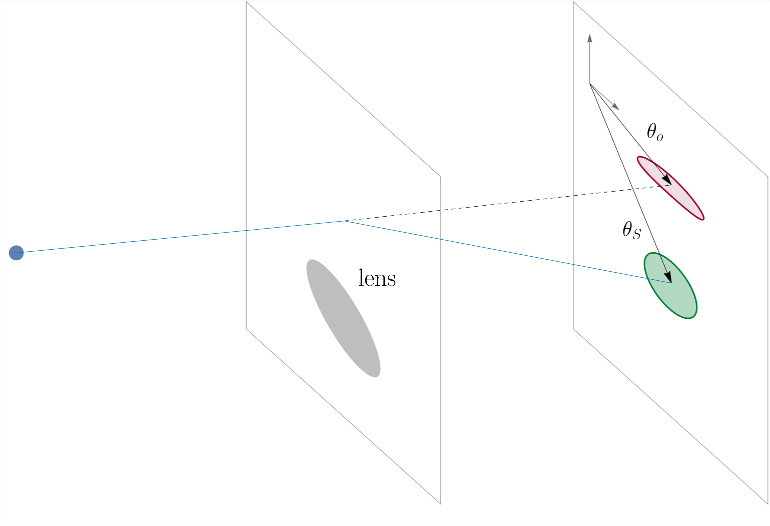
The angular correlation function is the Fourier transform of the angular power spectrum:

$$\xi_g(\theta) = \int \frac{d^2\ell}{(2\pi)^2} e^{i\boldsymbol{\ell}\cdot\boldsymbol{\theta}} C_g(\ell) = \int_0^\infty d\ell\, \ell C_g(\ell) J_0(\ell\theta)$$



DES Year 1 Results: galaxy-galaxy auto-correlation function for different photometric redshifts [1708.01536]

2. Weak lensing basics



θ_o
θ_S
lens

We use the perturbed metric in Newtonian Gauge

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1-2\Phi)\delta_{ij}dx^i dx^j.$$

With this convention, the weak field limit gives

$$\text{Geodesic equation:} \quad \ddot{\boldsymbol{x}} + H(t)\dot{\boldsymbol{x}} = -\frac{1}{a}\boldsymbol{\nabla}\Psi(\boldsymbol{x},t)$$

$$\text{Poisson equation:} \quad \frac{1}{a^2}\nabla^2\Phi(\boldsymbol{x},t) = 4\pi G\rho_m(t)\delta_m(\boldsymbol{x},t) \quad \text{(In GR)}$$

In the absence on anisotropic stresses (also assuming GR) $\Psi = \Phi$. e.g, this is the case of ΛCDM from the matter dominated epoch to today ($z \lesssim 100$),

Christoffel symbols:

$$\Gamma^0_{00} = \dot{\Psi}, \quad \Gamma^i_{00} = \frac{\partial\Psi}{\partial x_i}, \quad \Gamma^i_{j0} = \delta^i_j(H - \dot{\Phi}),$$

$$\Gamma^i_{00} = \delta_{ij}a^2\left[H + \dot{\Phi} - 2H(\Phi+\Psi)\right],$$

$$\Gamma^i_{jk} = \left(\delta_{jk}\frac{\partial}{\partial x_i} - \delta^i_j\frac{\partial}{\partial x^k} - \delta^i_k\frac{\partial}{\partial x^j}\right)\Phi.$$

$$\text{4-momentum of photons:} \quad g_{\mu\nu}P^\mu P^\nu = 0 \quad \text{with} \quad P^\mu = \frac{dx^\mu}{d\lambda}.$$

Hence, defining the generalized spatial momentum p,

$$p = g_{ij}P^i P^j,$$

and

$$P^0 = \frac{p}{\sqrt{1+2\Psi}} = p(1-\Psi).$$

This is the generalization to a perturbed FRW of the relativistic expression $E = p$. In the metric convention we are following, an overdense region has $\Psi < 0$. Therefore, photons loose energy (and redshift) as they move away from an overdense region.

Geodesic equation $P^\nu \nabla_\mu P^\nu = 0$:

$$\frac{d^2 x^i}{d\lambda^2} = -\Gamma^i_{\mu\nu} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}.$$

The LHS is, remind $x^i = \chi\theta^i = \chi(\theta^1, \theta^2, \chi)$,

$$\begin{aligned}
\frac{d^2 x^i}{d\lambda^2} &= \frac{dt}{d\lambda}\frac{d\chi}{dt}\frac{d}{d\chi}\left[\frac{d\chi\theta^i}{d\chi}\frac{dt}{d\lambda}\frac{d\chi}{dt}\right] = -\frac{p}{a}\frac{d}{d\chi}\left[-\frac{p}{a}\frac{d\chi\theta^i}{d\chi}\right] \\
&= p^2 \frac{d}{d\chi}\left[\frac{1}{a^2}\frac{d}{d\chi}(\chi\theta^i)\right] \\
&= \frac{p^2}{a^2}\left[\frac{d^2}{d\chi^2}(\chi\theta^i) + 2aH\frac{d}{d\chi}(\chi\theta^i)\right]
\end{aligned}$$

where we used the definition of the radial comoving distance $\chi = \int_a^{a_0} dt/a(t)$ and $P^0 \equiv dt/d\lambda = p(1-\Psi)$. We have assumed the deflection angle is small, and set $\theta \times \Psi = 0$. In the second line, we use the background evolution of the momentum $p \propto 1/a$, so at the lowest order ap is a constant that we can pull out of the derivative.

Using the Christoffel symbols we obtain

$$-\Gamma^i_{\mu\nu}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = -p^2\left[-\frac{\partial}{\partial x_i}(\Psi+\Phi) + 2\frac{H}{a}\frac{d}{d\chi}(\chi\theta^i)\right]$$

Hence, the geodesic equation becomes

$$\boxed{\frac{d^2}{d\chi^2}(\chi\theta_m) = -a^2\frac{\partial}{\partial x^m}(\Psi+\Phi)}$$

In the ΛCDM model at late times, the anisotropic stresses are neglibible, so $\Psi = \Phi$ and

$$\frac{d^2}{d\chi^2}(\chi\theta_m) = -2a^2\frac{\partial\Psi}{\partial x^m}$$

We can integrate twice to obtain the true source position $\theta_S^m = \theta^m(\chi)$, subject to the initial condition that the observed position is $\theta_O^m = \theta^m(\chi = 0)$

$$\theta_S^m = \theta^m(\chi) = \theta_O^m - \int_0^\chi d\chi' \, \frac{\chi - \chi'}{\chi} \, \frac{\partial}{\partial x^m} \Big[\Psi(\boldsymbol{x}) + \Phi(\boldsymbol{x}) \Big]$$

with $\boldsymbol{x} \equiv \boldsymbol{x}(\boldsymbol{\theta}(\chi'), \chi')$.

This is a non-linear relation between the observed and true source positions.

The first approximation considers $\boldsymbol{\theta}(\chi') = \boldsymbol{\theta}_O$ in the arguments of the gravitational potentials, corresponding to integrate the potential gradient along the unperturbed ray, which is called the Born approximation.

Defining the lensing potential ψ (from now on, we simplify the notation and name θ the observed position)

$$\psi(\boldsymbol{\theta}, \chi) = -\int_0^{\chi} \frac{d\chi'}{\chi'} \frac{\chi - \chi'}{\chi} \Big[\Psi(\boldsymbol{x}(\chi')) + \Phi(\boldsymbol{x}(\chi'))\Big],$$

the true and observed position are related by

$$\theta_S^m = \theta^m + \partial^m \psi(\theta),$$

where we used $\frac{\partial}{\partial x^m} = \frac{1}{\chi}\frac{\partial}{\partial \theta^m}$ inside the integral. And we denote $m, n = 1, 2$, so $\partial_m = \left(\frac{\partial}{\partial \theta^1}, \frac{\partial}{\partial \theta^2}\right)$.

This equation is valid for a single background source at, radial comoving distance χ, whose light travels toward us and deviates due to the foreground matter distribution

We are interested in small deflections of path light rays. In such a case, the relation between observed and true coordinates is linear:

$$A_{mn} \equiv \frac{\partial \theta^S_m}{\partial \theta^n} = I_{mn} + \partial_m \partial_n \psi \equiv \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}.$$

Assuming small deflection angles we expand

$$\theta^S_m(\theta) = \underbrace{A_{mn}\theta_n}_{\text{Convergence + Shear}} + \underbrace{\frac{1}{2} D_{mns}\theta_n\theta_s}_{\text{Flexion}} + \cdots$$

$$\text{Weak lensing:} \quad \boxed{\theta^S_m = A_{mn}\theta_n.}$$

That is, weak lensing is described by a linear map relating the observed and true positions of the sources.

Weak lensing assumes that the value of the derivatives of the lensing potential $\partial_m \partial_n \psi$ do not change across the source surface (e.g., through a galaxy subtended solid angle), otherwise we would have to account for $D_{mns} = \partial_m \partial_n \partial_s \psi$.

Weak lensing:

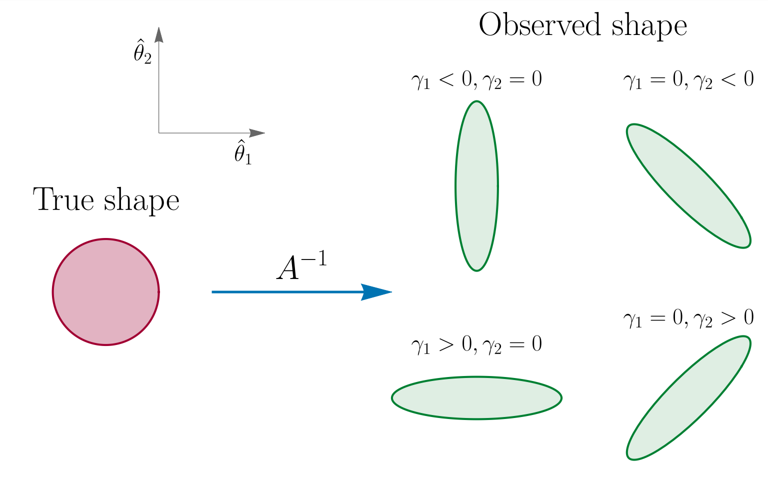
$$\begin{pmatrix} \theta_1^S \\ \theta_2^S \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} -\kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & -\kappa + \gamma_1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

Using $\theta_i^S = \theta_i + (\partial_i \partial_j \psi)\, \theta^j + \cdots$, with ψ the lensing potential

$$\kappa = -\frac{1}{2}(\partial_1\partial_1 + \partial_2\partial_2)\psi = -\frac{1}{2}\partial^2\psi$$
$$\gamma_1 = -\frac{1}{2}(\partial_1\partial_1 - \partial_2\partial_2)\psi, \qquad \gamma_2 = -\partial_1\partial_2\psi$$

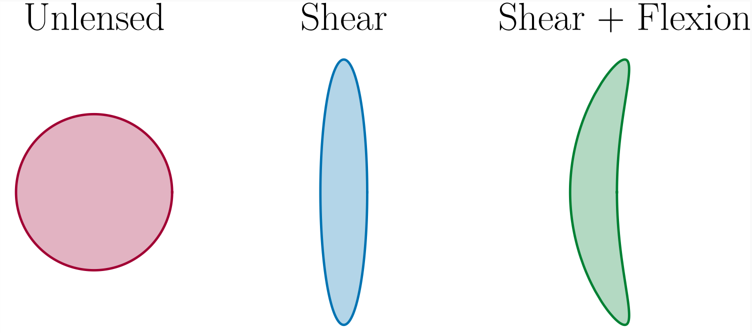
The inverse Jacobian A^{-1} describes the local mapping of the source light distribution to image coordinates.

The convergence κ is an isotropic increase or decrease of the observed size of a source image. The shear (γ_1, γ_2), the trace-free part of A, quantifies an anisotropic stretching, turning a circular into an elliptical light distribution.



$\hat{\theta}_2$
$\hat{\theta}_1$
True shape
A^{-1}
Observed shape
$\gamma_1 < 0, \gamma_2 = 0$
$\gamma_1 = 0, \gamma_2 < 0$
$\gamma_1 > 0, \gamma_2 = 0$
$\gamma_1 = 0, \gamma_2 > 0$

$$\boldsymbol{\theta}_s = \boldsymbol{A}\,\boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\theta}^T \boldsymbol{D}\,\boldsymbol{\theta}$$



In a right-handed orthonormal coordinate basis $(\hat{\theta}_1, \hat{\theta}_2, \hat{z} = \hat{\boldsymbol{n}})$

$$A_{mn} = \begin{pmatrix} 1-\kappa & 0 \\ 0 & 1-\kappa \end{pmatrix} + \begin{pmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix} \equiv (1-\kappa) I_{mn} + \Gamma_{mn}.$$

Notice that if the convergence is not observable what we actually measure is the reduced shear $g \equiv \gamma/(1-\kappa) \simeq \gamma$.

Against a rotation by an angle φ, Γ_{mn} transforms as

$$\Gamma'_{mn} = (R\Gamma R^{-1})_{mn} = \begin{pmatrix} -\gamma_1 \cos(2\varphi) + \gamma_2 \sin(2\varphi) & -\gamma_2 \cos(2\varphi) - \gamma_1 \sin(2\varphi) \\ -\gamma_2 \cos(2\varphi) - \gamma_1 \sin(2\varphi) & +\gamma_1 \cos(2\varphi) - \gamma_2 \sin(2\varphi) \end{pmatrix}$$
$$= \begin{pmatrix} -\gamma'_1 & -\gamma'_2 \\ -\gamma'_2 & \gamma'_1 \end{pmatrix}$$

That is, the shear components γ_1 and γ_2 transform between themselves against the rotation of coordinates $R(\varphi) : (\hat{\theta}_1, \hat{\theta}_2) \to (\hat{\theta}'_1, \hat{\theta}'_2)$ as

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \to \begin{pmatrix} \gamma'_1 \\ \gamma'_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 \cos(2\varphi) - \gamma_2 \sin(2\varphi) \\ \gamma_2 \cos(2\varphi) + \gamma_1 \sin(2\varphi) \end{pmatrix},$$

that is, they transform with the double angle.

Define the helicity basis

$$\boldsymbol{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}(\hat{\theta}_1 \pm i\hat{\theta}_2).$$

We use the components

$$\gamma \equiv \Gamma_{++} = \Gamma_{mn}\boldsymbol{\epsilon}_+^m \boldsymbol{\epsilon}_+^n = \gamma_1 + i\gamma_2$$
$$\bar{\gamma} \equiv \Gamma_{--} = \Gamma_{mn}\boldsymbol{\epsilon}_-^m \boldsymbol{\epsilon}_-^n = \gamma_1 - i\gamma_2$$

and $\Gamma_{+-} = \Gamma_{-+} = 0$.

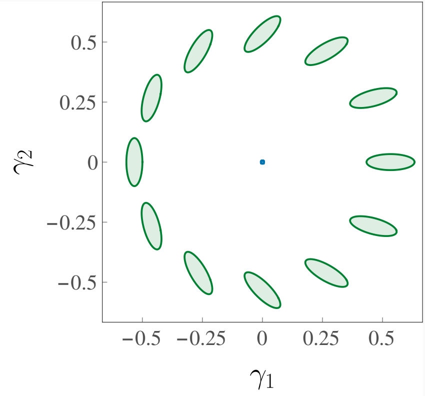
With this, the transformation becomes

$$\gamma_1' \pm i\gamma_2' = e^{\pm 2i\varphi}(\gamma_1 \pm i\gamma_2),$$

or, equivalently

$$\gamma \quad \to \quad \gamma' = e^{2i\varphi}\gamma.$$

That is, γ and $\bar{\gamma}$ transform as spin-2 functions against rotation.



Convergence as a projected density

$$\kappa(\boldsymbol{\theta}, \chi) = -\frac{1}{2}\partial^2\psi = \int_0^\chi \frac{d\chi'}{\chi'}\frac{\chi-\chi'}{\chi}\partial^2\Phi$$
$$= \int_0^\chi d\chi'\,\chi'\,\frac{\chi-\chi'}{\chi}\nabla^2_{\boldsymbol{x}}\Phi$$
$$= \frac{3}{2}\Omega_m H_0^2 \int_0^\chi \frac{d\chi'}{a(\chi')}\chi'\frac{\chi-\chi'}{\chi}\delta(\chi'\boldsymbol{\theta}, \chi'),$$

where we used $\partial^2 = \chi^2\nabla^2_{\boldsymbol{x}}$ in the third equality, and the Poisson equation to relate the gravitational potential Φ with the overdensity field δ.

Notice that the geometrical factor $\chi'(\chi - \chi')$ is a parabola with maximum at $\chi' = \chi/2$. Hence, structures at half the distance between the source and the observers are more efficient to produce lensing distortions.

The total convergence from a population of source galaxies is obtained by weighting $\kappa(\theta, \chi)$ expression with the galaxy probability distribution $W_g(\chi)$:

$$
\begin{aligned}
\kappa(\boldsymbol{\theta}) &= \int_0^\infty d\chi W_g(\chi) \kappa(\boldsymbol{\theta}, \chi) \\
&= \frac{3}{2}\Omega_m H_0^2 \int_0^\infty d\chi W_g(\chi) \int_0^\chi d\chi' \frac{1}{a(\chi')} \chi' \frac{\chi - \chi'}{\chi} \delta(\boldsymbol{\theta}, \chi') \\
&= \frac{3}{2}\Omega_m H_0^2 \int_0^\infty d\chi' \int_{\chi'}^\infty d\chi \, W_g(\chi) \frac{1}{a(\chi')} \chi' \frac{\chi - \chi'}{\chi} \delta(\boldsymbol{\theta}, \chi') \\
&= \int_0^\infty d\chi' q(\chi') \delta(\chi' \boldsymbol{\theta}, \chi')
\end{aligned}
$$

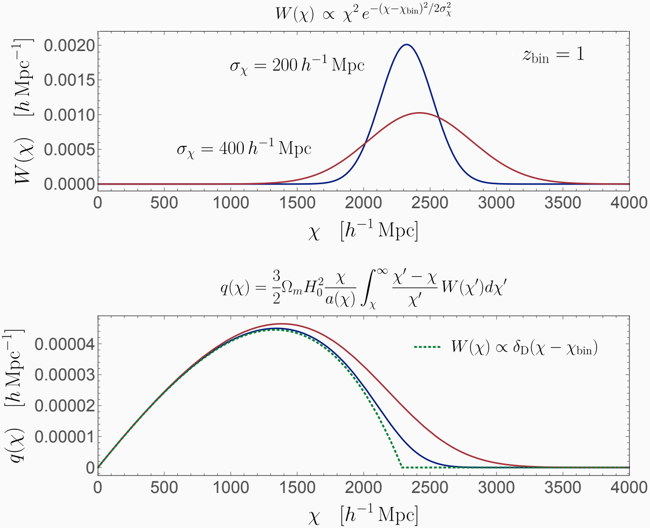
with the lens efficiency q defined as

$$
q(\chi) = \frac{3}{2}\Omega_m H_0^2 \frac{\chi}{a(\chi)} \int_\chi^\infty d\tilde{\chi} \, W_g(\tilde{\chi}) \frac{\tilde{\chi} - \chi}{\tilde{\chi}}
$$

The convergence becomes a linear measure of the total matter density, projected along the line of sight and weighted by the source galaxy distribution W_g.

q yields the lensing strength at a distance χ of the combined background galaxy distribution.

There are different conventions/definition for the lens efficiency


$W(\chi) \propto \chi^2 e^{-(\chi-\chi_{\rm bin})^2/2\sigma_\chi^2}$
$\sigma_\chi = 200\, h^{-1}\,\mathrm{Mpc}$
$\sigma_\chi = 400\, h^{-1}\,\mathrm{Mpc}$
$z_{\rm bin} = 1$
$W(\chi)\quad [h\,\mathrm{Mpc}^{-1}]$
0.0000
0.0005
0.0010
0.0015
0.0020
$\chi \quad [h^{-1}\,\mathrm{Mpc}]$
0
500
1000
1500
2000
2500
3000
3500
4000
$q(\chi) = \frac{3}{2}\Omega_m H_0^2 \frac{\chi}{a(\chi)} \int_\chi^\infty \frac{\chi' - \chi}{\chi'} W(\chi') d\chi'$
$W(\chi) \propto \delta_{\rm D}(\chi - \chi_{\rm bin})$
$q(\chi)\quad [h\,\mathrm{Mpc}^{-1}]$
0
0.00001
0.00002
0.00003
0.00004
$\chi \quad [h^{-1}\,\mathrm{Mpc}]$
0
500
1000
1500
2000
2500
3000
3500
4000

The convergence angular power spectrum is computed through the definition,

$$\langle \kappa(\boldsymbol{\ell})\kappa(\boldsymbol{\ell}')\rangle = (2\pi)^2 \delta_{\mathrm{D}}(\boldsymbol{\ell}+\boldsymbol{\ell}')C_\kappa(\ell),$$

under the Limber/flat sky approximation, we have

$$\boxed{C_\kappa(\ell) = \int_0^\infty \frac{d\chi}{\chi^2}\, q^2(\chi) P_\delta\left(\frac{\ell+1/2}{\chi}, \chi\right).}$$

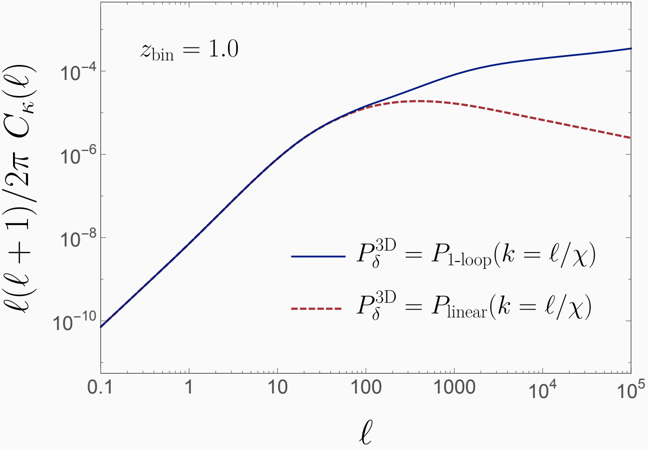
Compare it to the galaxy angular power spectrum in

$$C_g(\ell) = \int_0^\infty \frac{d\chi}{\chi^2}\, W_g^2(\chi) P_g\left(\frac{\ell+1/2}{\chi}, \chi\right)$$

$W_g(\chi) \longrightarrow q(\chi)$, but more important $P_g(\chi) \longrightarrow P_\delta(\chi)$. That is, weak lensing probes the whole power spectrum, including the dark matter.

Convergence angular power spectrum

$$C_\kappa(\ell) = \int_0^\infty \frac{d\chi}{\chi^2}\, q^2(\chi) P_\delta\Big(\frac{\ell + 1/2}{\chi}, \chi\Big).$$



Shear power spectra

We comeback to the equations relating the components of $\boldsymbol{A}$ with the lensing potential ψ:

$$\kappa(\boldsymbol{\theta}) = -\frac{1}{2}(\partial_1\partial_1 + \partial_2\partial_2)\psi(\boldsymbol{\theta}) = -\frac{1}{2}\nabla^2\psi(\boldsymbol{\theta})$$

$$\gamma_1(\boldsymbol{\theta}) = -\frac{1}{2}(\partial_1\partial_1 - \partial_2\partial_2)\psi(\boldsymbol{\theta}), \quad \gamma_2(\boldsymbol{\theta}) = -\partial_1\partial_2\psi(\boldsymbol{\theta})$$

using the first equation in the form $\psi(\boldsymbol{\theta}) = -2\nabla^{-2}\kappa(\boldsymbol{\theta})$, we have

$$\gamma(\boldsymbol{\theta}) = (\partial_1 + i\partial_2)^2\,\nabla^{-2}\kappa(\boldsymbol{\theta}),$$

or

$$\gamma(\boldsymbol{\theta}) = \int d^2\theta'\mathcal{D}(\boldsymbol{\theta} - \boldsymbol{\theta}')\kappa(\boldsymbol{\theta}'), \quad \text{with} \quad \mathcal{D}(\boldsymbol{\theta}) = \frac{\theta_2^2 - \theta_1^2 - 2i\theta_1\theta_2}{\theta^4}$$

$$\kappa(\boldsymbol{\theta}) = \int d^2\theta'\mathcal{D}^*(\boldsymbol{\theta} - \boldsymbol{\theta}')\gamma(\boldsymbol{\theta}')$$

Fourier space

Dual variables: $\boldsymbol{\theta} = (\theta_1, \theta_2) \quad \longrightarrow \quad \boldsymbol{\ell} = (\ell_1, \ell_2).$

$$f(\boldsymbol{\ell}) = \int d^2\theta \, e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\ell}} f(\boldsymbol{\theta}),$$
$$f(\boldsymbol{\theta}) = \int \frac{d^2\ell}{(2\pi)^2} e^{i\boldsymbol{\theta}\cdot\boldsymbol{\ell}} f(\boldsymbol{\ell}),$$

$$\partial_m f(\boldsymbol{\theta}) \longrightarrow i\ell_m f(\boldsymbol{\ell}) \qquad (m = 1, 2)$$
$$\nabla^2 f(\boldsymbol{\theta}) \longrightarrow -\ell^2 f(\boldsymbol{\ell}) \qquad (\ell^2 = \ell_1^2 + \ell_2^2)$$

Real field $f(\boldsymbol{\theta})$ implies $f^*(\boldsymbol{\ell}) = f(-\boldsymbol{\ell})$

Shear power spectra

We comeback to the equations relating the components of $\boldsymbol{A}$ with the lensing potential ψ:

$$\kappa(\boldsymbol{\theta}) = -\frac{1}{2}(\partial_1^2 + \partial_2^2)\psi(\boldsymbol{\theta}) \longrightarrow \kappa(\boldsymbol{\ell}) = \frac{1}{2}(\ell_1^2 + \ell_2^2)\psi = \frac{1}{2}\ell^2\psi(\boldsymbol{\ell})$$

$$\gamma_1(\boldsymbol{\theta}) = -\frac{1}{2}(\partial_1^2 - \partial_2^2)\psi(\boldsymbol{\theta}) \longrightarrow \gamma_1(\boldsymbol{\ell}) = \frac{1}{2}(\ell_1^2 - \ell_2^2)\psi(\boldsymbol{\ell})$$

$$\gamma_2(\boldsymbol{\theta}) = -\partial_1\partial_2\psi(\boldsymbol{\theta}) \longrightarrow \gamma_2(\boldsymbol{\ell}) = \ell_1\ell_2\psi(\boldsymbol{\ell})$$

$$\gamma(\boldsymbol{\ell}) = \gamma_1(\boldsymbol{\ell}) + i\gamma_2(\boldsymbol{\ell}) = (\ell_1 + i\ell_2)^2\psi(\boldsymbol{\ell})$$

In Fourier space: $$\gamma(\boldsymbol{\ell}) = \frac{(\ell_1 + i\ell_2)^2}{\ell^2}\kappa(\boldsymbol{\ell})$$

That is, In Fourier space,

$$\gamma(\boldsymbol{\ell}) = \gamma_1(\boldsymbol{\ell}) + i\gamma_2(\boldsymbol{\ell}) = \left(\frac{\ell_1}{\ell} + i\frac{\ell_2}{\ell}\right)^2 \kappa(\boldsymbol{\ell}) = e^{i2\phi_l}\kappa(\boldsymbol{\ell})$$

with $\phi_l = \tan^{-1}(\ell_2/\ell_1)$, and $\boldsymbol{\ell} = (\ell_1, \ell_2) = \ell(\cos\phi_l, \sin\phi_l)$. Then

$$\gamma_1(\boldsymbol{\ell}) = \cos(2\phi_l)\kappa(\boldsymbol{\ell}), \qquad \gamma_2(\boldsymbol{\ell}) = \sin(2\phi_l)\kappa(\boldsymbol{\ell})$$

We can immediately construct three angular spectra out of γ_1 and γ_2:

$$(2\pi)^2\delta(\boldsymbol{\ell}+\boldsymbol{\ell}')P_{\gamma_1}(\ell) = \langle\gamma_1(\boldsymbol{\ell})\gamma_1(\boldsymbol{\ell}')\rangle$$
$$(2\pi)^2\delta(\boldsymbol{\ell}+\boldsymbol{\ell}')P_{\gamma_2}(\ell) = \langle\gamma_2(\boldsymbol{\ell})\gamma_2(\boldsymbol{\ell}')\rangle$$
$$(2\pi)^2\delta(\boldsymbol{\ell}+\boldsymbol{\ell}')P_{\gamma_1,\gamma_2}(\ell) = \langle\gamma_1(\boldsymbol{\ell})\gamma_2(\boldsymbol{\ell}')\rangle$$

using the above relations between $\gamma_{1,2}$ and κ,

$$C_{\gamma_1}(\ell) = \cos^2(2\phi_l)\, C_\kappa(\ell),$$
$$C_{\gamma_2}(\ell) = \sin^2(2\phi_l)\, C_\kappa(\ell),$$
$$C_{\gamma_1,\gamma_2}(\ell) = \cos(2\phi_l)\sin(2\phi_l)\, C_\kappa(\ell).$$

These spectra are not independent. And further, they depend on how we choose the basis $\hat{\theta}_1, \hat{\theta}_2$.

A clever combination is

$$\mathcal{E}(\boldsymbol{\ell}) = \cos(2\phi_l)\gamma_1(\boldsymbol{\ell}) + \sin(2\phi_l)\gamma_2(\boldsymbol{\ell})$$
$$\mathcal{B}(\boldsymbol{\ell}) = -\sin(2\phi_l)\gamma_1(\boldsymbol{\ell}) + \cos(2\phi_l)\gamma_2(\boldsymbol{\ell})$$

A rapid computation of the correlations $\langle\mathcal{E}(\boldsymbol{\ell})\mathcal{E}(\boldsymbol{\ell}')\rangle$, $\langle\mathcal{E}(\boldsymbol{\ell})\mathcal{B}(\boldsymbol{\ell}')\rangle$ and $\langle\mathcal{B}(\boldsymbol{\ell})\mathcal{B}(\boldsymbol{\ell}')\rangle$ gives

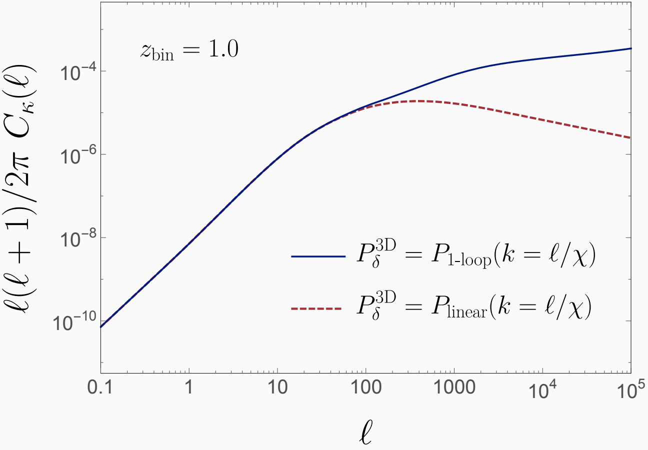
$$C_{\mathcal{E}}(\ell) = C_\kappa(\ell), \qquad C_{\mathcal{E},\mathcal{B}}(\ell) = 0, \qquad C_{\mathcal{B}}(\ell) = 0.$$

e.g.

$$\langle\mathcal{E}(\boldsymbol{\ell})\mathcal{E}(\boldsymbol{\ell}')\rangle = \langle\big(\cos(2\phi_l)\gamma_1(\boldsymbol{\ell}) + \sin(2\phi_l)\gamma_2(\boldsymbol{\ell})\big)\big(\cos(2\phi_l)\gamma_1(\boldsymbol{\ell}') + \sin(2\phi_l)\gamma_2(\boldsymbol{\ell}')\big)\rangle$$
$$= \cos^2(2\phi_l)\underbrace{\langle\gamma_1(\boldsymbol{\ell})\gamma_1(\boldsymbol{\ell}')\rangle}_{\cos^2(2\phi_l)C_\kappa} + 2\cos(2\phi_l)\sin(2\phi_l)\underbrace{\langle\gamma_1(\boldsymbol{\ell})\gamma_2(\boldsymbol{\ell}')\rangle}_{\cos(2\phi_l)\sin(2\phi_l)C_\kappa} + \sin^2(2\phi_l)\underbrace{\langle\gamma_2(\boldsymbol{\ell})\gamma_2(\boldsymbol{\ell}')\rangle}_{\sin^2(2\phi_l)C_\kappa}$$
$$= (2\pi)^2\delta(\boldsymbol{\ell}+\boldsymbol{\ell}')\big(\cos^2(2\phi_l) + \sin^2(2\phi_l)\big)^2 C_\kappa = (2\pi)^2\delta(\boldsymbol{\ell}+\boldsymbol{\ell}')C_\kappa(\boldsymbol{\ell})$$

Cosmic shear angular power spectrum

$$C_{\mathcal{E}}(\ell) = \int_0^\infty \frac{d\chi}{\chi^2}\, q^2(\chi) P_\delta\left(\frac{\ell + 1/2}{\chi}, \chi\right).$$



The matrix $\mathbf{\Gamma}$ becomes

$$\Gamma_{mn}(\boldsymbol{\ell}) = \begin{pmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix} = 2\left(\frac{\ell_m \ell_n}{\ell^2} - \frac{1}{2}\delta_{ab}\right)\mathcal{E}(\boldsymbol{\ell}) + \Gamma^{\perp}_{mn}(\boldsymbol{\ell}),$$

with the transverse piece $\ell_m \Gamma^{\perp}_{mn} = 0$. And $\mathcal{E} = \frac{\ell_m \ell_n}{\ell^2}\Gamma_{mn}$.

It is easy to see

$$\Gamma_{mn}(\boldsymbol{\ell}) = \begin{pmatrix} \cos(2\phi_\ell) & \sin(2\phi_\ell) \\ \sin(2\phi_\ell) & -\cos(2\phi_\ell) \end{pmatrix} \mathcal{E}(\boldsymbol{\ell}) + \begin{pmatrix} -\sin(2\phi_\ell) & \cos(2\phi_\ell) \\ \cos(2\phi_\ell) & \sin(2\phi_\ell) \end{pmatrix} \mathcal{B}(\boldsymbol{\ell}).$$

One can define the following vector:

$$\boldsymbol{u} = \begin{pmatrix} \partial_1\gamma_1 + \partial_2\gamma_2 \\ \partial_1\gamma_2 - \partial_2\gamma_1 \end{pmatrix}$$

But since the converge κ and shear γ are related through its derivatives, one finds

$$\boldsymbol{u} = \nabla\kappa$$

and hence $\nabla \times \boldsymbol{u} = \partial_1 u_2 - \partial_2 u_1 = 0$ since this vector is a gradient. That is,

$$-2\partial_1\partial_2\gamma_1 + \left(\partial_1^2 + \partial_2^2\right)\gamma_2 = 0$$

A shear field fulfilling this equation is called an $\mathcal{E}$-mode.

In other words, the field $\gamma = \gamma_1 + i\gamma_2$, written as

$$\gamma(\boldsymbol{\theta}) = -\int \frac{d^2\ell}{2\pi}\left(\mathcal{E}(\boldsymbol{\ell}) + i\mathcal{B}(\boldsymbol{\ell})\right)e^{-2i\phi_l}e^{i\boldsymbol{\ell}\cdot\boldsymbol{\theta}}$$

is an E-mode iff $\mathcal{B}(\ell) = 0$.

The shear field obtained from weak lensing is a pure $\mathcal{E}$-mode.

However, there are also $\mathcal{B}$-modes (rotational modes), arising from different origins: e.g. systematic errors in the measurements, higher order approximations in the light propagation, and intrinsic alignments.

Very briefly γ is not only due to weak lensing, and hence $\nabla \times u$ can be different from zero. It is customary to define $\kappa^{\mathcal{E}}$ and $\kappa^{\mathcal{B}}$ and potentials $\psi^{\mathcal{E}}$ and $\psi^{\mathcal{B}}$, and then combine the two modes into the complex fields $\kappa = \kappa^{\mathcal{E}} + i\kappa^{\mathcal{B}}$, $\psi = \psi^{\mathcal{E}} + i\psi^{\mathcal{B}}$. The complex shear $\gamma = \gamma_1 + i\gamma_2$ becomes

$$\gamma = -\frac{1}{2}(\partial_1 + i\partial_2)^2\psi$$

That is, in the case that there are other sources of shear, apart from weak lensing, one obtains B-mode correlations.

Cosmic shear correlation function

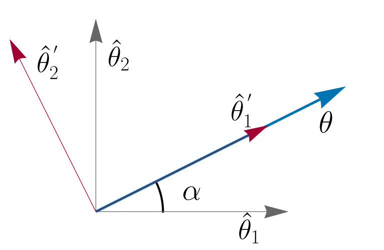
Remind that $\gamma(\boldsymbol{\ell}) = e^{2i\phi_l}\kappa(\boldsymbol{\ell})$, where ϕ_l is the polar angle of $\boldsymbol{\ell}$, that is $\boldsymbol{\ell} = \ell(\cos\phi_l, \sin\phi_l)$.

Then

$$
\begin{aligned}
\langle\gamma(\boldsymbol{\ell})\gamma^*(\boldsymbol{\ell}')\rangle &= (2\pi)^2\delta_{\rm D}(\boldsymbol{\ell}-\boldsymbol{\ell}')\,C_{\mathcal{E}}(\boldsymbol{\ell}), \\
\langle\gamma(\boldsymbol{\ell})\gamma(\boldsymbol{\ell}')\rangle &= (2\pi)^2\delta_{\rm D}(\boldsymbol{\ell}+\boldsymbol{\ell}')e^{4i\phi_l}\,C_{\mathcal{E}}(\boldsymbol{\ell}).
\end{aligned}
$$

Now, consider two galaxies at two positions in the projected plane, one at $\boldsymbol{\nu}$ and the other at $\boldsymbol{\nu} + \boldsymbol{\theta}$. We want to compute the correlation between their shears. We choose coordinates that are aligned to the separation vector $\boldsymbol{\theta}$. That is, we first rotate the shear by the angle $-\alpha$, where $\alpha = \hat{\boldsymbol{\theta}} \cdot \hat{\theta}_1$,

$$\gamma \quad \longrightarrow \quad \gamma' = \gamma e^{-2i\alpha}$$



Then, we take the real and imaginary parts to isolate the components of the shear in the new coordinate system:

$$\gamma_{\mathrm{t}} = -\mathrm{Re}\left[\gamma e^{-2i\alpha}\right], \quad \gamma_{\times} = -\mathrm{Im}\left[\gamma e^{-2i\alpha}\right],$$

named the tangential and cross components of the shear along the direction $\boldsymbol{\theta}$.

The shear correlation functions are defined as

$$\begin{aligned}
\xi_+(\theta) &= \langle\gamma_t(\boldsymbol{\nu})\gamma_t(\boldsymbol{\nu}+\boldsymbol{\theta})\rangle + \langle\gamma_\times(\boldsymbol{\nu})\gamma_\times(\boldsymbol{\nu}+\boldsymbol{\theta})\rangle \\
\xi_-(\theta) &= \langle\gamma_t(\boldsymbol{\nu})\gamma_t(\boldsymbol{\nu}+\boldsymbol{\theta})\rangle - \langle\gamma_\times(\boldsymbol{\nu})\gamma_\times(\boldsymbol{\nu}+\boldsymbol{\theta})\rangle \\
\xi_\times(\theta) &= \langle\gamma_t(\boldsymbol{\nu})\gamma_\times(\boldsymbol{\nu}+\boldsymbol{\theta})\rangle
\end{aligned}$$

We choose the coordinates $(\hat{\theta}_1', \hat{\theta}_2')$, on which $\gamma_1 = -\gamma_t$ and $\gamma_2 = -\gamma_\times$. Then we can simply compute

$$\begin{aligned}
\langle\gamma_1(0)\gamma_1(\boldsymbol{\theta})\rangle &= \int \frac{d^2\ell}{(2\pi)^2}\int\frac{d^2\ell'}{(2\pi)^2}\cos(2\phi_l)\cos(2\phi_{l'})\langle\mathcal{E}(\boldsymbol{\ell})\mathcal{E}(\boldsymbol{\ell}')\rangle e^{i\boldsymbol{\ell}\cdot\boldsymbol{\theta}} \\
&= \int \frac{d^2\ell}{(2\pi)^2} e^{il\theta\cos\phi_l}\cos^2(2\phi_l)\,C_{\mathcal{E}}(\ell)
\end{aligned}$$

and similarly for $\langle\gamma_1(0)\gamma_1(\boldsymbol{\theta})\rangle$ which gives a $\sin^2(2\phi_l)$ instead of a cosine function.

$$\begin{aligned}
\langle\gamma_2(0)\gamma_2(\boldsymbol{\theta})\rangle &= \int \frac{d^2\ell}{(2\pi)^2}\int\frac{d^2\ell'}{(2\pi)^2}\cos(2\phi_l)\cos(2\phi_{l'})\langle\mathcal{E}(\boldsymbol{\ell})\mathcal{E}(\boldsymbol{\ell}')\rangle e^{i\boldsymbol{\ell}\cdot\boldsymbol{\theta}} \\
&= \int \frac{d^2\ell}{(2\pi)^2} e^{il\theta\cos\phi_l}\sin^2(2\phi_l)\,C_{\mathcal{E}}(\ell).
\end{aligned}$$

The sum gives

$$\begin{aligned}\xi_+(\theta) &= \langle\gamma_1(0)\gamma_1(\boldsymbol{\theta})\rangle + \langle\gamma_2(0)\gamma_2(\boldsymbol{\theta})\rangle \\ &= \int \frac{d^2\ell}{(2\pi)^2} e^{il\theta\cos\phi_l}\, C_{\mathcal{E}}(\ell) = \int \frac{d\ell\,\ell}{2\pi} C_{\mathcal{E}}(\ell) \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi_l\, e^{il\theta\cos\phi_l} \\ &= \int \frac{d\ell\,\ell}{2\pi} C_{\mathcal{E}}(\ell) J_0(\ell\theta),\end{aligned}$$

and the difference

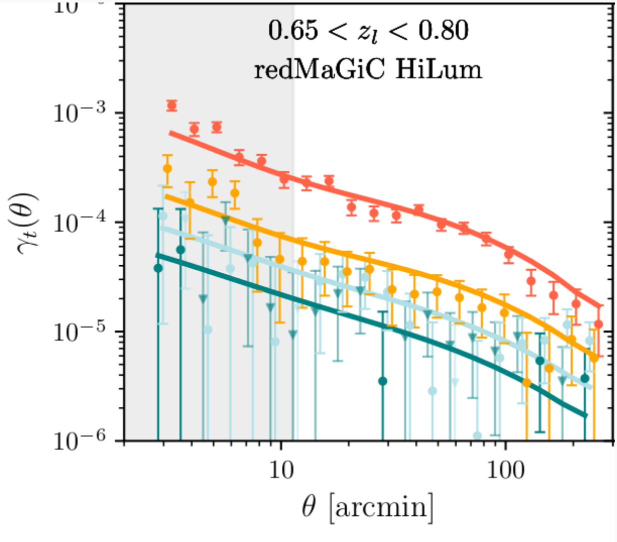
$$\begin{aligned}\xi_-(\theta) = \langle\gamma_1(0)\gamma_1(\boldsymbol{\theta})\rangle - \langle\gamma_2(0)\gamma_2(\boldsymbol{\theta})\rangle = \int \frac{d^2\ell}{(2\pi)^2} e^{il\theta\cos\phi_l}\, \cos(4\phi_l) C_{\mathcal{E}}(\ell) \\ = \int \frac{d\ell\,\ell}{2\pi} C_{\mathcal{E}}(\ell) J_4(\ell\theta).\end{aligned}$$

We used the integral form of the Bessel functions

$$J_n(x) = \frac{i^{-n}}{\pi}\int_0^{\pi} d\phi\, e^{ix\cos\phi} \cos(n\phi).$$

Analogously, one can show that $\xi_\times(\theta) = 0$

These correlation functions are the most important observables of weak lensing catalogs



DES year 1

2. FFTLog method

This method only works, as it stands, for linear theory. We have to change it for non-linear (halo models). But is a good start to understand how these FFTLog methods works

Let's compute the shear 2PCF cases for warming up. This method works only for linear field since one can factorize $P(k,t) = D_+^2(t)P(k)$. Later, I will try a different approach than this (but slower).

We want to integrate

$$\xi_+(\theta) = \int \frac{\ell d\ell}{2\pi} J_0(\theta\ell) C_\kappa(\ell)$$
$$\xi_-(\theta) = \int \frac{\ell d\ell}{2\pi} J_4(\theta\ell) C_\kappa(\ell)$$

but instead of doing it directly with $C_\kappa(\ell)$, We will use a different route:

$$\begin{aligned}
\xi_+(\theta) &= \int \frac{\ell d\ell}{2\pi} J_0(\theta\ell) C_\kappa(\ell) \\
&= \int \frac{\ell d\ell}{2\pi} J_0(\theta\ell) \int d\chi \frac{q_f(\chi) q_g(\chi)}{\chi^2} D_+^2(\chi) P_\delta\left(\frac{\ell}{\chi}\right) \\
&= \int d\chi \frac{q_f(\chi) q_g(\chi)}{\chi^2} D_+^2(\chi) \int \frac{d\ell}{2\pi}\, \ell J_0(\theta\ell) P_\delta\left(\frac{\ell}{\chi}\right)
\end{aligned}$$

That is, we perform first the ℓ integral, even though it looks slower than the standard route of first computing the χ integral. Defining $k = \ell/\chi$, we obtain

$$\xi_{+,-}(\theta) = \int d\chi \frac{q_f(\chi) q_g(\chi)}{\chi^2} \chi^2 D_+^2(\chi) \int \frac{dk}{2\pi}\, k J_{0,4}(\chi\theta k) P_\delta\left(k\right)$$

We expand the linear power spectrum as a sum of scale invariant spectra with complex powers as https://jila.colorado.edu/~ajsh/FFTLog/

$$\bar{P}_L(k) = \sum_{m=-N/2}^{N/2} c_m k^{\nu + i\eta_m},$$

$$\eta_m = \frac{N-1}{N} \frac{2\pi m}{\ln(k_{\max}/k_{\min})},$$

where we have split an interval $[k_{\min}, k_{\max}]$ in N logarithmic spaced wave-numbers. The coefficients c_m comes from the discrete log-Fourier transform

$$c_m = W_m k_{\min}^{-\nu - i\eta_m} \frac{1}{N} \sum_{l=0}^{N-1} P_L(k_l) \left(\frac{k_l}{k_{\min}} \right)^{-\nu} e^{-2\pi i m l/N},$$

with the weights $W_m = 1$, except for the end points, for which $W_{-N/2} = W_{N/2} = 1/2$. The so-called bias ν is in principle any real number, but its value is chosen to have a better convergence.

Substituting eq. (**??**) in eq. (**??**)

$$\xi_{+,-}(\theta) = \sum_{m=-N/2}^{N/2} c_m \int d\chi \frac{q_f(\chi) q_g(\chi)}{\chi^2} \chi^2 D_+^2(\chi) \int \frac{dk}{2\pi}\, k^{a_m} J_{0,4}(\chi\theta k) \tag{1}$$

$$\text{with} \qquad a_m = 1 + \nu + i\eta_m \tag{2}$$

Now,

$$\int \frac{dk}{2\pi}\, k^{a_m} J_n(xk) = \frac{2^{-1+a_m} x^{-a_m-1} \Gamma\left(\frac{1}{2}(a_m + n + 1)\right)}{\pi \Gamma\left(\frac{1}{2}(-a_m + n + 1)\right)} \tag{3}$$

for $\mathrm{Re}[a_m] < 1/2$ and $\mathrm{Re}[a_m + n] > -1$, or since $n = 0, 1, 2...$, $-1 < \mathrm{Re}[a_m] < 1/2$. Hence we have to choose the bias, for example e.g., $\nu = -1.5$, hence $a_m = -0.5 + i\eta_m$. Hence

$$\begin{aligned}
\xi_n(\theta) &= \sum_{m=-N/2}^{N/2} c_m \int d\chi \frac{q_f(\chi)q_g(\chi)}{\chi^2} D_+^2(\chi)\chi^2(\chi\theta)^{-a_m-1} \frac{2^{-1+a_m}\Gamma\left(\frac{1}{2}(a_m+n+1)\right)}{\pi\Gamma\left(\frac{1}{2}(-a_m+n+1)\right)} \\
&= \sum_{m=-N/2}^{N/2} c_m\theta^{-a_m-1} \frac{2^{-1+a_m}\Gamma\left(\frac{1}{2}(a_m+n+1)\right)}{\pi\Gamma\left(\frac{1}{2}(-a_m+n+1)\right)} \\
&\qquad \times \int d\chi\, q_f(\chi)q_g(\chi)D_+^2(\chi)\chi^{-a_m-1} \qquad (4)
\end{aligned}$$

or

$$\xi_+(\theta) = \sum_{m=-N/2}^{N/2} c_m A_m \theta^{-(a_m+1)} \frac{2^{-1+a_m}\Gamma\left(\frac{1}{2}(a_m+1)\right)}{\pi\Gamma\left(\frac{1}{2}(-a_m+1)\right)} \qquad (5)$$

$$\xi_-(\theta) = \sum_{m=-N/2}^{N/2} c_m A_m \theta^{-(a_m+1)} \frac{2^{-1+a_m}\Gamma\left(\frac{1}{2}(a_m+5)\right)}{\pi\Gamma\left(\frac{1}{2}(-a_m+5)\right)} \qquad (6)$$

with

$$A_m = \int d\chi\, q_f(\chi)q_g(\chi)\chi^{-a_m-1}D_+^2(\chi) \qquad (7)$$