UNIVERSIDAD SIMÓN BOLIVAR **26 de mayo de 2010**
CI-7521 - Matemáticas para la computación **Lic. Alejandro Claro Mosqueda**

Tarea #2

Problema #1: Calcular la suma $\sum_{1\le k\le n} \lfloor \log_2 k \rfloor$, donde a es la parte entera por debajo de a.

Solución

Siguiendo el procedimiento explicado en la sección **3.5**[1] se obtiene que

$$\begin{aligned}
\sum_{1\le k\le n} \lfloor \log_2 k \rfloor &= \sum_{k,m\ge 0} m\,[m = \lfloor \log_2 k \rfloor]\,[1\le k\le n] \\
&= \sum_{k,m\ge 0} m\,[1\le k\le n]\,[m\le \log_2 k < m+1] \\
&= \sum_{k,m\ge 0} m\,[1\le k\le n]\left[2^m \le k < 2^{m+1}\right]
\end{aligned}$$

Para cada entero positivo k tal que $2^m \le k < 2^{m+1}$, donde $m = \lfloor \log_2 k \rfloor$, se tiene que hay $2^{m+1} - 2^m = 2^m$ términos iguales a m siempre y cuando $m < \lfloor \log_2 n \rfloor$. También es fácil reconocer que hay $n - 2^{\lfloor \log_2 n \rfloor} + 1$ términos en la sumas que son iguales a $\lfloor \log_2 n \rfloor$. Por cual se obtiene que

$$\begin{aligned}
\sum_{1\le k\le n} \lfloor \log_2 k \rfloor &= \sum_{m\ge 0} m\,2^m\,[m < \lfloor \log_2 n \rfloor] + \lfloor \log_2 n \rfloor \left(n - 2^{\lfloor \log_2 n \rfloor} + 1\right) \\
&= \sum_{m=0}^{\lfloor \log_2 n \rfloor - 1} m\,2^m + \lfloor \log_2 n \rfloor \left(n - 2^{\lfloor \log_2 n \rfloor} + 1\right) \\
&= \sum_{x=0}^{\lfloor \log_2 n \rfloor} x\,2^x \delta x + \lfloor \log_2 n \rfloor \left(n - 2^{\lfloor \log_2 n \rfloor} + 1\right)
\end{aligned}$$

Encontrar una forma cerrada para la suma $\sum_0^{\lfloor \log_2 n \rfloor} x\,2^x \delta x$ es sencillo al realizar una suma por partes con $u(x) = x$ y $\Delta v = 2^x$,

$$\begin{aligned}
\sum_{x=0}^{\lfloor \log_2 n \rfloor} x\,2^x \delta x &= x\,2^x \Big|_{x=0}^{\lfloor \log_2 n \rfloor} - \sum_{x=0}^{\lfloor \log_2 n \rfloor} 2^{x+1} \\
&= x\,2^x - 2^{x+1} \Big|_{x=0}^{\lfloor \log_2 n \rfloor} \\
&= \lfloor \log_2 n \rfloor\, 2^{\lfloor \log_2 n \rfloor} - 2^{\lfloor \log_2 n \rfloor + 1} + 2
\end{aligned}$$

Remplazando esta suma se obtiene finalmente que

$$\sum_{1\le k\le n} \lfloor \log_2 k \rfloor \quad = \quad \lfloor \log_2 n \rfloor \, 2^{\lfloor \log_2 n \rfloor} - 2^{\lfloor \log_2 n \rfloor + 1} + 2 + \lfloor \log_2 n \rfloor \left(n - 2^{\lfloor \log_2 n \rfloor} + 1 \right)$$

$$\boxed{\sum_{k=1}^{n} \lfloor \log_2 k \rfloor = \lfloor \log_2 n \rfloor \,(n+1) - 2\left(2^{\lfloor \log_2 n \rfloor} - 1\right)}$$

Problema #2: Evalúe $\varnothing_n = \sum_{k=1}^{n} k^3$ utilizando el método 5 del libro **Concrete Mathematics**[1] como sigue: primero escriba $\varnothing_n + \Box_n = 2\sum_{1\le j\le k\le n} jk$; luego aplique (**2.33**)[1].

Solución

Primero observese que,

$$\begin{aligned}
\varnothing_n + \Box_n &= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \\
&= \sum_{k=1}^{n} \left\{ k^3 + k^2 \right\} \\
&= \sum_{k=1}^{n} \left\{ k^2(k+1) \right\} \\
&= \sum_{k=1}^{n} \left\{ 2\,k \left[\frac{k(k+1)}{2} \right] \right\} \\
&= 2\sum_{k=1}^{n} \left\{ k \sum_{j=1}^{k} j \right\} \\
&= 2 \sum_{1\le j\le k\le n} jk
\end{aligned}$$

Aprovechando este resultado y (**2.33**)[1]

$$S_{\triangledown} = \sum_{1\le j\le k\le n} a_j\, a_k = \frac{1}{2}\left[\left(\sum_{k=1}^{n} a_k \right)^2 + \sum_{k=1}^{n} a_k^2 \right]$$

se obtiene, al sustituir $a_j = j$ y $a_k = k$, que

$$
\begin{aligned}
\frac{\emptyset_n + \Box_n}{2} &= \sum_{1 \le j \le k \le n} jk \\
&= \frac{1}{2}\left[\left(\sum_{k=1}^{n} k\right)^2 + \sum_{k=1}^{n} k^2\right] \\
&= \frac{1}{2}\left\{\left[\frac{n(n+1)}{2}\right]^2 + \Box_n\right\} \\
\emptyset_n &= \left[\frac{n(n+1)}{2}\right]^2
\end{aligned}
$$

$$
\boxed{\emptyset_n = \frac{[n(n+1)]^2}{4}}
$$

Problema #3: Evalúe la suma $\sum\limits_{k=1}^{n} \frac{(2k+1)}{k(k+1)}$ usando sumas por partes.

Solución

Observese que,

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{(2k+1)}{k(k+1)} &= \sum_{j=0}^{n-1} \frac{(2j+3)}{(j+1)(j+2)} \qquad\qquad (j = k-1) \\
&= \sum_{j=0}^{n-1} (2j+3) j^{\underline{-2}} \\
&= \sum_{j=0}^{n} (2x^{\underline{1}} + 3) x^{\underline{-2}} \delta x
\end{aligned}
$$

Ahora, al sumar por partes con,

$$
\begin{aligned}
u(x) &= 2x^{\underline{1}} + 3 \\
\Delta u &= 2 \\
\Delta v &= x^{\underline{-2}} \\
v(x) &= -x^{\underline{-1}}
\end{aligned}
$$

se obtiene que

$$
\begin{aligned}
\sum_{j=0}^{n}(2x^{\underline{1}}+3)x^{\underline{-2}}\delta x &= -(2x^{\underline{1}}+3)x^{\underline{-1}}\Big|_{x=0}^{n} + 2\sum_{j=0}^{n}(x+1)^{\underline{-1}}\delta x \\
&= 2H_{x+1} - (2x^{\underline{1}}+3)x^{\underline{-1}}\Big|_{x=0}^{n} \\
&= 2H_{n+1} - \frac{2n+3}{n+1} - [2H_1 - 3] \\
&= 2H_{n+1} - \frac{n+2}{n+1} \\
&= 2H_n + \frac{2}{n+1} - \frac{n+2}{n+1} \\
&= 2H_n - \frac{n}{n+1}
\end{aligned}
$$

donde H_n es el (n)-ésimo número armónico.

$$
\boxed{\sum_{k=1}^{n}\frac{(2k+1)}{k(k+1)} = 2H_n - \frac{n}{n+1}}
$$

Problema #4: Determine una forma cerrada para la suma $\sum\limits_{0\le i\le n}\frac{i^2\,4^{i-1}}{(i+1)(i+2)}$.

Solución

$$
\begin{aligned}
\sum_{0\le i\le n}\frac{i^2\,4^{i-1}}{(i+1)(i+2)} &= \frac{1}{4}\sum_{0\le i\le n}\frac{i^2\,4^{i}}{(i+1)(i+2)} \\
&= \frac{1}{4}\sum_{0\le i\le n}4^i\left[\frac{i^2+4i+4-4i-4}{(i+1)(i+2)}\right] \\
&= \frac{1}{4}\sum_{0\le i\le n}4^i\left[\frac{(i+2)^2-4i-4}{(i+1)(i+2)}\right] \\
&= \frac{1}{4}\sum_{0\le i\le n}4^i\left[\frac{i+2}{i+1}-\frac{4i+4}{(i+1)(i+2)}\right] \\
&= \frac{1}{4}\sum_{0\le i\le n}4^i\left[\frac{i+2}{i+1}-\frac{4}{i+2}\right] \\
&= \frac{1}{4}\left[\sum_{0\le i\le n}4^i\left(\frac{i+2}{i+1}\right) - \sum_{1\le k\le n+1}4^k\left(\frac{1}{k+1}\right)\right] \qquad (k=i+1) \\
&= \frac{1}{4}\left[2+\sum_{1\le i\le n}4^i\left(\frac{i+2}{i+1}\right) - \sum_{1\le k\le n}4^k\left(\frac{1}{k+1}\right) - 4^{n+1}\left(\frac{1}{n+2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\sum_{0\le i\le n} \frac{i^2\,4^{i-1}}{(i+1)(i+2)} &= \frac{1}{4}\left[2+\sum_{1\le i\le n} 4^i - 4^{n+1}\left(\frac{1}{n+2}\right)\right] \\
&= \frac{1}{4}\left[2+\sum_{0\le i\le n} 4^i - 1 - 4^{n+1}\left(\frac{1}{n+2}\right)\right] \\
&= \frac{1}{4}\left[1+\left(\frac{1-4^{n+1}}{1-4}\right) - 4^{n+1}\left(\frac{1}{n+2}\right)\right] \\
&= \frac{1}{12}\left[2+4^{n+1} - 3\left(\frac{4^{n+1}}{n+2}\right)\right] \\
&= \frac{1}{3}\left[\frac{1}{2}+4^{n} - 3\left(\frac{4^{n}}{n+2}\right)\right] \\
&= \frac{1}{6} - \frac{4^n}{3}\left(\frac{n-1}{n+2}\right)
\end{aligned}
$$

$$
\boxed{\sum_{i=0}^{n} \frac{i^2\,4^{i-1}}{(i+1)(i+2)} = \frac{1}{6} - \frac{4^n}{3}\left(\frac{n-1}{n+2}\right)}
$$

Referencias

[1] R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, MA, 1989.