Categories

Definition. An object A in a category Cc is called an *initial object* if for every $X \in \mathrm{obj}\,(\mathbf{Sets})$ there is a unique morphism $A \to X$. Also A is initial in $\mathcal{C}$ if and only if A is terminal in $\mathcal{C}^{\mathrm{op}}$.

Examples. ☐ $A = \emptyset$ and $\mathcal{C} = \mathbf{Sets}$ (cf. Exercise 5.1).

☐ $A = 0$ and $\mathcal{C} = \mathbf{Mod}_R$, with the morphism $0 \mapsto 0$.

☐ $A = 0$ and $\mathcal{C} = (\mathbb{N}, \leq)$ because $0 \leq n$ for all $n \in \mathbb{N}$.

☒ In $\mathcal{C} = (\mathbb{Z}, \leq)$ there is no initial object, because if $n \in \mathbb{Z}$ were initial then $n \leq n - 1$ which is a contradiction.

Definition. An object A in a category Cc is called a *terminal object* if for every $X \in \mathrm{obj}\,(\mathbf{Sets})$ there is a unique morphism $X \to A$. Also, A es terminal in $\mathcal{C}$ if and only if A is initial in $\mathcal{C}^{\mathrm{op}}$.

Examples. (cf. Exercise 5.2)

☐ $A = \{e\}$ and $\mathbb{C} = \mathbf{Groups}$.

☐ $A = 0$ and $\mathcal{C} = \mathbf{Mod}_R$, with the morphism $0 \mapsto 0$.

☐ If $B = \{b_0\}$ then (B, b_0) is a zero object in $\mathbf{Sets}_*$

☐ If $X = \{x_0\}$ is given the discrete topology, then (X, x_0) is a zero object in $\mathbf{Top}_*$.

☒ **Sets** and **Top** have no zero object.

Definition 1. An object A of a category $\mathcal{C}$ is a zero object if it is both an initial and terminal object in $\mathcal{C}$.

Examples. ☐ Each singleton $\Omega = \{x_0\}$ is terminal in $\mathcal{C} = \mathbf{Sets}$ (cf. Exercise 5.1).

☐ $A = 0$ and $\mathcal{C} = \mathbf{Mod}_R$, with the morphism $0 \mapsto 0$.

☒ In $\mathcal{C} = (\mathbb{Z}, \leq)$ and $\mathcal{C} = (\mathbb{N}, \leq)$ there is no terminal object because there is no largest number.

Algebraic Geometry

Definition. Let X be a topological space viewed as a category, and $\mathcal{C}$ a category. A *presheaf* over X is a contravariant functor $\mathcal{F} : X \to \mathcal{C}$; more precisely it consists of the data:

(i) For every open $U \subseteq X$ there is an object $\mathcal{F}(U) \in \mathrm{obj}\,(\mathcal{C})$.

(ii) For every inclusion $\iota_U^V : V \to U$, there is a morphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$.

(iii) $\rho_U^U = \mathrm{Id}_{\mathcal{F}(U)}$

(iv) For every open subsets $W \subseteq V \subseteq U$, $\rho_U^W = \rho_U^V \circ \rho_V^W$.

Examples.

☐ Let X be a topological space and for an arbitrary open set $U \subseteq X$, set

$$\mathcal{F}(U) := \{f : U \to Y \mid f \text{ is continuous}\} \quad \text{and} \quad \rho_U^V(f) := f|_V$$

where Y is any topological space. I verify that $\mathcal{F} : \textbf{Top}(X) \to \textbf{Sets}$ is a contravariant functor: First observe that if $f \in \mathcal{F}(U)$ then $\rho_U^U(f) = f|_U = f$ and thus $\rho_U^U = \mathrm{Id}_{\mathcal{F}(U)}$. Furthurmore, if $W \subseteq V \subseteq U$ is a sequence of open subsets of X, then $f|_W = (f|_V)|_W$ so that

$$\rho_U^W(f) = f|_W = (f|_V)|_W = \rho_V^W(f|_V) = \rho_V^W(\rho_U^V(f)) = \big(\rho_V^W \circ \rho_U^V\big)(f).$$

In fact, if I give $\mathcal{F}(U)$ the compact-open topology, then the presheaf $\mathcal{F}$ becomes a presheaf $\mathcal{F} : \textbf{Top}(X) \to \textbf{Top}$. I only need to prove that $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is a continuous map for all open sets $V \subseteq U$ of X. Indeed, consider the subasic open set:

$$B(K, W) := \{f \in \mathcal{F}(V) \mid f[K] \subseteq W\}$$

where $K \subseteq V$ is compact and $W \subseteq Y$ is open. Since $K \subseteq V \subseteq U$, then for all $f \in \mathcal{F}(U)$ I have $f[K] = f|_V[K] = \rho_U^V(f)[K]$ so that $f|_V \in B(K, W)$ if and only if $f \in B(K, W) \subseteq \mathcal{F}(U)$ (because K is still compact in U). Thus the preimage of $B(K, W) \subseteq \mathcal{F}(V)$ is open and ρ_U^V is continuous.