1: Roots of Commutative Algebra

Exercise 1

Let M be an R-module. Prove that the following are equivalent:

1. **M is noetherian (ie. every submodule is finitely generated).**
2. **Every ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M terminates.**
3. **Every set of submodules of M has a maximal element with respect to inclusion.**
4. **Given any sequence $\{f_n\}$ of elements of M there exists an $m \in \mathbb{N}$ such that for all $n > m$ f_n is a linear combination of $\{f_1, \ldots, f_m\}$. That is**

$$f_n = \sum_{i=1}^{m} a_i f_i \quad (a_i \in R).$$

Proof.

$1 \implies 2$ Let $M_1 \subsetneq M_2 \subsetneq \cdots$ an ascending chain of submodules that doesn't terminate. This means that for all $n \in \mathbb{N}$ there is an element $f_n \in M_n - M_{n-1}$. Now consider the submodule $M' = \langle f_1, f_2, f_3, \ldots \rangle$. By hypothesis, M' is finitely generated so that there is an $N \in \mathbb{N}$ such that $M' = \langle f_1, \ldots, f_N \rangle$.

However, since $f_n \in M_n$ for all n, this implies that

$$M' = \langle f_1, f_2, \ldots \rangle = \langle f_1, \ldots, f_N \rangle \subseteq M_1 \cup \cdots \cup M_N \subseteq M_N,$$

but this is a contradiction because $f_{N+1} \notin M_N$ by construction. Thus the ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ must terminate.

$2 \implies 3$ Let Ω be a non-empty set of submodules of M that does not have maximal elements with respect to inclusion. With this hypothesis we will construct an ascending chain that doesn't terminate and thus prove the implication $(2 \implies 3)$ by contradiction.

Let $M_1 \in \Omega$ be any submodule of M. Since M_1 is not maximal in Ω, there is an $M_2 \in \Omega$ such that $M_1 \subsetneq M_2$. Likewise, since M_2 isn't maximal there is an $M_3 \in \Omega$ such that $M_2 \subsetneq M_3$. We can thus construct, inductively, an ascending chain $M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$ that doesn't terminate.

$3 \implies 4$ Given a sequence $\{f_n\}_{n=1}^{\infty}$ of elements of M, define $M_n := \langle f_1, \ldots, f_n \rangle$. By hypothesis, the set $\Omega := \{M_1, M_2, \ldots\}$ has a maximal element, say M_m. However, for $n > m$ we have by construction $M_m \subseteq M_n$, but since M_m is maximal, then $M_m = M_n$. In particular, $f_n \in M_m = \langle f_1, \ldots, f_m \rangle$ and thus f_n is a linear combination of the set $\{f_1, \ldots, f_m\}$.

$4 \implies 1$ Let M' be a submodule of M that is not finitely generated. Then if $f_1 \in M'$ we have that $\langle f_1 \rangle \neq M'$ because M' is not finitely generated. Thus there exists an $f_2 \in M' - \langle f_1 \rangle$. Again, since M' isn't finitely generated, we have $\langle f_1, f_2 \rangle \subsetneq M'$. Inductively we can construct a sequence $f_1, f_2, f_3, \dots$ such that

$$\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \langle f_1, f_2, f_3 \rangle \subsetneq \cdots .$$

However, by hypothesis there is an $m \in \mathbb{N}$ such that for all $n > m$, f_n is a linear combination of $\{f_1, \dots, f_m\}$ or in other words $f_n \in \langle f_1, \dots, f_m \rangle$ for all $n > m$. This contradicts the construction of the sequence $f_1, f_2, \dots$ becuase $f_{m+1} \notin \langle f_1, \dots, f_m \rangle$. Thus M' must be finitely generated.

□

Exercise 2

Let R be a noetherian ring and $I \leq R$ an ideal. Define $\mathbb{V}(I) := \{\mathfrak{p} \in \mathbf{Spec}(R) \mid I \subseteq \mathfrak{p}\}$ as the set of prime ideals that contain I. Prove that there are only finitely many prime ideals in $\mathbb{V}(I)$ that are minimal with respect to inclusion.

Proof. First denote $\min(\mathbb{V}(I))$ for the set of minimal prime ideales that contain I.

Suppose that there is an ideal I such that $\mathbb{V}(I)$ doesn't have a finite number of minimal primes, that is $|\min(\mathbb{V}(I))| = \infty$. Since R is noetherian, then the set Ω of such ideals that don't have a finitely many minimal prime ideals has a maximal element say J. Clearly J isn't a prime ideal because J would be the only minimal prime of $\mathbb{V}(J)$. Therefore there exist $f, g \notin J$ such that $fg \in J$.

Now consider the ideals $J_f = J + \langle f \rangle$ and $J_g = J + \langle g \rangle$ that strictly contain J (because $f, g \notin J$); this means $J_f, J_g \notin \Omega$ since $J \in \Omega$ is maximal. Thus $\mathbb{V}(J_f)$ and $\mathbb{V}(J_g)$ have finitely many minimal primes or in symbols $\min(\mathbb{V}(J_f))$ and $\min(\mathbb{V}(J_g))$ are finite.

Let $\mathfrak{p} \in \min(\mathbb{V}(J))$ be a minimal prime containing J and suppose that $\mathfrak{p}$ does not contain J_f nor J_g. This means that $f, g \notin \mathfrak{p}$ and since $\mathfrak{p}$ is prime, then $fg \notin \mathfrak{p}$. However $fg \in J \subseteq \mathfrak{p}$ which is a contradiction. Thus $\mathfrak{p}$ must contain either J_f or J_g. Furthermore, if $\mathfrak{q} \subseteq \mathfrak{p}$ is a prime ideal that contains J_f or J_g then clearly $\mathfrak{q}$ contains J and thus, by the minimality of $\mathfrak{p}$ we have $\mathfrak{p} = \mathfrak{q}$. Therefore $\mathfrak{p}$ is a minimal prime containing J_f or J_g.

We have just proven that if $\mathfrak{p} \in \min(\mathbb{V}(J))$ then $\mathfrak{p} \in \min(\mathbb{V}(J_f)) \cup \min(\mathbb{V}(J_g))$ or equivalently

$$\min(\mathbb{V}(J)) \subseteq \min(\mathbb{V}(J_f)) \cup \min(\mathbb{V}(J_g)).$$

Since both $\min(\mathbb{V}(J_f))$ and $\min(\mathbb{V}(J_g))$ are finite then $\min(\mathbb{V}(J))$ is also finite which contradicts the definition of J. Thus every ideal of R has finitely many minimal prime ideals that contain it. □

Exercise 3

Let N be a submodule of the R-module M. Prove that

$$M \textbf{ is noetherian} \quad \iff \quad N \textbf{ and } \frac{M}{N} \textbf{ are noetherian}.$$

Proof.
($\Longrightarrow$) Suppose M is noetherian. Let $N' \leq N$ be a submodule of N, then clearly N' is a submodule of M and thus it is finitely generated. Now let N' be a submodule of M/N. By the Correspondence Theorem, there is a submodule $N'' \leq M$ that contains N such that $N'' \mapsto N'$ under the canonical projection $M \twoheadrightarrow M/N$. Since M is noetherian, N'' is finitely generated, say $N'' = \langle f_1, \dots, f_n \rangle$.

Now let $f + N \in N' = N''/N$ with $f \in N''$ so that there exist $\lambda_1, \dots, \lambda_n \in R$ such that $f = \lambda_1 f_1 + \cdots + \lambda_n f_n$. By projecting this equality via $M \twoheadrightarrow M/N$ we conclude that $f + N = (\lambda_1 f_1 + N) + \cdots + (\lambda_n f_n + N)$

and thus $f+N \in \langle f_1+N, \dots, f_n+N\rangle$. Therefore $N' = N''/N$ is finitely generated by $\{f_1+N, \dots, f_n+N\}$ and M/N is noetherian.

($\Longleftarrow$) Suppose that both N and M/N are noetherian and let $M' \leq M$ be a submodule. By the Second Isomorphism Theorem we have that

$$\frac{M'}{M' \cap N} \cong \frac{M'+N}{N}.$$

The right hand side is a submodule of M/N so that it is finitely generated and thus $M'/(M' \cap N) = \langle f_1 + (M' \cap N), \dots, f_n + (M' \cap N)\rangle$ for some $f_1, \dots, f_n \in M'$.

Now $M' \cap N$ is a submodule of N and thus finitely generated so that $M' \cap N = \langle g_1, \dots, g_m\rangle$ for some $g_1, \dots, g_m \in M' \cap N \subseteq M'$. We confirm that $\{f_1, \dots, f_n, g_1, \dots, g_m\}$ generate M'.

Let $f \in M'$. If we project onto $M' \cap N$ then $f + (M' \cap N)$ is generated by $\{f_1 + (M' \cap N), \dots, f_n + (M' \cap N)\}$. That is there are $\lambda_1, \dots, \lambda_n \in R$ such that:

$$f + (M' \cap N) = \lambda_1 f_1 + (M' \cap N) + \cdots + \lambda_n f_n + (M' \cap N).$$

This equality means that $f - \lambda_1 f_1 - \cdots - \lambda_n f_n \in M' \cap N = \langle g_1, \dots, g_m\rangle$ and thus there exist $\mu_1, \dots, \mu_m \in R$ such that $f - \lambda_1 f_1 - \cdots - \lambda_n f_n = \mu_1 g_1 + \cdots + \mu_m g_m$ or equivalently:

$$f = \lambda_1 f_1 + \cdots + \lambda_n f_n + \mu_1 g_1 + \cdots + \mu_m g_m.$$

We can therefore conclude that $f \in \langle f_1, \dots, f_n, g_1, \dots, g_m\rangle$ and we are done. $\square$

Exercise 5

Let R be a subring of S. Prove that if there exists a retraction $S \xrightarrow{\varphi} R$, that is φ is a homomorphism such that $\varphi|_R = \mathbf{Id}_R$, then if S is noetherian this implies that R is noetherian. Give an example of rings $R \subseteq S$ such that S is noetherian, but R isn't.

Proof. Let $I \leq R$ be an ideal and consider the ideal $J = \varphi^{-1}[I]$ of S. Since S is noetherian by hypothesis, J is finitely generated, say $J = \langle f_1, \dots, f_n\rangle_S$. By construction, we have that $\varphi(f_i) \in I \subseteq R$ for all $i = 1, \dots, n$ and we state that $I = \langle \varphi(f_1), \dots, \varphi(f_n)\rangle_R$.

Let $g \in I$. Since φ is a retraction, then $\varphi(g) = g \in I$ so that $g \in J$. Thus there exist scalars $\lambda_1, \dots, \lambda_n \in S$ such that $g = \lambda_1 f_1 + \cdots + \lambda_n f_n$. If we apply φ to this equation we get:

$$g = \varphi(g) = \varphi(\lambda_1)\varphi(f_1) + \cdots + \varphi(\lambda_n)\varphi(f_n) \quad (\varphi(\lambda_i) \in R).$$

Therefore g is an R-linear combination of $\{\varphi(f_1), \dots, \varphi(f_n)\}$ and we can conclude that $I = \langle \varphi(f_1), \dots, \varphi(f_n)\rangle_R$. Thus every ideal of R is finitely generated which implies that R is noetherian.

Next we give an example where this exercise fails if there isn't a retraction from S to R. Let $R = k[x_1, x_2, \dots]$ be the polynomial ring over a field k with an infinite amount of variables. Clearly R isn't noetherian because

$$\langle x_1\rangle \subsetneq \langle x_1, x_2\rangle \subsetneq \langle x_1, x_2, x_3\rangle \subsetneq \cdots$$

is an ascending chain of ideals that doesn't terminate (because $x_{n+1} \notin \langle x_1, \dots, x_n\rangle$ for all n). Since R is an integral domain then the zero ideal $\mathbf{0}$ is prime and we may localize with respect to the multiplicatively closed set $R - \mathbf{0}$. This localization is K, the field of fractions of R. The canonical localization homomorphism:

$$R \xrightarrow{\ell} K \quad f \mapsto \frac{f}{1}$$

is thus injective because

$$\frac{f}{1} = \frac{g}{1} \quad \Longleftrightarrow \quad \exists h \in R - \mathbf{0} \text{ such that } h(f-g) = 0$$

and since R is an integral domain, this last equality is equivalent to $f = g$.

We may therefore conclude that the image of ℓ is isomorphic to R and we may think of R as a subring of K. Since K is a field, it is clearly noetherian. Thus the noetherian property is not always preserved for subrings of a noetherian ring.

We finish by observing that the existence of a retraction $S \xrightarrow{\varphi} R$ is equivalent to R being a summand of S, that is there exists another subring $T \subseteq S$ such that $S = R \oplus T$.

Indeed, if $S \xrightarrow{\varphi} R$ is a retraction then the composition map $R \hookrightarrow S \xrightarrow{\varphi} R$ is the identity map because the image of the inclusion $R \hookrightarrow S$ is clearly R and $\varphi|_R = \mathrm{Id}_R$. This means that the short exact sequence

$$0 \to \ker\varphi \hookrightarrow S \xrightarrow{\varphi} R \to 0$$

is split and thus $S = R \oplus \ker\varphi$ so that R is a summand of S.

Conversly if $S = R \oplus T$ for some subring T of S, then the projection map

$$S = R \oplus T \xrightarrow{\pi} R \quad f = (r, t) \mapsto r$$

is clearly a retraction.

$\square$

2: Localization