3: Analytic Functions

Exercise 4. *Show that*

$$\frac{d}{dz}\cos z = -\sin z \quad \textit{and} \quad \frac{d}{dz}\sin z = cosz.$$

Proof. Since both $\cos z$ and $\sin z$ are entire, with Taylor series:

$$\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} \quad \text{and} \quad \sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

their derivatives can be calculated by differentiating under the summation sign (cf. 2.5, chapter 3, pg 35):

$$\frac{d}{dz}\cos z = \frac{d}{dz}\left\{\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}\right\} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{d}{dz}\left\{z^{2n}\right\} = -\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!} = -\sin z.$$

The other calculation is done in exactly the same manner. □

Exercise 8. *Define*

$$\tan z := \frac{\sin z}{\cos z}.$$

Where is this function defined and analytic?

Proof. By the chain rule we have

$$\frac{d}{dz}\frac{1}{f(z)} = -\frac{f'(z)}{f(z)^2}$$

provided $f(z) \neq 0$. Thus if f is analytic, then $1/f$ is analytic wherever $f(z) \neq 0$. In this case we have $f(z) = \cos z$ so that $\tan z$ is well-defined and analytic wherever $\cos z \neq 0$ which is when $z \neq \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$. □

Exercise 9. *Let $G = \mathbb{C} - \mathbb{R}_{\leq 0}$ and $\{z_n\} \subseteq G$ a sequence given by $z_n = r_n e^{i\theta_n}$ where $0 < r_n$ and $-\pi < \theta_n < \pi$. If another point $z \in G$ is given by $z = re^{i\theta}$, prove that*

$$z_n \to z \quad \Longrightarrow \quad r_n \to r \quad \textit{and} \quad \theta_n \to \theta$$

.

Proof. The complex plane is cannonically isometric to $\mathbb{R}^2$, so we can identify:

$$z = re^{i\theta} \quad \longleftrightarrow \quad (r\cos\theta, r\sin\theta)$$

Thus our hypothesis is reduced to:

$$z_n \to z \quad \Longleftrightarrow \quad \forall\varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } |z_n - z| < \varepsilon \text{ for } n > N.$$

Since

$$\begin{aligned} |z_n - z| &= |(r_n\cos\theta_n, r_n\sin\theta_n) - (r\cos\theta, r\sin\theta)| \\ &= \sqrt{(r_n\cos\theta_n - r\cos\theta)^2 + (r_n\sin\theta_n - r\sin\theta)^2} \\ &= \sqrt{r_n(\cos^2\theta_n + \sin^2\theta_n) + r(\cos^2\theta + \sin^2\theta) - 2r_n r\cos(\theta - \theta_n)}. \end{aligned}$$

Thus

$$z_n \to z \quad \Longleftrightarrow \quad \forall\varepsilon > 0 \quad r^2 + r_n^2 - 2r_n r\cos(\theta - \theta_n) < \varepsilon^2 \quad \text{for sufficiently large } n$$

□