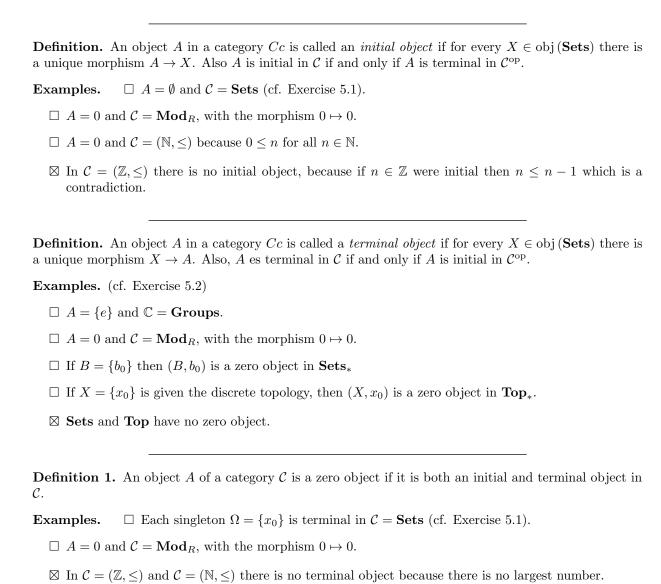
Categories



Algebraic Geometry

Definition. Let X be a topological space viewed as a category, and \mathcal{C} a category. A *presheaf* over X is a contravariant functor $\mathcal{F}: X \to \mathcal{C}$; more precisely it consists of the data:

- (i) For every open $U \subseteq X$ there is an object $\mathcal{F}(U) \in \text{obj}(\mathcal{C})$.
- (ii) For every inclusion $\iota_U^V: V \to U$, there is a morphism $\rho_V^U: \mathcal{F}(U) \to \mathcal{F}(V)$.
- (iii) $\rho_U^U = \mathrm{Id}_{\mathcal{F}(U)}$
- $(iv) \ \ \text{For every open subsets} \ W \subseteq V \subseteq U, \, \rho_U^W = \rho_U^V \circ \rho_V^W.$

Examples.

 \square Let X be a topological space and for an arbitrary open set $U \subseteq X$, set

$$\mathcal{F}(U) := \{ f : U \to Y \mid f \text{ is continuous} \} \text{ and } \rho_U^V(f) := f|_V$$

where Y is any topological space. I verify that $\mathcal{F}: \mathbf{Top}(X) \to \mathbf{Sets}$ is a contravariant functor:

First observe that if $f \in \mathcal{F}(U)$ then $\rho_U^U(f) = f|_U = f$ and thus $\rho_U^U = \mathrm{Id}_{\mathcal{F}(U)}$. Furthurmore, if $W \subseteq V \subseteq U$ is a sequence of open subsets of X, then $f|_W = (f|_V)|_W$ so that

$$\rho_U^W(f) = f|_W = (f|_V)|_W = \rho_V^W(f|_V) = \rho_V^W(\rho_U^V(f)) = \left(\rho_V^W \circ \rho_U^V\right)(f).$$

In fact, if I give $\mathcal{F}(U)$ the compact-open topology, then the presheaf \mathcal{F} becomes a presheaf \mathcal{F} : $\mathbf{Top}(X) \to \mathbf{Top}$. I only need to prove that $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is a continuous map for all open sets $V \subseteq U$ of X. Indeed, consider the subasic open set:

$$B(K,W) := \{ f \in \mathcal{F}(V) \mid f[K] \subseteq W \}$$

where $K \subseteq V$ is compact and $W \subseteq Y$ is open. Since $K \subseteq V \subseteq U$, then for all $f \in \mathcal{F}(U)$ I have $f[K] = f|_V[K] = \rho_U^V(f)[K]$ so that $f|_V \in B(K,W)$ if and only if $f \in B(K,W) \subseteq \mathcal{F}(U)$ (because K is still compact in U). Thus the preimage of $B(K,W) \subseteq \mathcal{F}(V)$ is open and ρ_U^V is continuous.