2: Schemes

Exercise 1.22

Glueing Sheaves. Suppose X is a topological space with an open covering $\{U_i\}_{i\in\Lambda}$ where we denote the intersections by: $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$. Suppose we are given, for each $i \in \Lambda$ a sheaf $\mathfrak{F}_i$ on U_i together with the family of sheaf isomorphisms:

$$\left\{\varphi_{ij} : \mathfrak{F}_i|_{U_{ij}} \longrightarrow \mathfrak{F}_j|_{U_{ij}}\right\}_{i,j\in\Lambda}.$$

Prove that we can glue the family of sheaves $\{F_i\}$ into a sheaf on X by proving that there exists a sheaf $\mathfrak{F}$ on X such that $\mathfrak{F}\,|_{U_i} = \mathfrak{F}_i$. For this to be possible we require that the family of sheaf isomorphisms $\{\varphi_{ij}\}$ satisfy the following functorial properties:

1. **$\varphi_{ii} = \mathbf{Id}_{\mathfrak{F}_i|_{U_i}}$ for all $i \in \Lambda$.**
2. **For all $i, j, k \in \Lambda$ the representations of $\varphi_{ij}, \varphi_{jk}$ and φ_{ik} over the open set U_{ijk} satisfy $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ or equivalently we have the commutative diagram:**

Proof. First we write explicitly what condition 2 means. The representation of φ_{ij} the open set $U = U_{ijk} \subseteq U_{ij}$ is the morphism

$$\mathfrak{F}_i|_{U_{ij}}(U) \xrightarrow{(\varphi_{ij})_U} \mathfrak{F}_j|_{U_{ij}}(U).$$

At first glance it might not be to clear how we can compose these morphisms, but there is a natural equality that can help us deduce the commutative diagram for condition 2:

$$\mathfrak{F}_i|_{U_{ij}}(U) = \mathfrak{F}_i|_{U_{ik}}(U).$$

This equality is indeed true because if $s \in \mathfrak{F}_i|_{U_{ij}}(U)$ is a section and since $U_{ij} \cap U_{ik} = U_{ijk} =$

$$\begin{tikzcd}
\mathfrak{F}_i|_{U_{ij}}(U_{ijk}) \arrow[r, "(\varphi_{ij})_{U_{ijk}}"] \arrow[d, equal] & \mathfrak{F}_j|_{U_{ij}}(U_{ijk}) \arrow[r, equal] & \mathfrak{F}_j|_{U_{jk}}(U_{ijk}) \arrow[r, "(\varphi_{jk})_{U_{ijk}}"] & \mathfrak{F}_k|_{U_{jk}}(U_{ijk}) \arrow[d, equal] \\
\mathfrak{F}_i|_{U_{ik}}(U_{ijk}) \arrow[rrr, "(\varphi_{ij})_{U_{ijk}}"'] & & & \mathfrak{F}_k|_{U_{ik}}(U_{ijk})
\end{tikzcd} \tag{1}$$

$\square$