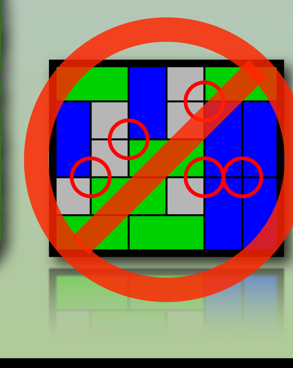
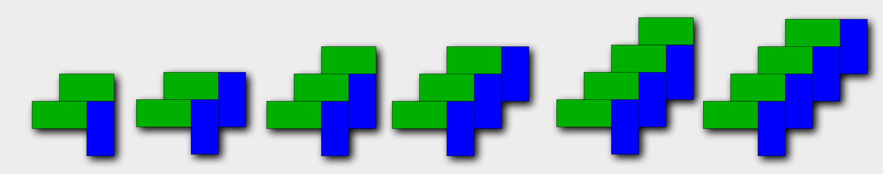
Tatami mat arrangements of square grids with *v* vertical dimers

by Alejandro Erickson and Frank Ruskey

☹ not tatami



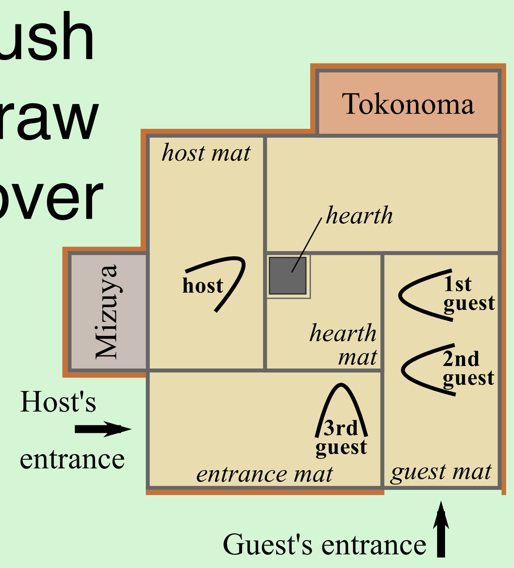
A tatami tiling is a covering of the *n x n* grid with monomer and dimer tiles, in which no four tiles meet at any point. Its structure is characterized by the *ray*, which propagates itself to the boundary of the grid.



Japanese tatami Mats

Rice straw core

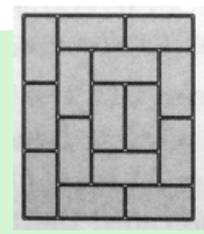
Rush straw cover



NO FOUR MATS MEET

From Don Knuth's TAOCP Vol 4.

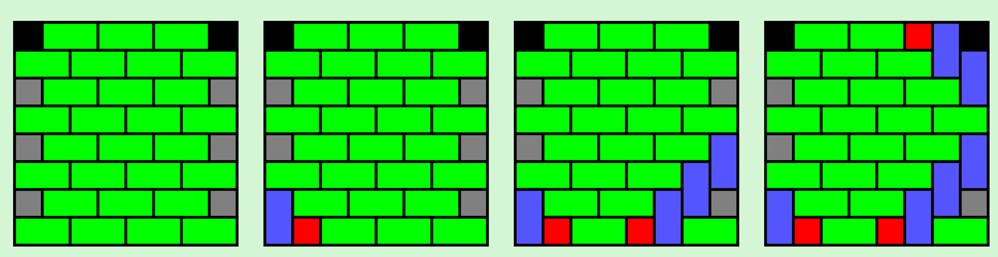
Fig. 29(a) shows a 6 × 5 pattern from the 1641 edition of Mitsuyoshi Yoshida's *Jinkōki*, a book first published in 1627.



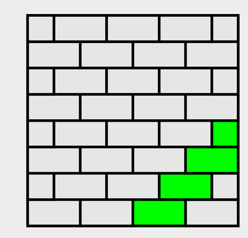
We discuss tatami tilings of the *n x n* grid with *n* monomers.

A few facts:

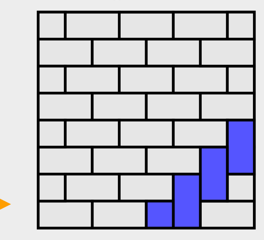
- *n* is the maximum possible number of monomers.
- there are $n2^{n-1}$ of them.
- they can all be obtained from a brick tiling via *diagonal flips.*



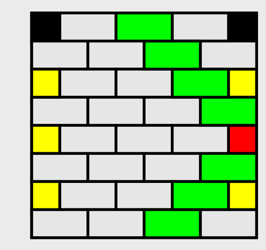
unflipped → *flipped diagonal*



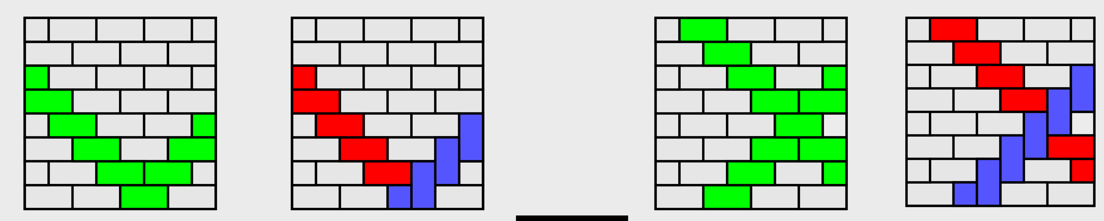
3 dimers become vertical



Every (non-black) monomer is in exactly two diagonals, but only one can be flipped



can't flip both *conflicting* diagonals



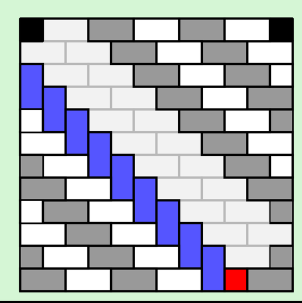
different sides   same side

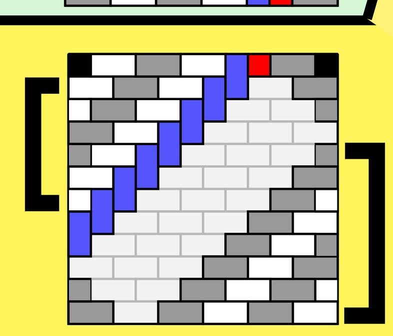
Lemma: For even *n*, the number (up to rotation) of *n x n* tatami tilings with *n* monomers and exactly *k* vertical dimers is *V(n,k)*:=

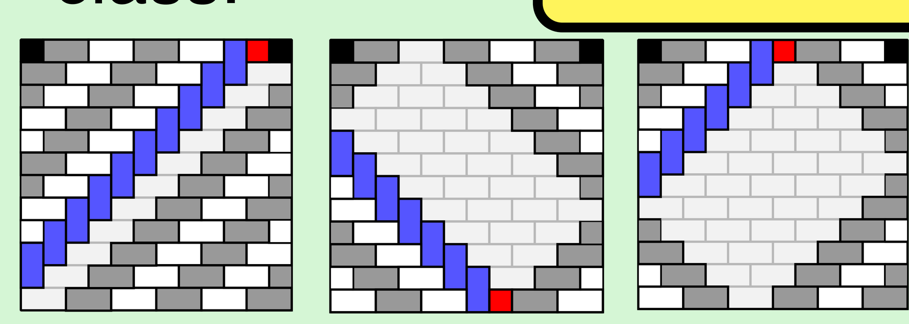
$$2\sum_{i=1}^{\lfloor (n-1)/2\rfloor}\left(\sum_{\substack{k_1+k_2=\\k-(n-i-1)}}|S(n-i-2,k_1)||S(i-1,k_2)|\right)$$
$$+\sum_{k_1+k_2=k}\left|S\left(\left\lfloor\frac{n-2}{2}\right\rfloor,k_1\right)\right|\left|S\left(\left\lfloor\frac{n-2}{2}\right\rfloor,k_2\right)\right|.$$

Partition in Pictures:

Blue is longest flipped diagonal. **Grey** and **white** diagonals can contribute 2^{n-3} tilings to each class.



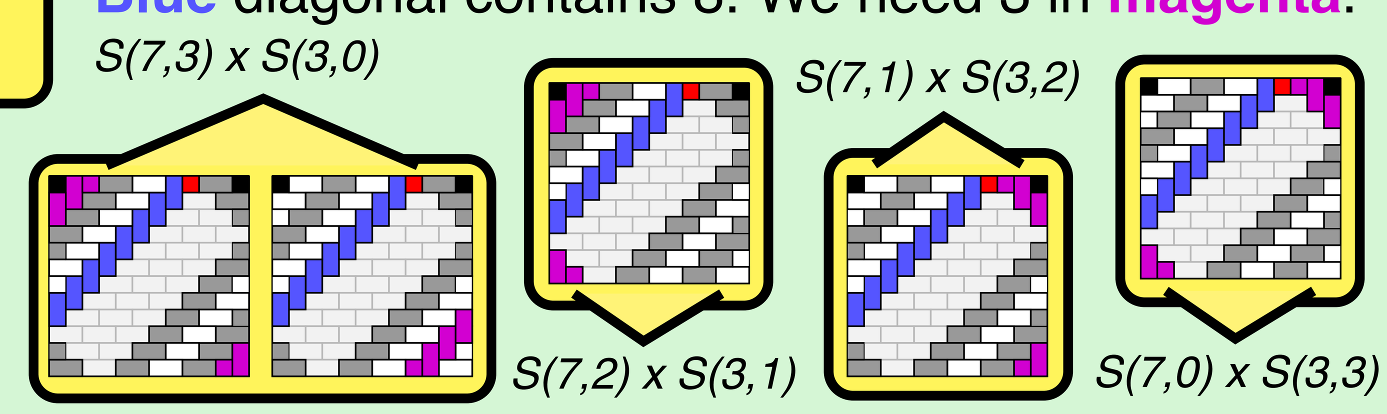




Flipping one of these diagonals changes *1, 2, 3, 4, 5, 6*, or *7* horizontal dimers into vertical dimers, and they can be flipped *independently of each other.*

Let *S(n,k)* be the set of subsets of $\{1,2,\ldots,n\}$ whose members sum to *k*.

Bijection in Pictures: tilings with 11 vertical dimers. **Blue** diagonal contains 8. We need 3 in **magenta**.



Let $V_n(z)=\sum_{k\geq 0}V(n,k)z^k$. Then

$$V_n(z)=P_n(z)\prod_{j\geq 1}\left(\Phi_{2j}(z)\right)^{\left\lfloor\frac{n-2}{2j}\right\rfloor},$$

where $\Phi_i(z)$ is the *i*th cyclotomic polynomial. These factor out of the generating polynomial for *S(n,k)*, but the origin of $P_n(z)$ is mostly mysterious. For $2<n<200$:

- $P_n(z)$ is irreducible over the integers,
- for some *N* and *k*, the first *k* coefficients of $P_n(z)$ are the same for all n > N; and

$$\sum_{n\geq 2}P_n(-1)z^{n-2}=\frac{(1+z)(1-2z)}{(1-2z^2)\sqrt{1-4z^2}}$$

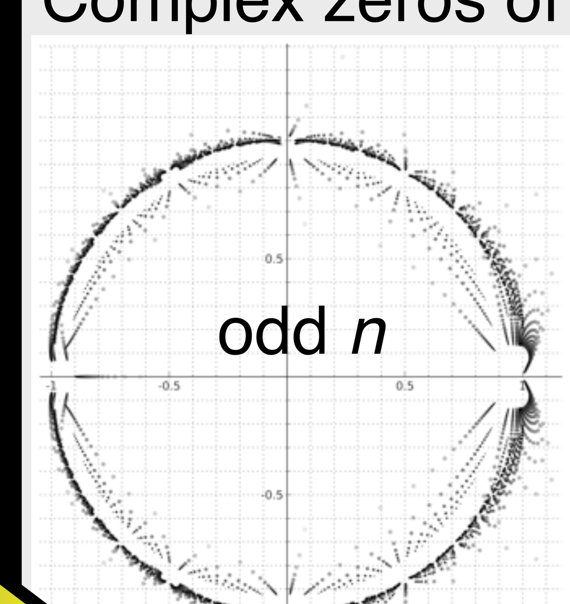
which is an interleaving of the sequences

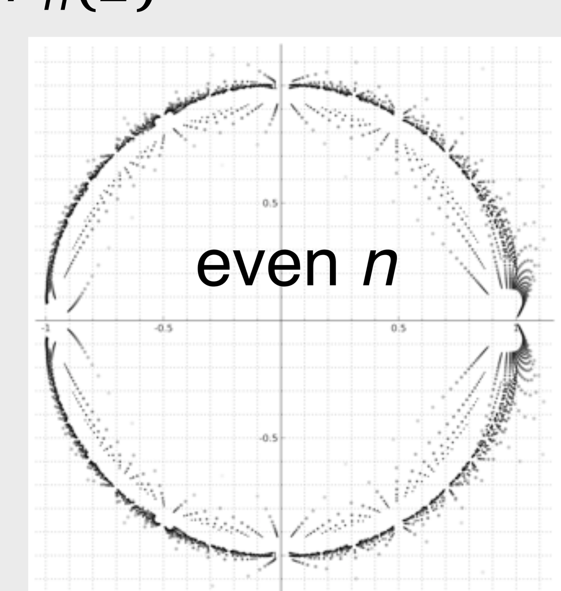
$$-\sum_{i=0}^{k}2^{k-i}\binom{2i}{i}\quad\text{and}\quad\binom{2k}{2}$$

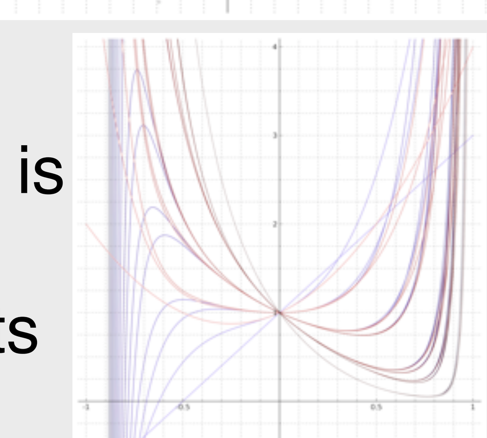
Table of coefficients of $P_n(z)$

3	1	2																		
4	1	1	2																	
5	1	1	2	4	0	2														
6	1	0	1	2	2	-2	2													
7	1	0	1	2	2	4	-2	4	0	2	-2	2								
8	1	0	1	1	2	3	4	-2	2	0	4	-2	2	-2	2					
9	1	0	1	1	2	3	4	6	-2	6	0	8	-2	4	-4	6	-2	4	-2	2
10	1	-1	1	0	1	1	1	2	2	-6	6	-2	6	-6	4	-4	6	-6	6	-4
11	1	-1	1	0	1	1	1	2	2	4	-8	10	-4	10	-8	8	-8	10	-10	12
12	1	-1	1	0	1	0	2	1	2	3	3	-7	8	-2	6	-6	12	-10	10	-14
13	1	-1	1	0	1	0	2	1	2	3	3	5	-8	12	-4	12	-10	20	-14	16
14	1	-1	1	-1	2	-1	2	0	2	1	2	3	3	-9	7	-4	16	-14	18	-16
15	1	-1	1	-1	2	-1	2	0	2	1	2	3	3	5	-11	12	-8	24	-20	28
16	1	-1	1	-1	2	-1	2	-1	3	0	3	1	4	3	5	-10	9	-4	19	-15
17	1	-1	1	-1	2	-1	2	-1	3	0	3	1	4	3	5	6	-11	14	-7	27
18	1	-2	2	-2	3	-3	3	-3	4	-3	3	-2	3	-1	2	1	1	-15	21	-14
19	1	-2	2	-2	3	-3	3	-3	4	-3	3	-2	3	-1	2	1	1	3	-19	28
20	1	-2	2	-2	3	-3	3	-3	4	-4	5	-4	5	-4	5	-2	4	-1	4	-19
$n\backslash z^k$	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
9	-2	2																		
10	4	-4	2																	
11	-8	10	-12	10	-6	6	-6	6	-4	4	-4	2								
12	16	-12	14	-14	16	-14	12	-12	12	-10	10	-8	6	-4	4	-4	2			
13	-18	22	-20	26	-22	26	-24	26	-28	28	-24	26	-24	24	-20	20	-20	18	-14	12
14	18	-26	26	-24	34	-36	38	-36	40	-44	42	-40	42	-42	40	-34	36	-36	30	-26
15	-24	30	-36	38	-34	44	-48	56	-50	60	-66	66	-68	72	-74	78	-70	76	-80	76
16	19	-17	32	-32	34	-40	50	-56	62	-62	80	-80	84	-90	98	-104	106	-108	118	-120
17	-21	31	-26	48	-44	52	-54	70	-74	82	-82	108	-102	116	-124	138	-144	152	-160	176
18	26	-36	37	-38	54	-68	70	-76	92	-108	120	-128	154	-176	186	-198	220	-242	252	-262
19	-22	38	-51	56	-58	78	-94	100	-108	128	-144	156	-164	190	-214	236	-250	276	-308	328
20	26	-17	30	-40	41	-41	57	-71	90	-100	112	-139	162	-168	194	-234	270	-290	326	-364
$n\backslash z^k$	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
13	-10	10	-8	6	-4	4	-4	2												
14	26	-24	20	-16	14	-12	10	-6	4	-4	2									
15	-70	72	-72	68	-62	60	-58	56	-46	42	-42	36	-28	26	-24	20	-16	14	-12	10
16	120	-116	124	-122	116	-116	116	-110	104	-98	96	-90	82	-74	72	-64	54	-50	46	-38
17	-182	192	-190	208	-208	212	-218	224	-224	224	-220	224	-220	216	-208	210	-200	190	-184	178
18	284	-306	318	-322	340	-356	358	-362	376	-386	382	-376	380	-382	372	-358	356	-352	334	-316
19	-346	378	-414	438	-450	484	-516	536	-552	580	-612	626	-632	654	-678	686	-684	698	-712	710
20	398	-426	460	-514	562	-592	636	-688	732	-764	810	-860	904	-934	968	-1014	1046	-1062	1090	-1126
$n\backslash z^k$	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
15	-6	4	-4	2																
16	32	-28	24	-18	14	-12	10	-6	4	-4	2									
17	-166	156	-146	138	-126	116	-106	98	-86	78	-70	60	-52	46	-38	32	-28	24	-18	14
18	308	-296	274	-254	242	-228	206	-186	174	-158	138	-122	110	-98	84	-70	60	-52	42	-32
19	-700	702	-704	690	-670	662	-656	634	-606	592	-576	548	-516	492	-472	444	-410	386	-366	336
20	1148	-1154	1168	-1182	1188	-1180	1174	-1178	1170	-1144	1126	-1112	1084	-1048	1016	-986	952	-906	864	-830
$n\backslash z^k$	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99
17	-12	10	-6	4	-4	2														
18	26	-22	16	-10	8	-6	2													
19	-302	278	-260	234	-204	186	-170	146	-124	110	-98	84	-70	60	-52	42	-32	26	-22	16
20	788	-736	692	-654	608	-560	520	-480	438	-398	360	-326	294	-258	228	-204	176	-150	132	-114
$n\backslash z^k$	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119
19	-10	8	-6	2																
20	94	-78	66	-54	42	-32	26	-22	16	-10	8	-6	2							

Complex zeros of $P_n(z)$







Plots of $P_n(z)$. Blue is odd. Smaller and darker lines and dots represent larger *n*.

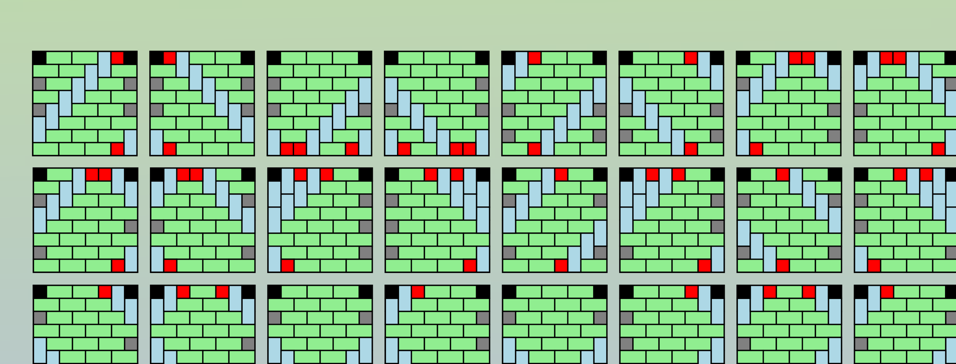
Proven for all n:

- $\deg(P_n(z))=\sum_{i=1}^{n-2}\mathrm{Od}(i)$, where $\mathrm{Od}(i)$ is the largest odd divisor of *i*.
- $P_n(1)=n2^{\nu(n-2)-1}$, where $\nu(i)$ is the number of *1*s in the binary representation of *i*.

Two Questions:

- geometric interpretation for factorization of $V_n(z)$?
- how do we calculate $P_n(z)$ independently of $V_n(z)$?

The construction of *V(n,k)* yields an algorithm to generate tilings in constant amortized time. Here is output for *V(8,7)*.



Based on: Erickson, A. and Ruskey, F. 2012: Enumerating tatami mat arrangements of square grids with v vertical dimers, in preparation.

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