An Expository Paper on "A Theorem on Paths in Planar Graphs" by Carsten Thomassen

Alejandro Erickson
aerickso@sfu.ca
Department of Mathematics, Simon Fraser University,
8888 University Drive, Burnaby, BC, Canada V5A 1S6.
Colour version at *http://www.sfu.ca/˜aerickso/project/*

January 10, 2007

Contents

1 Introduction

Much work has been done to find classes of graphs with long and ultimately, Hamiltonian paths. Hassler Whitney and W. T. Tutte pioneered in this area with [2] and [4]. The strategy in many of the papers cited here is to prove the existence of certain paths and show that high connectivity forces them to be long.

The purposes of this paper are to give a detailed explanation and verification of Carsten Thomassen's 1983 paper, [5], "A theorem on paths in planar graphs" and also to briefly look at papers related to [5]. The proof has been broken down so that an undergraduate with a basic knowledge of graph theory can understand it and be satisfied that it is correct. Many diagrams have been included and the author recommends viewing them in colour. The proof here is complete, however, it is recommended that the reader look carefully at [5].

2 History: Whitney and Tutte

Hassler Whitney was a pioneer in the area of finding Hamiltonian cycles in planar graphs. His 1931 paper, [2], which proved that a 4-connected planar triangulation is Hamiltonian is cited in [5, 3, 4, 6]. Moreover, he recognizes himself as likely being the first to find a large class of hamiltonian graphs.

Whitney uses interesting language, describing his results without many of the modern terms we have now. The statement of his theorem ([2], page 378 lines 16-23) is as follows:

> 2. The fundamental theorem of this paper is the following: THEOREM I. Given a planar graph composed of elementary triangles, in which there are no circuits of 1,2, or 3 edges other than these elementary triangles, there exists a circuit which passes through every vertex of the graph.

An elementary triangle is a face that is a cycle of length 3. This graph is referred to in the more recent papers as a 4-connected planar triangulation and the cycle as a Hamilton cycle. Tutte describes this as a "well-known theorem" in [4].

Further work was published in 1956 by W. T. Tutte in [4] where he proved that **every** *4-connected planar graph is Hamiltonian.* Whitney's theorem is a special case of this.

In 1977, Tutte also proved in [3] that a *vertically 4-connected planar graph with at least three vertices is Hamiltonian.* To appreciate this theorem we give a definition and then prove the existence of an example to show that it is an improvement over [4].

Definition: For a graph G and a non-negative integer n, we say that G is **vertically n-separated** if $G = H \cup K$ where H and K each have at least one edge and obey the following properties:

- $E(H) \cap E(K) = \emptyset$
- $|V(H) \cap V(K)| = n$
- $\exists x \in V(H) \setminus V(K)$ and $\exists y \in V(K) \setminus V(H)$

G is **vertically k-connected** if it not vertically n-separated for any $n < k$, see figure 1.

A graph that is 4-connected is vertically 4-connected, however the converse is not true. Consider a graph G which is 3-connected and not 4-connected as in figure 2. Select a 3-separation, $\{v_1, v_2, v_3\}$, of G and add the edges v_1v_2 and v_2v_3, repeating this process until all 3-separations have these edges for some ordering of their vertices. The resulting graph is vertically 4-connected but not 4-connected and furthermore this process need not

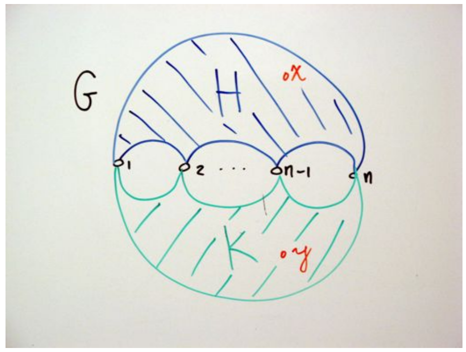


Figure 1: A vertically n-separated graph

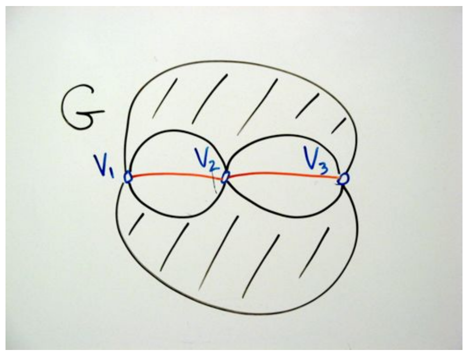


Figure 2: Constructing a graph that is vertically 4-connected but not 4-connected.

affect planarity. Thus, the theorem adds to the growing class of planar graphs which are Hamiltonian.
To prove the theorem in [3] we first suppose that the theorem is false by assuming there is a minimal counterexample in the number of edges. If G is not simple we delete a multiple edge or loop which does not affect connectivity. Thus we have a smaller counterexample which contradicts the assumption that G was minimal. The other case builds on many things in Tutte's paper. As a final note, he mentions two conjectures, due to Plummer, both of which have now been proven. Let u and v be distinct vertices in a 4-connected planar graph G,

Conjecture 1: There is a Hamilton path connecting u and v.

Conjecture 2: $G - u - v$ is Hamiltonian.

The first of these is a corollary of Thomassen in [5] and the second was proven later by Thomas and Yu in [6].

3 Thomassen

Definition : Let G be a graph and let H be a subgraph of G. An H **- component**, H' of G is either a copy of K_2 with only its endpoints in H or it is a connected component of $G - V(H)$ plus all the edges and vertices connecting it to H, called **vertices of attachment** or simply **attachments**.

The statement of Thomassen's theorem follows.

Theorem (Figure 3) : Let G be a 2-connected plane graph with outer cycle C. Let v and e be a vertex and edge, respectively, of C and let u be any vertex of G distinct from v. Then G has a v, u - path P containing e such that

1. Each P-component has at most three vertices of attachment and
2. Each P-component containing an edge of C has at most two vertices of attachment.

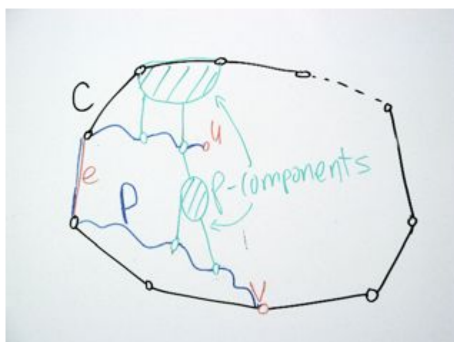


Figure 3: Thomassen's theorem

A path satisfying 1 and 2 in the statement of the theorem will be called a v, u, e **- Tutte path** or a v, u **- Tutte path** (through e) as in [7], see figure 3.

The true power of Thomassen's theorem is seen in Plummer's (first) conjecture. Let G be a 4-connected and P be a u, v - Tutte path in G. Suppose H is a P-component then, since G is 4-connected, H must have 4 vertices of attachment in P which is a contradiction because P is a Tutte path.

Before the proof, we need another definition.

Definition: Let G be as in the statement of the theorem with $x, y \in V(C)$ where xy is not an edge of C. We will call $\{x, y\}$ a v, e **- separating set** if G has two subgraphs, G_1, G_2 such that

- $G = G_1 \cup G_2$
- $V(G_1) \cap V(G_2) = \{x, y\}$
- $E(G_1) \cap E(G_2) = \emptyset$
- $e \in E(G_2)$
- $v \in V(G_1)$
- $V(G_2) \setminus \{x, y\} \neq \emptyset$
- $V(G_1) \setminus \{v, x, y\} \neq \emptyset$, (it may be that $v \in \{x, y\}$)

See figure 4.

3.1 Proof

The proof is inductive on the number of vertices and treats the cases where there is and isn't a v, e - separating set.
We begin with some trivial cases.

1. (a) $2 \leq |V(G)| \leq 3$. All u, v - paths are Tutte paths and G is 2-connected, thus there is a u, v - Tutte path through e. **We may now assume that** $|V(G)| \geq 4$**.**

   (b) $uv = e$. Let P be the path on u and v through e. A path of length 1 is a Tutte path. Without loss of generality we may assume e **is not incident with** u.

2. We now look at the non trivial cases.

   (a) *G has a v, e - separating set,* $\{x, y\}$ which we choose so that $|V(G_2)|$ is minimized. Let $G_1' = G_1 + e_1$ and $G_2' = G_2 + e_2$ where $e_1 = xy = e_2$ in their respective graphs and let G_2'' be G_2 with a new vertex v' adjacent to x and y. Each of these are 2-connected, see figure 4. A number of subcases are considered here.

      i. $u \in V(G_1)$. By the induction hypothesis, there is a u, v - Tutte path through e_1 in G_1' as well as an x, y - Tutte path through e in G_2'. Joining these two graphs on xy gives the required path in G, see figure 5.

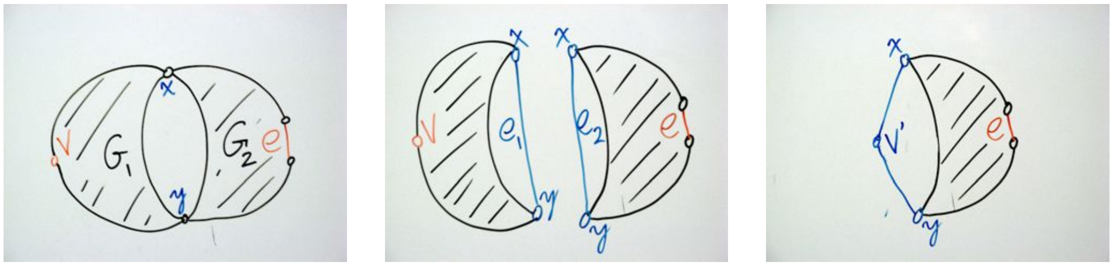


Figure 4: Graphs G, G_1', G_2', G_2''

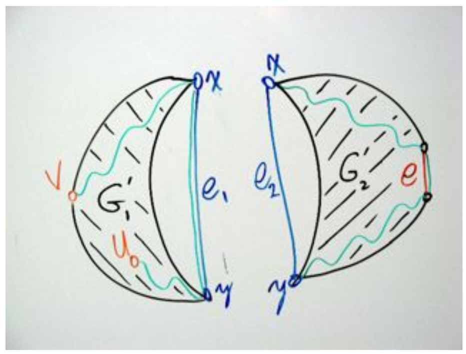
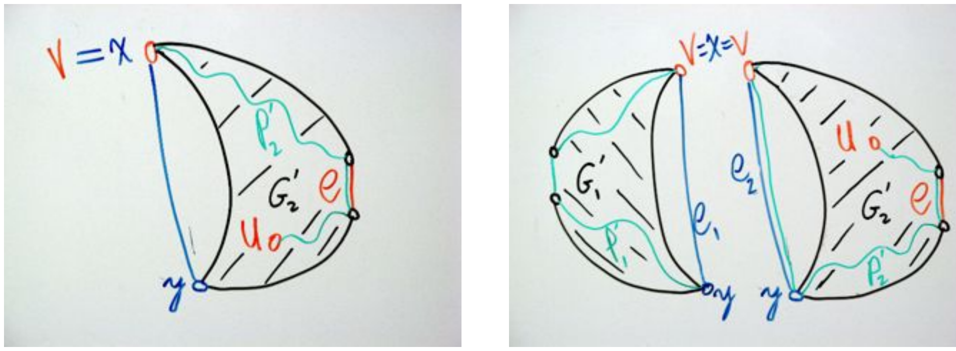


Figure 5: $u \in V(G_1)$

ii. $u \notin V(G_1)$ *and* $v \in \{x, y\}$ (WLOG let $v = x$). By induction G_2' has a Tutte path P_2', and, if it doesn't pass through e_2, the path is also in G as in figure 6(a). Otherwise we replace e_2 by an x, y - Tutte path in G_1', through an edge of the outer cycle of G_1'. This path also exists by induction and taking them together gives the required path in G.

iii. $u \notin V(G_1)$ *and* $v \notin \{x, y\}$. G_2'' contains a v', u, e - Tutte path, P_2'', and G_1' contains a v, x, xy - Tutte path, P_1'', by induction. WLOG suppose $v'y$ is on P_2''. If x is on P_2'' we have a v, u, e - Tutte path in G by removing x from P_1'' and removing v' from P_2'' as in figure 7(a).

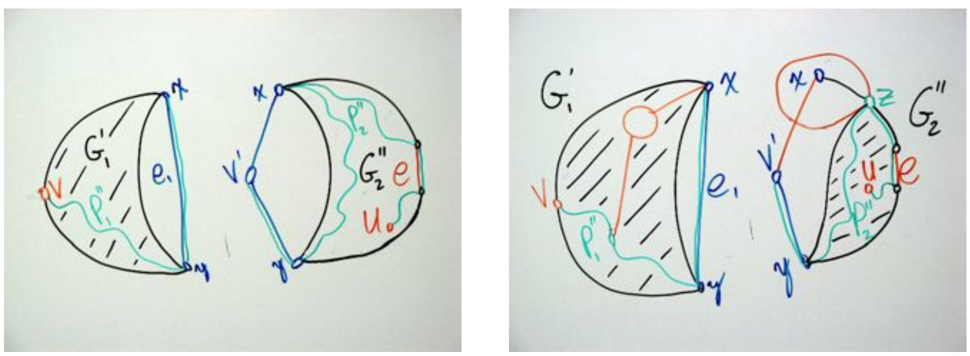
If $x \notin P_2''$ combining the paths in G may not yield a Tutte path. Why was it okay to recombine similar looking paths previously? In all such cases the vertices in the separation, x and y, have been on the paths and thus it was impossible for a component to gain a vertex of attachment in the recombination. We fix this case using the minimality of G. Consider the P_2'' component containing x. Since G_2'' is 2-connected and $v'x$ is on its outer cycle, there is a unique vertex of attachment, $z \neq v'$ which also lies on the



(a) (b)

Figure 6: $u \notin V(G_1)$ and $v \in \{x, y\}$

outer cycle. We can see in figure 7(b) that $\{y, z\}$ is a v, e - separating set which contradicts the minimality of G_2.



(a) (b)

Figure 7: $u \notin V(G_1)$ and $v \notin \{x, y\}$

(b) *G has no v, e - separating set.* Here we lay out some definitions and assumptions that we will use throughout the rest of the proof, see figure 8. Some of them will be repeated when called upon.

- G: A 2-connected graph with no v, e - separating set with $|V(G)| \geq 4$.
- C: The outer cycle of G.
- v: A vertex on C.
- e: An edge of C
- u: A vertex of G and not an endpoint of e.

- P_1: The segment of C from v to an endpoint of e which does not include e or u.
- $P_2 := C - V(P_1)$. Note that e is not in P_2.
- $H := G - V(P_1)$. H is not necessarily connected. Much of the remainder of the proof is spent considering the cases arising from its (2-connected) blocks.
- v_1: The endpoint of e on P_2.
- e_1: The edge incident to v_1 sharing a face (of G) with e and not the infinite face. It is necessary to show that e_1 exists. Since G is 2-connected, e is on exactly two faces and v_1 has degree at least 2 but e is incident with v_1 so its degree is greater than 2, otherwise G would have a v, e - separating set.
- B: The block of H containing v_1.
- C': The outer cycle of B ($C' = B$ if $B = K_2$).
- e_1': The edge of C' incident with v_1 and $e_1' \neq e_1$ ($e_1' = e_1$ if $B = K_2$).
- v_2: The end of P_2 distinct from v_1 (unless P_2 has length zero).

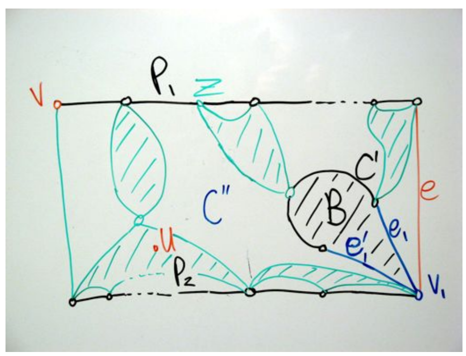


Figure 8: G displaying P_1, P_2, H, B, C''

Before we come to the final set of subcases we need to show that P_2 is contained in B.

Suppose e_1' is not on P_2 . Let C'' be the facial cycle containing e_1' which is not inside C'. We claim that C'' contains a vertex, z, on P_1. If not then either C'' is simply the outer cycle of B ($C'' = C'$) and v_1 is a cut vertex of G, contradicting G's 2-connectivity or else $B \cup C''$ is 2-connected in H which contradicts the maximality of B (as a block). This is a contradiction because $\{z, v_1\}$ is a v, e - separating set and we assumed there wouldn't be one. **Thus B is the only block of H which contains v_1 and e_1' is on P_2.**

Suppose $P_2 \not\subseteq B$. We saw in the previous case that P_2 shares at least one edge with B, now we suppose that it is not contained in B. Let H' be the component of H which contains P_2. H' contains a cut vertex, w, on P_2 since we are assuming that P_2 is contained in more than one block. Consider the subgraphs of H' joined at w. In symbols, H_1, H_2 are subgraphs of H' such that $H' = H_1 \cup H_2$ and $H_1 \cap H_2 = \{w\}$ with $v_1 \in V(H_1)$. Notice that $B \subseteq H_1$ and H_1 and H_2 are not necessarily 2-connected. Let r be the vertex of P_1 connected to H_2 and furthest from v (on P_1). This exists since at worst $r = v$. Since w is a cut vertex of H and by the maximality of the position of r, both vertices are on the same facial cycle of G and thus form a v, e - separating set as in figure 9. **Thus B contains all of P_2.**

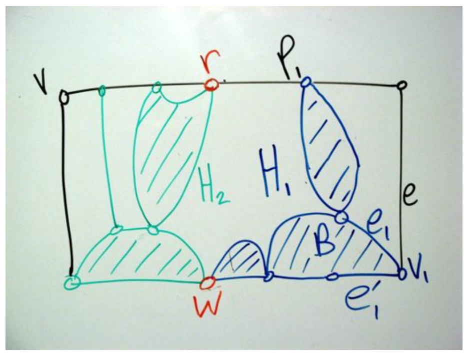


Figure 9: $P_2 \not\subseteq B$

Suppose u is not in the same component as P_2 . Assume u is in a C-component of G with vertices of attachment only in P_1. Let P_1 be the other v, e - segment of C, thus **we may assume that u and P_2 are in the same component of H.**

i. *H is not connected.* Let $H'' \subseteq H$ such that it is disconnected from H'. H'' must have all its vertices of attachment (as a C - component) in P_1. Let H''' be the subgraph depicted in figure 10(a). The graph obtained from G by replacing H''' with the edge xy is 2-connected so it has a v, u, e - Tutte path by induction. If the path contains xy we can replace xy by an x, y - Tutte path in H''', also obtained by induction since it is 2-connected.

ii. *H is connected.* If u is in B define $u' = u$, otherwise let $u' \in V(C')$ be the unique vertex that is on every path (in H) from u to a vertex of B. Recall that v_2 is the end of P_2 distinct from v_1 (or $v_2 = v_1$ if P_2 has length zero). B is 2-connected so by induction B contains a v_1, u' - Tutte path, P', that

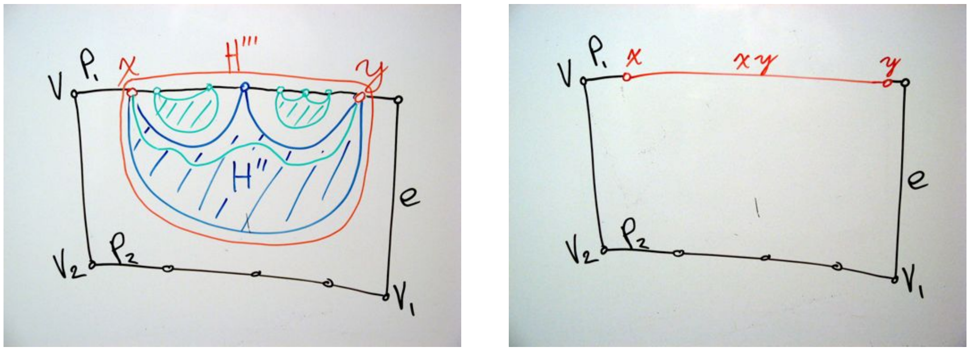


Figure 10: H is not connected

visits v_2.

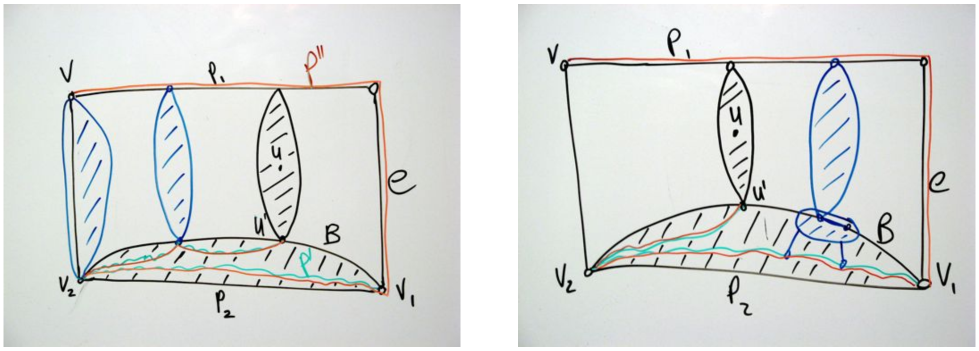
Let P'' be the union of the vu' path along P_1, the edge e and P'. It is this path we will modify to obtain a v, u, e - Tutte path.

Let J be an arbitrary $B \cup C$ - component of G whose properties will be specified in each case. Since H is connected and B is a block of H, J must be connected to B at a cut vertex, b_J, and the rest of its vertices of attachments are on P_1. Two different $B \cup C$ - components don't share more than one vertex of attachment in P_1. A few more subcases arise, some require a modification and others do not.

A. *J has one attachment in P_1 and b_J is on P'* . This cannot cause a problem because J is already a P'' component with exactly two attachments, see figure 11(a).

B. *J has one attachment in P_1 and b_J is* **not** *on P'* . Since v_1 and v_2 are on P', J cannot contain an edge of C and since b_J is not on P', J is part of a P'' - component which contains an edge of C' which can have up to two attachments to P' for a total of three to P'', see figure 11(b).

C. *J has more than one vertex of attachment in P_1, b_J is on P' and u is not in $J - b_J$.* Let $P(J)$ be the minimal segment of P_1 containing vertices in J and P_1. Let x, y be the endpoints of $P(J)$. The graph $P(J) \cup J + yb_J$ is 2-connected so, by induction, it has an x, b_J - Tutte path through yb_J. We modify P'' by replacing $P(J)$ with the x, y segment of this path. Figure 12 shows us that this fixes any problems caused by J.



(a) J has one attachment in P_1 and b_J is on P'

(b) J has one attachment in P_1 and b_J is **not** on P'

Figure 11: J has one attachment in P_1

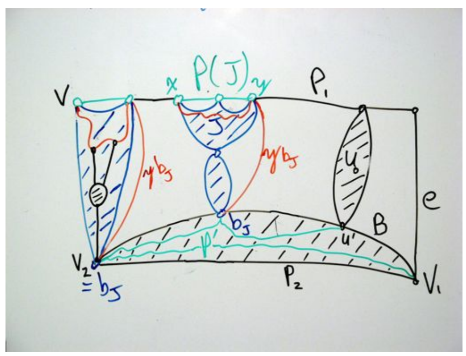
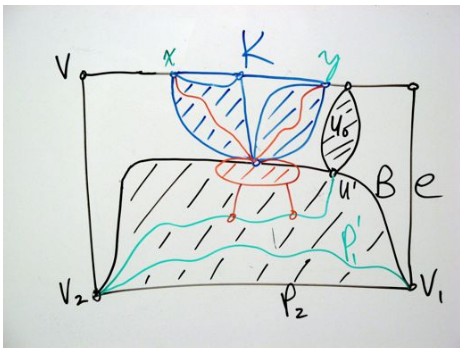


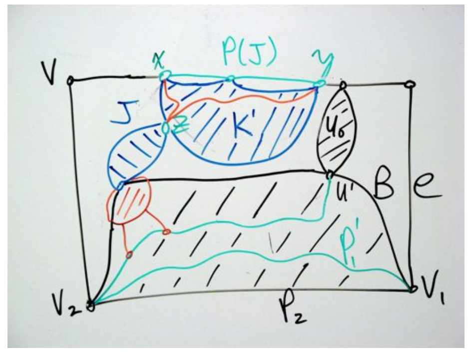
Figure 12: Configurations where $|J \cap P_1| \geq 2$, $b_J \in P'$ and $u \notin J - b_J$.

D. *J has more than one vertex of attachment in P_1, or b_J is the cut vertex for more than one $B \cup C$ - component, J, and b_J is* **not** *on P'.* We notice first that u cannot be in J. Let K be the set of all blocks of H attached to b_J (other than B) together with the minimal segment of P_1 containing all their vertices of attachment with x and y being the endpoints of this segment. If K is 2-connected we modify P'' by replacing the segment with an x, y - Tutte path through b_J in K, obtained by induction, see figure 13(a).

Otherwise $K = J \cup P(J)$. Since J is connected, $P(J)$ contains no cut vertex of K. Let K' be the block of K containing $P(J)$ and let z be the (unique) cut vertex of K in K'. We modify P'' by replacing the x, y segment of P_1 with an x, y - Tutte path in K' through z (obtained by induction and the fact that K' is 2-connected). Figure 13(b) shows how this fixes P''.



(a) K is 2-connected



(b) K is not 2-connected then $K = J \cup P(J)$

Figure 13: b_J is **not** on P'

If u is in B then we are done and P'' is the desired v, u, e - Tutte path.

If u is in $J - b_J$ then $u \neq b_J = u'$. Let x and y be the endpoints of $P(J)$ with x closer to v on P_1. Let J_u be the graph of J union $P(J)$ plus the edge yu'. J_u is 2-connected so by induction it has an x, u, yu' - Tutte path. We obtain the final v, u, e - Tutte path by taking the segment of P'' from v to x together with the Tutte path in J_u and we replace the edge yu' with the segment of P'' from y to u' as in figure 14.

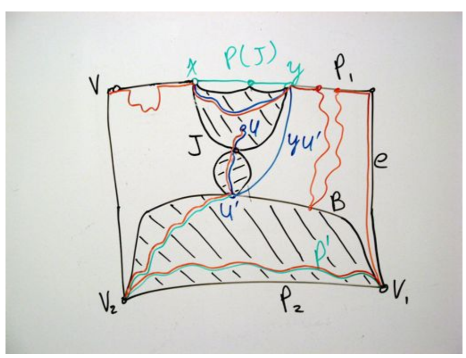
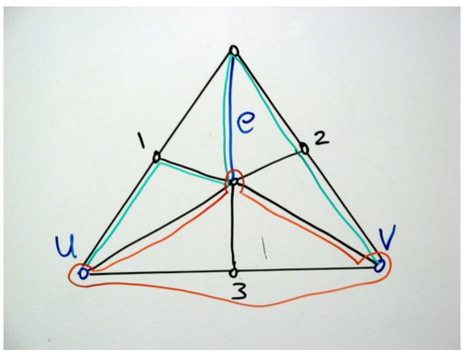


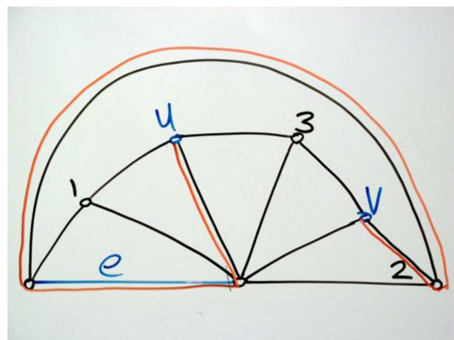
Figure 14: The final v, u, e - Tutte path in red.

4 Related Papers and Generalizations

Daniel P. Sanders of Ohio State University, published [7] in 1997 where he removes the restriction on the placement of v in Thomassen's theorem. In other words, if G is a 2-connected graph with arbitrary, distinct vertices u and v, and e is an edge of the outer cycle C then G contains a u, v, e - Tutte path. He gives an example to show that the edge e cannot be removed from the outer cycle C (figure 15(a)). Each of the vertices 1,2 and 3 are adjacent to exactly three of u, v or the endpoints of e and are also on C, thus they must all be on the u, v, e - Tutte path. This is impossible because we can only travel to and from 1,2 or 3 by going from u to e and then e to v which only gives us two opportunities to visit three vertices. We see in figure 15(b) that another planar imbedding of G with e in C does yield a u, v, e - Tutte path.



(a) An embedding of G where e is not in C as in [7]



(b) Another imbedding G with e in C and a u, v, e - Tutte path in red

Figure 15: Sander's Theorem

Sanders' implies that a 4-connected graph has a Hamilton cycle through any two of its edges.

Thomas and Yu prove Plummer's other conjecture (if G is 4-connected and planar then $G - u - v$ is Hamiltonian) and that 4-connected projective planar graphs are Hamiltonian in [6]. Their proof relies heavily on Thomassen's theorem.

These last two papers, along with Thomassen's and Tutte's work, are all related by their common goal to find Hamiltonian paths in planar graphs and by the strategies they employ.

References

[1] Douglas B. West, Introduction to Graph Theory,
PRENTICE HALL, Upper Saddle River, NJ 07458, (2001)

[2] H. Whitney, A theorem on graphs, Ann. of Math. 32 (1931) 378-390.

[3] W. T. Tutte, Bridges and Hamiltonian circuits in planar graphs, Aequationes Math. 15 (1977) 1-33

[4] W. T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956), 99-116.

[5] C. Thomassen, A theorem on paths in planar graphs, J. Graph Theory, 7 (1983), 169-176.

[6] Robin Thomas, Xingxing Yu, J. Combin. Theory Ser. B 62 (1994), no. 1, 114-132

[7] Daniel P. Sanders, On Paths in Planar Graphs, Journal of Graph Theory Vol. 24, No. 4, 341 345 (1997)