Schwarzschild Geometry: Metric

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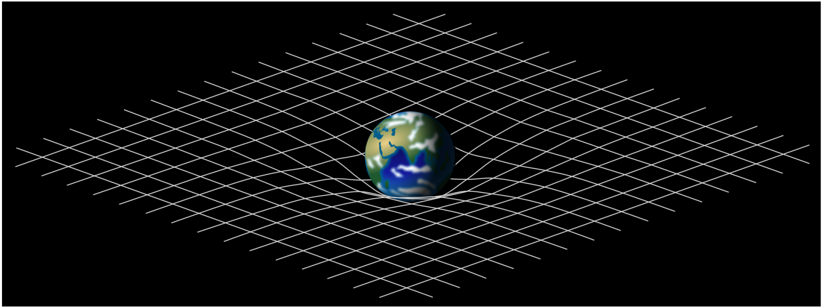
Introduction

the Schwarzschild metric is the most general spherically symmetric vacuum solution of the Einstein field equations



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Static spacetime

To construct the most general metric for a static spatially isotropic spacetime, we start with the line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Properties of a static spacetime

1. All the metric components are independent of x^0.
2. The line element is invariant under $x^0 \to -x^0$

If the spacetime only satisfies 1, it's called **stationary**

To get an isotropic metric

Isotropic metric

It's obtained when ds^2 depends only on rotational invariants of x^i and dx^i

The only rotational invariants and their differentials:

$$\vec{x}\cdot\vec{x}, \quad d\vec{x}\cdot d\vec{x}, \quad \vec{x}\cdot d\vec{x}$$

Starting with the most general form of a spatially isotropic metric

$$ds^2 = A(t,r)dt^2 - B(t,r)dt\vec{x}\cdot d\vec{x} - C(t,r)(\vec{x}\cdot d\vec{x})^2 - D(t,r)d\vec{x}^2$$

To get an isotropic metric

Transforming to spherical polar coordinates, we get:

$$\vec{x}\cdot\vec{x} = r^2, \quad \vec{x}\cdot d\vec{x} = rdr, \quad d\vec{x}\cdot d\vec{x} = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

Then, the metric takes the form:

$$\begin{aligned} ds^2 =& A(t,r)dt^2 - B(t,r)rdtdr - C(t,r)r^2dr^2 \\ &- D(t,r)\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) \end{aligned},$$

where it can also be written as

$$ds^2 = A(t,r)dt^2 - B(t,r)dtdr - C(t,r)dr^2 - D(t,r)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

To get an isotropic metric

However, we can express it in terms of a new radial coordinate

$$ds^2 = A(t,\bar{r})dt^2 - B(t,\bar{r})dtd\bar{r} - C(t,\bar{r})d\bar{r}^2 - \bar{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Introducing a new timelike coordinate.

Using

$$d\bar{t} = \Phi(t,\bar{r})\left[A(t,\bar{r})dt - \frac{1}{2}B(t,\bar{r})d\bar{r}\right]$$

we can find

$$Adt^2 - Bdtd\bar{r} = \frac{1}{A\Phi^2}d\bar{t}^2 - \frac{B}{4A}d\bar{r}^2$$

and redefine the functions A and B

To get an isotropic metric

The functions take the new form

$$\bar{A} = 1/\left(A\Phi^2\right) \text{ and } \bar{B} = C + B/(4A)$$

from which we get the isotropic metric

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

The stationary isotropic metric

To get the form of the metric for a general static spatially isotropic spacetime, we need the metric functions to be independent of the timelike coordinate. This means:

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\Downarrow$$

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Solving the empty-space field equations

To solve the empty-space field equations, we must have a Ricci tensor that vanishes. This is:

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} = 0$$

having in mind that

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right)$$

To solve this, we start with the non-zero elements of the metric $g_{\mu\nu}$

$$\begin{aligned}
g_{00} &= A(r), & g^{00} &= 1/A(r) \\
g_{11} &= -B(r), & g^{11} &= -1/B(r) \\
g_{22} &= -r^2, & g^{22} &= -1/r^2 \\
g_{33} &= -r^2\sin^2\theta, & g^{33} &= -1/\left(r^2\sin^2\theta\right)
\end{aligned}$$

Solving the empty-space field equations

The connections coeffients are found as shown

$$\Gamma^0_{00} = 0$$

$$\Gamma^i_{00} = -\frac{1}{2} g^{i\rho}\partial_\rho g_{00} \quad \Rightarrow \quad \Gamma^1_{00} = \frac{1}{2B(r)}\frac{dA(r)}{dr}$$

$$\Gamma^0_{0i} = \frac{1}{2} g^{0\rho}\left(\partial_i g_{\rho 0} + \partial_0 g_{\rho i} - \partial_\rho g_{0i}\right) = \frac{1}{2} g^{00}\partial_i g_{00} \quad \Rightarrow \quad \Gamma^0_{01} = \frac{1}{2A(r)}\frac{dA(r)}{dr}$$

$$\Gamma^0_{ij} = 0$$

$$\Gamma^i_{ii} = \frac{1}{2} g^{i\rho}\left(\partial_i g_{\rho i} + \partial_i g_{\rho i} - \partial_\rho g_{ii}\right) = \frac{1}{2} g^{ii}\partial_i g_{ii} \quad \Rightarrow \quad \Gamma^1_{11} = \frac{1}{2B(r)}\frac{dB(r)}{dr}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right) \quad \Rightarrow \quad \Gamma^1_{22} = -\frac{r}{B(r)}$$

$$\Gamma^1_{33} = -\frac{1}{2} g^{11}\partial_1 g_{33} \quad \Rightarrow \quad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B(r)}$$

Solving the empty-space field equations

Only nine of the connection coefficients are non-zero:

$$
\begin{array}{lll}
\Gamma^0_{01} = A'/(2A), & \Gamma^1_{00} = A'/(2B), & \Gamma^1_{11} = B'/(2B) \\
\Gamma^1_{22} = -r/B, & \Gamma^1_{33} = -\left(r\sin^2\theta\right)/B, & \Gamma^2_{12} = 1/r \\
\Gamma^2_{33} = -\sin\theta\cos\theta, & \Gamma^3_{13} = 1/r, & \Gamma^3_{23} = \cot\theta
\end{array}
$$

We'll use these to get Ricci's tensor

Solving the empty-space field equations

Diagonal components of the Ricci tensor

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right)$$

$$R_{33} = R_{22}\sin^2\theta$$

Solving the empty-space field equations

Since the tensor must vanish, we can get the relationship

$$A'B + AB' = 0$$

Showing that $AB = constant$, so we can use $B = \alpha/A \rightarrow A + rA' = \alpha$

$$\frac{d(rA)}{dr} = \alpha$$

Integrating, we get

$$A(r) = \alpha\left(1 + \frac{k}{r}\right) \quad \text{and} \quad B(r) = \left(1 + \frac{k}{r}\right)^{-1}$$

Solving the empty-space field equations

We can get the constants k and α for a spherically symmetric mass M as:

$$k = -\frac{2GM}{c^2} \quad \text{and} \quad \alpha = c^2$$

Schwarzschild metric for the empty spacetime outside a spherical body of mass M

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

Birkhoff's theorem

For a non-stationary metric

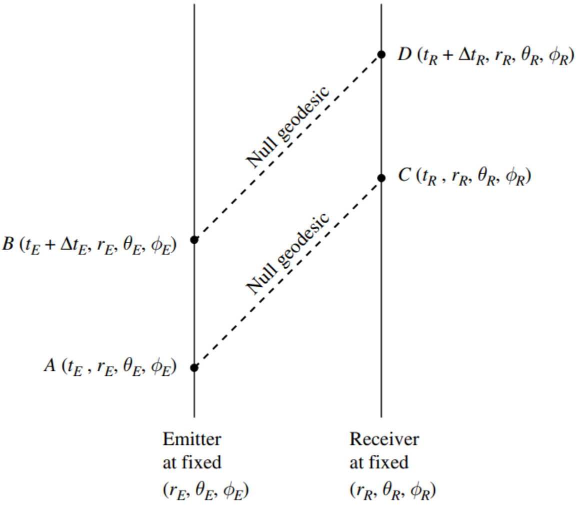
$$ds^2 = A(t, r)dt^2 - B(t, r)dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

but solving the Einstein's empty space field equations $R_{\mu\nu} = 0$ with this expression leads to the same metric

Birkhoff's theorem

The spacetime geometry outside a general spherically symmetric matter distribution is the Schwarzschild geometry

Emision and receptions of two light signals



Gravitational redshift in a null curve

In a null curve $ds^2 = 0$ at all points. This means

$$c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

using an affine parameter σ

$$\frac{dt}{d\sigma} = \frac{1}{c}\left(1 - \frac{2\mu}{r}\right)^{-1/2} \left[-g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}\right]^{1/2}$$

and integrating

$$t_R - t_E = \frac{1}{c}\int_{\sigma_E}^{\sigma_R} \left(1 - \frac{2\mu}{r}\right)^{-1/2} \left[-g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}\right]^{1/2} d\sigma$$

But we have that $\Delta t_R = \Delta t_E$ and $dr = d\theta = d\phi = 0$, then:

$$c^2 d\tau^2 \equiv ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2$$

and since r is constant, we can integrate to obtain

$$\Delta\tau_E = \left(1 - \frac{2\mu}{r_E}\right)^{1/2} \Delta t_E \quad \text{and} \quad \Delta\tau_R = \left(1 - \frac{2\mu}{r_R}\right)^{1/2} \Delta t_R$$

which leads to

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left(\frac{1 - 2\mu/r_R}{1 - 2\mu/r_E}\right)^{1/2}$$

that is the basis of the formula for the gravitational redshift

The frequencies of the photon follow the relation

$$\frac{\nu_R}{\nu_E} = \left[\frac{1 - 2GM/\left(r_E c^2\right)}{1 - 2GM/\left(r_R c^2\right)}\right]^{1/2}$$

This can be generalized as

$$ds^2 = g_{00}(\vec{x})dt^2 + g_{ij}(\vec{x})dx^i dx^j$$

where we find that

$$\frac{\nu_R}{v_E} = \left[\frac{g_{00}\left(\vec{x}_E\right)}{g_{00}\left(\vec{x}_R\right)}\right]^{1/2}$$

Thanks for your attention