Schwarzschild Geometry: Geodesics for massive particles

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Schwarzschild metric

Suppose a body of mass M. Let $\mu = GM/c^2$

$$c = 3 \cdot 10^8 \mathrm{m/s} \qquad G = 6.67 \cdot 10^{-11} \mathrm{m^3 kg^{-1} s^{-2}}$$

.

Line element

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

$$ds^2 = g_{\mu\nu} x^\mu x^\nu$$

Geodesic equations

The length-minimizing curves within the geometry and are given by

$$\frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\sigma} \frac{dx^\rho}{d\sigma} = 0$$

where σ is the affine parameter.

Euler-Lagrange formalism

Equivalent form of computing the geodesics

$$\mathcal{L} \equiv g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \longrightarrow \frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$$

Geodesic equations

The Lagrangian is given by

$$\mathcal{L} = c^2 \left(1 - \frac{2\mu}{r}\right) \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2 \left(\dot{\theta}^2 + sen^2\theta\right) \dot{\phi}^2$$

- 4 differential equations from Euler-Lagrange equations.
- Variables: $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.
- $\mathcal{L}$ is cyclic on ct and ϕ

Geodesic equations

Confining our attention to particles moving in the equatorial plane $(\theta = \pi/2)$

- $x^0 = ct$

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k$$

- $x^1 = r$

$$\left(1 - \frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{2\mu}{r^2}\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\dot{\phi}^2 = 0$$

- $x^3 = \phi$

$$r^2\dot{\phi} = h$$

Physical interpretation of k and h

- $h = r^2\dot{\phi}$ is the angular momentum per unit test-mass.
- k is related to the energy of the test-particle stored in its orbit.

$$k = \left(1 - \frac{2\mu}{r}\right)\dot{t} = g_{00}\,\dot{t}$$

$$k = g_{00}\frac{m_0 c\,\dot{t}}{m_0 c} = g_{00}\frac{p^0}{m_0 c}$$

$$k = \frac{p_0}{m_0 c} = \frac{E}{m_0 c^2}$$

First Integral

Let the proper time τ be our affine parameter, so the following holds

$$g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = c^2$$

which is equivalent to the r-equation obtained before. Thus,

$$\begin{aligned} \left(1-\frac{2\mu}{r}\right)\dot{t} &= k \\ c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r\dot{\phi}^2 &= c^2 \\ r^2\dot{\phi} &= h \end{aligned}$$

First Integral

Using the new set of differential equations, we obtain

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2(k^2 - 1)$$

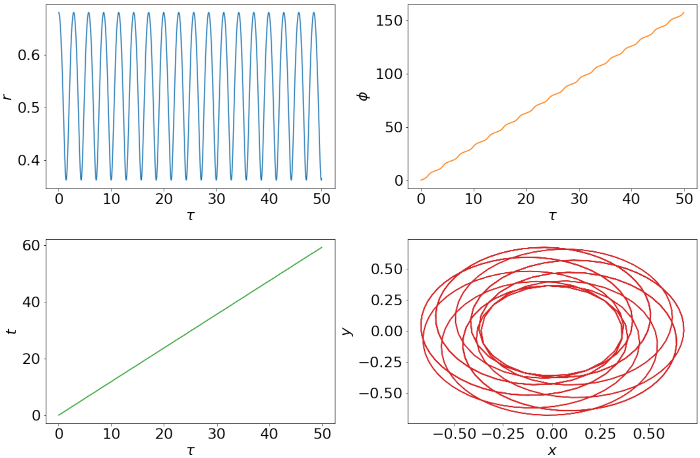
Differentiating...

$$\ddot{r} = \frac{h^2}{r^3} - \frac{3\mu h^2}{r^4} - \frac{\mu c^2}{r^2}$$

We can solve this equation and find $\phi(\tau)$ from

$$\frac{d\phi}{d\tau} = \frac{h^2}{r^2}$$

Trajectories



Trajectories

Analogously, we could use the chain rule $\frac{dr}{d\tau} = \frac{h}{r^2}\frac{dr}{d\phi}$ and define $u = 1/r$ to convert

$$\ddot{r} = \frac{h^2}{r^3} - \frac{3\mu h^2}{r^4} - \frac{\mu c^2}{r^2}$$

into

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu c^2}{h^2} + 3\mu u^2$$

This is a closed differential equation for the shape of the orbits.

Stability

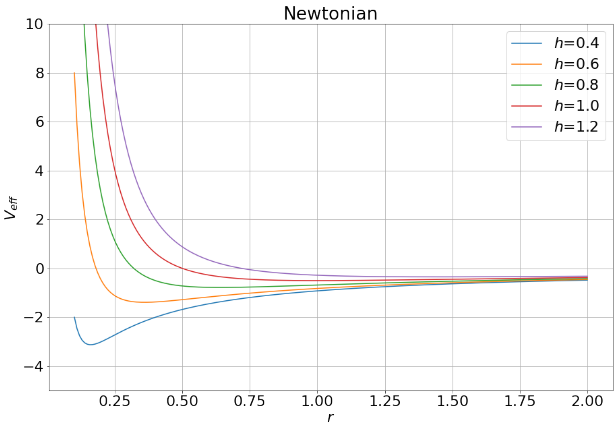
In Newtonian mechanics,

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \underbrace{\frac{h^2}{2r^2} - \frac{\mu c^2}{r}}_{V_{\text{eff}}(r)} = E$$

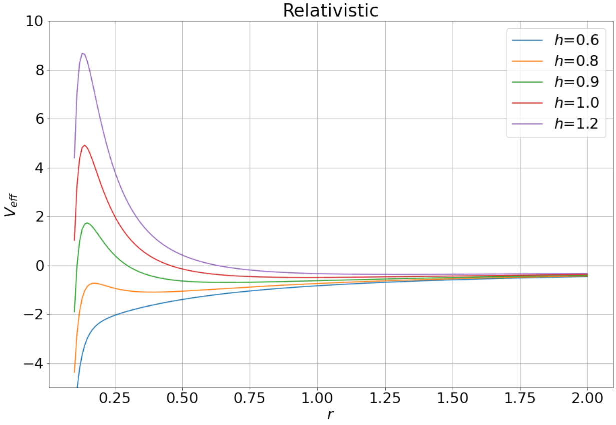
whereas in Relativity,

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\frac{h^2}{2r^2} - \frac{\mu c^2}{r} - \frac{\mu h^2}{r^3}}_{V_{\text{eff}}(r)} = \frac{c^2}{2}(k^2 - 1)$$

Newtonian V_{eff}



Relativistic V_{eff}



Relativistic V_{eff}

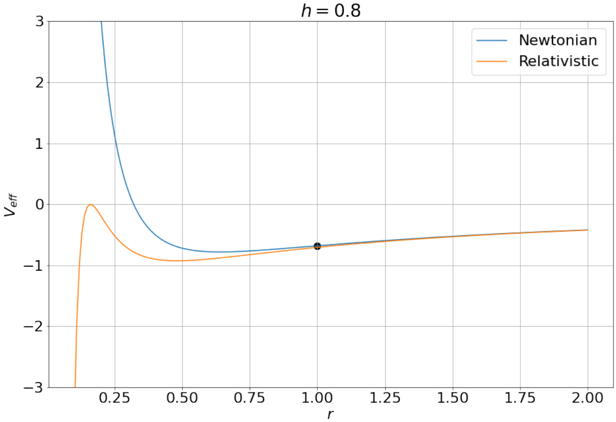
$$\left.\frac{dV_{eff}}{dr}\right|_{r=r^*} = 0$$

leads us to

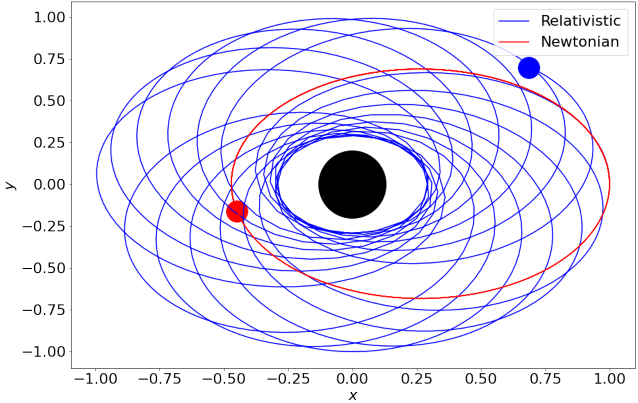
$$r^* = \frac{h}{2\mu c^2}\left(h \pm \sqrt{h^2 - 12\mu^2 c^2}\right)$$

We have a stable (+) and unstable (-) circular orbits.
If $h = 2\sqrt{3}\mu c$ there exists only one stable orbit with $r^* = 6\mu$

Classical and Relativistic



Classical and Relativistic



Radial motion

There is no angular momentum, so $h = 0$

$$\ddot{r} = \frac{h^2}{r^3} - \frac{3\mu h^2}{r^4} - \frac{\mu c^2}{r^2}$$

becomes

$$\ddot{r} = -\frac{GM}{r^2}$$

which has the same form of the classical solution. They are not the same, though.

Circular motion

There is no radial momentum, so $\dot{r} = 0$

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu c^2}{h^2} + 3\mu u^2$$

becomes

$$u = \frac{\mu c^2}{h^2} + 3\mu u^2$$

Then,

$$h = \left(\frac{\mu c^2 r^2}{r - 3\mu}\right)^{1/2}$$

There exists no valid circular orbit below $r = 3\mu$.

Out of the Equator

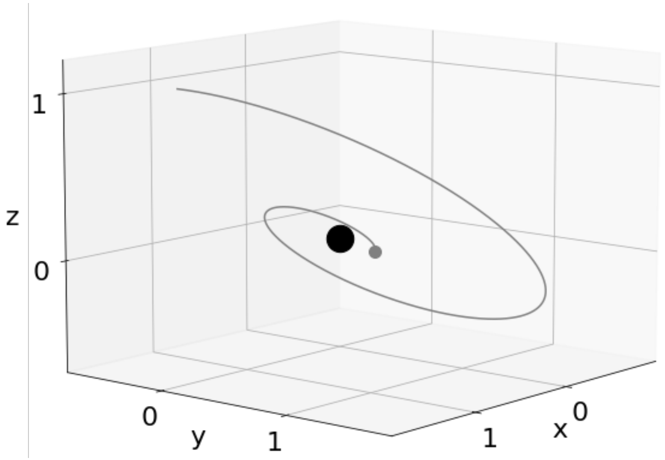
If the condition $\theta = \pi/2$ is not imposed, the equations read

$$\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \frac{\cos\theta}{r^4\sin^3\theta}h^2$$

$$\ddot{r} = -\frac{\mu c^2 k^2}{r^2} + \left(1 - \frac{2\mu}{r}\right)^{-1}\frac{\mu}{r^2}\dot{r}^2 + r^2\dot{\theta}^2 + \frac{h^2}{r^2\sin^2\theta}$$

which can be easily solved numerically.

Outside the Equator



Problem Statements

Conceptual

- What happens if we add another body? Would the spherical symmetry be conserved?

Analytical

- Differentiate $g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = c^2$ and compare with the initial set of differential equations.

Numerical

- Obtain the graphs shown throughout this presentations and play with the parameters.

`github.com/alejandrogm668/Schwarzschild-Geodesics.git`