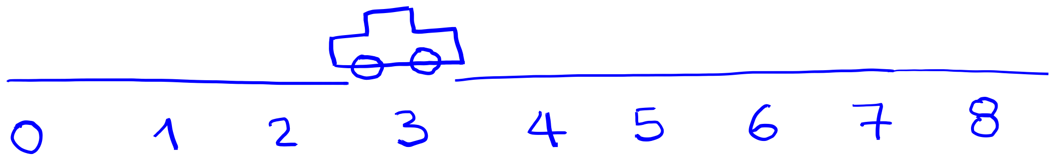
Día 2

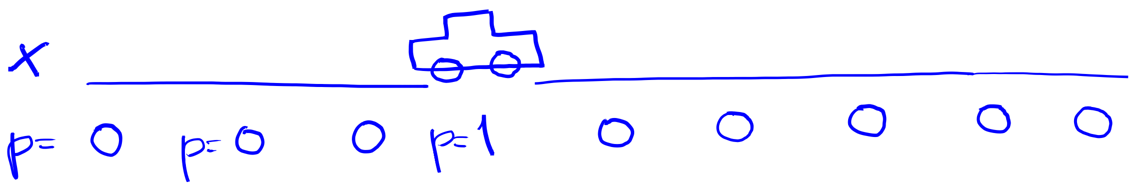
1. Postulado del vector de estado

Objetivo: Entender qué nos propone la cuántica para seguir el estado de un sistema.



→ posición $x = 3$

α, β



→ Amplitud de probabilidad

$x = 3$

$$|x\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad 8 = 2^3 \qquad n \quad 2^n$$

→ 3 qubits

2. Esfera de Bloch

Objetivo: Representar un qubit gráficamente

2.1. Fases globales

$$|\alpha|^2 + |\beta|^2 = 1$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \alpha, \beta \in \mathbb{C}$$

$$|\psi\rangle = \alpha \cdot \gamma |0\rangle + \beta \cdot \gamma |1\rangle \qquad |\gamma| = 1$$

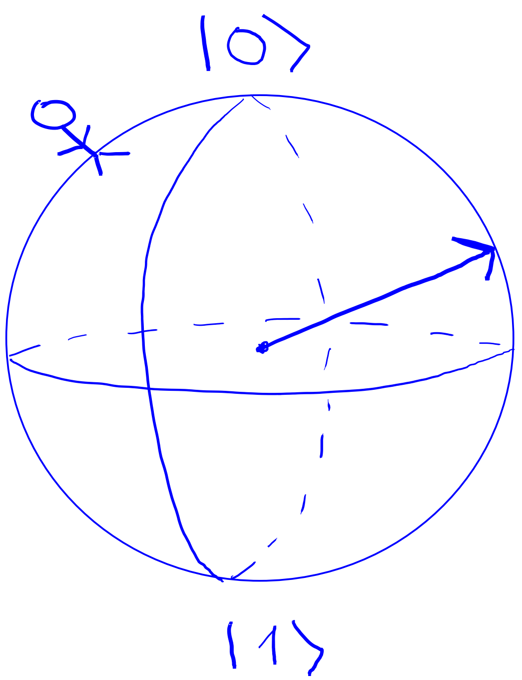
$$P_T = |\alpha\gamma|^2 + |\beta\gamma|^2 = |\alpha|^2 |\gamma|^2 + |\beta|^2 |\gamma|^2 = |\alpha|^2 + |\beta|^2 = 1$$

$\alpha, \beta \qquad \alpha\gamma, \beta\gamma$

Probabilidades iguales!

Estados equivalentes

2.2. La esfera



¿Cuántas direcciones tenemos para movernos sobre una esfera? z

Variables = 2 ✓

$\alpha, \beta \in \mathbb{C}$ $\qquad \theta, \varphi$

Recordemos que $\quad |\alpha|^2 + |\beta|^2 = 1$

$$\sin^2 x + \cos^2 x = 1$$

$$\downarrow \qquad\qquad \downarrow$$

$$|\alpha|^2 \qquad |\beta|^2 = 1$$

$|\gamma| = 1$ ✓

Polar

$$\alpha = |\alpha| \underbrace{e^{i\varphi_\alpha}}_{} \qquad \beta = |\beta| \underbrace{e^{i\varphi_\beta}}_{}$$

fases

$$|e^{i\varphi_\alpha}|^2 = 1$$

$$e^{i\varphi_\alpha} \cdot e^{-i\varphi_\alpha}$$

$$= e^{i\varphi_\alpha - i\varphi_\alpha} = e^0 = 1$$

$$|\psi\rangle = \sin x\, |0\rangle + e^{i\varphi} \cos x\, |1\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = |\alpha| e^{i\varphi_\alpha} |0\rangle + |\beta| e^{i\varphi_\beta} |1\rangle$$

$$= \sin x\, \underbrace{e^{i\varphi_\alpha}}_{} |0\rangle + \cos x\, e^{i\varphi_\beta} |1\rangle$$

$$= \cancel{e^{i\varphi_\alpha}} \left(\sin x\, |0\rangle + \cos x\, \underbrace{e^{i\varphi_\beta} e^{-i\varphi_\alpha}}_{e^{i\varphi}} |1\rangle \right)$$

γ

Fase global

$$\varphi = \varphi_\beta - \varphi_\alpha$$

Fase relativa

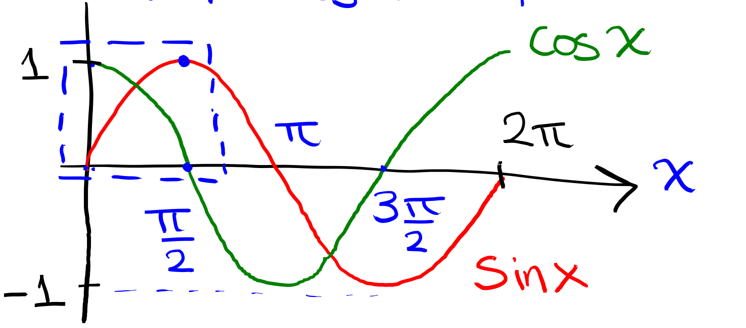
Rangos del ángulo:

$$0 \le x \le \frac{\pi}{2} \qquad x = \frac{\theta}{2}$$

$$0 \le \frac{\theta}{2} \le \frac{\pi}{2}$$

$$\boxed{0 \le \theta \le \pi}$$

$|\alpha| = \cos x \qquad |\beta| = \sin x$



$$|\alpha|^2 = \cos^2 \frac{\theta}{2} \qquad |\beta|^2 = \sin^2 \frac{\theta}{2} \underbrace{|e^{i\varphi}|^2}_{1} = \sin^2 \frac{\theta}{2}$$

$$|\alpha|^2 + |\beta|^2 = \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} = 1$$

$e^{i\varphi} = 1$ $\quad e^{i\pi} = -1$ $\quad e^{i\varphi} = \cos\varphi + i\sin\varphi$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\varphi}|1\rangle$$

$\cos\frac{\pi}{2} = 0$

$\theta = \frac{\pi}{2}$ $\quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4}$

$\sin\frac{\pi}{2} = 1$

- $\theta = 0 \rightarrow |\psi\rangle = |0\rangle$

$= \cos\frac{\theta}{2}|0\rangle$

$|-\rangle$ $+\sin\frac{\theta}{2} e^{i\varphi}|1\rangle$ $\theta = \pi \rightarrow |\psi\rangle = |1\rangle$

$\theta = \frac{\pi}{2}$ $\quad \varphi = \pi$

$|\pm\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle$

$|+\rangle$

Pregunta! z

Para qué ángulos se tiene $|+\rangle, |-\rangle$?

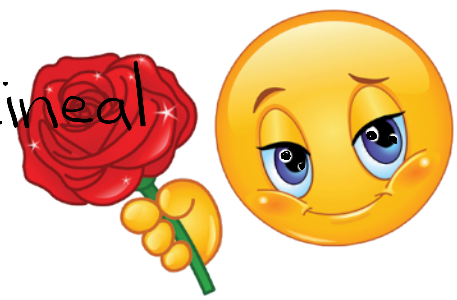
$\theta = \frac{\pi}{2}$ $\quad \varphi = \pi$

3. Compuertas de un qubit

Objetivo: Transformar estados en la esfera de Bloch.

Nuevamente . . .

Lineal



3.1 Álgebra lineal

3.1.1 Operadores y matrices

Transforman vectores $\longrightarrow$ $A|u\rangle = |v\rangle$

- Linealidad
  - $A(|u\rangle + |v\rangle) = A|u\rangle + A|v\rangle$
  - $A(c|u\rangle) = c(A|u\rangle)$

- Probabilidades suman 1

$$|\alpha|^2 + |\beta|^2 = 1 = \langle\psi|\psi\rangle = \||\psi\rangle\|^2 \qquad \|A|\psi\rangle\| = 1$$

$$\||\psi\rangle\| = 1 \qquad A \text{ es unitario}$$

3.1.1.1 Producto exterior

KetBra : $\langle a|b\rangle$

$$|a\rangle = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \langle b| = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$$

$$|a\rangle\langle b| = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} & \cdots & b_n \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{bmatrix} \qquad \langle a|b\rangle \rightarrow \#$$

$$|a\rangle\langle b| = \begin{bmatrix} b_1a_1 & \cdots & b_na_1 \\ \vdots & \ddots & \vdots \\ b_1a_n & \cdots & b_na_n \end{bmatrix}$$

Una matriz!

3.1.1.2 Matrices de Pauli

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} I &= |0\rangle\langle 0| + |1\rangle\langle 1| \\ X &= |0\rangle\langle 1| + |1\rangle\langle 0| \\ Y &= -i|0\rangle\langle 1| + i|1\rangle\langle 0| \\ Z &= |0\rangle\langle 0| - |1\rangle\langle 1| \end{aligned}\right\} \text{Operadores de Pauli}$$

$$A|\psi\rangle = \cdots$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$I|\psi\rangle = (|0\rangle\langle 0| + |1\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$$

$$= \alpha|0\rangle\cancelto{1}{\langle 0|0\rangle} + \beta|0\rangle\cancelto{0}{\langle 0|1\rangle} + \alpha|1\rangle\cancelto{0}{\langle 1|0\rangle} + \beta|1\rangle\cancelto{1}{\langle 1|1\rangle}$$

$$= \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$$

$$X|\psi\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle)$$

$$= \alpha|0\rangle\cancelto{0}{\langle 1|0\rangle} + \beta|0\rangle\langle 1|1\rangle + \alpha|0\rangle\langle 0|0\rangle$$

$$= \beta|0\rangle + \alpha|1\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

X : NOT

Ahora, con matrices

$$|0\rangle\langle 0| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1\begin{pmatrix}1\\0\end{pmatrix} & 0\begin{pmatrix}1\\0\end{pmatrix}\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$|0\rangle\langle 1| = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

$$|1\rangle\langle 0| = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$$|1\rangle\langle 1| = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

Operadores de Pauli $\longrightarrow$ Matrices de Pauli

$$I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \qquad Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$$

$|0\rangle\langle 0| + |1\rangle\langle 1|$

$|0\rangle$
$|1\rangle$

$$\underbrace{Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}_{\text{Mat. Pauli}}$$

Preguntas! $Y|\psi\rangle, Z|\psi\rangle$

3.1.1.3 Matrices unitarias

$$\left\{ \text{En Computación Cuántica, los operadores son unitarios} \right\}$$

Tomemos $\underline{\mathcal{U}}$ $\quad$ $\underline{\mathcal{U}^{+}}$: Adjunta, hermítica conjugada

$$\mathcal{U}\mathcal{U}^{+} \overset{!}{=} I$$

Recordemos que $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}^{+} = \begin{bmatrix} a_1^* & \cdots & a_n^* \end{bmatrix}$

$$|a\rangle^{+} = \langle a|$$

$\mathcal{U}^{+}$:

$$\mathcal{U} = \begin{bmatrix} \boxed{\begin{matrix} u_{11} \\ \vdots \\ u_{n1} \end{matrix}} & \cdots & u_{1n} \\ & \ddots & \vdots \\ & \cdots & u_{nn} \end{bmatrix}$$

$$\mathcal{U}\mathcal{U}^{+} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}$$

$$\mathcal{U}^{+} = \begin{bmatrix} \boxed{u_{11}^* \quad \cdots \quad u_{n1}^*} \\ \vdots \quad \ddots \quad \vdots \\ u_{1n}^* \quad \cdots \quad u_{nn}^* \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Por ejemplo, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\quad Y = Y^{\dagger} = \begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix}$

$i \to -i$

$$Y Y^{\dagger} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ i \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} -i \\ 0 \end{pmatrix} \end{bmatrix}$$

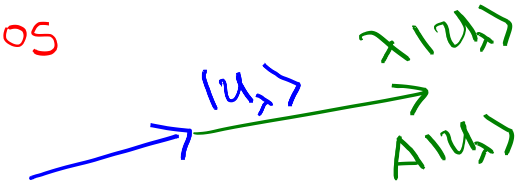
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$Y^{\dagger} Y = I$ ✓

Z

X

3.1.1.4 Valores y vectores propios



$$A\,|u_\lambda\rangle = \lambda\,|u_\lambda\rangle$$

Operador

Vector propio /autovector

Valor propio /autovalor

Cómo los encontramos?

$$A|u_\lambda\rangle = \lambda|u_\lambda\rangle \qquad \lambda|u_\lambda\rangle = \lambda I|u_\lambda\rangle$$

$$A|u_\lambda\rangle - \lambda|u_\lambda\rangle = 0$$

$$A|u_\lambda\rangle - \lambda I|u_\lambda\rangle = 0$$

$$(A - \lambda I)|u_\lambda\rangle = 0$$

$$\underline{\det(A - \lambda I) = 0}$$

$$f(\lambda) = 0$$

Ecuación característica $\longrightarrow$ $f(\lambda) = \det(A - \lambda I) = 0$

Sea $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad \det A = ad - bc \checkmark$

$\lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\qquad \mathrm{Tr}\, A = a + d \checkmark$

Traza

$$\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0 \quad = f(\lambda)$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \underline{ad} - a\underline{\lambda} - \underline{\lambda d} + \lambda^2 - \underline{bc}$$

$$= \underbrace{(ad - bc)}_{\det A} - \underbrace{(a+d)}_{\mathrm{Tr}\, A}\lambda + \lambda^2$$

Por ejemplo, para X $\qquad f(\lambda) = \underline{\lambda}^2 - \mathrm{Tr}A\, \underline{\lambda} + \det A = 0$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad \mathrm{Tr}\, X = 0 \qquad \det X = -1$

$\lambda^2 - 1 = 0 \implies \lambda^2 = 1 \checkmark$

$\lambda_1 = 1 \qquad \lambda_2 = -1$

$X|+\rangle = |+\rangle \qquad X|-\rangle = -|-\rangle$

$$X|u_1\rangle = |u_1\rangle \qquad X|u_{-1}\rangle = -|u_{-1}\rangle$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = |-\rangle$$

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$b = a$$

$$\langle u_1 | u_1 \rangle = 1$$

$$|u_1\rangle = \begin{bmatrix} a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

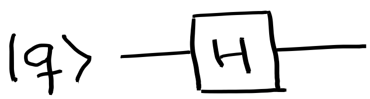
$$\begin{bmatrix} a^* & a^* \end{bmatrix}\begin{bmatrix} a \\ a \end{bmatrix} = 2a^2 = 1$$

$$a^2 = \frac{1}{2}$$

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$$

$$a = \frac{1}{\sqrt{2}}$$

Compuerta Hadamard



$$H|0\rangle = |+\rangle$$

$$H|1\rangle = |-\rangle$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Preguntas! Hallar los autovalores/vectores de $\boxed{Y}$ y ubicarlos en la esfera de Bloch.