BUILDING SPARSE XX HAMILTONIAN

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The XX Hamiltonian is given by

$$H = J\sum_{j=1}^{N-1}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y\right) - h\sum_{j=1}^{N}\sigma_j^z = H_{xy} + H_z, \tag{1}$$

where

$$\sigma_j^x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_j^y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_j^z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2}$$

Notice that they come in pairs in the xy-Hamiltonian, such that

$$\sigma_j^x\sigma_{j+1}^x = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}, \qquad \sigma_j^y\sigma_{j+1}^y = \begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}. \tag{3}$$

Therefore, the components of H are always real. Now, we can try to find a pattern. Let's assume $J = h = 1$ and start with the 2-site Hamiltonian.

$$H^{(2)} = \sigma^x\otimes\sigma^x + \sigma^y\otimes\sigma^y - \sigma^z\otimes\sigma^0 - \sigma^0\otimes\sigma^z = \begin{pmatrix}0&0&0&0\\0&0&2&0\\0&2&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}2&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-2\end{pmatrix}. \tag{4}$$

H_z is always going to have a single σ^z matrix with $N-1$ identity matrices at the remaining sites. Then, we can write it as

$$H_z = -\sum_{i=0}^{N-1}\sigma^0\otimes\cdots\otimes\sigma_i^z\otimes\cdots\otimes\sigma^0. \tag{5}$$

The following pattern is observed:

- If σ^z is at the N-th site position, then $1, -1$ alternate in the diagonal, namely
$$\sigma^0\otimes\cdots\otimes\sigma^z = \mathrm{diag}(1,-1,1,-1,\cdots,1,-1)$$
- If σ^z is at the (N-1)-th site position, then $1, 1, -1, -1$ alternate in the diagonal, namely
$$\sigma^0\otimes\cdots\otimes\sigma^z\otimes\sigma^0 = \mathrm{diag}(1,1,-1,-1,\cdots,1,1,-1,-1)$$
- If σ^z is at the first site position, then $\underbrace{1,\cdots,1}_{2^{N-1}\text{ times}},\underbrace{-1,\cdots,-1}_{2^{N-1}\text{ times}}$ alternate in the diagonal, namely
$$\sigma^z\otimes\cdots\otimes\sigma^0 = \mathrm{diag}(1,1,-1,-1,\cdots,1,1,-1,-1)$$

Therefore, the j-th diagonal component of the i-th term of $-H_z$ is given by

$$F_{ij} = (-1)^{\lfloor \frac{j}{2^i} \rfloor}. \tag{6}$$

In this terms, the j-th diagonal component of $-H_z$ is

$$G_j = \sum_{i=0}^{N-1} F_{ij} = \sum_{i=0}^{N-1} (-1)^{\lfloor \frac{j}{2^i} \rfloor} \tag{7}$$

and finally,

$$\boxed{H_z = -h \operatorname{diag}\left(G_0, G_1, \cdots, G_{2^N-1}\right) = -h \operatorname{diag}\left(N, \sum_{i=0}^{N-1}(-1)^{\lfloor \frac{1}{2^i} \rfloor}, \cdots, \sum_{i=0}^{N-1}(-1)^{\lfloor \frac{2^N-2}{2^i} \rfloor}, -N\right)}. \tag{8}$$

Now, we can work on the planar Hamiltonian H_{xy}

$$H_{xy}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$H_{xy}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

$$H_{xy}^{(4)} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&2&0&0&2&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0 \\ 0&0&2&0&0&0&0&0&2&0&0&0&0&0&0&0 \\ 0&0&0&2&0&0&2&0&0&2&0&0&0&0&0&0 \\ 0&0&0&0&0&2&0&0&0&0&2&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0 \\ 0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&2&0&0&0&0&2&0&0&0&0&0 \\ 0&0&0&0&0&0&2&0&0&2&0&0&2&0&0&0 \\ 0&0&0&0&0&0&0&2&0&0&0&0&0&2&0&0 \\ 0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&2&0&0&2&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}. \tag{11}$$

We can build $H^{(n)}$ recursively by making $I \otimes H^{(n-1)}$ and then filling $(\pm 2^{n-2})$-th diagonal (the main diagonal is taken as the 0-th diagonal) with 2s from its (2^{n-2})-th to its $(2^{n-1}-1)$-th component.

As our approach is recursive, we start from the non-zero elements of $H_{xy}^{(2)}$ and build the non-zero elements of the next matrices. These matrices are symmetric, so we can work only with the lower triangular part.

From $H_{xy}^{(2)}$, $(N = 2)$ the non-zero entries are

$$L_2 = \{(2,1)\}. \tag{12}$$

For the next matrix, $(N = 3)$,

$$L_3 = \{(2,1), (2+2^{N-1}, 1+2^{N-1}), (2^{N-1}, 2^{N-2}), (2^{N-1}+1, 2^{N-2}+1)\},$$

$$L_3 = \{(2,1), (6,5), (4,2), (5,3)\}. \tag{13}$$

For the next matrix, $(N = 4)$,

$$L_4 = \{(2,1), (6,5), (4,2), (5,3), (10,9), (14,13), (12,10), (13,11), (8,4), (9,5), (10,6), (11,7)\}. \tag{14}$$

The red color indicates copying the matrix in the bottom right part of the identity whereas blue accounts for filling the -2^{n-1} diagonal.

In general, the non-zero elements of $H_{xy}^{(N)}$, which have a value of $2J$, are located at

$$\boxed{L_N = \{L_{N-1}, \{L_{N-1} \oplus (2^{N-1}, 2^{N-1})\}, \{(2^{N-1}+j, 2^{N-2}+j)\}_{j=0,1,\cdots,2^{N-2}}\}.} \tag{15}$$

Finally, the planar Hamiltonian can be built as

$$\boxed{[H_{xy}]_{ij} = 2J\delta_{(i,j),L_N}} \tag{16}$$