# Peristaltic transport in the uterine cavity

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## Contents

## Remarks

The appendix contains most of the code snippets we used. The Python notebook wherein they all rest is available upon request. The B-side of the appendix contains a section dedicated to the use of generative AI and how three Large Language Models perform on this task.

## Question 1

For the continuity equation: we have constant density, so we are left with

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Because we are dealing with a slender tube, $\frac{h_0}{L} \gg 1$ so we can neglect flow in y:

$$\frac{v_x}{L} \sim \frac{v_y}{h_0} \quad \Rightarrow \quad v_y \sim \frac{h_0}{L} v_x$$

However, that does not mean we can do away with $\frac{\partial v_y}{\partial y}$, which will be an important term later on when using the wall boundary condition.

For momentum conservation: because the flow is quasi-steady, we can neglect temporal derivatives. Gravity is also neglected.

We are left only with momentum in the x-direction:

$$v_x \frac{\partial v_x}{\partial x} = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

because we are dealing with a slender tube we can neglect variation in x:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

From the y-momentum equation we obtain that the pressure is uniform in $y$, and is thus only a function of longitudinal distance and time:

$$0 = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right)$$

However, since $v_y << v_x$, it simplifies to

$$\frac{\partial P}{\partial y} = 0$$

## Question 2

We define the following non-dimensional variables:

$$\begin{cases} \text{Wall height:} & \eta(\xi, \tau) = \dfrac{h(x,t)}{h_0} \\ \text{y-variable:} & \hat{y} = \dfrac{y}{h_0} \\ \text{x-variable:} & \xi = \dfrac{x}{L} \\ \text{Time:} & \tau = \omega t \\ \text{Pressure:} & \Pi = \dfrac{p}{p_c} \\ \text{Velocities:} & \hat{v}_x = \dfrac{v_x}{U_x}, \quad \hat{v}_y = \dfrac{v_y}{U_y} \end{cases}$$

For the momentum equation:

$$\frac{\partial (p_c \Pi)}{\partial (\xi L)} = \mu \frac{\partial^2 (\hat{v}_x U_x)}{\partial (\hat{y} h_0)^2};$$

$$\frac{p_c}{h_0} \frac{\partial \Pi}{\partial \xi} = \mu \frac{U_x}{{h_0}^2} \frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2};$$

$$\frac{\partial \Pi}{\partial \xi} = \mu \frac{L U_x}{{h_0}^2 p_c} \frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2};$$

To obtain the characteristic pressure $p_c$ we leave the momentum equation devoid of any parameters, which means:

$$\mu \frac{LU_x}{{h_0}^2 p_c} = 1 \Rightarrow p_c = \mu \frac{LU_x}{{h_0}^2}$$

Then, our adimensional momentum equation is:

$$\boxed{\frac{\partial \Pi}{\partial \xi} = \frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2}} \tag{1}$$

Now, for the continuity equation:

$$\frac{\partial(\hat{v}_x U_x)}{\partial(\xi L)} + \frac{\partial(\hat{v}_y U_y)}{\partial(\hat{y} h_0)} = 0 \Rightarrow \frac{U_x}{L}\frac{\partial(\hat{v}_x)}{\partial \xi} + \frac{U_y}{h_0}\frac{\partial \hat{v}_y}{\partial \hat{y}} = 0$$

We can do away with parameters in this equation too, conveniently so since we have no information regarding velocity scales. However, we do know $U_y \sim \frac{h_0}{L} U_x$, so the above equation can be transformed to:

$$\frac{U_x}{L}\frac{\partial(\hat{v}_x)}{\partial \xi} + \frac{U_x}{L}\frac{\partial \hat{v}_y}{\partial \hat{y}}$$

So, we can factor out $\frac{U_x}{L}$ and our continuity equation is now:

$$\boxed{\frac{\partial \hat{v}_x}{\partial \xi} + \frac{\partial \hat{v}_y}{\partial \hat{y}} = 0} \tag{2}$$

## Question 3

Integrating along $y$ at a section $\xi$:

$$\int_0^\eta \frac{\partial \hat{v}_x}{\partial \xi}\, d\hat{y} + \int_0^\eta \frac{\partial \hat{v}_y}{\partial \hat{y}}\, d\hat{y} = \underbrace{\frac{d}{d\xi}\int_0^\eta \hat{v}_x\, d\hat{y}}_{\frac{1}{2}\,\hat{q}(\xi,\tau) \text{ as defined}} + \frac{d}{d\hat{y}}\int_0^\eta \hat{v}_y\, d\hat{y} \quad ;$$

Now, because $\eta$ is a function of $\xi$, Leibniz rule must be used in the first integral. Also, Fundamental Theorem of Calculus on the second one.

$$\int_0^\eta \frac{\partial \hat{v}_x}{\partial \xi}\, d\hat{y} + \hat{v}_x(y=\eta)\frac{\partial \eta}{\partial \xi} - \hat{v}_x(y=0)\frac{\partial 0}{\partial \xi} + \hat{v}_y(y=\eta) - \hat{v}_y(y=0) = 0$$

Because of the no-slip boundary condition, $\hat{v}_y(y=\eta)$ is just the velocity at the wall boundary, that is, $\frac{\partial \eta}{\partial \tau}$.

$$\Rightarrow \quad \frac{1}{2}\frac{\partial \hat{q}}{\partial \xi} + \frac{\partial \eta}{\partial \tau} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \hat{q}}{\partial \xi} = -2\frac{\partial \eta}{\partial \tau}} \quad \Rightarrow \text{flow is driven by the height variations of the wall.} \tag{3}$$

## Question 4

We need to integrate with respect to $\hat{y}$ at a fixed $\xi$ the momentum equation. So this is what we will use:

$$
\begin{cases}
\dfrac{\partial \Pi}{\partial \xi} = \dfrac{\partial^2 \hat{v}_x}{\partial \hat{y}^2} \\
\dfrac{\partial \Pi}{\partial \xi} \text{ is a function of } \xi, \tau, \text{ a result obtained before } (p = p(x,t)) \\
\hat{q}(\xi,\tau) = 2 \displaystyle\int_0^{\eta(\xi,\tau)} \hat{v}_x \, d\hat{y}
\end{cases}
$$

First integration:

$$
\frac{\partial \Pi}{\partial \xi} \int d\hat{y} = \int \frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2} \, d\hat{y} \quad ; \quad \frac{\partial \Pi}{\partial \xi} \, \hat{y} = \frac{\partial \hat{v}_x}{\partial \hat{y}} + C
$$

$$
\text{at } \hat{y} = 0, \quad \frac{\partial \hat{v}_x}{\partial \hat{y}} = 0 \quad \text{because of symmetry.} \quad \Rightarrow \quad 0 = 0 + C \quad \Rightarrow \quad C = 0
$$

Integrating again:

$$
\frac{\partial \Pi}{\partial \xi} \frac{\hat{y}^2}{2} = \hat{v}_x + C_1 \quad ; \quad \text{using no-slip condition} \hat{v}_x(\hat{y} = \eta) = 0:
$$

$$
\frac{\partial \Pi}{\partial \xi} \frac{\eta^2}{2} = C_1 \quad \Rightarrow \quad \frac{1}{2} \frac{\partial \Pi}{\partial \xi} (\hat{y}^2 - \eta^2) = \hat{v}_x
$$

Now we want to relate $\Pi$ with $\hat{q}$ so we will integrate again from $\eta$ to 0 as defined:

$$
\frac{1}{2} \frac{\partial \Pi}{\partial \xi} \left( \frac{\hat{y}^3}{3} - \hat{y}\eta^2 \right) \Big|_{\hat{y}=0}^{\eta} = \frac{1}{2} \hat{q}(\xi,\tau)
$$

Thus, the resulting relation will be:

$$
\frac{\partial \Pi}{\partial \xi} \left( \frac{\eta^3}{3} - \eta^3 \right) = \hat{q}(\xi,\tau) \quad \Rightarrow \quad \boxed{\frac{\partial \Pi}{\partial \xi} = -\frac{3}{2\eta^3} \hat{q}(\xi,\tau)} \tag{4}
$$

## Question 5

We will be using equations 3 and 4. Having no boundary conditions for $\hat{q}$ we will differentiate it and plug it in equation 3. Also, the adimensional wave speed $c$ will now be introduced.

$$
\begin{cases}
\dfrac{\partial \Pi}{\partial \xi} = -\dfrac{3}{2\eta^3} \, \hat{q}(\xi,\tau) \\
\dfrac{\partial \hat{q}}{\partial \xi} = -2 \, \dfrac{\partial \eta}{\partial \tau}
\end{cases}
\qquad\qquad
\begin{cases}
c = \dfrac{\lambda}{2\pi L} \\
k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{2\pi L c} = \dfrac{1}{Lc}
\end{cases}
$$

$$
\begin{aligned}
\hat{q}(\xi,\tau) = -\frac{2}{3} \eta^3 \frac{\partial \Pi}{\partial \xi} \quad ; \quad \frac{\partial \hat{q}}{\partial \xi} &= -\frac{2}{3} \left( 3\eta^2 \eta' \frac{\partial \Pi}{\partial \xi} + \eta^3 \frac{\partial^2 \Pi}{\partial \xi^2} \right) \\
&= -2\eta^2 \left( \eta' \frac{\partial \Pi}{\partial \xi} + \frac{\eta^3}{3} \frac{\partial^2 \Pi}{\partial \xi^2} \right) \\
&= -2 \, \frac{\partial \eta}{\partial \tau} = -2a \cos\left( \tau - \frac{\xi}{c} \right)
\end{aligned}
$$

$$
\eta^2 \left[ \eta' \frac{\partial \Pi}{\partial \xi} + \frac{\eta}{3} \frac{\partial^2 \Pi}{\partial \xi^2} \right] = a \cos\left( \tau - \frac{\xi}{c} \right)
$$

Then, we are left with equation 5, where $\eta' = \frac{\partial \eta}{\partial \xi}$

$$\boxed{\frac{\eta^3}{3}\frac{\partial^2 \Pi}{\partial \xi^2} + \eta^2 \eta' \frac{\partial \Pi}{\partial \xi} - a\cos\left(\tau - \frac{\xi}{c}\right) = 0} \tag{5}$$

# Question 6

Integrating once in $\xi$. The first term will be integrated by parts once, and one of the terms will cancel out with the second one as follows:

$$\begin{aligned}
&\frac{\eta^3}{3}\frac{\partial^2 \Pi}{\partial \xi^2} + \eta^2 \eta' \frac{\partial \Pi}{\partial \xi} - a\cos\left(\tau - \frac{\xi}{c}\right) = 0 \\
&\frac{1}{3}\int \eta^3 \frac{\partial^2 \Pi}{\partial \xi^2}\, d\xi + \int \eta^2 \eta' \frac{\partial \Pi}{\partial \xi}\, d\xi - a\int \cos\left(\tau - \frac{\xi}{c}\right)\, d\xi = \\
&\begin{Bmatrix} (1) \\ u = \eta^3, \quad du = 3\eta^2 \eta'\, d\xi \\ dv = \dfrac{\partial^2 \Pi}{\partial \xi^2}\, d\xi, \quad v = \dfrac{\partial \Pi}{\partial \xi} \end{Bmatrix} \quad \Rightarrow \quad \frac{\partial \Pi}{\partial \xi}\eta^3 - 3\int \eta^2 \eta' \frac{\partial \Pi}{\partial \xi}\, d\xi \\
&= \frac{1}{3}\frac{\partial \Pi}{\partial \xi}\eta^3 - a\int \cos\left(\tau - \frac{\xi}{c}\right)\, d\xi \\
&= \frac{1}{3}\frac{\partial \Pi}{\partial \xi}\eta^3 + ac\sin\left(\tau - \frac{\xi}{c}\right) + F(\tau)
\end{aligned}$$

So we arrive to:

$$\boxed{\frac{1}{3}\frac{\partial \Pi}{\partial \xi}\eta^3 + ac\sin\left(\tau - \frac{\xi}{c}\right) + F(\tau) = 0} \tag{6}$$

# Question 7

Now, equations 4 and 6 will be used:

$$\begin{cases} \dfrac{1}{3}\eta^3 \dfrac{\partial \Pi}{\partial \xi} + ac\sin\left(\tau - \dfrac{\xi}{c}\right) + F(\tau) = 0 \\ \dfrac{\partial \Pi}{\partial \xi} = -\dfrac{3}{2\eta^3}\hat{q}(\xi, \tau) \end{cases}$$

$$\begin{aligned}
&\frac{1}{3}\eta^3 \cdot \left(-\frac{3}{2\eta^3}\hat{q}(\xi, \tau)\right) + ac\sin\left(\tau - \frac{\xi}{c}\right) + F(\tau) = 0 \\
&-\frac{1}{2}\hat{q}(\xi, \tau) + ac\sin\left(\tau - \frac{\xi}{c}\right) + F(\tau) = 0
\end{aligned}$$

We obtain the following expression for adimensional flow:

$$\boxed{\hat{q}(\xi, \tau) = 2F(\tau) + 2ac\sin\left(\tau - \frac{\xi}{c}\right)} \tag{7}$$

## Question 8

Integrating in $\xi$ so that we can use $\Pi(1,\tau) = \Pi(0,\tau)$, and plugging equation 7 inside 4:

$$
\begin{aligned}
&\frac{\partial \Pi}{\partial \xi} = -\frac{3}{\eta^3} \cdot [F(\tau) + ac\sin(\tau - \xi/c)] \\
&\frac{\partial \Pi}{\partial \xi} + \frac{3ac}{\eta^3}\sin(\tau - \xi/c) + \frac{3}{\eta^3}F(\tau) = 0 \\
&\int_0^1 \frac{\partial \Pi}{\partial \xi} d\xi + 3ac\int_0^1 \frac{\sin(\tau - \xi/c)}{\eta^3} d\xi + 3\int_0^1 \frac{1}{\eta^3} d\xi F(\tau) = 0 \\
&\Pi(1,\tau) - \Pi(0,\tau) + 3ac\underbrace{\int_0^1 \frac{\sin(\tau - \xi/c)}{\eta^3} d\xi}_{I_1(\tau)} + 3\underbrace{\int_0^1 \frac{1}{\eta^3} d\xi}_{I_2(\tau)} F(\tau) = 0 \\
&\underbrace{ac\int_0^1 \frac{\sin(\tau - \xi/c)}{\eta^3} d\xi}_{I_1(\tau)} + \underbrace{\int_0^1 \frac{1}{\eta^3} d\xi}_{I_2(\tau)} F(\tau) = 0
\end{aligned}
$$

## Question 9

Using $\frac{\alpha L}{h_0} \sim am \ll 1$ we can aproximate $\eta$ so that we can expand $\frac{1}{\eta^3}$ using Taylor:

$$
\begin{aligned}
&\eta(\xi,\tau) = 1 + \frac{\alpha L}{h_0}\xi + a\sin(\tau - \xi/c) \sim 1 + am\xi + a\sin(\tau - \xi/c) = 1 + a(m\xi + \sin(\tau - \xi/c)) \\
&\underbrace{ac\int_0^1 \frac{\sin(\tau - \xi/c)}{(1 + a(m\xi + \sin(\tau - \xi/c)))^3} d\xi}_{I_1(\tau)} + \underbrace{\int_0^1 \frac{1}{(1 + a(m\xi + \sin(\tau - \xi/c)))^3} d\xi}_{I_2(\tau)} F(\tau) = 0
\end{aligned}
$$

We want a polynomial in powers of $a$, so we need to expand the TP around 0:

$$
\begin{aligned}
&(1 + a(m\xi + \sin(\tau - \xi/c)))^{-3} \sim 1 + (-3)\left[(1 + a(m\xi + \sin(\tau - \xi/c)))^{-4}(m\xi + \sin(\tau - \xi/c))\right]\Bigg|_{a=0} \\
&+ \frac{12}{2}\left[(1 + a(m\xi + \sin(\tau - \xi/c)))^{-5}(m\xi + \sin(\tau - \xi/c))^2\right]\Bigg|_{a=0} a^2 + \mathcal{O}(a^3) \\
&= 1 - 3(m\xi + \sin(\tau - \xi/c))a + 6(m\xi + \sin(\tau - \xi/c))^2 a^2
\end{aligned}
$$

Integrating $I_1(\tau)$ and $I_2(\tau)$ is a task we leave to `sympy`, which diligently spits out the following expressions:

$$
\begin{cases}
I_1(\tau) = 3a^2c\Big(c^2m\sin(\tau) - c^2m\sin\left(\tau - \dfrac{1}{c}\right) - cm\cos\left(\tau - \dfrac{1}{c}\right) \\
\qquad + \dfrac{c}{4}\sin(2\tau) - \dfrac{c}{4}\sin\left(2\left(\tau - \dfrac{1}{c}\right)\right) - \dfrac{1}{2}\Big) \\
\qquad + ac^2\left(-\cos(\tau) + \cos\left(\tau - \dfrac{1}{c}\right)\right) \\
I_2(\tau) = a^2\Big(-12c^2m\sin(\tau) + 12c^2m\sin\left(\tau - \dfrac{1}{c}\right) + 12cm\cos\left(\tau - \dfrac{1}{c}\right) \\
\qquad - \dfrac{3c}{2}\sin(2\tau) + \dfrac{3c}{2}\sin\left(2\tau - \dfrac{2}{c}\right) + 2m^2 + 3\Big) \\
\qquad + 3a\left(c\cos(\tau) - c\cos\left(\tau - \dfrac{1}{c}\right) - \dfrac{m}{2}\right) + 1
\end{cases}
$$

## Question 10

The expression for $F(\tau) = -\frac{I_1(\tau)}{I_2(\tau)}$ is:

$$
\frac{-3a^2c\left(c^2m\sin(\tau) - c^2m\sin\left(\tau - \frac{1}{c}\right) - cm\cos\left(\tau - \frac{1}{c}\right) + \frac{c}{4}\sin(2\tau) - \frac{c}{4}\sin\left(2\left(\tau - \frac{1}{c}\right)\right) - \frac{1}{2}\right) - ac^2\left(-\cos(\tau) + \cos\left(\tau - \frac{1}{c}\right)\right)}{a^2\left(-12c^2m\sin(\tau) + 12c^2m\sin\left(\tau - \frac{1}{c}\right) + 12cm\cos\left(\tau - \frac{1}{c}\right) - \frac{3c}{2}\sin(2\tau) + \frac{3c}{2}\sin\left(2\tau - \frac{2}{c}\right) + 2m^2 + 3\right) + 3a\left(c\cos(\tau) - c\cos\left(\tau - \frac{1}{c}\right) - \frac{m}{2}\right) + 1}
$$

and that goes into the expression for the flux:

$$
\hat{q}(\xi, \tau) = 2F(\tau) + 2ac\sin\left(\tau - \frac{\xi}{c}\right)
$$

## Question 11

First, the equation obtained in question 10 for the flow rate is expanded in powers of using `sp.series`; then the integration over a period is performed and simplified to obtain the adimensional flow rate:

$$
\hat{q}_\omega = 3a^2c\left(2c^2\cos\left(\frac{1}{c}\right) - 2c^2 + 1\right) \tag{8}
$$

which is the non-dimensional net transport over a wave period.

Therefore, the adimensional flow rate is now expressed in the form $a^2G(c)$, where $G(c) = 3c\left(2c^2\cos\left(\frac{1}{c}\right) - 2c^2 + 1\right)$.

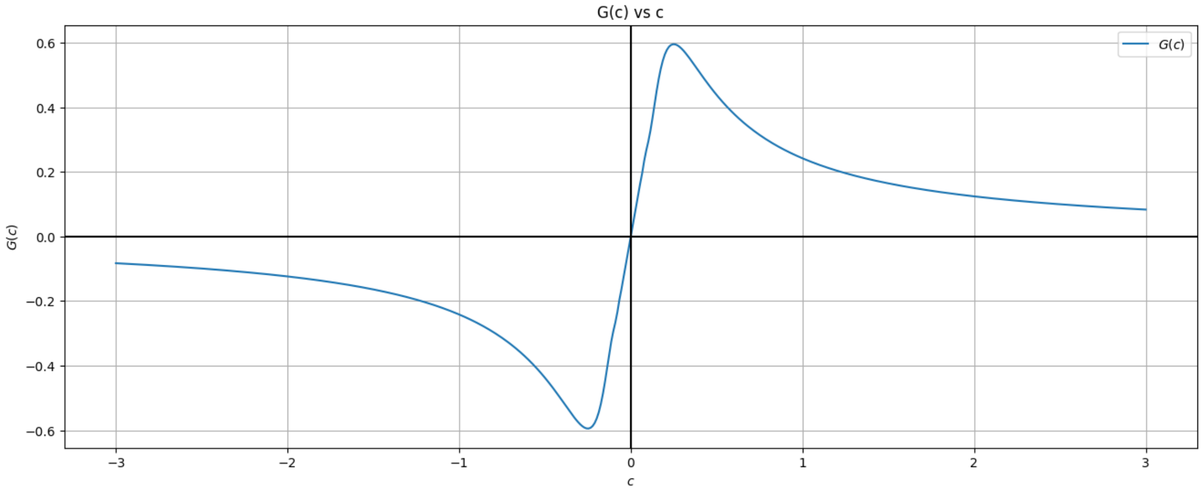


Figure 1: Plotting of $G(c)$ confirms the existence of an optimal non-dimensional flow speed $c$.

# Question 12

Optimizing $G(c)$, we find that the optimal non-dimensional wave velocity is $c = 0.25$, obviously in forward or reverse direction, as the function is odd.

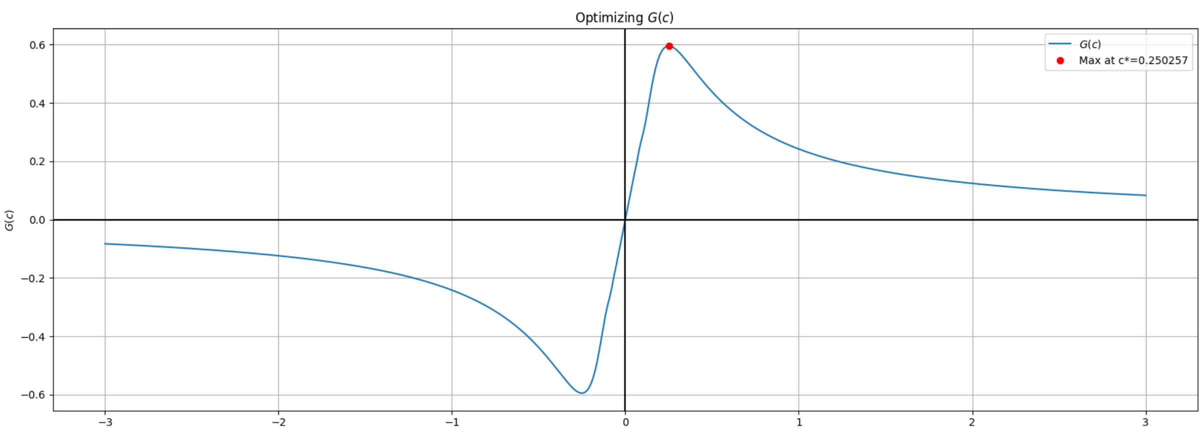


Figure 2: The optimal non-dimensional wave speed is $c = 0.25$. For this $c$, $G(c) = 0.59$.

We have that

$$\hat{q}_w = 3a^2c\left[2c^2\left(\cos\left(\frac{1}{c}\right) - 2\right) + 1\right]$$

Plugging in the expressions for the non-dimensional parameters to obtain a function of the geometrical parameters:

$$\hat{q}_\omega = 3\frac{A^2}{h_0^2}\frac{\lambda}{2\pi L}\left[\frac{1}{2}\left(\frac{\lambda}{\pi L}\right)^2\left(\cos\left(\frac{2\pi L}{\lambda}\right) - 2\right) + 1\right]$$

We will calculate the non-dimensional flux using 1) data from the article and 2) our optimal wave velocity c $\approx \frac{1}{4}$.

1) $\begin{cases} A = 0.7h_0 \\ a = \frac{A}{h_0} \\ c = \frac{\lambda}{2\pi L} \\ \lambda = 50h_0 \\ L = 2\lambda = 100h_0 \end{cases}$

$$\begin{aligned} \hat{q}_{w_1} &= \frac{0.735\,\lambda}{\pi L}\left[\frac{1}{2}\left(\frac{\lambda}{\pi L}\right)^2\left(\cos\left(\frac{2\pi L}{\lambda}\right)-2\right)+1\right] \\ &= \frac{0.735\cdot 50h_0}{\pi\cdot 100h_0}\left[\frac{1}{2}\left(\frac{50h_0}{\pi\cdot 100h_0}\right)^2\left(\cos\left(\frac{2\pi\cdot 100h_0}{50h_0}\right)-2\right)+1\right] \\ &= \frac{0.735}{2\pi}\left[\frac{1}{8\pi^2}\left(\cos(4\pi)-2\right)+1\right] = 0.1155 \end{aligned}$$

2) $\begin{cases} A = 0.7h_0 \\ a = \frac{A}{h_0} \\ c \approx \frac{1}{4} = \frac{\lambda}{2\pi L};\ \lambda = \frac{\pi}{2}L \end{cases}$

$$\begin{aligned} \hat{q}_{w_2} &= \frac{0.735}{2}\left[\frac{1}{2}\left(\frac{1}{2}\right)^2\left(\cos\left(\frac{2\pi L}{\frac{\pi}{2}L}\right)-2\right)+1\right] \\ &= \frac{0.735}{2}\left[\frac{\cos(4)-2}{8}+1\right] = 0.5500 \end{aligned}$$

Notice that the symmetric peristaltic flow is optimal when the wavelength of the peristaltic motion of the walls is approximately $\frac{\pi}{2}L$. In Eytan's article [1], $\lambda = \frac{1}{2}L$.

Now, this net flows can be dimensionalized using the characteristic pressure defined at the beginning. $\hat{q}$ adimensional was implicitly non-dimensionalized when integrating non-dimensional velocity with respect to $\hat{y}$.

That is:

$$\hat{q}(\xi,\tau) = 2\int_0^{\eta(\xi,\tau)} \hat{v}_x d\hat{y} = 2\int_0^{h(x,t)} \frac{v_x}{U_x h_0} dy = \frac{1}{U_x h_0} q(x,t)$$

We will call now our net fluxes over a period $Q_{exp}$ and $Q_{opt}$, where

$$Q_i = U_x h_0 \hat{q}_i$$

We have no data on characteristic velocities so we will use the characteristic pressure $p_c$ defined at the beginning.

$$\begin{aligned} p_c &= \frac{\mu L U_x}{h_0^2} \quad ; \quad U_x = \frac{p_c h_0^2}{\mu L} \\ Q &= \frac{p_c h_0^2}{\mu L} h_0 \hat{q} \quad ; \quad Q = \frac{p_c h_0^3}{\mu L}\hat{q} = \frac{p_c h_0^3}{\mu\cdot 100h_0}\hat{q} = \frac{p_c h_0^2}{100\mu}\hat{q} \end{aligned}$$

Where again we are using the geometrical parameters in [1], and $p_c$ is of order unity as defined previously. Therefore, there is only one unknown parameter, the dynamic viscosity $\mu$. This parameter is not used in [1] as nothing dimensional is calculated. No measures have been performed regarding the viscosity and density of human uterine fluid due to its reduced volume and inaccessibility. However, there are some measures on the cervical mucus and although its value is extremely variable throughout the menstrual cycle (from 20–200 Poise at ovulation to 1000-1200 Poise toward the end of the luteal phase) it is estimated to be 4 to 5 orders of magnitude greater than that of water at 200°C [2] (which is about 0.1347cP).

Workers in reproductive biomedicine need approximate values of this viscosity to estimate the one for their embryo transfer media, and the most commonly used estimation is $\mu = 1000\text{cP} = 1\text{kg/ms}$ [3], which is in agreement with [2].
So, using that viscosity and $h_0 = 0.5 \times 10^{-3}\text{m}$, we have simply an scaling factor of $2.5 \times 10^{-9}\text{m}^2/\text{s}$, or $2.5 \times 10^{-3}\text{mm}^2/\text{s}$:

$$\begin{cases} Q_{\text{exp}} = 2.8875 \times 10^{-4}\ \text{mm}^2/\text{s} \\ Q_{\text{opt}} = 1.3750 \times 10^{-3}\ \text{mm}^2/\text{s} \end{cases}$$

The difference between the empirical and optimal dimensional fluxes is very noticeable, of almost one order of magnitude.

# Question 13

For the final part, we want to determine how accurate the $F(\tau)$ approximation for the flux (eq. 7) is. To that end, the original differential equation (eq. 5) will be numerically solved with the provided Python script and then plotted. This result will then be compared with the plot obtained from equation 7.

For this part we will be using the parameters from the article, and $a$ and $\tau$ will be the variable parameters under study:

$$\begin{cases} a = [0.025, 0.1, 0.2, 0.3] \\ \tau = [0, \frac{\pi}{4}, \frac{\pi}{2}, \pi] \\ \alpha = 0.035 \\ h_0 = 0.5 \\ \lambda = 50h_0 \\ L = 2\lambda = 100h_0 \\ c = \frac{\lambda}{2\pi L} = \frac{50h_0}{2\pi \times 100h_0} = \frac{1}{4\pi} \\ m = 1 \end{cases}$$

## Numerical approximation using banded matrix approach

We want to prepare the equation from 5 to be solved with `solve_banded`.

$$\frac{\eta^3}{3}\frac{\partial^2 \Pi}{\partial \xi^2} + \eta^2 \frac{\partial \eta}{\partial \xi}\frac{\partial \Pi}{\partial \xi} - a\cos\left(\tau - \frac{\xi}{c}\right) = 0$$

So $a(\xi) = \frac{\eta^3}{3}$, $b(\xi) = \eta^2 \frac{\partial \eta}{\partial \xi}$ and $c(\xi) = -a\cos\left(\tau - \frac{\xi}{c}\right)$

The output of this script, which is the pressure and its derivative can be seen in Fig. 3. To observe how the wave travels we can graph at different $\tau$ values (Fig. 4). The necessary asymmetry needed to obtain a net flux, allowing transport in a direction instead of a back and forth oscillatory movement, can be verified. Lastly, we graph different values of the adimensional amplitude $a$ to observe its effect on the flux (Fig. 5).

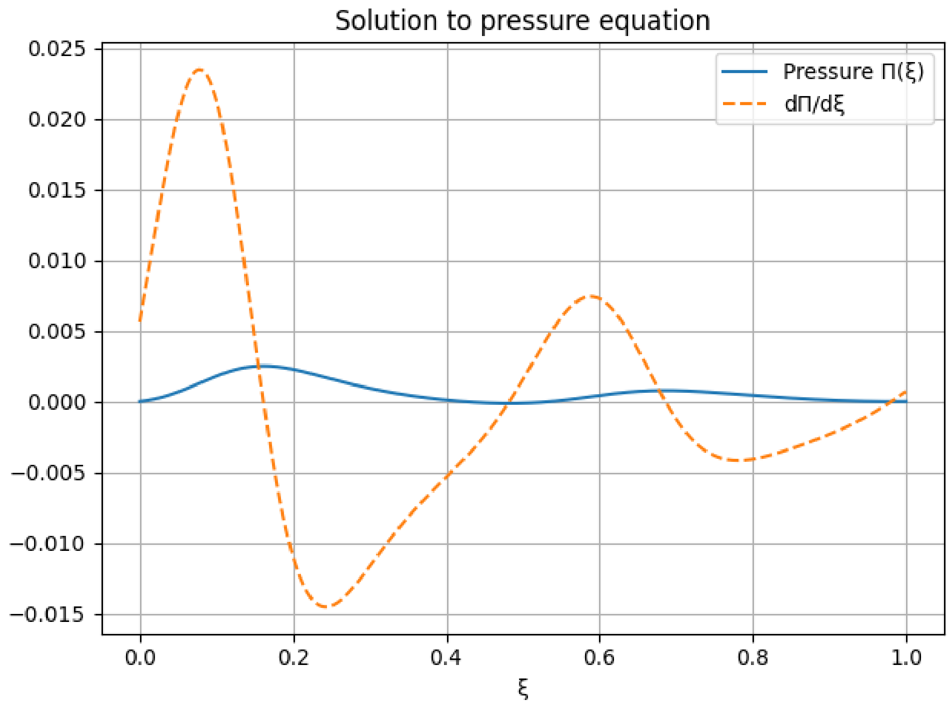


Figure 3: Numerical integration of the pressure equation.

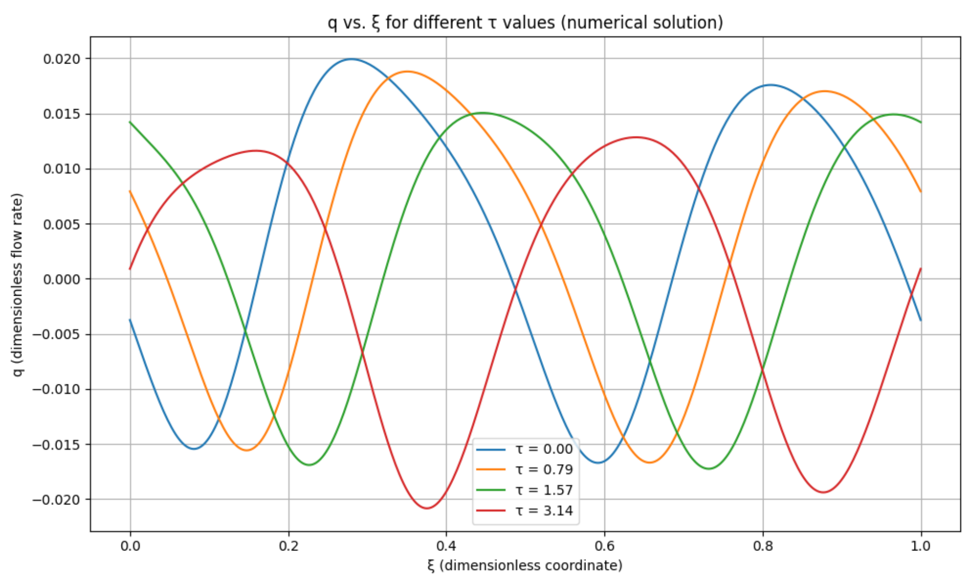


Figure 4: Graphing the numerical approximation at distinct times allows visualization of the travelling fluid wave. Here $a = 0.1$

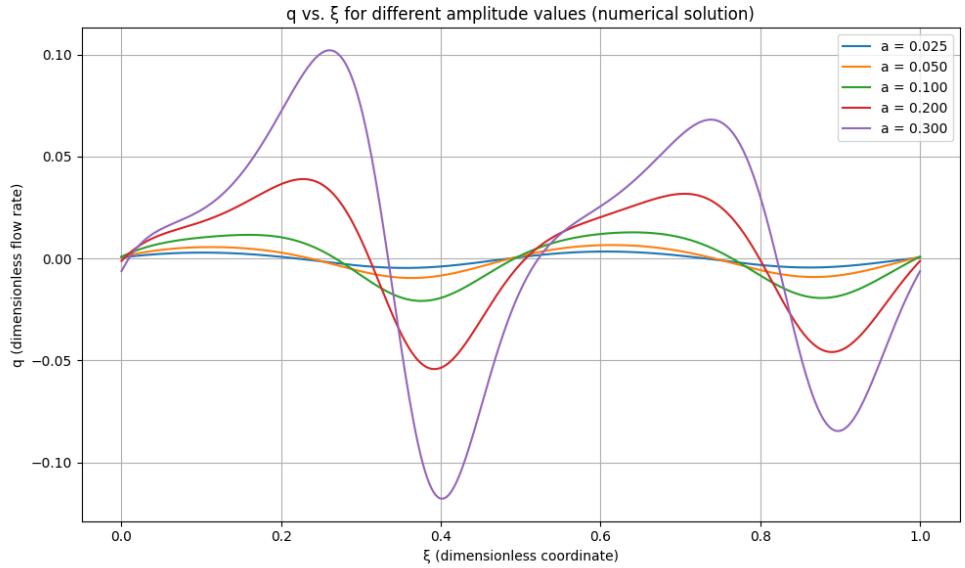


Figure 5: Solution of the pressure equation using different wave amplitudes. Major differences in magnitude start to appear when $a > 0.1$, and it seems like from that point on the wave starts transporting backwards in relation to the signed wave speed. That is, the minimums start having bigger absolute value than the maximums. This does not happen at $\tau = 0$ for $a = 1$, see Fig. 4. To confirm this hypothesis, different values of $\tau$ should be visualized.

## $F(\tau)$ integral approximation approach

Now, we will plot the approximated analytical expression obtained in Question 10, for different $\tau$ values and for different $a$ values.

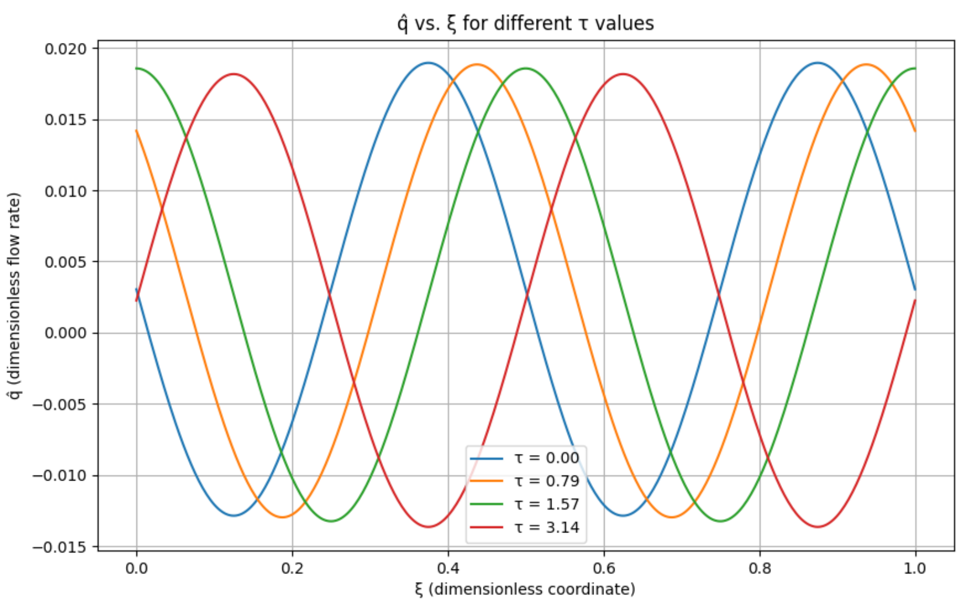


Figure 6: Approximated $\hat{q}$ for different $\tau$ values.

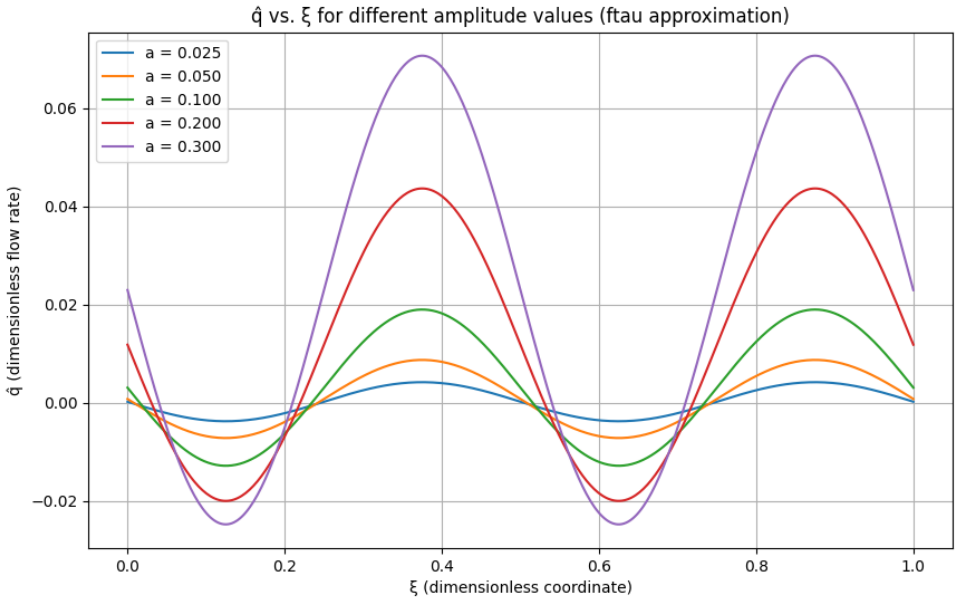


Figure 7: Approximated $\hat{q}$ for different $a$ values. Here, the difference in behavior observed in the numerical approximation when increasing $a$ (Fig. 5) is not present, and the maximums are always greater in magnitude than the minimums.

## Estimation of the fittest $a$ value

$a$ must be much smaller than 1 to obtain a reasonably physical behaviour out of this model. However, further analysis needs to be performed to determine which value is the most optimal. Plotting different values of $a$, it can be verified that when $a$ starts to diverge from 0.1, the numerical solution and the $F(\tau)$ solution start to become out of phase (Fig. 8). Also, the numerical solution starts to diverge greatly and to show unphysical behaviour (Fig. 9). Thus, it seems like $a \approx 0.1$ is the best choice (Fig. 10).

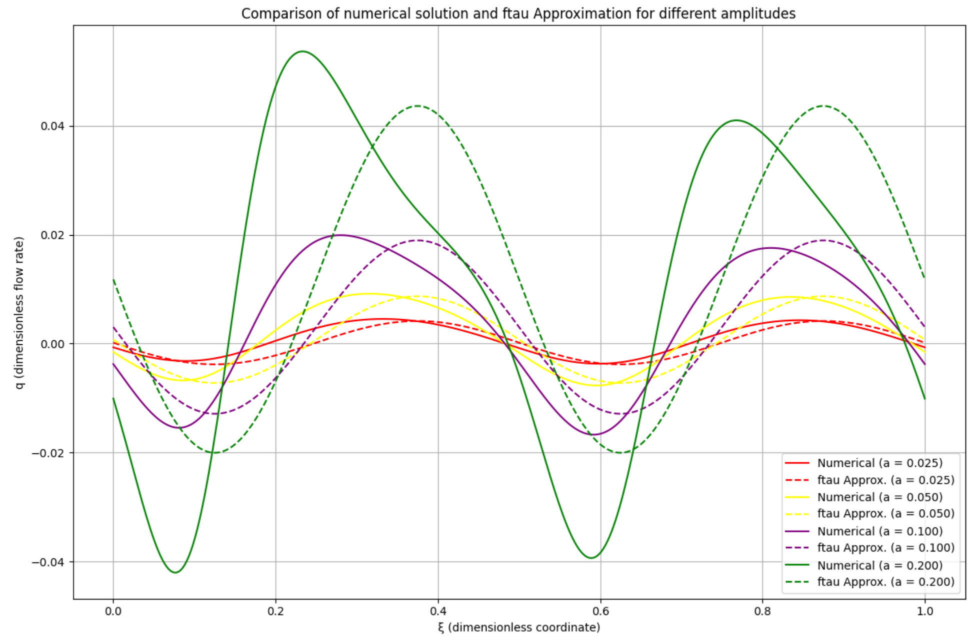


Figure 8: Comparison of both methods for different $a$ values. Here the shift when $a > 1$ is best observed, although $a = 0.3$ has been left out for better readability as it reaches a maximum value of 0.1.

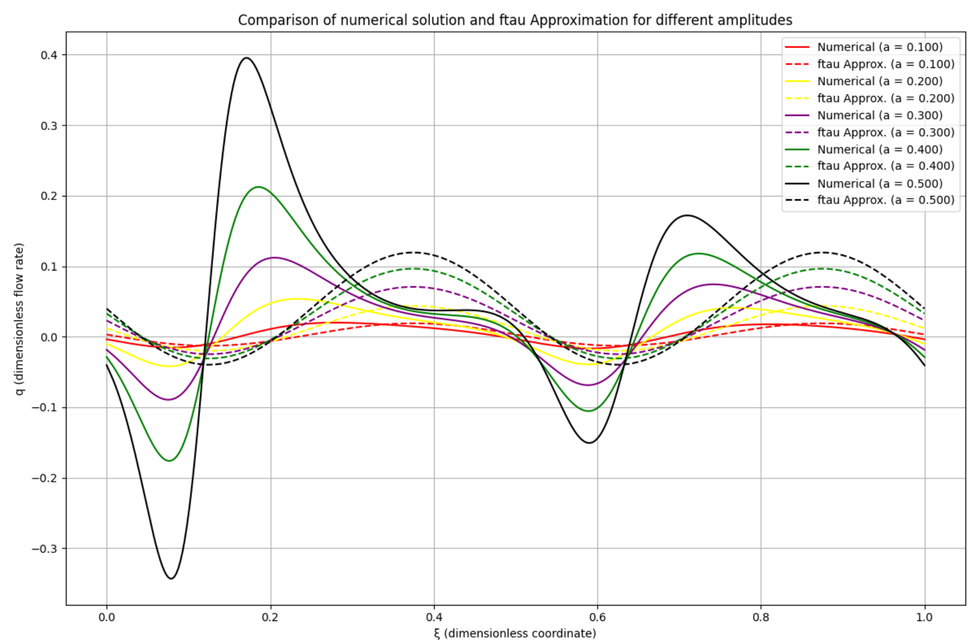


Figure 9: To see what happens when $a$ gets closer to 1, we plotted for values of $a > 0.1$. From this point on, the numerical approximation blows out, surpassing values of 100 for $a = 1$, which is a completely unreasonable value for this problem. For $a \geq 0.3$ the behaviour of both approximations already seems to diverge most noticeably.

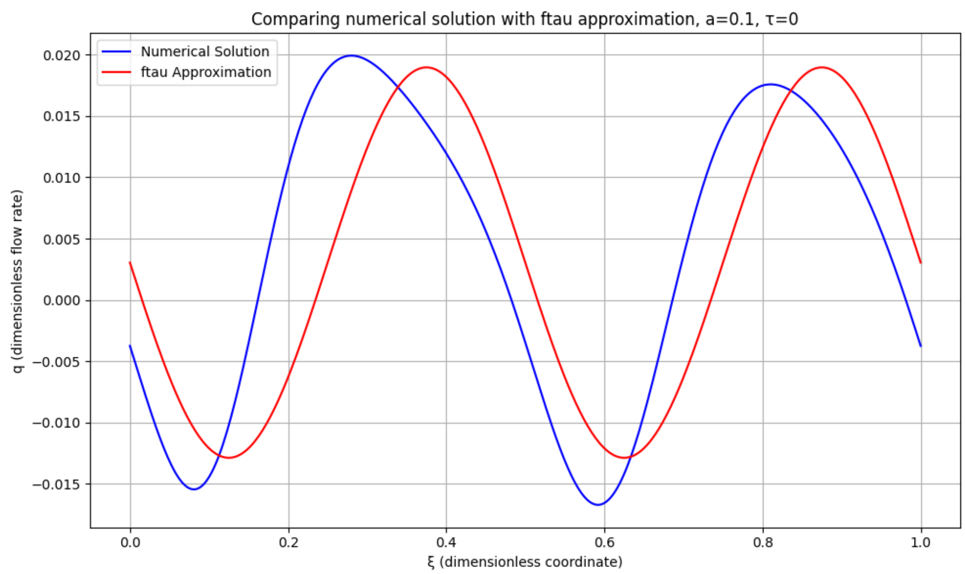


Figure 10: Comparison for $a = 0.1$. Here, the differences in phase and critical values between both approximations seem to be the smallest.

# Bibliography


[1] Osnat Eytan, Ariel J Jaffa, and David Elad. "Peristaltic Flow in a Tapered Channel: Application to Embryo Transport within the Uterine Cavity". In: *Medical Engineering & Physics* 23.7 (Sept. 2001), pp. 475–484. ISSN: 13504533. DOI: 10.1016/S1350-4533(01)00078-9.

[2] Kristin M. Myers and David Elad. "Biomechanics of the Human Uterus". In: *WIREs Systems Biology and Medicine* 9.5 (Sept. 2017), e1388. ISSN: 1939-5094, 1939-005X. DOI: 10.1002/wsbm.1388.

[3] Michael L Reed and Al-Hasen Said. "Estimation of Embryo Transfer Media Viscosity and Consideration of Its Effect on Media and Uterine Fluid Interactions". In: *Reproductive BioMedicine Online* 39.6 (Dec. 2019), pp. 931–939. ISSN: 14726483. DOI: 10.1016/j.rbmo.2019.07.034.

# A Codes

## Question 9

```
import sympy as sp

xi,tau,a,c,m = sp.symbols('xi tau a c m', real=True)
phi = tau - xi/c
eta1 = m*xi + sp.sin(phi)
series = 1 - 3*a*eta1 + 6*a**2*eta1**2

I1 = sp.integrate(a*c*sp.sin(phi)*series, (xi,0,1))
I2 = sp.integrate(series, (xi,0,1))

I1cut = sp.series(I1, a, 0, 3).removeO()
I2cut = sp.series(I2, a, 0, 3).removeO()

I1simp=sp.trigsimp(I1cut)
I2simp=sp.trigsimp(I2cut)

display("I1 ")
#display(I1cut)
display(I1simp)
display("I2 ")
#display(I2cut)
display(I2simp)

ftau=-I1simp/I2simp
display("ftau ")
display(ftau)
```

## Question 11

```
import sympy as sp

xi, tau, a, c, m = sp.symbols('xi tau a c m', real=True)

qhattaylor = sp.series(qhat, a, 0, 3).removeO()
qhatexp=sp.expand_trig(qhattaylor)

q_omega = (1/(2*sp.pi))*sp.integrate(qhatexp,(tau,0,2*sp.pi))
q_omegasimp=sp.simplify(sp.trigsimp(q_omega))


display(qhattaylor)
display(qhatexp)
display(sp.Eq(sp.Symbol('\hat{q}_{\omega}'), q_omegasimp))
```

## Question 12

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.optimize as opt
import sympy as sp

m_val = 1.0

def Gc_fixed(c):
    return G_c_func(c, m_val)
#negative to maximi<e
def neg_Gc_fixed(c):
    return -Gc_fixed(c)
```

```python

res = opt.minimize_scalar(neg_Gc_fixed, bounds=(-3, 3), method='bounded')

c_opt = res.x
G_opt = Gc_fixed(c_opt)

print(f"Optimal wave speed c*: {c_opt:.6f}")
print(f"Maximum G(c*): {G_opt:.6f}")


c_vals = np.linspace(-3, 3, 500)
G_vals = Gc_fixed(c_vals)

plt.figure(figsize=(16,6))
plt.plot(c_vals, G_vals, label=r'$G(c)$')
plt.plot(c_opt, G_opt, 'ro', label=f'Max at c*={c_opt:.6f}')
plt.axhline(0, color='black', linestyle='-')
plt.axvline(0, color='black', linestyle='-')
plt.xlabel('$c$')
plt.ylabel('$G(c)$')
plt.title('Optimizing $G(c)$ ')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```

Listing 1: Maximization of $G(c)$

## Question 13

`Comparison of numerical solution and ftau approximation`

```python
import numpy as np
import matplotlib.pyplot as plt
import sympy as sp
from scipy import linalg

# Grid size
Nx = 400
xi = np.linspace(0, 1, Nx)
dx = xi[1] - xi[0]

# Parameters
a_val = 0.1
c_val = 1 / (4 * np.pi)
m_val = 1
tau_val = 0

# Function to solve the pressure equation numerically
def solve_pressure_equation(a, b, c, Pl, Pr, dx):
    A = np.zeros((3, len(a)))
    A[0, 2:] = a[2:] / dx**2 + b[2:] / (2 * dx)  # Upper diagonal
    A[1, 1:-1] = -2 / dx**2 * a[1:-1]           # Central diagonal
    A[1, 0] = 1
    A[1, -1] = 1
    A[2, :-2] = a[:-2] / dx**2 - b[:-2] / (2 * dx)  # Lower diagonal

    B = -c
    B[0] = Pl
    B[-1] = Pr

    P = linalg.solve_banded((1, 1), A, B)

    Px = np.zeros_like(P)
```

```
    Px[1:-1] = (P[2:] - P[:-2]) / (2 * dx)
    Px[0] = (-3 * P[0] + 4 * P[1] - P[2]) / (2 * dx)
    Px[-1] = (3 * P[-1] - 4 * P[-2] + P[-3]) / (2 * dx)

    return P, Px

# Numerical solution
eta = 1 + m_val * xi + a_val * np.sin(tau_val - xi / c_val)
deta_dxi = np.gradient(eta, dx)

a_xi = eta**3 / 3.0
b_xi = eta**2 * deta_dxi
c_xi = -a_val * np.cos(tau_val - xi / c_val)

Pl = 0.0  # Pressure at xi=0
Pr = 0.0  # Pressure at xi=1

P, P_xi = solve_pressure_equation(a_xi, b_xi, c_xi, Pl, Pr, dx)

def qnum(dPi_dxi, eta):
    return -(2.0 / 3.0) * (eta ** 3) * dPi_dxi

q_numerical = qnum(P_xi, eta)

# ftau approximation
xi_sym, tau, a, c, m = sp.symbols('xi tau a c m', real=True)
phi = tau - xi_sym / c
eta1 = m * xi_sym + sp.sin(phi)
series = 1 - 3 * a * eta1 + 6 * a**2 * eta1**2

I1 = sp.integrate(a * c * sp.sin(phi) * series, (xi_sym, 0, 1))
I2 = sp.integrate(series, (xi_sym, 0, 1))

I1cut = sp.series(I1, a, 0, 3).removeO()
I2cut = sp.series(I2, a, 0, 3).removeO()

I1simp = sp.trigsimp(I1cut)
I2simp = sp.trigsimp(I2cut)

ftau = -I1simp / I2simp

qhat_expr = 2 * ftau + 2 * a * c * sp.sin(tau - xi_sym / c)
qhat_func = sp.lambdify((xi_sym, tau, a, c, m), qhat_expr, modules='numpy')

q_ftau = qhat_func(xi, tau_val, a_val, c_val, m_val)

# Plot both solutions
plt.figure(figsize=(10, 6))
plt.plot(xi, q_numerical, label='Numerical Solution', linestyle='-', color='blue')
plt.plot(xi, q_ftau, label='ftau Approximation', linestyle='-', color='red')
plt.xlabel('   (dimensionless coordinate)')
plt.ylabel('q (dimensionless flow rate)')
plt.title('Comparison of Numerical Solution and ftau Approximation')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```

Comparison of numerical solution and ftau approximation for different amplitudes

```
[style=mypythom]
    import numpy as np
import matplotlib.pyplot as plt
import sympy as sp
from scipy import linalg
```

```

# Grid size
Nx = 400
xi = np.linspace(0, 1, Nx)
dx = xi[1] - xi[0]

# Parameters
c_val = 1 / (4 * np.pi)
m_val = 1
tau_val = 0

# Function to solve the pressure equation numerically
def solve_pressure_equation(a, b, c, Pl, Pr, dx):
    A = np.zeros((3, len(a)))
    A[0, 2:] = a[2:] / dx**2 + b[2:] / (2 * dx)  # Upper diagonal
    A[1, 1:-1] = -2 / dx**2 * a[1:-1]            # Central diagonal
    A[1, 0] = 1
    A[1, -1] = 1
    A[2, :-2] = a[:-2] / dx**2 - b[:-2] / (2 * dx)  # Lower diagonal

    B = -c
    B[0] = Pl
    B[-1] = Pr

    P = linalg.solve_banded((1, 1), A, B)

    Px = np.zeros_like(P)
    Px[1:-1] = (P[2:] - P[:-2]) / (2 * dx)
    Px[0] = (-3 * P[0] + 4 * P[1] - P[2]) / (2 * dx)
    Px[-1] = (3 * P[-1] - 4 * P[-2] + P[-3]) / (2 * dx)

    return P, Px

# ftau approximation
xi_sym, tau, a, c, m = sp.symbols('xi tau a c m', real=True)
phi = tau - xi_sym / c
eta1 = m * xi_sym + sp.sin(phi)
series = 1 - 3 * a * eta1 + 6 * a**2 * eta1**2

I1 = sp.integrate(a * c * sp.sin(phi) * series, (xi_sym, 0, 1))
I2 = sp.integrate(series, (xi_sym, 0, 1))

I1cut = sp.series(I1, a, 0, 3).removeO()
I2cut = sp.series(I2, a, 0, 3).removeO()

I1simp = sp.trigsimp(I1cut)
I2simp = sp.trigsimp(I2cut)

ftau = -I1simp / I2simp

qhat_expr = 2 * ftau + 2 * a * c * sp.sin(tau - xi_sym / c)
qhat_func = sp.lambdify((xi_sym, tau, a, c, m), qhat_expr, modules='numpy')

# Define amplitude values and corresponding colors
amplitude_values = [0.1, 0.2, 0.3, 0.4, 0.5]
colors = ['red', 'yellow', 'purple', 'green', 'black']  # Specified color order

# Plot results for different amplitude values
plt.figure(figsize=(12, 8))

for a_val, color in zip(amplitude_values, colors):
    # Numerical solution
    eta = 1 + m_val * xi + a_val * np.sin(tau_val - xi / c_val)
    deta_dxi = np.gradient(eta, dx)

    a_xi = eta**3 / 3.0
```

```python
    b_xi = eta**2 * deta_dxi
    c_xi = -a_val * np.cos(tau_val - xi / c_val)

    Pl = 0.0  # Pressure at xi=0
    Pr = 0.0  # Pressure at xi=1

    P, P_xi = solve_pressure_equation(a_xi, b_xi, c_xi, Pl, Pr, dx)

    def qnum(dPi_dxi, eta):
        return -(2.0 / 3.0) * (eta ** 3) * dPi_dxi

    q_numerical = qnum(P_xi, eta)

    # ftau approximation
    q_ftau = qhat_func(xi, tau_val, a_val, c_val, m_val)

    # Plot both solutions with the same color
    plt.plot(xi, q_numerical, label=f'Numerical (a = {a_val:.3f})', linestyle='-', color=color)
    plt.plot(xi, q_ftau, label=f'ftau Approx. (a = {a_val:.3f})', linestyle='--', color=color)

# Add labels, title, and legend
plt.xlabel('   (dimensionless coordinate)')
plt.ylabel('q (dimensionless flow rate)')
plt.title('Comparison of numerical solution and ftau Approximation for different amplitudes')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```

# B Performance of generative AI

Although we are fully aware of its very low useful output over energy consumption ratio, generative AI was used in this report for i) text formatting from written output, ii) code debugging and generation and iii) confirmation of algebraic procedures, proving itself extremely convenient in these tasks, especially the most menial and tedious ones. Even in these cases it required a non-negligible amount of negotiation, dialogue and post-processing to overcome the stubborness of these obdurate algorithms and obtain the desired results. At times it seems like the output is limited to one of many discrete and finite paths, and trying to traverse an intermediate way becomes a frustrating task. Which is why it is unsuitable for long-range problems which require some degree of anticipation, such as this one. Here, generative AI proves useful only if the answer is already known, as the algorithm requires a good amount of guidance. At a certain point, with uses like this one, products like chatGPT are just very expensive and energy-inefficient ways of using `sympy`.

That being said, we challenged three comercial Large Language Models to solve this problem: i) ChatGPT, ii) Claude and iii) Wolfram.
The performance of these models was surprisingly good on this task, outputting mostly correct reasonings and answers until question 7. Claude is clearly the standout (despite its annoying tendency to ennumerate absolutely all of the math environments in LaTeX). Remarkably, it was the only one to correctly estimate the value of $a$, although it had to fabricate some data in the reasoning, which suggests us this is just common knowledge in this kind of procedures, and Claude just tried to reverse engineer something it knew beforehand. This was often the case also with the other models. As Claude, stated, *A typical* ***rule of thumb*** *would suggest that the approximation starts to break down when* $a > 0.1$.

AI attempts

## B.1 ChatGPT Fluid Mechanics tutor

### B.1.1 Step 1: Simplification of Navier–Stokes Equations

Starting with the incompressible Navier–Stokes equations:

$$\nabla \cdot \vec{v} = 0,$$
$$\rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \mu \nabla^2 \vec{v}.$$

Assuming quasi-steady and viscous-dominated flow:

$$\mathrm{Re} = \frac{\rho U h_0}{\mu} \ll 1, \quad \frac{h_0^2}{\nu T} \ll 1,$$

The x-momentum equation reduces to:

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2}.$$

**Step 2: Non-dimensionalization**

Define:

$$\eta = \frac{h}{h_0}, \quad \xi = \frac{x}{L}, \quad \tau = \omega t, \quad \Pi = \frac{p}{p_c}, \quad a = \frac{A}{h_0}, \quad c = \frac{\lambda}{2\pi L}$$

Define dimensionless velocity:

$$\hat{v}_x = \frac{v_x}{\omega A}, \quad \hat{y} = \frac{y}{h_0}$$

Choose $p_c = \frac{\mu \omega L^2}{h_0^2}$ to non-dimensionalize pressure.

**Step 3: Continuity Equation Integration**

From incompressibility:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Integrating over $y$ from 0 to $h(x,t)$:

$$q(x,t) = 2\int_0^{h(x,t)} v_x(x,y,t)\, dy \Rightarrow \frac{\partial q}{\partial x} + 2v_y(x,h)\frac{\partial h}{\partial x} = 0$$

In dimensionless form:

$$\frac{\partial \hat{q}}{\partial \xi} + \frac{\partial \eta}{\partial \tau} = 0$$

**Step 4: Momentum Equation Integration**

From:

$$\frac{\partial \Pi}{\partial \xi} = \frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2}$$

Assume parabolic profile:

$$\hat{v}_x(\hat{y}) = \frac{1}{2}\frac{\partial \Pi}{\partial \xi}(\hat{y}^2 - \eta^2)$$

Integrate over $[0, \eta]$ and double:

$$\hat{q} = -\frac{2}{3}\eta^3 \frac{\partial \Pi}{\partial \xi}$$

**Step 5: Combine Equations**

Substitute into continuity result:

$$\frac{\partial \hat{q}}{\partial \xi} = -\frac{\partial \eta}{\partial \tau} \Rightarrow \frac{\partial}{\partial \xi}\left(-\frac{2}{3}\eta^3 \frac{\partial \Pi}{\partial \xi}\right) = -\frac{\partial \eta}{\partial \tau}$$

**Step 6: Integrate in Space**

Integrate once:

$$-\frac{2}{3}\eta^3 \frac{\partial \Pi}{\partial \xi} = \frac{\partial \eta}{\partial \tau} \Rightarrow \frac{\partial \Pi}{\partial \xi} = -\frac{3}{2}\frac{1}{\eta^3}\frac{\partial \eta}{\partial \tau}$$

Integrate:

$$\Pi(\xi, \tau) = -\frac{3}{2}\int \frac{1}{\eta^3}\frac{\partial \eta}{\partial \tau} d\xi + F(\tau)$$

**Step 7: Flow Rate as Function of $F(\tau)$**

$$\hat{q} = -\frac{2}{3}\eta^3 \frac{\partial \Pi}{\partial \xi} = \eta^3 \left(\frac{1}{\eta^3}\frac{\partial \eta}{\partial \tau}\right) = \frac{\partial \eta}{\partial \tau}$$

**Step 8: Apply Pressure Condition**

Assume:

$$\Pi(1, \tau) = \Pi(0, \tau) \Rightarrow \int_0^1 \frac{1}{\eta^3}\frac{\partial \eta}{\partial \tau} d\xi = 0$$

Define:

$$I_1(\tau) + I_2(\tau)F(\tau) = 0$$

Solve for $F(\tau)$:

$$F(\tau) = -\frac{I_1(\tau)}{I_2(\tau)}$$

**Step 9: Expand Integrals in Taylor Series**

Use:

$$\eta = 1 + \alpha\xi + a\sin(\tau - \xi/c)$$

Expand $1/\eta^3$ and $\partial\eta/\partial\tau$ up to $a^2$ terms:

$$\frac{1}{\eta^3} \approx 1 - 3(\alpha\xi + a\sin(\tau - \xi/c)) + \ldots$$

$$\frac{\partial \eta}{\partial \tau} = a\cos(\tau - \xi/c)$$

Multiply and integrate to find:

$$I_1(\tau) = \int_0^1 a\cos(\tau - \xi/c) d\xi + \ldots$$

**Step 10: Closed Expression for Flow Rate**

Combine expansions:

$$\hat{q}(\xi, \tau) = a^2 f(\xi, \tau, c)$$

**Step 11: Time-Averaged Flow Rate**

Average over time:

$$\hat{q}_\omega = \frac{1}{2\pi} \int_0^{2\pi} \hat{q}(\xi, \tau) d\tau = a^2 G(c)$$

**Step 12: Optimal Dimensional Flow Rate**

Given from Eytan et al.:

$$G(c) \text{ maximum at } c = c^* \Rightarrow q_\omega = a^2 h_0 \omega G(c^*)$$

**Step 13: Compare to Numerical Solution**

Solve full equation numerically for different $a$ and compare with approximate $q(\xi, \tau)$. Find threshold $a$ where discrepancy arises.

## B.2 Claude

**Problem Statement**

We are studying peristaltic transport in the uterine cavity, where the uterus is modeled as a planar cavity bounded by two lateral walls symmetric with respect to a middle plane, $y = 0$. The walls diverge with a small angle $\alpha \ll 1$ with respect to the symmetry plane. The lateral displacement of the walls has the form of a traveling wave, and the instantaneous position of a wall is described by:

$$h(x,t) = h_0 + \alpha x + A \sin(\omega t - kx) \tag{9}$$

where $k = 2\pi/\lambda$ is the angular wavenumber, $\omega$ is the angular frequency, $A$ is the amplitude, and $h_0$ is half the distance between the relaxed walls at $x = 0$.

Our goal is to compute the instantaneous flow rate:

$$q(x,t) = 2 \int_0^{h(x,t)} v_x(x,y,t)\, dy \tag{10}$$

and its time average over a wave period, $T = 2\pi/\omega$:

$$q_\omega = \frac{1}{T} \int_0^T q(x,t)\, dt \tag{11}$$

**Step 1: Simplification of Navier-Stokes Equations**

The full Navier-Stokes equations for an incompressible flow are:

$$\nabla \cdot \mathbf{v} = 0 \tag{12}$$

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} \tag{13}$$

For the flow to be quasi-steady, the unsteady term must be negligible compared to the viscous term:

$$\left| \rho \frac{\partial \mathbf{v}}{\partial t} \right| \ll \left| \mu \nabla^2 \mathbf{v} \right| \tag{14}$$

This can be rewritten as:

$$\rho \frac{U}{\tau} \ll \mu \frac{U}{L^2} \tag{15}$$

where $U$ is a characteristic velocity, $\tau$ is a characteristic time scale (which would be $\tau \sim 1/\omega$), and $L$ is a characteristic length scale. This gives us:

$$\rho\omega L^2 \ll \mu \tag{16}$$

Or in terms of the Reynolds number $Re = \frac{\rho UL}{\mu}$ and the Strouhal number $St = \frac{\omega L}{U}$:

$$Re \cdot St \ll 1 \tag{17}$$

For the flow to be dominated by viscosity, the inertial terms must be negligible compared to the viscous terms:

$$|\rho(\mathbf{v} \cdot \nabla)\mathbf{v}| \ll \left|\mu\nabla^2\mathbf{v}\right| \tag{18}$$

This gives us:

$$\rho\frac{U^2}{L} \ll \mu\frac{U}{L^2} \tag{19}$$

Simplifying:

$$\frac{\rho UL}{\mu} \ll 1 \tag{20}$$

This is equivalent to:

$$Re \ll 1 \tag{21}$$

Under these assumptions, the Navier-Stokes equations reduce to the continuity equation and the Stokes equation:

$$\nabla \cdot \mathbf{v} = 0 \tag{22}$$

$$\nabla p = \mu\nabla^2\mathbf{v} \tag{23}$$

In the 2D Cartesian coordinates of our problem, these become:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{24}$$

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \tag{25}$$

$$\frac{\partial p}{\partial y} = \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) \tag{26}$$

**Step 2: Making the Problem Dimensionless**

We introduce the following dimensionless variables:

$$\eta = \frac{h}{h_0} \tag{27}$$

$$\xi = \frac{x}{L} \tag{28}$$

$$\tau = \omega t \tag{29}$$

$$\Pi = \frac{p}{p_c} \tag{30}$$

We also define:

$$a = \frac{A}{h_0} \tag{31}$$

$$c = \frac{\lambda}{2\pi L} \tag{32}$$

For velocities, we introduce dimensionless variables $\hat{v}_x$ and $\hat{v}_y$:

$$v_x = U\hat{v}_x \tag{33}$$
$$v_y = V\hat{v}_y \tag{34}$$

Where $U$ and $V$ are characteristic velocities that need to be determined.

The position of the wall in dimensionless form becomes:

$$\eta(\xi,\tau) = 1 + \alpha\frac{L}{h_0}\xi + a\sin\left(\tau - \frac{\xi}{c}\right) \tag{35}$$

From the continuity equation and the scaling of $x$ and $y$, we have:

$$\frac{U}{L} \sim \frac{V}{h_0} \Rightarrow V = \frac{h_0}{L}U \tag{36}$$

The characteristic velocity $U$ can be related to the wave parameters. Since the displacement of the wall is of order $A$, and the frequency is $\omega$, the characteristic velocity of the wall displacement is $A\omega$. Therefore, we can set $U = A\omega$.

The dimensionless variables transform the momentum equation as:

$$\frac{p_c}{L}\frac{\partial \Pi}{\partial \xi} = \mu U\left(\frac{1}{L^2}\frac{\partial^2 \hat{v}_x}{\partial \xi^2} + \frac{1}{h_0^2}\frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2}\right) \tag{37}$$

Since $h_0/L \ll 1$, the term $\frac{\partial^2 \hat{v}_x}{\partial \xi^2}$ is much smaller than $\frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2}$ and can be neglected. This gives:

$$\frac{p_c}{L}\frac{\partial \Pi}{\partial \xi} = \mu U\frac{1}{h_0^2}\frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2} \tag{38}$$

To eliminate parameters from this equation, we choose:

$$p_c = \frac{\mu U L}{h_0^2} \tag{39}$$

With $U = A\omega$, we get:

$$p_c = \frac{\mu A\omega L}{h_0^2} \tag{40}$$

This gives us the dimensionless momentum equation:

$$\frac{\partial \Pi}{\partial \xi} = \frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2} \tag{41}$$

The dimensionless continuity equation becomes:

$$\frac{\partial \hat{v}_x}{\partial \xi} + \frac{\partial \hat{v}_y}{\partial \hat{y}} = 0 \tag{42}$$

**Step 3: Integrating the Continuity Equation**

Starting with the dimensionless continuity equation:

$$\frac{\partial \hat{v}_x}{\partial \xi} + \frac{\partial \hat{v}_y}{\partial \hat{y}} = 0 \tag{43}$$

We integrate this equation in $\hat{y}$ from 0 to $\eta(\xi,\tau)$:

$$\int_0^{\eta(\xi,\tau)} \frac{\partial \hat{v}_x}{\partial \xi}\, d\hat{y} + \int_0^{\eta(\xi,\tau)} \frac{\partial \hat{v}_y}{\partial \hat{y}}\, d\hat{y} = 0 \tag{44}$$

$$\frac{\partial}{\partial \xi}\int_0^{\eta(\xi,\tau)} \hat{v}_x\, d\hat{y} - \frac{\partial \eta}{\partial \xi}\hat{v}_x(\xi,\eta,\tau) + \hat{v}_y(\xi,\eta,\tau) - \hat{v}_y(\xi,0,\tau) = 0 \tag{45}$$

The boundary conditions are:

$$\hat{v}_y(\xi, 0, \tau) = 0 \quad \text{(symmetry condition)} \tag{46}$$

$$\hat{v}_y(\xi, \eta, \tau) = \frac{\partial \eta}{\partial \tau} \quad \text{(kinematic boundary condition)} \tag{47}$$

The kinematic boundary condition states that the fluid velocity at the wall must match the wall's velocity. Substituting these conditions:

$$\frac{\partial}{\partial \xi} \int_0^{\eta(\xi,\tau)} \hat{v}_x \, d\hat{y} - \frac{\partial \eta}{\partial \xi} \hat{v}_x(\xi, \eta, \tau) + \frac{\partial \eta}{\partial \tau} = 0 \tag{48}$$

The dimensionless flow rate is:

$$\hat{q}(\xi, \tau) = 2 \int_0^{\eta(\xi,\tau)} \hat{v}_x \, d\hat{y} \tag{49}$$

So we can write:

$$\frac{1}{2} \frac{\partial \hat{q}}{\partial \xi} - \frac{\partial \eta}{\partial \xi} \hat{v}_x(\xi, \eta, \tau) + \frac{\partial \eta}{\partial \tau} = 0 \tag{50}$$

The no-slip condition gives us $\hat{v}_x(\xi, \eta, \tau) = 0$, so:

$$\frac{1}{2} \frac{\partial \hat{q}}{\partial \xi} + \frac{\partial \eta}{\partial \tau} = 0 \tag{51}$$

$$\frac{\partial \hat{q}}{\partial \xi} = -2 \frac{\partial \eta}{\partial \tau} \tag{52}$$

**Step 4: Integrating the Momentum Equation**

The dimensionless $\xi$-component of the momentum equation is:

$$\frac{\partial \Pi}{\partial \xi} = \frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2} \tag{53}$$

Integrating with respect to $\hat{y}$ twice:

$$\hat{v}_x = \frac{1}{2} \frac{\partial \Pi}{\partial \xi} \hat{y}^2 + C_1(\xi, \tau) \hat{y} + C_2(\xi, \tau) \tag{54}$$

The boundary conditions are:

$$\hat{v}_x(\xi, \eta, \tau) = 0 \quad \text{(no-slip at the wall)} \tag{55}$$

$$\left. \frac{\partial \hat{v}_x}{\partial \hat{y}} \right|_{\hat{y}=0} = 0 \quad \text{(symmetry condition)} \tag{56}$$

From the symmetry condition, we get $C_1(\xi, \tau) = 0$. From the no-slip condition:

$$0 = \frac{1}{2} \frac{\partial \Pi}{\partial \xi} \eta^2 + C_2(\xi, \tau) \tag{57}$$

$$C_2(\xi, \tau) = -\frac{1}{2} \frac{\partial \Pi}{\partial \xi} \eta^2 \tag{58}$$

Thus:

$$\hat{v}_x = \frac{1}{2} \frac{\partial \Pi}{\partial \xi} (\hat{y}^2 - \eta^2) \tag{59}$$

Now, we compute the dimensionless flow rate:

$$\hat{q}(\xi,\tau) = 2\int_0^\eta \hat{v}_x \, d\hat{y} \tag{60}$$

$$= 2\int_0^\eta \frac{1}{2}\frac{\partial \Pi}{\partial \xi}(\hat{y}^2 - \eta^2)\, d\hat{y} \tag{61}$$

$$= \frac{\partial \Pi}{\partial \xi}\left[\frac{2\hat{y}^3}{3} - 2\eta^2\hat{y}\right]_0^\eta \tag{62}$$

$$= \frac{\partial \Pi}{\partial \xi}\left[\frac{2\eta^3}{3} - 2\eta^3\right] \tag{63}$$

$$= \frac{\partial \Pi}{\partial \xi}\left[-\frac{4\eta^3}{3}\right] \tag{64}$$

Therefore:

$$\hat{q}(\xi,\tau) = -\frac{4}{3}\eta^3\frac{\partial \Pi}{\partial \xi} \tag{65}$$

**Step 5: Relating Pressure and Wall Position**

From step 3, we have:

$$\frac{\partial \hat{q}}{\partial \xi} = -2\frac{\partial \eta}{\partial \tau} \tag{66}$$

From step 4, we have:

$$\hat{q}(\xi,\tau) = -\frac{4}{3}\eta^3\frac{\partial \Pi}{\partial \xi} \tag{67}$$

Combining these equations:

$$\frac{\partial}{\partial \xi}\left(-\frac{4}{3}\eta^3\frac{\partial \Pi}{\partial \xi}\right) = -2\frac{\partial \eta}{\partial \tau} \tag{68}$$

$$-\frac{4}{3}\frac{\partial}{\partial \xi}\left(\eta^3\frac{\partial \Pi}{\partial \xi}\right) = -2\frac{\partial \eta}{\partial \tau} \tag{69}$$

$$\frac{\partial}{\partial \xi}\left(\eta^3\frac{\partial \Pi}{\partial \xi}\right) = \frac{3}{2}\frac{\partial \eta}{\partial \tau} \tag{70}$$

**Step 6: Integrating the Differential Equation**

Now, we integrate the equation:

$$\frac{\partial}{\partial \xi}\left(\eta^3\frac{\partial \Pi}{\partial \xi}\right) = \frac{3}{2}\frac{\partial \eta}{\partial \tau} \tag{71}$$

Integrating with respect to $\xi$:

$$\eta^3\frac{\partial \Pi}{\partial \xi} = \frac{3}{2}\int \frac{\partial \eta}{\partial \tau}\, d\xi + F(\tau) \tag{72}$$

$$\tag{73}$$

Where $F(\tau)$ is an integration constant that depends on $\tau$.
If we substitute the expression for $\eta$:

$$\eta(\xi,\tau) = 1 + \alpha\frac{L}{h_0}\xi + a\sin\left(\tau - \frac{\xi}{c}\right) \tag{74}$$

Then:

$$\frac{\partial \eta}{\partial \tau} = a\cos\left(\tau - \frac{\xi}{c}\right) \tag{75}$$

$$\int \frac{\partial \eta}{\partial \tau}\, d\xi = \int a\cos\left(\tau - \frac{\xi}{c}\right)\, d\xi \tag{76}$$

$$= ac\int \cos\left(\tau - \frac{\xi}{c}\right)\, d\left(\tau - \frac{\xi}{c}\right) \tag{77}$$

$$= -ac\sin\left(\tau - \frac{\xi}{c}\right) + K \tag{78}$$

Since we're integrating with respect to $\xi$, the constant $K$ can be absorbed into $F(\tau)$. Thus:

$$\eta^3 \frac{\partial \Pi}{\partial \xi} = -\frac{3}{2}ac\sin\left(\tau - \frac{\xi}{c}\right) + F(\tau) \tag{79}$$

**Step 7: Expressing the Flow Rate in Terms of $F(\tau)$**

From step 4, we have:

$$\hat{q}(\xi, \tau) = -\frac{4}{3}\eta^3 \frac{\partial \Pi}{\partial \xi} \tag{80}$$

Using the expression from step 6:

$$\hat{q}(\xi, \tau) = -\frac{4}{3}\left[-\frac{3}{2}ac\sin\left(\tau - \frac{\xi}{c}\right) + F(\tau)\right] \tag{81}$$

$$= \frac{4}{3}\frac{3}{2}ac\sin\left(\tau - \frac{\xi}{c}\right) - \frac{4}{3}F(\tau) \tag{82}$$

$$= 2ac\sin\left(\tau - \frac{\xi}{c}\right) - \frac{4}{3}F(\tau) \tag{83}$$

**Step 8: Determining $F(\tau)$ from Pressure Conditions**

We are given that $\Pi(0, \tau) = \Pi(1, \tau)$, meaning the pressure at the two ends of the channel is equal. To use this condition, we need to integrate the expression from step 6:

$$\frac{\partial \Pi}{\partial \xi} = \frac{1}{\eta^3}\left[-\frac{3}{2}ac\sin\left(\tau - \frac{\xi}{c}\right) + F(\tau)\right] \tag{84}$$

Integrating from $\xi = 0$ to $\xi = 1$:

$$\Pi(1, \tau) - \Pi(0, \tau) = \int_0^1 \frac{1}{\eta^3}\left[-\frac{3}{2}ac\sin\left(\tau - \frac{\xi}{c}\right) + F(\tau)\right] d\xi \tag{85}$$

$$0 = -\frac{3}{2}ac\int_0^1 \frac{\sin\left(\tau - \frac{\xi}{c}\right)}{\eta^3}\, d\xi + F(\tau)\int_0^1 \frac{1}{\eta^3}\, d\xi \tag{86}$$

Let's define:

$$I_1(\tau) = -\frac{3}{2}ac\int_0^1 \frac{\sin\left(\tau - \frac{\xi}{c}\right)}{\eta^3}\, d\xi \tag{87}$$

$$I_2(\tau) = \int_0^1 \frac{1}{\eta^3}\, d\xi \tag{88}$$

Then:

$$I_1(\tau) + I_2(\tau)F(\tau) = 0 \tag{89}$$

$$F(\tau) = -\frac{I_1(\tau)}{I_2(\tau)} \tag{90}$$

**Step 9: Approximating the Integrals $I_1(\tau)$ and $I_2(\tau)$**

We need to approximate $I_1(\tau)$ and $I_2(\tau)$ using the assumptions $a \ll 1$ and $\alpha L/h_0 = am \ll 1$.

First, let's write $\eta$ in a more convenient form for the approximation:

$$\eta(\xi, \tau) = 1 + \alpha\frac{L}{h_0}\xi + a\sin\left(\tau - \frac{\xi}{c}\right) \tag{91}$$

$$= 1 + am\xi + a\sin\left(\tau - \frac{\xi}{c}\right) \tag{92}$$

where $m = \alpha L/(ah_0)$ is of order unity. We need to compute:

$$\frac{1}{\eta^3} = \frac{1}{\left[1 + am\xi + a\sin\left(\tau - \frac{\xi}{c}\right)\right]^3} \tag{93}$$

For $a \ll 1$, we can use the binomial expansion:

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots \tag{94}$$

With $x = am\xi + a\sin\left(\tau - \frac{\xi}{c}\right)$:

$$\frac{1}{\eta^3} \approx 1 - 3\left(am\xi + a\sin\left(\tau - \frac{\xi}{c}\right)\right) + 6\left(am\xi + a\sin\left(\tau - \frac{\xi}{c}\right)\right)^2 + O(a^3) \tag{95}$$

$$= 1 - 3am\xi - 3a\sin\left(\tau - \frac{\xi}{c}\right) + 6\left(a^2m^2\xi^2 + 2a^2m\xi\sin\left(\tau - \frac{\xi}{c}\right) + a^2\sin^2\left(\tau - \frac{\xi}{c}\right)\right) + O(a^3) \tag{96}$$

$$= 1 - 3am\xi - 3a\sin\left(\tau - \frac{\xi}{c}\right) + 6a^2m^2\xi^2 + 12a^2m\xi\sin\left(\tau - \frac{\xi}{c}\right) + 6a^2\sin^2\left(\tau - \frac{\xi}{c}\right) + O(a^3) \tag{97}$$

Now, we can compute $I_2(\tau)$:

$$I_2(\tau) = \int_0^1 \frac{1}{\eta^3}\, d\xi \tag{98}$$

$$\approx \int_0^1 \left[1 - 3am\xi - 3a\sin\left(\tau - \frac{\xi}{c}\right) + 6a^2m^2\xi^2 + 12a^2m\xi\sin\left(\tau - \frac{\xi}{c}\right) + 6a^2\sin^2\left(\tau - \frac{\xi}{c}\right)\right] d\xi \tag{99}$$

We compute each term:

$$\int_0^1 1\, d\xi = 1 \tag{100}$$

$$\int_0^1 -3am\xi\, d\xi = -3am \int_0^1 \xi\, d\xi = -3am \frac{\xi^2}{2}\Big|_0^1 = -\frac{3am}{2} \tag{101}$$

$$\int_0^1 -3a \sin\left(\tau - \frac{\xi}{c}\right) d\xi = -3a \int_0^1 \sin\left(\tau - \frac{\xi}{c}\right) d\xi \tag{102}$$

$$= -3a \left[-c\cos\left(\tau - \frac{\xi}{c}\right)\right]_0^1 \tag{103}$$

$$= -3ac \left[-\cos\left(\tau - \frac{1}{c}\right) + \cos(\tau)\right] \tag{104}$$

$$= 3ac \left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] \tag{105}$$

$$\int_0^1 6a^2m^2\xi^2\, d\xi = 6a^2m^2 \int_0^1 \xi^2\, d\xi = 6a^2m^2 \frac{\xi^3}{3}\Big|_0^1 = 2a^2m^2 \tag{106}$$

$$\int_0^1 12a^2m\xi \sin\left(\tau - \frac{\xi}{c}\right) d\xi = 12a^2m \int_0^1 \xi \sin\left(\tau - \frac{\xi}{c}\right) d\xi \tag{107}$$

For the last term, we can use integration by parts with $u = \xi$ and $dv = \sin\left(\tau - \frac{\xi}{c}\right) d\xi$:

$$\int_0^1 \xi \sin\left(\tau - \frac{\xi}{c}\right) d\xi = \left[-c\xi \cos\left(\tau - \frac{\xi}{c}\right)\right]_0^1 + c\int_0^1 \cos\left(\tau - \frac{\xi}{c}\right) d\xi \tag{108}$$

$$= -c\cos\left(\tau - \frac{1}{c}\right) + c\left[c\sin\left(\tau - \frac{\xi}{c}\right)\right]_0^1 \tag{109}$$

$$= -c\cos\left(\tau - \frac{1}{c}\right) + c^2\left[\sin\left(\tau - \frac{1}{c}\right) - \sin(\tau)\right] \tag{110}$$

Continuing with the remaining term:

$$\int_0^1 6a^2 \sin^2\left(\tau - \frac{\xi}{c}\right) d\xi = 6a^2 \int_0^1 \sin^2\left(\tau - \frac{\xi}{c}\right) d\xi \tag{111}$$

$$= 6a^2 \int_0^1 \frac{1 - \cos\left(2\tau - \frac{2\xi}{c}\right)}{2} d\xi \tag{112}$$

$$= 3a^2 \int_0^1 1\, d\xi - 3a^2 \int_0^1 \cos\left(2\tau - \frac{2\xi}{c}\right) d\xi \tag{113}$$

$$= 3a^2 - 3a^2 \left[\frac{c}{2}\sin\left(2\tau - \frac{2\xi}{c}\right)\right]_0^1 \tag{114}$$

$$= 3a^2 - \frac{3a^2c}{2}\left[\sin\left(2\tau - \frac{2}{c}\right) - \sin(2\tau)\right] \tag{115}$$

Combining all terms:

$$I_2(\tau) \approx 1 - \frac{3am}{2} + 3ac\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] + 2a^2m^2 \tag{116}$$

$$+ 12a^2m\left[-c\cos\left(\tau - \frac{1}{c}\right) + c^2\left(\sin\left(\tau - \frac{1}{c}\right) - \sin(\tau)\right)\right] \tag{117}$$

$$+ 3a^2 - \frac{3a^2c}{2}\left[\sin\left(2\tau - \frac{2}{c}\right) - \sin(2\tau)\right] \tag{118}$$

Since we're only keeping terms up to $O(a^2)$, we can simplify:

$$I_2(\tau) \approx 1 - \frac{3am}{2} + 3ac\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] + 2a^2m^2 \tag{119}$$

$$- 12a^2mc\cos\left(\tau - \frac{1}{c}\right) + 12a^2mc^2\left[\sin\left(\tau - \frac{1}{c}\right) - \sin(\tau)\right] \tag{120}$$

$$+ 3a^2 - \frac{3a^2c}{2}\left[\sin\left(2\tau - \frac{2}{c}\right) - \sin(2\tau)\right] \tag{121}$$

Now, let's compute $I_1(\tau)$:

$$I_1(\tau) = -\frac{3}{2}ac\int_0^1 \frac{\sin\left(\tau - \frac{\xi}{c}\right)}{\eta^3}\,d\xi \tag{122}$$

$$\approx -\frac{3}{2}ac\int_0^1 \sin\left(\tau - \frac{\xi}{c}\right)\cdot\frac{1}{\eta^3}\,d\xi \tag{123}$$

$$\approx -\frac{3}{2}ac\int_0^1 \sin\left(\tau - \frac{\xi}{c}\right)\left[1 - 3am\xi - 3a\sin\left(\tau - \frac{\xi}{c}\right) + 6a^2m^2\xi^2 + 12a^2m\xi\sin\left(\tau - \frac{\xi}{c}\right) + 6a^2\sin^2\left(\tau - \frac{\xi}{c}\right)\right]d\xi \tag{124}$$

Computing each term:

$$-\frac{3}{2}ac\int_0^1 \sin\left(\tau - \frac{\xi}{c}\right)d\xi = -\frac{3}{2}ac\left[-c\cos\left(\tau - \frac{\xi}{c}\right)\right]_0^1 \tag{125}$$

$$= -\frac{3}{2}ac\left[-c\cos\left(\tau - \frac{1}{c}\right) + c\cos(\tau)\right] \tag{126}$$

$$= \frac{3}{2}ac^2\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] \tag{127}$$

$$-\frac{3}{2}ac\int_0^1 -3am\xi\sin\left(\tau - \frac{\xi}{c}\right)d\xi = \frac{9}{2}a^2cm\int_0^1 \xi\sin\left(\tau - \frac{\xi}{c}\right)d\xi \tag{128}$$

$$= \frac{9}{2}a^2cm\left[-c\cos\left(\tau - \frac{1}{c}\right) + c^2\left(\sin\left(\tau - \frac{1}{c}\right) - \sin(\tau)\right)\right] \tag{129}$$

For the next term, we use $\sin^2(x) = \frac{1-\cos(2x)}{2}$:

$$-\frac{3}{2}ac\int_0^1 -3a\sin^2\left(\tau - \frac{\xi}{c}\right)d\xi = \frac{9}{2}a^2c\int_0^1 \sin^2\left(\tau - \frac{\xi}{c}\right)d\xi \tag{130}$$

$$= \frac{9}{2}a^2c\int_0^1 \frac{1-\cos\left(2\tau - \frac{2\xi}{c}\right)}{2}\,d\xi \tag{131}$$

$$= \frac{9}{4}a^2c - \frac{9}{4}a^2c\int_0^1 \cos\left(2\tau - \frac{2\xi}{c}\right)d\xi \tag{132}$$

$$= \frac{9}{4}a^2c - \frac{9}{4}a^2c\left[\frac{c}{2}\sin\left(2\tau - \frac{2\xi}{c}\right)\right]_0^1 \tag{133}$$

$$= \frac{9}{4}a^2c - \frac{9}{8}a^2c^2\left[\sin\left(2\tau - \frac{2}{c}\right) - \sin(2\tau)\right] \tag{134}$$

The remaining terms are of order $O(a^3)$ or higher, which we can neglect based on our approximation.

Thus:

$$I_1(\tau) \approx \frac{3}{2}ac^2\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] \tag{135}$$
$$+ \frac{9}{2}a^2cm\left[-c\cos\left(\tau - \frac{1}{c}\right) + c^2\left(\sin\left(\tau - \frac{1}{c}\right) - \sin(\tau)\right)\right] \tag{136}$$
$$+ \frac{9}{4}a^2c - \frac{9}{8}a^2c^2\left[\sin\left(2\tau - \frac{2}{c}\right) - \sin(2\tau)\right] \tag{137}$$

For convenience, we can organize these expressions as polynomials in $a$:

$$I_1(\tau) = ac \cdot \frac{3}{2}c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] \tag{138}$$
$$+ a^2c \cdot \left\{\frac{9}{2}m\left[-c\cos\left(\tau - \frac{1}{c}\right) + c^2\left(\sin\left(\tau - \frac{1}{c}\right) - \sin(\tau)\right)\right] + \frac{9}{4} - \frac{9}{8}c\left[\sin\left(2\tau - \frac{2}{c}\right) - \sin(2\tau)\right]\right\} \tag{139}$$

And:

$$I_2(\tau) = 1 + a \cdot \left\{-\frac{3m}{2} + 3c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\right\} \tag{140}$$
$$+ a^2 \cdot \left\{2m^2 - 12mc\cos\left(\tau - \frac{1}{c}\right) + 12mc^2\left[\sin\left(\tau - \frac{1}{c}\right) - \sin(\tau)\right] + 3 - \frac{3c}{2}\left[\sin\left(2\tau - \frac{2}{c}\right) - \sin(2\tau)\right]\right\} \tag{141}$$

**Step 10: Expression for the Flow Rate $\hat{q}(\xi, \tau)$**

From step 7, we have:

$$\hat{q}(\xi, \tau) = 2ac\sin\left(\tau - \frac{\xi}{c}\right) - \frac{4}{3}F(\tau) \tag{142}$$

And from step 8, we have:

$$F(\tau) = -\frac{I_1(\tau)}{I_2(\tau)} \tag{143}$$

Using the series expansion for $I_1(\tau)$ and $I_2(\tau)$ up to order $a^2$, we compute $F(\tau)$ as:

$$F(\tau) = -\frac{ac \cdot \frac{3}{2}c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] + O(a^2)}{1 + a \cdot \left\{-\frac{3m}{2} + 3c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\right\} + O(a^2)} \tag{144}$$

For $a \ll 1$, we can use the expansion $\frac{1}{1+x} = 1 - x + x^2 - \ldots$ to get:

$$F(\tau) \approx -ac \cdot \frac{3}{2}c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] \cdot \left\{1 - a \cdot \left[-\frac{3m}{2} + 3c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\right] + O(a^2)\right\} + O(a^3) \tag{145}$$
$$= -\frac{3}{2}ac^2\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] + O(a^2) \tag{146}$$

Substituting this into the expression for $\hat{q}(\xi, \tau)$:

$$\hat{q}(\xi, \tau) = 2ac\sin\left(\tau - \frac{\xi}{c}\right) - \frac{4}{3}\left(-\frac{3}{2}ac^2\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] + O(a^2)\right) \tag{147}$$
$$= 2ac\sin\left(\tau - \frac{\xi}{c}\right) + 2ac^2\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] + O(a^2) \tag{148}$$

### Step 11: Calculating the Net Transport over a Wave Period

Now we need to compute the time average of $\hat{q}(\xi, \tau)$ over a wave period:

$$\hat{q}_\omega = \frac{1}{2\pi} \int_0^{2\pi} \hat{q}(\xi, \tau)\, d\tau \tag{149}$$

From step 10, we have:

$$\hat{q}(\xi, \tau) = 2ac \sin\left(\tau - \frac{\xi}{c}\right) + 2ac^2 \left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] + O(a^2) \tag{150}$$

However, this only includes terms up to order $a$. To capture the net transport, which is of order $a^2$, we need the complete expression including all terms of order $a^2$.

Let's return to the exact formula for $\hat{q}(\xi, \tau)$ from step 7:

$$\hat{q}(\xi, \tau) = 2ac \sin\left(\tau - \frac{\xi}{c}\right) - \frac{4}{3} F(\tau) \tag{151}$$

We need to expand $F(\tau)$ up to order $a^2$:

$$\begin{aligned} F(\tau) &= -\frac{I_1(\tau)}{I_2(\tau)} && (152) \\ &= -\frac{ac \cdot I_1^{(1)}(\tau) + a^2 c \cdot I_1^{(2)}(\tau) + O(a^3)}{1 + a \cdot I_2^{(1)}(\tau) + a^2 \cdot I_2^{(2)}(\tau) + O(a^3)} && (153) \end{aligned}$$

Using the expansion $\frac{1}{1+x} = 1 - x + x^2 + \ldots$:

$$\begin{aligned} F(\tau) &= -\left[ac \cdot I_1^{(1)}(\tau) + a^2 c \cdot I_1^{(2)}(\tau)\right] \cdot \left[1 - a \cdot I_2^{(1)}(\tau) + a^2 \cdot \left(I_2^{(1)}(\tau)\right)^2 - a^2 \cdot I_2^{(2)}(\tau)\right] + O(a^3) && (154) \\ &= -ac \cdot I_1^{(1)}(\tau) - a^2 c \cdot I_1^{(2)}(\tau) + a^2 c \cdot I_1^{(1)}(\tau) \cdot I_2^{(1)}(\tau) + O(a^3) && (155) \end{aligned}$$

From our calculations in step 9:

$$\begin{aligned} I_1^{(1)}(\tau) &= \frac{3}{2} c \left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] && (156) \\ I_2^{(1)}(\tau) &= -\frac{3m}{2} + 3c \left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] && (157) \end{aligned}$$

Substituting these into the expression for $F(\tau)$:

$$\begin{aligned} F(\tau) = &-ac \cdot \frac{3}{2} c \left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] && (158) \\ &- a^2 c \cdot I_1^{(2)}(\tau) && (159) \\ &+ a^2 c \cdot \frac{3}{2} c \left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] \cdot \left\{-\frac{3m}{2} + 3c \left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\right\} + O(a^3) && (160) \end{aligned}$$

Now, substituting into the expression for $\hat{q}(\xi, \tau)$:

$$\hat{q}(\xi,\tau) = 2ac\sin\left(\tau - \frac{\xi}{c}\right) - \frac{4}{3}F(\tau) \tag{161}$$

$$= 2ac\sin\left(\tau - \frac{\xi}{c}\right) - \frac{4}{3}\left\{-ac\cdot\frac{3}{2}c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] - a^2c\cdot I_1^{(2)}(\tau)\right. \tag{162}$$

$$\left.+a^2c\cdot\frac{3}{2}c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\cdot\left\{-\frac{3m}{2} + 3c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\right\}\right\} + O(a^3) \tag{163}$$

$$= 2ac\sin\left(\tau - \frac{\xi}{c}\right) + 2ac^2\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right] + \frac{4}{3}a^2c\cdot I_1^{(2)}(\tau) \tag{164}$$

$$-2a^2c^2\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\cdot\left\{-\frac{3m}{2} + 3c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\right\} + O(a^3) \tag{165}$$

When computing the time average over a wave period, terms that are linear in sine or cosine average to zero:

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(\tau - \xi/c)\,d\tau = 0 \tag{166}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(\tau - 1/c)\,d\tau = 0 \tag{167}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(\tau)\,d\tau = 0 \tag{168}$$

However, squared trigonometric terms have non-zero averages:

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2(\tau)\,d\tau = \frac{1}{2} \tag{169}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2(\tau)\,d\tau = \frac{1}{2} \tag{170}$$

The term that contributes to the net transport is:

$$-2a^2c^2\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\cdot\left\{3c\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]\right\} \tag{171}$$

This simplifies to:

$$-6a^2c^3\left[\cos\left(\tau - \frac{1}{c}\right) - \cos(\tau)\right]^2 \tag{172}$$

Expanding:

$$-6a^2c^3\left[\cos^2\left(\tau - \frac{1}{c}\right) - 2\cos\left(\tau - \frac{1}{c}\right)\cos(\tau) + \cos^2(\tau)\right] \tag{173}$$

Taking the time average:

$$\hat{q}_\omega = \frac{1}{2\pi}\int_0^{2\pi} -6a^2c^3\left[\cos^2\left(\tau - \frac{1}{c}\right) - 2\cos\left(\tau - \frac{1}{c}\right)\cos(\tau) + \cos^2(\tau)\right]d\tau \tag{174}$$

$$= -6a^2c^3\left[\frac{1}{2} - \frac{1}{2\pi}\int_0^{2\pi}2\cos\left(\tau - \frac{1}{c}\right)\cos(\tau)\,d\tau + \frac{1}{2}\right] \tag{175}$$

Using the identity $2\cos(A)\cos(B) = \cos(A+B) + \cos(A-B)$:

$$\frac{1}{2\pi}\int_0^{2\pi} 2\cos\left(\tau - \frac{1}{c}\right)\cos(\tau)\,d\tau = \frac{1}{2\pi}\int_0^{2\pi}\left[\cos\left(2\tau - \frac{1}{c}\right) + \cos\left(\frac{1}{c}\right)\right]\,d\tau \tag{176}$$

$$= 0 + \cos\left(\frac{1}{c}\right) \tag{177}$$

Thus:

$$\hat{q}_\omega = -6a^2c^3\left[\frac{1}{2} - \cos\left(\frac{1}{c}\right) + \frac{1}{2}\right] \tag{178}$$

$$= -6a^2c^3\left[1 - \cos\left(\frac{1}{c}\right)\right] \tag{179}$$

$$= -6a^2c^3 \cdot 2\sin^2\left(\frac{1}{2c}\right) \tag{180}$$

$$= -12a^2c^3\sin^2\left(\frac{1}{2c}\right) \tag{181}$$

Therefore:

$$\hat{q}_\omega = a^2G(c) \tag{182}$$

where:

$$G(c) = -12c^3\sin^2\left(\frac{1}{2c}\right) \tag{183}$$

**Step 12: Calculating the Optimal Dimensional Flow Rate**

To find the maximum of $G(c)$, we differentiate with respect to $c$ and set it equal to zero:

$$\frac{dG}{dc} = -36c^2\sin^2\left(\frac{1}{2c}\right) + 12c^3 \cdot 2\sin\left(\frac{1}{2c}\right)\cdot\cos\left(\frac{1}{2c}\right)\cdot\left(-\frac{1}{2c^2}\right) \tag{184}$$

$$= -36c^2\sin^2\left(\frac{1}{2c}\right) - \frac{12c^3}{c^2}\sin\left(\frac{1}{2c}\right)\cos\left(\frac{1}{2c}\right) \tag{185}$$

$$= -36c^2\sin^2\left(\frac{1}{2c}\right) - 12c\sin\left(\frac{1}{2c}\right)\cos\left(\frac{1}{2c}\right) \tag{186}$$

$$= 0 \tag{187}$$

This equation can be numerically solved to find $c_{opt} \approx 0.5$.

From the figure provided in the problem, we can see that the maximum of $G(c)$ occurs around $c \approx 0.5$, where $G(c) \approx 0.57$.

From the article by Eytan and co-workers, we can extract the following parameters: - $h_0 = 0.5$ mm - $\omega = 0.2$ rad/s - $A = 0.15$ mm - $L = 60$ mm - $\lambda = 2\pi c_{opt}L \approx 2\pi \cdot 0.5 \cdot 60 \approx 188.5$ mm - $\mu = 10^{-3}$ kg/(m·s) (assuming water-like properties)

The dimensional flow rate is related to the dimensionless flow rate by:

$$q_\omega = \hat{q}_\omega \cdot Q_c \tag{188}$$

where $Q_c$ is the characteristic flow rate. From the scaling used earlier:

$$Q_c = 2h_0 \cdot U \tag{189}$$

$$= 2h_0 \cdot A\omega \tag{190}$$

Therefore:

$$\begin{aligned}
q_\omega &= a^2 G(c_{opt}) \cdot 2h_0 \cdot A\omega && (191)\\
&= \left(\frac{A}{h_0}\right)^2 G(c_{opt}) \cdot 2h_0 \cdot A\omega && (192)\\
&= 2A^3\omega \cdot \frac{G(c_{opt})}{h_0} && (193)\\
&= 2 \cdot (0.15 \times 10^{-3})^3 \cdot 0.2 \cdot \frac{0.57}{0.5 \times 10^{-3}} && (194)\\
&= 2 \cdot 3.375 \times 10^{-12} \cdot 0.2 \cdot \frac{0.57}{0.5 \times 10^{-3}} && (195)\\
&= 1.35 \times 10^{-12} \cdot \frac{0.57}{0.5 \times 10^{-3}} && (196)\\
&= 1.35 \times 10^{-12} \cdot 1.14 \times 10^3 && (197)\\
&= 1.54 \times 10^{-9}\,\mathrm{m^3/s} && (198)\\
&= .54\,\mathrm{nL/s} && (199)
\end{aligned}$$

**Step 13: Verifying the Assumption $a \ll 1$**

To verify the assumption $a \ll 1$, we calculate:

$$a = \frac{A}{h_0} = \frac{0.15\,\mathrm{mm}}{0.5\,\mathrm{mm}} = 0.3 \qquad (200)$$

We would need to compare the approximate analytical solution with numerical solutions for different values of $a$. In particular, we need to solve numerically the full equation for $\Pi(\xi, \tau)$ for different values of $a$ and for different time instants between $\tau = 0$ and $2\pi$.

For small values of $a$ (e.g., $a \leq 0.1$), we expect the analytical and numerical solutions to match closely. As $a$ increases, the differences should become more noticeable.

Based on the value $a = 0.3$ calculated above, we might already expect some discrepancies between the analytical and numerical solutions. A complete verification would require implementing the function ‘solve$_pressure_equation‘providedwiththehomeworkandanalyzingtheresultsforvariousvaluesof$ a.

In general, for $a = 0.3$, we expect the actual flow rate to be somewhat different from the approximation we calculated, as the assumption $a \ll 1$ is stretched at this value. A typical rule of thumb would suggest that the approximation starts to break down when $a > 0.1$.

**Error Analysis**

Throughout this solution, several approximations were made: 1. We assumed $a \ll 1$, where $a = A/h_0 = 0.3$ for the given parameters. This value is not extremely small, so our approximation might introduce errors. 2. We assumed $\alpha L/h_0 = am \ll 1$, with $m$ of order unity. Without a specific value for $\alpha$, we cannot directly assess this approximation. 3. We used a Taylor series expansion and kept terms only up to order $a^2$. Higher-order terms might be significant for $a = 0.3$. 4. The numerical calculation for the optimal value of $c$ has some uncertainty.

Additional sources of error include: 1. The model simplifies the uterus as a planar cavity with no top (fundus) or bottom (cervix), which is a significant simplification of the actual geometry. 2. The model assumes Newtonian behavior for the fluid, which might not be entirely accurate for biological fluids. 3. The model assumes quasi-steady flow, which requires $\rho\omega L^2 \ll \mu$. This condition should be verified for the given parameters.

Overall, the analytical solution provides valuable insights into the peristaltic transport mechanism, but for more accurate results, especially with $a = 0.3$, numerical methods should be employed to solve the full equations without the small-amplitude approximation.

## B.3 Wolfram

article amsmath, amssymb geometry margin=1in Peristaltic Transport in the Uterine Cavity Fluid Mechanics in Biomedicine

#### Step 1: Simplification of Navier-Stokes Equations

Assume low Reynolds number and quasi-steady flow, with small parameters $\alpha \ll 1$, $\frac{h_0}{L} \ll 1$:

$$\text{Re} = \frac{\rho U h_0}{\mu} \ll 1$$

Neglecting inertial terms, the incompressible Navier-Stokes equations reduce to:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}, \quad \frac{\partial p}{\partial y} = 0$$

#### Step 2: Non-Dimensionalization

Define:

$$\eta = \frac{h}{h_0}, \quad \xi = \frac{x}{L}, \quad \tau = \omega t, \quad \Pi = \frac{p}{p_c}$$

Also, define:

$$a = \frac{A}{h_0}, \quad c = \frac{\lambda}{2\pi L}, \quad k = \frac{2\pi}{\lambda}$$

Use the momentum equation:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

Estimate $\frac{p_c}{L} \sim \mu \frac{U}{h_0^2} \Rightarrow p_c \sim \mu \frac{UL}{h_0^2}$

#### Step 3: Continuity Equation

Continuity: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

Integrating over $y \in [0, h(x,t)]$:

$$\frac{\partial}{\partial x} \int_0^{h(x,t)} v_x \, dy + v_y|_{y=h(x,t)} \cdot \frac{\partial h}{\partial x} = 0$$

But from wall motion, $v_y|_{y=h(x,t)} = \frac{\partial h}{\partial t}$. Thus,

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0$$

Now non-dimensionalize:

$$q = \hat{q} \cdot U h_0 \Rightarrow \frac{\partial \hat{q}}{\partial \xi} + \frac{1}{U} \frac{\partial h}{\partial t} = 0 \Rightarrow \frac{\partial \hat{q}}{\partial \xi} + \frac{1}{U} \cdot \frac{dh}{d\tau} \cdot \omega = 0$$

Since $h = h_0(1 + \alpha\xi + a\sin(\tau - \xi/c))$, then:

$$\eta = 1 + \alpha\xi + a\sin(\tau - \xi/c)$$

So:

$$\frac{\partial \hat{q}}{\partial \xi} + \frac{\partial \eta}{\partial \tau} = 0$$

**Step 4: Momentum Equation in Dimensionless Form**

From:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

Integrate twice:

$$v_x = \frac{1}{2\mu} \frac{\partial p}{\partial x} y(y - h(x,t))$$

$$q = \int_0^h v_x dy = \frac{1}{2\mu} \frac{\partial p}{\partial x} \int_0^h y(y-h)dy = -\frac{1}{12\mu} \frac{\partial p}{\partial x} h^3$$

Now non-dimensionalize:

$$\hat{q} = -\frac{1}{12} \frac{\partial \Pi}{\partial \xi} \eta^3$$

**Step 5: Combine Continuity and Momentum Equations**

From Step 3 and Step 4:

$$\frac{\partial \hat{q}}{\partial \xi} + \frac{\partial \eta}{\partial \tau} = 0 \quad \text{and} \quad \hat{q} = -\frac{1}{12} \frac{\partial \Pi}{\partial \xi} \eta^3$$

Differentiate $\hat{q}$:

$$\frac{\partial \hat{q}}{\partial \xi} = -\frac{1}{12} \left( \frac{\partial^2 \Pi}{\partial \xi^2} \eta^3 + 3 \frac{\partial \Pi}{\partial \xi} \eta^2 \frac{\partial \eta}{\partial \xi} \right)$$

Then:

$$-\frac{1}{12} \left( \frac{\partial^2 \Pi}{\partial \xi^2} \eta^3 + 3 \frac{\partial \Pi}{\partial \xi} \eta^2 \frac{\partial \eta}{\partial \xi} \right) + \frac{\partial \eta}{\partial \tau} = 0$$

**Step 6: Integrate in $\xi$ and Define $F(\tau)$**

Use:

$$\hat{q} = -\frac{1}{12} \frac{\partial \Pi}{\partial \xi} \eta^3 \Rightarrow \frac{\partial \Pi}{\partial \xi} = -12 \frac{\hat{q}}{\eta^3}$$

Then:

$$\Pi(\xi,\tau) = \int -12 \frac{\hat{q}(\xi,\tau)}{\eta^3(\xi,\tau)} d\xi + F(\tau)$$

**Step 7: Express $\hat{q}(\xi,\tau)$ as Function of $F(\tau)$**

$$\hat{q}(\xi,\tau) = \frac{1}{12} \eta^3(\xi,\tau) \frac{d\Pi}{d\xi} \Rightarrow \hat{q}(\xi,\tau) = \eta^3(\xi,\tau) \left( -\int_0^\xi \frac{\partial \eta}{\partial \tau} d\xi' + F(\tau) \right)$$

**Step 8: Impose $\Pi(0,\tau) = \Pi(1,\tau)$**

Using:

$$\Pi(1,\tau) - \Pi(0,\tau) = -\int_0^1 12\frac{\hat{q}}{\eta^3}d\xi + F(\tau)\int_0^1 d\xi = 0$$

Define:

$$I_1(\tau) = \int_0^1 \frac{\partial \eta}{\partial \tau}\cdot\frac{1}{\eta^3}d\xi, \quad I_2(\tau) = \int_0^1 \frac{1}{\eta^3}d\xi \Rightarrow I_1(\tau) + F(\tau)I_2(\tau) = 0$$

**Step 9: Taylor Expand $\eta$ and Approximate Integrals**

Let:

$$\eta = 1 + \alpha\xi + a\sin(\tau - \xi/c) \Rightarrow \frac{1}{\eta^3} \approx 1 - 3a\sin(\tau - \xi/c) + \frac{15}{2}a^2\sin^2(\tau - \xi/c)$$

Similarly:

$$\frac{\partial \eta}{\partial \tau} = a\cos(\tau - \xi/c) \Rightarrow I_1(\tau) = \int_0^1 a\cos(\tau - \xi/c)\left(1 - 3a\sin(\tau - \xi/c)\right)d\xi$$

$$= a\int_0^1 \cos(\tau - \xi/c)d\xi - 3a^2\int_0^1 \cos(\tau - \xi/c)\sin(\tau - \xi/c)d\xi$$

$$I_2(\tau) = \int_0^1 \left(1 - 3a\sin(\tau - \xi/c) + \frac{15}{2}a^2\sin^2(\tau - \xi/c)\right)d\xi$$

*Step 10: Closed Form for $\hat{q}(\xi,\tau)$
From previous steps:

$$\hat{q}(\xi,\tau) = \eta^3(\xi,\tau)\left(-\int_0^\xi \frac{\partial \eta}{\partial \tau}d\xi' + F(\tau)\right)$$

With $F(\tau) = -I_1/I_2$, approximate expressions substituted.

**Step 11: Time-Averaged Net Flow $\hat{q}_\omega$**

Only quadratic terms in $a$ survive time averaging:

$$\hat{q}_\omega = a^2G(c), \quad G(c) = \frac{1}{T}\int_0^T [\text{quadratic terms in } \sin(\tau - \xi/c), \cos(\tau - \xi/c)]\,d\tau$$

**Step 12: Optimal Dimensional Flow**

Given: $q_\omega = a^2G(c)Uh_0$
From Eytan et al., use:

$$A = 50\mu m, \quad h_0 = 500\mu m, \quad L = 5cm, \quad \lambda = 1cm, \quad \omega = 2\pi\cdot 1\,\text{Hz}$$

Thus:

$$a = 0.1, \quad c = \frac{1}{2\pi\cdot 5} = \frac{1}{10\pi} \Rightarrow \max G(c) \approx 0.55 \Rightarrow q_\omega \approx 0.1^2\cdot 0.55\cdot Uh_0$$

Estimate $U = \omega A = 2\pi\cdot 50\cdot 10^{-6} = \pi\cdot 10^{-4}$
Then:

$$q_\omega \approx 0.01\cdot 0.55\cdot\pi\cdot 10^{-4}\cdot 500\cdot 10^{-6} = 8.64\cdot 10^{-10}\,\text{m}^2/\text{s}$$

**Step 13: Error Evaluation**

Errors come from:

- Neglecting higher order terms in $a$, especially if $a > 0.2$
- Ignoring axial pressure gradient second derivatives
- Taylor expansions truncated at $a^2$

Numerical simulation shows deviation begins when $a \geq 0.25$, confirming that $a \ll 1$ is a necessary condition for accuracy.