A recurrent relationship for evaluating G/G/1 queues

- $x_i =$ service time for client i. (INPUT)
- $\theta_i =$ exit time instant from W.S. . for client i
- $t_i^S =$ arrival time instant to the service system.
- $t_i =$ entrance time instant to the W.S.; it can be obtained through the sequence of interarrival times $\tau_i = t_{i+1} - t_i$: $t_i = \sum_{\ell=1}^{i-1} \tau_\ell$. (It can be considered an INPUT)

- $w_i = \theta_i - t_i =$ sojourn time of client i in W.S.
- $w_{q,i} = t_i^S - t_i$ sojourn time of client i in queue.
- $\mathcal{L}_i = w_i$: contribution of i-th client to the occupancy.

The following recurrent relationships allow for the generation of arrival time instants in service system t_i^S and the exit times from the W.S. θ_i for i=1,2,3,4, n clients:

Inicialitzation: $\mathcal{L} = 0; W = 0\ \mathcal{L}_q = 0; W_q = 0; \theta_0 = -\infty,\ t_1 = 0$

For $i = 1, 2, 3, ...n$:

1. $t_i^S = max\{\theta_{i-1}, t_i\}$
2. Generate a service time x_i from the specified distribution.
3. $\theta_i = t_i^S + x_i$
4. If $i < n$ then: Generate τ_i from the specified distribution; $t_{i+1} = t_i + \tau_i$.
5. Printing statistics:

   a) $\mathcal{L}_i = w_i = \theta_i - t_i; \mathcal{L} = \mathcal{L} + \mathcal{L}_i; \mathcal{L}_{T_i} = \dfrac{\mathcal{L}}{t_i - t_1}; W = W + w_i$

   b) $\mathcal{L}_{q,i} = w_{q,i} = t_i^S - t_i; \mathcal{L}_q = \mathcal{L}_q + \mathcal{L}_{q,i}; W_q = W_q + w_{q,i}$

After client n report:

$$W = W/n\,;\ W_q = W_q/n\,;\ \mathcal{L} = \frac{\mathcal{L}}{t_n - t_1}\ ;\ \mathcal{L}_q = \frac{\mathcal{L}_q}{t_n - t_1}$$

In order to see if the previous values are stable, plot $\mathcal{L}_{T_i}$ versus t_i. It may be also useful to collect the generated samples w_i and $w_{q,i}$ for statistical analysis (plot of histograms, basic statistics, ...)

COMPARING G/G/1 AND G/G/2 SYSTEMS UNDER LONG-TAILED SERVICE TIMES

The objective of this work is twofold:

- To compare results obtained by simulation for a given G/G/1 system with the Allen-Cuneen's approximation formula. The student will be assigned a specified arrival process and a family of service time distributions (i.e.: service times following a Weibull distribution with parameters to be specified by the student accordingly to the required loading factors)
- Analyze the performance of queueing systems by simulation when service times correspond to a long-tailed distribution of probability.

1. **A typical long-tailed distribution**

   The Weibull distribution will be used in this exercise as example of a long tailed distribution for the service time x. Its probability distribution function, mean and variance are given by:

$$F_X(x) = 1 - \exp(-(x/b)^a),$$

$$E[x] = b\Gamma(\frac{a+1}{a}),\ Var[x] = b^2\left(\Gamma(\frac{a+2}{a}) - \Gamma^2(\frac{a+1}{a})\right)$$

   (where $\Gamma(\cdot)$ is the Euler function with $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(1/2) = \sqrt{\pi}$)

   Analysis of the service time . Generate a sample of service times corresponding to 10,000 clients with the only purpose of capturing a sample of values for the service times and analyze the sample. Report basic statistics (mean, variance, coefficient of variation, ...) and a histogram for the sample. Compare it with the theoretical values for the service time distribution.

2. **The effect of long tailed service times on G/G/1**

   The arrival process will be specified to the students by an assignment list. The number of clients will be 100.000

   For the sequence of traffic factors ρ =0.4, 0,7, 0.85, 0.925, select for each of them a value of the parameter b so that the corresponding loading factor ρ is achieved. Evaluate average waiting times in queue (W_q) using Allen Cuneen's approximation formula for each of the previous loading factors.

   Write a small program for evaluating a G/G/1 system implementing the previous recurrence relationships in order to evaluate the resulting waiting system in which arrivals are given to the student by the assignment list and service times are given by a Weibull distribution so that a is given in the assignment list and b depends varies accordingly to the loading factor ρ to be evaluated. The student will be assigned the value for a (in all cases $a < 1$). Write the corresponding functions for generating random numbers for the specified arrival process and for a Weibull distribution (for service times).

   Using different initial seeds carry out 10 simulations and obtain a confidence interval for the mean waiting time W_q and queue length L_q. Plot the evolution of the average occupancy $\mathcal{L}_{T_i}$ versus t_i in the previous simulations to see if stable values have been reached using 100.000 clients. Compare the waiting times at queues obtained by simulation with those obtained using Allen-Cuneen's Approximation formula.

3. **Comparison with a G/G/2 queueing system**

   For the same sequence of values for the traffic factor ρ =0.4, 0,7, 0.85, 0.925, set the appropriate values of the parameter b of the individual service time (necessarily they should be double than those previously used). Use SimQueue simulator in order to evaluate a G/G/2 system using a) equal interarrival time distribution than in section 2, for the arrival process and b) individual service times for servers, distributed following a Weibull law with b parameter stated as above.

- Obtain values for the basic magnitudes L, L_q, W, W_q and compare them to case in section 2 with a G/G/1 queue.

- Compare L, L_q, W, W_q with the ones obtained using Allen-Cuneen's formula.

Make any appropriate comments you consider in order at the view of the previous results.

To deliver:

- a report of no more than 20 pages answering/commenting all the previous aspects and
- the computer program developed in 2. (sources, executables and input data files as well as instructions for its use.)

No exercise will be positively evaluated without the delivery of a running program.