CHAPTER 4

FEM for Nonlinear Materials

Introduction

- Elastic material: a strain energy is differentiated by strain to obtain stress
  - History-independent, potential exists, reversible, no permanent deformation
- **Elastoplastic** material:
  - Permanent deformation for a force larger than elastic limit
  - No one-to-one relationship between stress and strain
  - Constitutive relation is given in terms of the rates of stress and strain (**Hypo-elasticity**)
  - Stress can only be calculated by integrating the stress rate over the past load history (**History-dependent**)
- Important to separate elastic and plastic strain
  - **Only elastic strain** generates stress

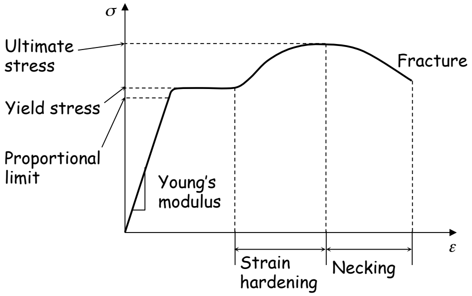
1D material models for steel

Plasticity

- Elasticity – A material deforms under stress, but then returns to its original shape when the stress is removed
- Plasticity - deformation of a material undergoing **non-reversible changes of shape** in response to applied forces
  - Plasticity in metals is usually a consequence of **dislocations**
  - Rough nonlinearity
- Found in most metals, and in general is a good description for a large class of materials
- **Perfect plasticity** – a property of materials to undergo irreversible deformation without any increase in stresses or loads
- **Hardening** - need increasingly higher stresses to result in further plastic deformation

Behavior of a Ductile Material

Terms	Explanation
Proportional limit	The greatest stress for which the stress is still proportional to the strain
Elastic limit	The greatest stress without resulting in any permanent strain on release of stress
Young's Modulus	Slope of the linear portion of the stress-strain curve
Yield stress	The stress required to produce 0.2% plastic strain
Strain hardening	A region where more stress is required to deform the material
Ultimate stress	The maximum stress the material can resist
Necking	Cross section of the specimen reduces during deformation



Elastoplasticity

- Most metals have both elastic and plastic properties
  - Initially, the material shows elastic behavior
  - After yielding, the material becomes plastic
  - By removing loading, the material becomes elastic again
- We will assume **small (infinitesimal) deformation** case
  - Elastic and plastic strain can be **additively decomposed** by

$$\boxed{\varepsilon = \varepsilon_e + \varepsilon_p}$$

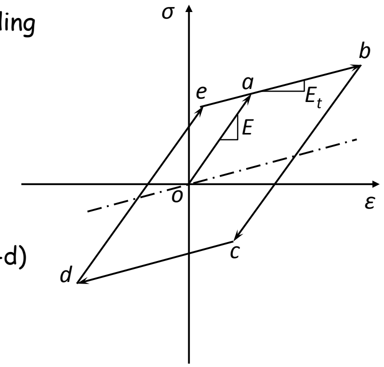
  - Strain energy density exists in terms of elastic strain

$$U_0 = \tfrac{1}{2}E(\varepsilon_e)^2$$

  - Stress is related to the elastic strain, not the plastic strain
- The plastic strain will be considered as an internal variable, which evolves according to plastic deformation

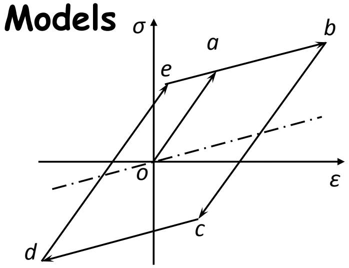
1D Elastoplasticity

- Idealized elastoplastic stress-strain behavior
  - Initial elastic behavior with slope E (**elastic modulus**) until yield stress σ_Y (line o–a)
  - After yielding, the plastic phase with slope E_t (**tangent modulus**) (line a–b).
  - Upon removing load, elastic unloading with slope E (line b-c)
  - Loading in the opposite direction, the material will eventually yield in that direction (point c)
  - **Work hardening** – more force is required to continuously deform in the plastic region (line a-b or c-d)

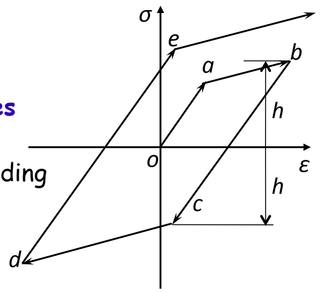


Work Hardening Models

- Kinematic hardening
  - **Elastic range remains constant**
  - Center of the elastic region moves parallel to the work hardening line
  - bc = de = 2oa
  - Use the center of elastic domain as an evolution variable



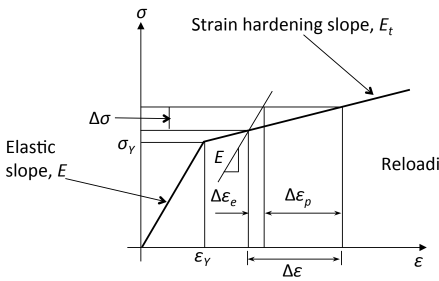
- Isotropic hardening $\sigma_Y \geq \sigma_Y^0 > 0$
  - **Elastic range (yield stress) increases proportional to plastic strain**
  - The yield stress for the reversed loading is equal to the previous yield stress
  - Use plastic strain as an evolution variable

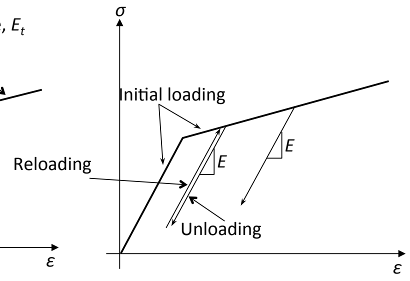


- No difference in proportional loading (line o-a-b)

Elastoplastic Analysis

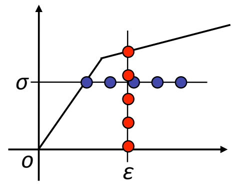
- Additive decomposition
  - Only elastic strain contributes to stress (but we don't know how much of the total strain corresponds to the elastic strain)
  - Let's consider an increment of strain: $\boxed{\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p}$
  - Elastic strain increases stress by $\boxed{\Delta\sigma = E\Delta\varepsilon_e}$
  - Elastic strain disappears upon removing loads or changing direction





Elastoplastic Analysis cont.

- Additive decomposition (continue)
  - Plastic strain remains constant during unloading
  - The effect of load-history is stored in the plastic strain
  - The yield stress is determined by the magnitude of plastic strain
  - **Decomposing elastic and plastic part of strain is an important part of elastoplastic analysis**
- For given stress σ, strain cannot be determined.
  - Complete history is required (path- or history-dependent)
  - History is stored in evolution variable (plastic strain)



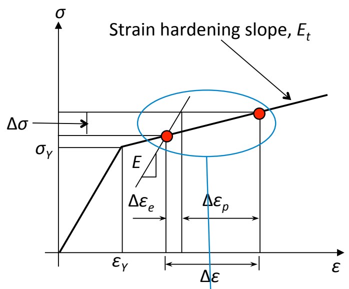
Plastic Modulus

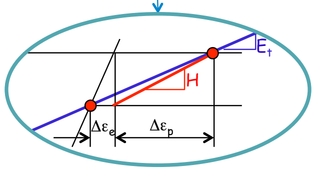
- Strain increment $\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$
- Stress increment $\Delta\sigma = E\Delta\varepsilon_e$
- **Plastic modulus** $H = \dfrac{\Delta\sigma}{\Delta\varepsilon_p}$
- Relation between moduli

$$\boxed{\Delta\sigma = E\Delta\varepsilon_e = H\Delta\varepsilon_p = E_t\Delta\varepsilon}$$

$$\frac{\Delta\sigma}{E_t} = \frac{\Delta\sigma}{E} + \frac{\Delta\sigma}{H} \quad\Rightarrow\quad \frac{1}{E_t} = \frac{1}{E} + \frac{1}{H}$$

$$H = \frac{EE_t}{E - E_t} \qquad E_t = \frac{EH}{E+H} = E\left(1 - \frac{E}{E+H}\right)$$





- Both kinematic and isotropic hardenings have the same plastic modulus

Analysis Procedure

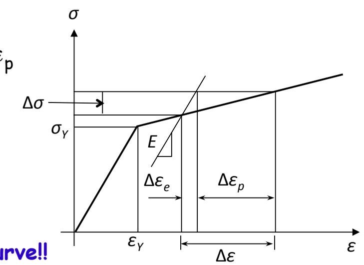
- Analysis is performed with a given incremental strain
  - N-R iteration will provide $\Delta\mathbf{d} \rightarrow \Delta\varepsilon$
  - But, we don't know $\Delta\varepsilon_e$ or $\Delta\varepsilon_p$
- When the material is in the initial elastic range, regular elastic analysis procedure can be used ($\Delta\varepsilon_e = \Delta\varepsilon$, $\Delta\varepsilon_p = 0$)
- When the material is in the plastic range, we have to **determine incremental plastic strain**

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p = \frac{\Delta\sigma}{E} + \Delta\varepsilon_p = \frac{H\Delta\varepsilon_p}{E} + \Delta\varepsilon_p$$

$$= \Delta\varepsilon_p\left(\frac{H}{E} + 1\right)$$

$$\Rightarrow \quad \boxed{\Delta\varepsilon_p = \frac{\Delta\varepsilon}{1 + H/E}}$$

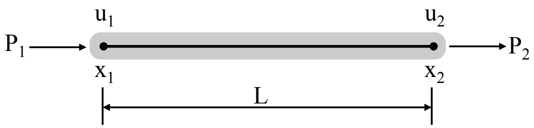
Only when the material is on the plastic curve!!



1D Finite Element Formulation

- Load increment
  - applied load is divided by N increments: $[t^1, t^2, \ldots, t^N]$
  - analysis procedure has been completed up to load increment t^n
  - a new solution at t^{n+1} is sought using the Newton-Raphson method
  - iteration k has been finished and the current iteration is k+1
- Displacement iterations
  - From last **increment** t^n: $\Delta \mathbf{d}_{k+1} = \mathbf{d}_{k+1}^{n+1} - \mathbf{d}^n$
  - From previous **iteration**: $\delta \mathbf{d}_{k+1} = \mathbf{d}_{k+1}^{n+1} - \mathbf{d}_k^{n+1}$

$$\mathbf{d} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$



1D FE Formulation cont.

- Interpolation

$$\Delta u(x) = [N_1 \ N_2] \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix} = \mathbf{N} \cdot \Delta \mathbf{d}$$

$$\Delta \varepsilon = \frac{d}{dx}(\Delta u) = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix} = \mathbf{B} \cdot \Delta \mathbf{d}$$

$$\delta u = \mathbf{N} \cdot \delta \mathbf{d}$$
$$\delta \varepsilon = \mathbf{B} \cdot \delta \mathbf{d}$$

- Weak form (1 element)
  - Internal force = external force

$$\int_0^L \mathbf{B}^T \sigma_{k+1}^{n+1} A \, dx = \mathbf{f}^{n+1}$$

1D FE Formulation cont.

- Stress-strain relationship (Incremental)

$$\sigma_{k+1}^{n+1} \approx \sigma_k^{n+1} + \frac{\partial \sigma}{\partial \varepsilon} \delta \varepsilon = \sigma_k^{n+1} + D^{ep} \delta \varepsilon$$

  - Elastoplastic tangent modulus

$$\boxed{D^{ep} = \begin{cases} E & \text{if elastic} \\ E_t & \text{if plastic} \end{cases}}$$

- Linearization of weak form

$$\underbrace{\left[\int_0^L \mathbf{B}^T D^{ep} \mathbf{B} A \, dx \right]}_{\text{Tangent stiffness}} \delta \mathbf{d}_{k+1} = \underbrace{\mathbf{f}^{n+1} - \int_0^L \mathbf{B}^T \sigma_{k+1}^{n+1} A \, dx}_{\text{Residual}}$$

1D FE Formulation cont.

- Tangent Stiffness

$$\boxed{\mathbf{K}_T = \frac{AD^{ep}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

- Residual

$$\mathbf{r}_k^{n+1} = \mathbf{f}^{n+1} - \int_0^L \mathbf{B}^T \sigma_k^{n+1} A \, dx = \begin{Bmatrix} f_1^{n+1} + \sigma_k^{n+1} A \\ f_2^{n+1} - \sigma_k^{n+1} A \end{Bmatrix}$$

- State Determination: $\sigma_k^{n+1} = f(\sigma^n, \varepsilon_p^n, \Delta \varepsilon_k, \ldots)$

  Will talk about next slides

- Incremental Finite Element Equation
  - N-R iteration until the residual vanishes

$$\boxed{\mathbf{K}_T \cdot \delta \mathbf{d}_{k+1} = \mathbf{r}_k^{n+1}}$$

Isotropic Hardening Model

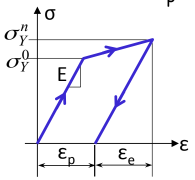
- Yield strength gradually increases proportional to the plastic strain
  - Yield strength is always positive for both tension or compression

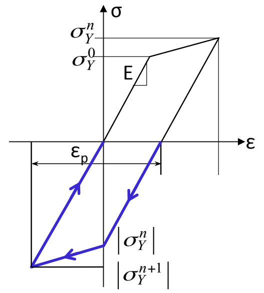
$$\boxed{\sigma_Y^n = \sigma_Y^0 + H\varepsilon_p^n}$$

Total plastic strain

Initial yield stress

  - Plastic strain is always positive and **continuously accumulated** even in cycling loadings $\varepsilon_p^n \geq 0$





State Determination (Isotropic Hardening)

- How to determine stress
  - Given: strain increment ($\Delta\varepsilon$) and all variables in load step n

$$(E, H, \sigma_Y^0, \sigma^n, \varepsilon_p^n)$$

1. Compute current yield stress (point d)

$$\sigma_Y^n = \sigma_Y^0 + H\varepsilon_p^n$$

2. Elastic predictor (point c)

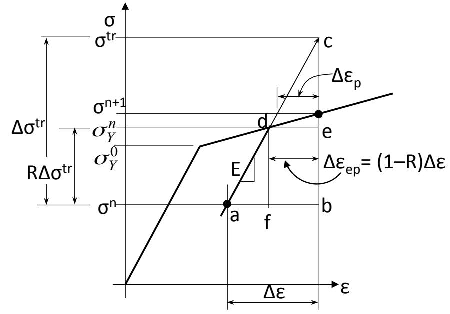
$$\boxed{\Delta\sigma^{tr} = E\Delta\varepsilon \qquad \sigma^{tr} = \sigma^n + \Delta\sigma^{tr}}$$

3. Check yield status

Trial yield function (c – e)

$$\boxed{f^{tr} = |\sigma^{tr}| - \sigma_Y^n}$$

$$f^{tr} = (1 - R)E\Delta\varepsilon$$



R: Fraction of $\Delta\sigma^{tr}$ to the yield stress

State Determination (Isotropic Hardening) cont.

- If $f^{tr} \leq 0$, material is elastic

$$\boxed{\sigma^{n+1} = \sigma^{tr}}$$

Either initial elastic region or unloading

- If $f^{tr} > 0$, material is plastic (yielding)

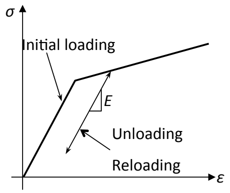
Either transition from elastic to plastic or continuous yielding

  - Stress update (return to the yield surface)

$$\boxed{\sigma^{n+1} = \sigma^{tr} - sgn(\sigma^{tr})E\Delta\varepsilon_p}$$

  - Update plastic strain

$$\boxed{\varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p}$$



Plastic strain increment is unknown

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$$

For a given strain increment, how much is elastic and plastic?

State Determination (Isotropic Hardening) cont.

- Plastic consistency condition
  - to determine plastic strain increment $\Delta\varepsilon_p$

$$\boxed{f^{n+1} = |\sigma^{n+1}| - \sigma_Y^{n+1} = 0}$$

  - Stress must be on the yield surface after plastic deformation

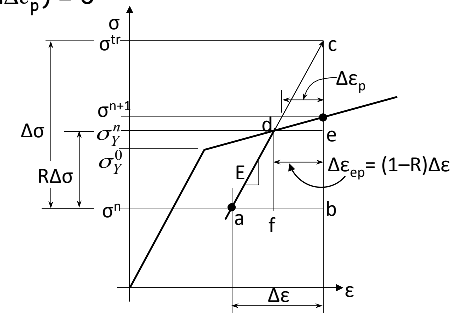
$$\Rightarrow \left|\sigma^{tr} - sgn(\sigma^{tr})E\Delta\varepsilon_p\right| - (\sigma_Y^n + H\Delta\varepsilon_p) = 0$$

$$\Rightarrow |\sigma^{tr}| - \sigma_Y^n - (E + H)\Delta\varepsilon_p = 0$$

$$\boxed{\Delta\varepsilon_p = \frac{|\sigma^{tr}| - \sigma_Y^n}{E + H} = \frac{f^{tr}}{E + H}}$$

$$\Delta\varepsilon_p = (1 - R)\frac{E}{E + H}\Delta\varepsilon$$

$$R = 1 - \frac{f^{tr}}{|\Delta\sigma^{tr}|}$$



Note: $\Delta\varepsilon_p$ is always positive!!

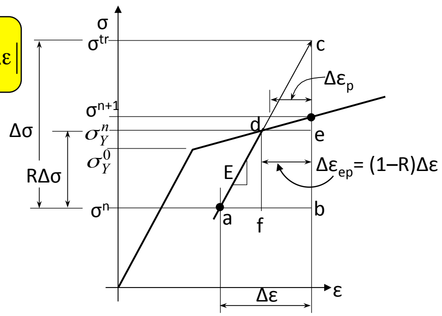
State Determination (Isotropic Hardening) cont.

- Update stress

$$\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\sigma^{tr}) E \Delta\varepsilon_p$$

$$\boxed{\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\sigma^{tr}) \frac{(1-R)E^2}{E+H} |\Delta\varepsilon|}$$

Elastic trial — Plastic compensation (return mapping)



- Algorithm
  1) Elastic trial
  2) Plastic return mapping
  - **No iteration is required in linear hardening models**

Algorithmic Tangent Stiffness

- Continuum tangent modulus
  - The slope of stress-strain curve $D^{ep} = \begin{cases} E & \text{if elastic} \\ E_t & \text{if plastic} \end{cases}$
- Algorithmic tangent modulus
  - Differentiation of the state determination algorithm

$$D^{alg} = \frac{\partial \Delta\sigma}{\partial \Delta\varepsilon} = \frac{\partial \sigma^{tr}}{\partial \Delta\varepsilon} - \mathrm{sgn}(\sigma^{tr}) E \frac{\partial \Delta\varepsilon_p}{\partial \Delta\varepsilon}$$

$$\frac{\partial \Delta\varepsilon_p}{\partial \Delta\varepsilon} = \frac{1}{E+H}\frac{\partial f^{tr}}{\partial \Delta\varepsilon} = \mathrm{sgn}(\sigma^{tr})\frac{E}{E+H}$$

$$\boxed{D^{alg} = \begin{cases} E & \text{if elastic} \\ E_t & \text{if plastic} \end{cases}}$$

- $D^{alg} = D^{ep}$ for 1D plasticity!!
  - In general they are different for multi-dimension

Algorithm for Isotropic Hardening

- Given: $\Delta\varepsilon, E, H, \sigma_Y^0, \sigma^n, \varepsilon_p^n \longrightarrow \sigma^{n+1}, \varepsilon_p^{n+1}$

1. Trial state $\sigma^{tr} = \sigma^n + E\Delta\varepsilon$

$$\sigma_Y^n = \sigma_Y^0 + H\varepsilon_p^n \qquad H = \frac{EE_t}{E - E_t}$$

$$f^{tr} = |\sigma^{tr}| - \sigma_Y^n$$

2. If $f^{tr} \le 0$ (elastic)
   - Remain elastic: $\boxed{\sigma^{n+1} = \sigma^{tr}, \quad \varepsilon_p^{n+1} = \varepsilon_p^n}$; exit
3. If $f^{tr} > 0$ (plastic)
   a. Calculate plastic strain: $\boxed{\Delta\varepsilon_p = \dfrac{f^{tr}}{E+H}}$
   b. Update stress and plastic strain (store them for next increment)

$$\boxed{\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\sigma^{tr}) E\Delta\varepsilon_p} \qquad \boxed{\varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p}$$

Ex) Elastoplastic Bar (Isotropic Hardening)

- $E = 200\text{GPa}$, $H = 25\text{GPa}$, ${}^0\sigma_y = 250\text{MPa}$
- $\sigma^n = 150\text{MPa}$, $\varepsilon^n_p = 0.0001$, $\Delta\varepsilon = 0.002$
- Yield stress: $\sigma_y^n = \sigma_0^n + H\varepsilon_p^n = 252.5\text{MPa}$
  - Material is elastic at t^n
- Trial stress: $\Delta\sigma^{tr} = E\Delta\varepsilon = 400\text{MPa}$

$$\sigma^{tr} = \sigma^n + \Delta\sigma^{tr} = 550\text{MPa}$$

Now material is plastic

- Plastic consistency condition

$$\Delta\varepsilon_p = \frac{f^{tr}}{E+H} = 1.322 \times 10^{-3}$$

- State update

$$\sigma^{n+1} = \sigma^{tr} - \mathrm{sgn}(\sigma^{tr}) E\Delta\varepsilon_p = 285.6\text{MPa}$$

$$\varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p = 1.422 \times 10^{-3}$$