Métodos de Resolución:

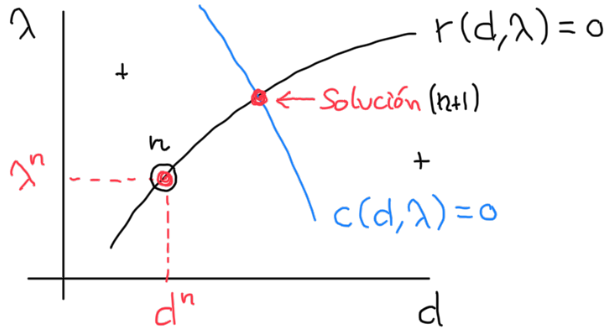
- Ecuación de equilibrio $\quad r(d,\lambda) = f(\lambda) - p(d) = 0$

  (N ecuaciones, N+1 incógnitas)

  Fuerzas externas $\equiv \; f(\lambda) = \lambda \, \underline{\underline{f_{ext}}}$ $\quad \lambda \equiv$ parámetro incremental de carga (escalar)

  $\to \; f_{ext} \equiv$ vector de carga (constante)

- Espacio de estados



Dimensión $\quad N+1$

Control general:

$$\left.\begin{aligned} d^{n+1} &= d^n + \underline{\Delta d} \\ \lambda^{n+1} &= \lambda^n + \underline{\Delta \lambda} \end{aligned}\right\} \qquad \left.\begin{aligned} r(\Delta d, \Delta\lambda) &= f(\Delta\lambda) - p(\Delta d) = 0 \\ c(\Delta d, \Delta\lambda) &= 0 \end{aligned}\right\}$$

Resolver para $(\Delta d, \Delta\lambda)$

Tipos de control:

- Control en carga: $\quad c(\Delta d, \Delta\lambda) = \Delta\lambda - \ell = 0$
- Control hiperesférico:

  $$c(\Delta d, \Delta\lambda) = \Delta d^T \Delta d + \Delta\lambda^2 - \ell^2 = 0$$

- Control hiperelíptico: . . .

Procedimiento de resolución:

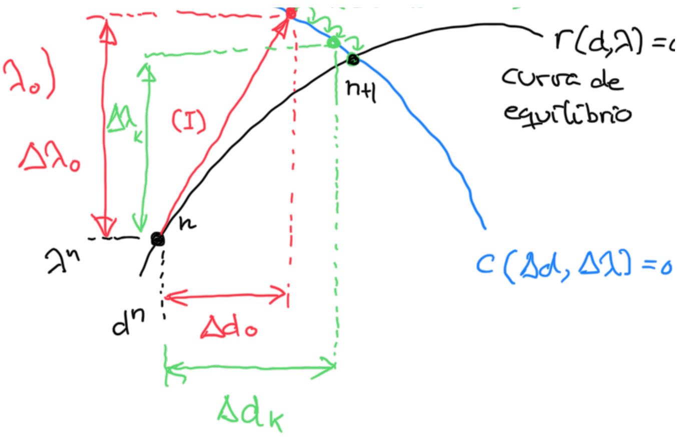
k subiteraciones

(I) Predicción $(\Delta d_0, \Delta \lambda_0)$

(II) Corrección

$(\Delta d_k, \Delta \lambda_k)$

$k = 1, 2, 3, \dots$



(I) Predicción:

$$dr(d,\lambda) = 0 = \underbrace{\frac{\partial r}{\partial d}\Big|_n}\,\Delta d_0 + \underbrace{\frac{\partial r}{\partial \lambda}\Big|_n}\,\Delta \lambda_0 = 0$$

$\rightarrow$ $r = f(\lambda) - p(d)$

$$\frac{\partial r}{\partial d} = -\underline{\frac{\partial p}{\partial d}} \qquad \frac{\partial f(\lambda)}{\partial \lambda} = f_{ext}\ (\text{cte}) \equiv \text{vector de carga (cte)}$$

$$K_T\big|_n \Delta d_0 = \Delta \lambda_0 f_{ext} \Rightarrow \Delta d_0 = \underbrace{\left\{K_T\big|_n^{-1} f_{ext}\right\}}\Delta \lambda_0$$

V = vector de velocidad incremental

$$\boxed{\Delta d_0 = \Delta \lambda_0 V}$$

Tangente a la curva de eq.

$$V = K_{T_n}^{-1} \underline{f_{ext}}$$

└ vector de inc. carga

Control hiperesférico:

$$c(\Delta d_0, \Delta \lambda_0) = \Delta d_0^T \Delta d_0 + \Delta \lambda_0^2 - \ell^2 = 0$$

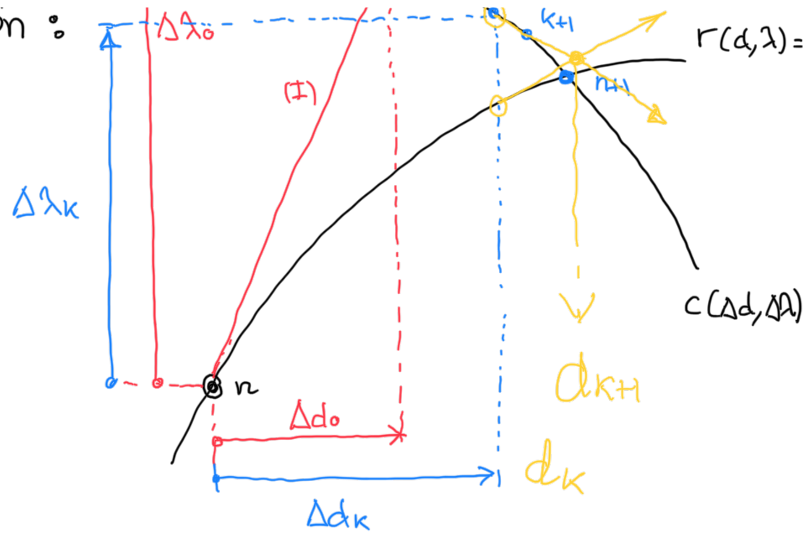
$\uparrow$
$\Delta d_0 = \Delta \lambda_0 V$

$$c(\Delta d_0, \Delta \lambda_0) = \Delta \lambda_0^2 V^T V + \Delta \lambda_0^2 - \ell^2 = 0$$

$$\Delta \lambda_0^2 = \frac{\ell^2}{1 + V^T V} \Rightarrow \boxed{\Delta \lambda_0 = \pm \frac{\ell}{\sqrt{1 + V^T V}}}$$

$$\text{Predicción} \quad (\Delta d_0, \Delta \lambda_0) = \pm \frac{\ell}{\sqrt{1 + V^T V}} (V, 1)$$

(II) Corrección:



- Avanzar desde subiteración "K" $(\Delta d_K, \Delta\lambda_K) \to (\Delta d_{K+1}, \Delta\lambda_{K+1})$

$$r_{K+1} = r_K + \underbrace{\left.\frac{\partial r}{\partial d}\right|_K}_{-K_T|_K} \underbrace{(d_{K+1} - d_K)}_{\delta d_K} + \underbrace{\left.\frac{\partial r}{\partial \lambda}\right|_K}_{f_{ext}} \underbrace{(\lambda_{K+1} - \lambda_K)}_{\delta\lambda_K} = 0$$

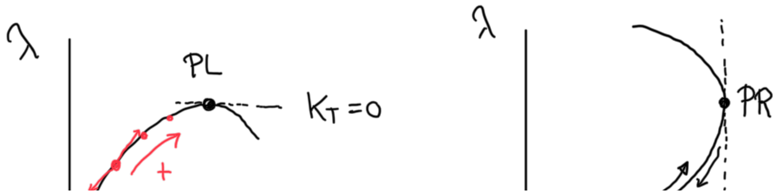
$\rightarrow$

$$c_{K+1} = c_K + \underbrace{\left.\frac{\partial c}{\partial d}\right|_K}_{a^T} \underbrace{(d_{K+1} - d_K)}_{\delta d_K} + \underbrace{\left.\frac{\partial c}{\partial \lambda}\right|_K}_{g} \underbrace{(\lambda_{K+1} - \lambda_K)}_{\delta\lambda_K} = 0$$

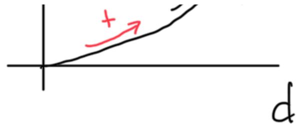
$$\begin{bmatrix} K_T|_K & -f_{ext} \\ a^T & g \end{bmatrix} \begin{bmatrix} \delta d_K \\ \delta\lambda_K \end{bmatrix} = \begin{bmatrix} r_K \\ -c_K \end{bmatrix} \quad \begin{matrix} (N \text{ ecs eq}) \\ (1 \text{ ec. restr}) \end{matrix}$$

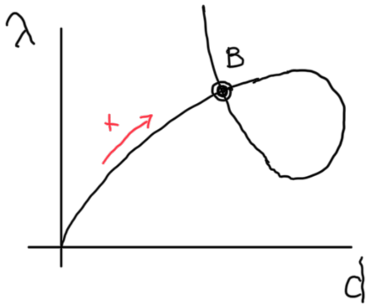
$$\text{Resolvemos} \quad \begin{bmatrix} \delta d_K \\ \delta\lambda_K \end{bmatrix} \Rightarrow \begin{cases} d_{K+1} = d_K + \delta d_K \\ \lambda_{K+1} = \lambda_K + \delta\lambda_K \end{cases}$$

Recorrido de la curva de equilibrio:



K_T singular





$f_{ext} \perp V$

① Criterio de trabajo (de las fuerzas externas) positivo:

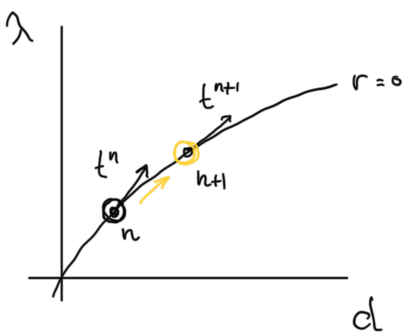
$$\Delta W_{ext} = f_{ext}^T \Delta d > 0 \qquad f_{ext} V \Delta\lambda_0 > 0$$

$$\boxed{Sgn(\Delta\lambda_0) = Sgn(f_{ext}^T V)}$$

Efectivo en PL. $\qquad V = K_T^{-1} f_{ext}$

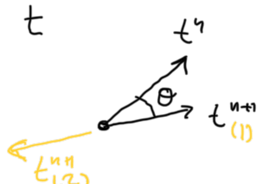
No efectivo en PR, B.

② Criterio del ángulo:



$dr = 0 \Rightarrow t$

$t^{n+1\,T} t^n > 0$



$$t^n = \frac{1}{\sqrt{1+v^{nT}v^n}} \begin{bmatrix} v^n \\ 1 \end{bmatrix} \qquad t^{n+1\,T} t^n > 0$$

$$(\Delta d_0^{n+1}, \Delta\lambda_0^{n+1})^T (\Delta d_0^n, \Delta\lambda_0^n) > 0$$

$$\Delta d_0^{n+1\,T} \Delta d_0^n + \Delta\lambda_0^{n+1} \Delta\lambda_0^n > 0 \qquad \Delta d_0^{n+1} = \Delta\lambda_0^{n+1} v^{n+1}$$

$$\Delta\lambda_0^{n+1} (v^{n+1\,T} \Delta d_0^n + \Delta\lambda_0^n) > 0$$

$$\boxed{sgn(\Delta\lambda_0^{n+1}) = sgn(\Delta\lambda_0^n + V^{n+1\,T}\Delta d_0^n)}$$

Método arc-length linealizado

(I) Predicción:

$$\Delta d_0 = \Delta\lambda_0 V^n \qquad V^n = K_T^{-1}\big|_{d^n} f_{ext} \qquad (1)$$

Control hiperesférico:

$$c(\Delta d, \Delta\lambda) = \underline{\Delta d^T \Delta d} + \underline{\Delta\lambda^2 \underline{\underline{\psi^2 f_{ext}^T f_{ext}}}} - \ell^2 = 0 \quad (2$$

ψ ≡ parámetro conocido.

$$\Delta\lambda_0 = \pm \frac{\ell}{(V^{nT}V^n + \psi^2 f_{ext}^T f_{ext})^{1/2}}$$

C1 : Trabajo externo positivo : $\pm \equiv sgn(V^{nT} f_{ext})$

C2 : Criterio del ángulo : $\pm \equiv sgn(\Delta\lambda_0^n + V^{nT}\Delta d_0^n)$

sol: $(\Delta\lambda_0 V^n, \Delta\lambda_0)$ $\qquad \Delta d_0 = \Delta\lambda_0 V^n$

(II) Corrección: $(d_k, \lambda_k) \rightarrow (d_{k+1}, \lambda_{k+1})$

$$\begin{array}{c} (N) \\ \rightarrow (1) \end{array} \begin{bmatrix} K_T & -f_{ext} \\ 2\Delta d^T & 2\Delta\lambda\psi^2 f_{ext}^T f_{ext} \end{bmatrix} \begin{bmatrix} \delta d_k \\ \delta\lambda_k \end{bmatrix} = \begin{bmatrix} r_k \\ -c_k \end{bmatrix} \begin{array}{c} (a) \\ (b) \end{array}$$

$$\frac{\partial c}{\partial d} = 2\Delta d^T$$

$$r_k + \frac{\partial r}{\partial d}\delta d + \frac{\partial r}{\partial \lambda}\delta\lambda = 0$$

∂a

$$\frac{\partial c}{\partial \lambda} = 2\Delta\lambda\Psi^2 f_{ext}^T f_{ext}$$

$r = \lambda f_{ext} - p(d)$

$\frac{\partial r}{\partial d} = -K_T \qquad \frac{\partial r}{\partial \lambda} = f_{ext}$

$K_T \delta d - f_{ext}\delta\lambda = r_k$

(a) $\delta d_k = K_T^{-1}\left(r_k + f_{ext}\delta\lambda_k\right)$

(b) $2\Delta d^T K_T^{-1}\left(r_k + f_{ext}\,\underline{\underline{\delta\lambda_k}}\right) + 2\Delta\lambda\Psi^2 f_{ext}^T f_{ext}\,\underline{\underline{\delta\lambda_k}} = -c_k$

$$\begin{cases} \delta\lambda_k = \dfrac{-c_k - 2\Delta d^T K_T^{-1} r_k}{2\Delta d^T K_T^{-1} f_{ext} + 2\Delta\lambda\Psi^2 f_{ext}^T f_{ext}} & (*) \\[2ex] \delta d_k = K_T^{-1}\left(r_k + \delta\lambda_k f_{ext}\right) & (**) \end{cases}$$

$$\begin{cases} K_T X = r_k \rightarrow X = K_T^{-1} r_k \leftarrow \\ K_T Y = f_{ext} \rightarrow Y = K_T^{-1} f_{ext} \leftarrow \end{cases}$$

$$\begin{cases} \underline{\delta\lambda_k} = \dfrac{-c_k - 2\Delta d^T X}{2\Delta d^T Y + 2\Delta\lambda\Psi^2 f_{ext}^T f_{ext}} & (*) \\[2ex] \underline{\delta d_k} = X + \delta\lambda_k Y & (**) \end{cases}$$

$$\begin{cases} \lambda_{k+1} = \lambda_k \,\delta\lambda_k \\ d_{k+1} = d_k + \delta d_k. \end{cases}$$

<u>Comprobación de la convergencia:</u>

- Residuo: $\dfrac{\|r_k\|}{\|r_0\|} < \varepsilon \Rightarrow$ Convergencia

- En desplazamiento: $\dfrac{\|\delta d_k\|}{\|\Delta d_0\|} < \varepsilon \Rightarrow$ Convergencia.

Solución es (d_{k+1}, λ_{k+1})

Implementación del método arc-length:

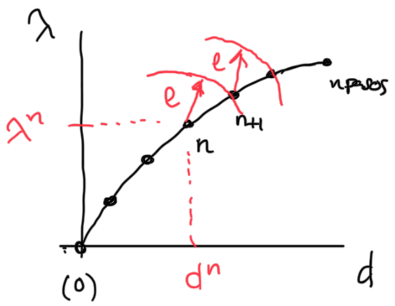
Datos:

data.forces $\longrightarrow$ f_{ext} (cte) $\qquad f(\lambda) = \lambda f_{ext}$.

data.ℓ $\longrightarrow$ longitud de arco

data.ψ $\longrightarrow$ $\begin{cases} \psi = 1 & \text{Hiperesférico} \\ \psi = 0 & \text{Hipercilíndrico.} \end{cases}$

npasos.



for $i = 1$: npasos

⋮

$[d^{n+1}, \lambda^{n+1}] = \text{Arclength}(d^n, \lambda^n, \text{data})$

⋮

end.

Pseudocódigo de la función Arc-Length:

$k_{max} = 100$ nº max de subiteraciones

$\varepsilon = 1 \cdot 10^{-5}$ epsilon

$\ell = 0.1$

$\psi = 0$ control hipercilíndrico.

f = data.forces ($f_{ext.}$ en la teoría.)

$K_T(d^n)$ Matriz de rig. tangente inicial

(Predicción)

$$K_T \, v^n = f \longrightarrow v^n$$ (no olvidar cond. cont.)

$$\Delta\lambda_0 = \mathrm{sgn}(f^T v^n) \frac{\ell}{\left(v^{nT} v^n + \psi^2 f^T f\right)^{1/2}}$$

Trabajo. ext. positivo.

$$\Delta d_0 = \Delta\lambda_0 \, v^n$$

$$\begin{cases} d^{n+1} = d^n + \Delta d_0 \\ \lambda^{n+1} = \lambda^n + \Delta\lambda_0 \end{cases}$$

$$P_K = P_{int}(d^{n+1})$$

$$r_K = \lambda^{n+1} f - P_K$$ (no olvidar eliminar reacciones)

$$r_0 = \| r_K \| ,$$

$$c_K = 0$$

$\Delta d_k = \Delta d_0$
$\Delta \lambda_k = \Delta \lambda_0$

(Corrección)

$k = 1$

while $\left((k \leq k\,max) \; \&\& \left(\frac{\|r_k\|}{\|r_0\|} \geq \varepsilon \right) \right)$

$K_T(d^{n+1})$

$K_T\, y = f \longrightarrow y$ (aplicar c.c.)

$K_T\, x = r_k \longrightarrow x$ (aplicar c.c.)

$$\delta\lambda_k = \frac{-c_k - 2\,\Delta d_k^T x}{2\,\Delta d_k^T y + 2\,\Delta\lambda_k \psi^2 f^T f}$$

$$\delta d_k = x + \delta\lambda_k\, y$$

$$\Delta\lambda_k = \Delta\lambda_k + \delta\lambda_k$$

$$\Delta d_k = \Delta d_k + \delta d_k$$

$$c_k = \Delta d_k^T \Delta d_k + \Delta\lambda_k^2 \psi^2 f^T f - \ell^2$$

$$d^{n+1} = d^{n+1} + \delta d_k$$
$$\lambda^{n+1} = \lambda^{n+1} + \delta\lambda_k$$

$$P_k = P_{int}(d^{n+1})$$

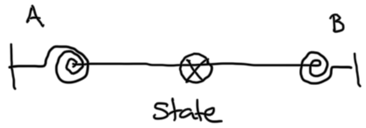
$$r_k = \lambda^{n+1} q - P_k$$ (eliminar reacciones)

$$\frac{\|r_k\|}{r_0} < \varepsilon$$

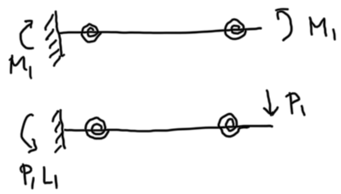
$$k = k+1$$

end

Recapitulación:



$\boxed{\begin{array}{l} \text{State:} \\ M_A,\ \theta_{rA} \\ M_B,\ \theta_{rB} \\ k_A,\ k_B \end{array}}$ $\quad k_A = \frac{M_A}{\theta_{rA}} \quad k_B = \frac{M_B}{\theta_{rB}}$



Elástico lineal $\left\langle \begin{array}{l} \theta_{rA} = \theta_{rB} = 0 \\ \text{Sin } P\Delta \end{array} \right.$

new_state = Connection2D(material, deps, prev_state)

$$\text{deps} = [\Delta\theta_A \quad \Delta\theta_B]$$

$$\left.\begin{array}{l} \Delta M_A = \frac{EI}{L}[4\Delta\theta_A + 2\Delta\theta_B] \\ \Delta M_B = \frac{EI}{L}[2\Delta\theta_A + 4\Delta\theta_B] \end{array}\right\}$$

$$\left\{\begin{array}{l} M_A = \text{prev\_state}.M_A + \Delta M_A \\ M_B = \text{prev\_state}.M_B + \Delta M_B \end{array}\right.$$

KN·m , MN·m

$$\theta_{rA} = C_1 K M_A + C_2 (K M_A)^3 + C_3 (K M_A)^5 \qquad N.m$$

$$\theta_{rB} = C_1 K M_B + C_2 (K M_B)^3 + C_3 (K M_B)^5$$

new_state. $M_A = M_A$
new_state. $M_B = M_B$
new_state. $\theta_{rA} = \theta_{rA}$
new_state. $\theta_{rB} = \theta_{rB}$
new_state. $K_A = M_A / \theta_{rA}$
new_state. $K_B = M_B / \theta_{rB}$

end

$$\theta_{rA} = C_1 K M_A + C_2 (K M_A)^3 + C_3 (K M_A)^5$$

$$\frac{\partial \theta_r}{\partial M_A} = C_1 K + 3 C_2 (K M_A)^2 K + \dots$$

$$\lim_{M_A \to 0} \frac{\partial \theta_r}{\partial M_A} = C_1 K \qquad \boxed{K_A \Big|_{M_A = 0} = \frac{1}{C_1 K}}$$

Ke = KeBeam2D (material, nodes, element_state)

K_A = element_state. K_A
K_B = element_state. K_B
$K_R = K_R(K_A, K_B) \quad r_{ii}(K_A, K_B) \quad r_{jj}(K_A, K_B) \quad r_{ij}(K_A, K_B)$

$$Ke = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & r_{ii} & r_{ij} & r_{ij} & & \\ & & & & & \end{bmatrix}$$

end

Pe = PeBeam2D (material, nodes, element_state, element_disp)

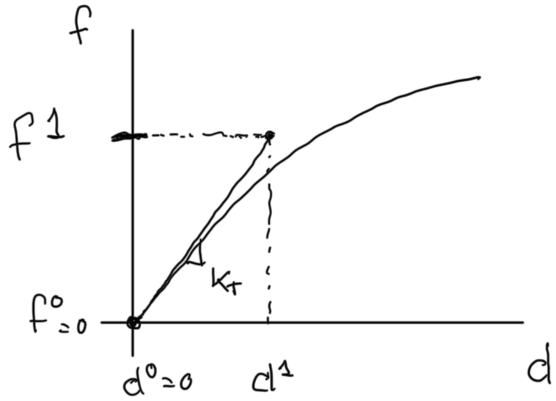
P = K × element_disp

$fe = Ke * element\_disp.$

$$\begin{bmatrix} N_A \\ V_A \\ M_A \\ N_B \\ V_B \\ M_B \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} u_{xA} \\ u_{yA} \\ \theta_A \\ u_{xB} \\ u_{yB} \\ \theta_B \end{bmatrix}$$

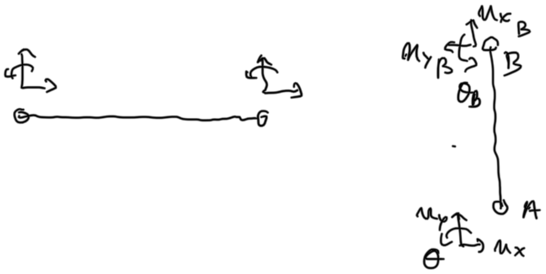
end.

$$f = \begin{bmatrix} F_{xA} \\ F_{yA} \\ M_A \\ F_{xB} \\ F_{yB} \\ M_B \end{bmatrix}$$



$$\Delta d^1 = d^1 - d^0$$

$$K_T \Delta d = r \rightarrow \Delta d$$



L

$$\hat{Ke} = L^T Ke L$$

Práctica:

OJO: $r = p(d) - \lambda f_{ext}$!! $\begin{cases} \frac{\partial r}{\partial d} = K_T \\ \frac{\partial r}{\partial \lambda} = -f_{ext} = q \end{cases}$

d^{n+1} λ^{n+1} d^n λ^n

function [u, lambda] = ArcLength(u0, lambda0, data)

maxiter	nº máx. de iteraciones (data)
epsilon	epsilon (data)
incl	longitud de arco (data)
psi = 0	parámetro de escalado (data) $\psi = 0$

q = data.forces

$$f(\lambda) = \lambda q = \lambda f_{ext}$$

q2 = $q^T q$

Kt0 = Ktangente(data, u0)

deltauq = solveLS(Kt0, q) $\qquad v^n = K_T^{-1} q$

deltauq2 = deltauq^T deltauq $\qquad v^{nT} v^n$

signo = sign(q^T deltauq)

deltalambda0 = signo * incl / sqrt(deltauq2 + $\psi^2 q^2$)

$$\Delta\lambda_0 = \operatorname{sgn}(q^T v) \frac{\ell}{\sqrt{v^T v + \psi^2}}$$

inclambda = deltalambda0

incu = deltalambda0 * deltauq $\qquad \Delta u_0 = \Delta\lambda_0 v^n$

u = u0 + incu $\qquad u^{n+1} = u^n + \Delta u_0$

lambda = lambda0 + inclambda $\qquad \lambda^{n+1} = \lambda^n + \Delta\lambda_0$

p = assembleP(data, u)

r = p − lambda * q

r(data.fixed) = 0 $\qquad r_0$

rnorm = norm(r) $\qquad \|r_0\|$

c = 0 $\qquad c_0 = 0$

iter = 1

while iter <= maxiter && rnorm >= epsilon

KT = asembleK(data, u)

deltauq = solveLS(data, q) $\qquad y = K_T^{-1} q$

deltaug = solveLS(data, −r) $\qquad x = -K_T^{-1} r$

deltalambda = (−c/2 − incu^{T}deltaug) / (incu^T deltauq + inclambda $\psi^2 q^2$)

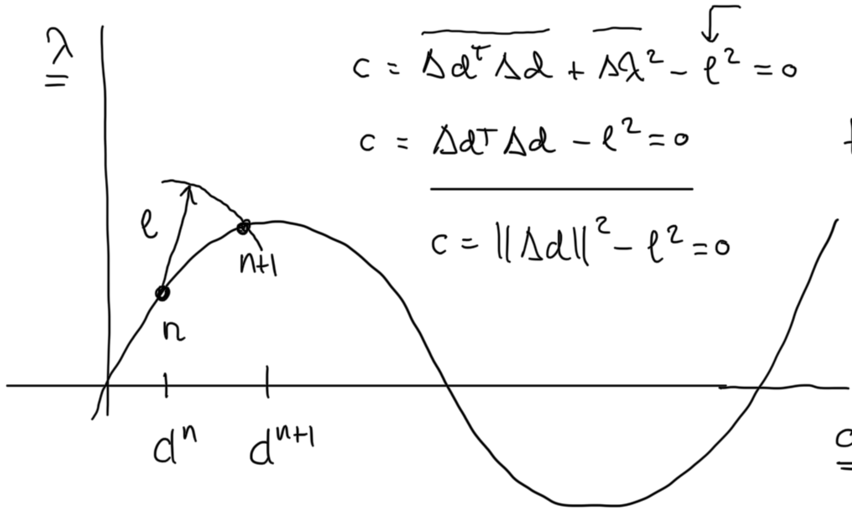
$$\delta\lambda = \frac{-c_k - 2\Delta u^T x}{\quad} = -c_k/2 - \Delta u^T x$$

$$\delta\lambda = \frac{\quad}{2\Delta u^T y + 2\Delta\lambda\psi^2 q^2} = \frac{\quad}{\Delta u^T y + \Delta\lambda\psi^2 q^2}$$

$\delta u = x + \delta\lambda y$

```
deltau = deltau_g + deltalambda deltau_q
inclamba = inclambda + deltalambda
incu = incu + deltau
c = incu^T incu + (inclamba)^2 ψ^2 q^2 - incl^2
u = u0 + incu
p = assemble p (data, u)
lambda = lambda0 + inc lambda
r = p - lambda * q
r(data.fixed) = 0
r norm = ||r||
iter = iter + 1
```

end



$c = \overline{\Delta d^T \Delta d} + \overline{\Delta\lambda}^2 - \ell^2 = 0$ **Hiperesférico** $\psi \neq 0$

$c = \Delta d^T \Delta d - \ell^2 = 0$ Hipercilíndrico $\psi = 0$

$c = \|\Delta d\|^2 - \ell^2 = 0$

$\ell = \|\Delta d\| = \|d^{n+1} - d^n\|$ (Hipercilíndrico)

$$\begin{cases} \ell = \sqrt{\|\Delta d\|^2 + \Delta\lambda^2} \\ \ell = \|\Delta d\| \end{cases}$$

$\|a\|_2 = \left(\sum_i a_i^2\right)^{1/2}$