Technical Note

Active versus Passive Rotations

Tomás Soler, Ph.D., M.ASCE[1]

Abstract: Active and passive rotations are used in many applied scientific fields, such as engineering, geodesy, and geophysics, just to name a few. However, a source of confusion could arise when both types of rotations are performed sequentially. The author is not aware of any publication where, in a tutorial manner, the relationship between active and passive rotations is coherently described, untangling in the process some of the most typical misconceptions. This technical note is a modest attempt to remedy this void by clarifying, as much as possible, some essential points that may help the understanding of these two varieties of commonly used rotations and how to properly apply them in some practical situations. **DOI:** 10.1061/(ASCE)SU.1943-5428.0000247.

Introduction

Rotations of coordinate frame axes (passive rotations) and physical vectors (active rotations) are profusely invoked in countless scientific disciplines [e.g., Millot and Man (2012)]. These days, frame rotations are a major part of geodesy and surveying triggered by the advent of global navigation satellite system (GNSS) techniques mainly focused on the procurement of accurate positioning. Currently, the three-dimensional (3D) coordinates of a point can be determined with accuracies of only a few centimeters anywhere around the globe when the proper GNSS hardware and software are accessible. Because GNSS-processed results depend on 3D time-variant reference frames, the transformation of coordinates between these frames becomes of critical importance if one is interested in comparing results that were archived at different epochs (Soler 1998).

In contrast, the fields of mechanics and, in particular, dynamics constantly study the interaction of rotating bodies. For example, in the physical phenomenon of plate tectonics, points located on crustal plates rotate with respect to a body-fixed frame, and the study of their displacements assumes the understanding of the theory behind body rotations.

Therefore, to avoid any possible misunderstandings, it is imperative to make a clear distinction between rotation of frame axes and rotation of vectors. This technical note introduces how these two kinds of rotations interrelate, reintroducing some basic definitions that will make the comprehension of this topic clear enough to avoid common mistakes and, at the same time, permitting the eradication of the most puzzling part of the problem, which is how to deal with the sign conventions implicit in any presumed rotation.

A note of caution, at times the arguments may seem repetitive. However, this was done intentionally to avoid possible mix-ups and to reinforce the full understanding of the crucial concepts.

Definitions

First, some basic definitions related to rotations are introduced. Restricting this discussion to the 3D Euclidean space (E^3), rotations are generally taken about one axis, which is called the axis of rotation. Rotations in E^3 always conserve angles and distances—or rephrased differently, spatial angles and baseline lengths are invariant under rotation. If one assumes a 3D Cartesian frame denoted by (x, y, z), there is always the possibility of taking rotations about each and every one of these three frame axes. As is explained later, three consecutive rotations about the three axes could be transformed into a single rotation about a unique axis going through the origin of the frame. In plate tectonics jargon, this axis of rotation is called the Euler axis. The direction in which the Euler rotation axis is pointing is called the Euler pole.

The coordinates of arbitrary points that refer to the aforementioned general frame are transformed through the application of rotation matrices. However, because a rotation about any axis could be clockwise or counterclockwise (anticlockwise), it is mandatory to select beforehand the sign convention of the rotation; hence, the following definition is introduced: all counterclockwise rotations about axes executed herein are assumed positive. Then, the following corollary can be stated: all clockwise rotations will have an opposite sign to those performed counterclockwise.

In mathematical verbiage, rotations are transformations executed through so-called rotation matrices. The three fundamental rotation matrices of dimension 3×3 about the three Cartesian axes (1, 2, 3) by an arbitrary angle ($0 \leq \theta \leq 2\pi$) could be explicitly written as follows (Kaula 1966, p. 13; Mueller 1969, p. 43):

$$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix};$$

$$\mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix};$$

$$\mathbf{R}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1)$$

The previous three rotations are consistent with the previously mentioned sign convention that all counterclockwise rotations of axes (frames) are positive. Then, if clockwise rotations of axes are desired, the following rotation matrix should be used: $\mathbf{R}_i(-\theta) =$

[1]Retired, National Geodetic Survey, National Oceanic and Atmospheric Administration, 1315 East-West Highway, Silver Spring, MD 20910-3282. E-mail: tomsolerngs@gmail.com

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$\mathbf{R}_i^T(\theta)$; $i = 1, 2, 3$, where the superindex T stands for matrix transpose. Furthermore, being the rotation matrices positive orthogonal matrices with determinant equal to $+1$, then $\mathbf{R}_i^T(\theta) = \mathbf{R}_i^{-1}(\theta)$. Finally, rotations about the same axis are additive: $\mathbf{R}_i(\alpha) + \mathbf{R}_i(\beta) = \mathbf{R}_i(\alpha + \beta)$.

At this point, it should be emphasized that, in reference to a 3D frame, one can introduce (represent) abstract concepts, such as points and vectors. In fact, as usual, a point will be defined by three coordinates and a vector by their three components. Moreover, in the context of this discussion, one is only going to be concerned with vectors whose starting points are at the origin of the frame, and whose tips (ending point of the vector) are at the location of the point in question (*position vectors*). If the origin of the frame is assumed at the Earth's center of mass (CM), the vectors associated with the points could be called geocentric vectors.

Passive Rotations (Rotation Matrix $\mathbf{R}$)

A passive rotation is a positive counterclockwise rotation around any of the coordinate axes of the frame (i.e., frame rotation) while the points referring to this frame remain fixed in space (axis rotates, points remain static). Consequently, once the frame has been rotated, an arbitrary point (which did not move during the rotation) will have two sets of coordinates, one set referring to the original frame before the rotation took place, and a second set referring to the frame that resulted after the rotation was performed. The rotation matrix involved in this assumed counterclockwise rotation of frames is represented by any of the three matrices ($\mathbf{R}$) in Eq. (1) (passive rotation $\Rightarrow$ frame rotates, points remain fixed in space).

Active Rotations (Rotation Matrix $\Re$)

An active rotation implies a positive counterclockwise rotation around an arbitrary axis that rotates 3D points (the tips of the vectors from the origin of the frame) while the frame remains fixed during the rotation. These classes of rotations are also called body or vector rotations. The rotation matrix involved in the counterclockwise rotation of vectors is represented by $\Re$ [active rotation $\Rightarrow$ points (vectors) rotate, frame remains fixed in space].

The matrix equivalents to Eq. (1) when counterclockwise rotations of vectors are considered positive are

$$\Re_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix};$$

$$\Re_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix};$$

$$\Re_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

If clockwise rotations of vectors are then desired, the following rotation matrix should be used:

$$\Re_i(-\theta) = \Re_i^T(\theta); \quad i = 1, 2, 3$$

It is important to stress here that passive and active rotations can be performed sequentially when the appropriate sign precautions are taken. As can be seen graphically below, active and passive rotations are opposite in sign, although by definition both types of rotation are assumed positive if counterclockwise.

Proof That Active and Passive Rotations Have Opposite Signs

As mentioned earlier, the basic definition of the three rotation matrices given in Eq. (1) is consistent with positive counterclockwise rotation of axes (frames). By a simple geometric argument that is very straightforward, it is easy to prove that a counterclockwise rotation of axes is equal to a clockwise rotation of vectors if the basic rotation matrix definitions presented in Eqs. (1) and (2) are followed.

Assume a 3D Cartesian frame (x, y, z) where, for simplicity's sake, Point A with coordinates (1, 1, 0) is selected on the x,y-plane, as depicted in Fig. 1. The z-axis is normal to the plane of the figure. Then, the following question is stated: what will be the new coordinates of this point if a counterclockwise rotation around the frame third-axis z by a value of $\theta = \pi/4 = 45°$ is performed? Calling the rotated frame (x', y', z'), and knowing that $\sin\theta = \cos\theta = 1/\sqrt{2}$ immediately follows (Fig. 1)

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \mathbf{R}_3(\theta) \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$
$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sqrt{2} \\ 0 \\ 0 \end{Bmatrix} \tag{3}$$

Exactly the same result is obtained if the vector $\mathbf{v} = (v_x, v_y, v_z) = (1, 1, 0)$ is rotated clockwise by the same angle θ (Fig. 1). Because the vector is rotated clockwise, and by definition counterclockwise rotation of vectors is positive, then $\theta = -45°$; therefore

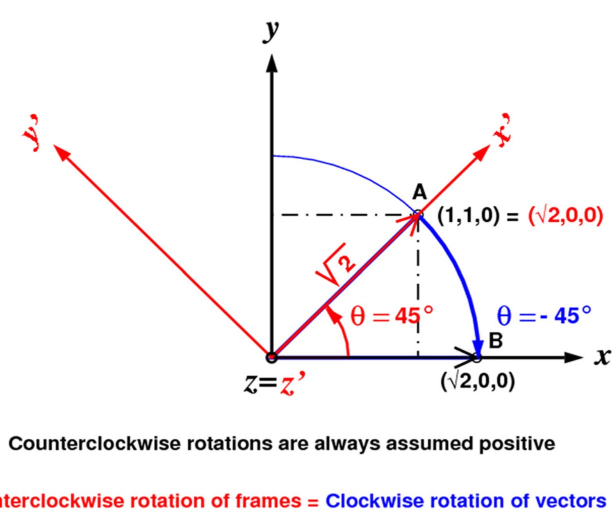

Fig. 1. Schematic graphical depiction showing that counterclockwise rotation of frame axes is equivalent to clockwise rotation of vectors

$$\Re_3(-\theta)\begin{Bmatrix}1\\1\\0\end{Bmatrix}=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\-1/\sqrt{2} & 1/\sqrt{2} & 0\\0 & 0 & 1\end{bmatrix}\begin{Bmatrix}1\\1\\0\end{Bmatrix}=\begin{Bmatrix}\sqrt{2}\\0\\0\end{Bmatrix} \quad (4)$$

confirming that a counterclockwise rotation of the frame axis is identical to a clockwise rotation of vectors.

Consequently, the three basic rotation matrices in Eqs. (1) and (2) could be used to transform coordinates or vectors when counterclockwise rotations are assumed positive. The final coordinates of the transformed point are the same, although two different types of rotation matrix definitions are implemented. Similarly, by using the same logic, it can be affirmed that a clockwise rotation of axes is equal to a counterclockwise rotation of vectors, or in mathematical terms (Fig. 2)

$$\begin{Bmatrix}\bar{x}\\\bar{y}\\\bar{z}\end{Bmatrix}=\mathbf{R}_3(-\theta)\begin{Bmatrix}1\\1\\0\end{Bmatrix}=\mathbf{R}_3^T(\theta)\begin{Bmatrix}1\\1\\0\end{Bmatrix}$$
$$=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}\begin{Bmatrix}1\\1\\0\end{Bmatrix}=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2} & 0\\1/\sqrt{2} & 1/\sqrt{2} & 0\\0 & 0 & 1\end{bmatrix}$$
$$\times\begin{Bmatrix}1\\1\\0\end{Bmatrix}=\begin{Bmatrix}0\\\sqrt{2}\\0\end{Bmatrix}=\Re_3(\theta)\begin{Bmatrix}1\\1\\0\end{Bmatrix} \quad (5)$$

Therefore, if a clockwise rotation of the axis is performed, the transpose of the matrices given in Eq. (1) should be used. Equivalently, if counterclockwise rotations of vectors are executed, the matrices in Eq. (2) should be implemented. This sign convention dichotomy has generated some confusion when defining rotations of geodetic frames and the sense of the rotations is not explicitly indicated in advance (Soler 1997).

In summary, Eqs. (1) and (2) are the three fundamental rotation matrices to rotate counterclockwise axes (frames) and vectors, respectively. Herein, to enforce clarity, all rotations (of axes and vectors) are assumed positive counterclockwise. However, under this hypothesis, two matrix representations were introduced, resulting in

$$\mathbf{R}_3(\theta)=\Re_3(-\theta) \quad (6)$$

Thus, counterclockwise rotation of frames is equal to clockwise rotation of vectors. In contrast, clockwise rotation of frames and counterclockwise rotation of vectors are equivalent rotation operations (Fig. 2)

$$\mathbf{R}_3(-\theta)=\Re_3(\theta) \quad (7)$$

Accordingly, the previously introduced definitions are consistent; counterclockwise rotations of frames and vectors around an arbitrary axis are always assumed positive, and they are enforced by the matrix rules explicitly given by Eqs. (1) and (2).

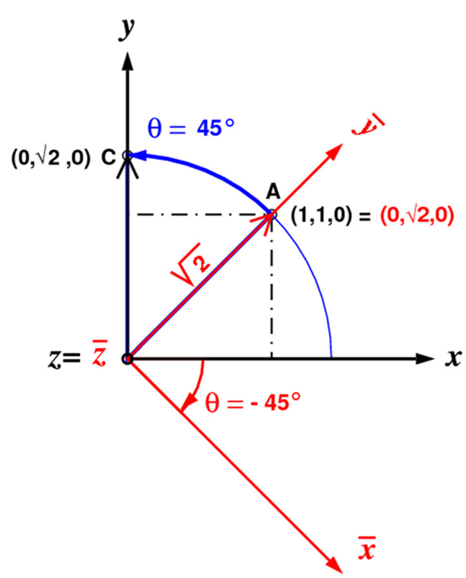

Fig. 2. Schematic graphical depiction showing that counterclockwise rotation of vectors is equivalent to clockwise rotation of frame axes

General Matrix Form of a Rotation about an Arbitrary Axis

The theory formulating the positive counterclockwise rotation of a vector (active or body rotation) around an arbitrary axis was originally introduced by Euler (1775) and revived a century later by Thomson and Tait (1879, p. 71). These equations are available in the literature in different forms [e.g., Pars (1965), p. 97; Soler (1977), Appendix A]. However, using modern matrix symbolic nomenclature currently very familiar to engineers, this type of rotation matrix could be written as the following, easy to remember, mathematical expression:

$$\Re_\ell(\theta)=[1]+\sin\theta\,[\underline{\ell}]+(1-\cos\theta)\,[\underline{\ell}]^2 \quad (8)$$

where ℓ = direction cosines (Fig. 3); and matrix [1] = identity matrix, and, explicitly, the skew-symmetric matrix of the direction cosines and its squared symmetric matrix are

$$[\underline{\ell}]=\begin{bmatrix}0 & -\ell_3 & \ell_2\\\ell_3 & 0 & -\ell_1\\-\ell_2 & \ell_1 & 0\end{bmatrix} \quad \text{and}$$
$$[\underline{\ell}]^2=\begin{bmatrix}-(\ell_2^2+\ell_3^2) & \ell_1\ell_2 & \ell_1\ell_3\\\ell_1\ell_2 & -(\ell_1^2+\ell_3^2) & \ell_2\ell_3\\\ell_1\ell_3 & \ell_2\ell_3 & -(\ell_1^2+\ell_2^2)\end{bmatrix} \quad (9)$$

According to the parametric representation depicted in Fig. 3, it easily follows by definition that the direction cosines are given by

$$\begin{Bmatrix}\ell_1\\\ell_2\\\ell_3\end{Bmatrix}=\begin{Bmatrix}\cos\phi\,\cos\lambda\\\cos\phi\,\sin\lambda\\\sin\phi\end{Bmatrix} \quad (10)$$

where $0\le\lambda\le 2\pi$ (assumed positive counterclockwise) and $-\pi/2\le\phi\le\pi/2$ = angles defining the orientation of a general rotation about any arbitrary axis in space.

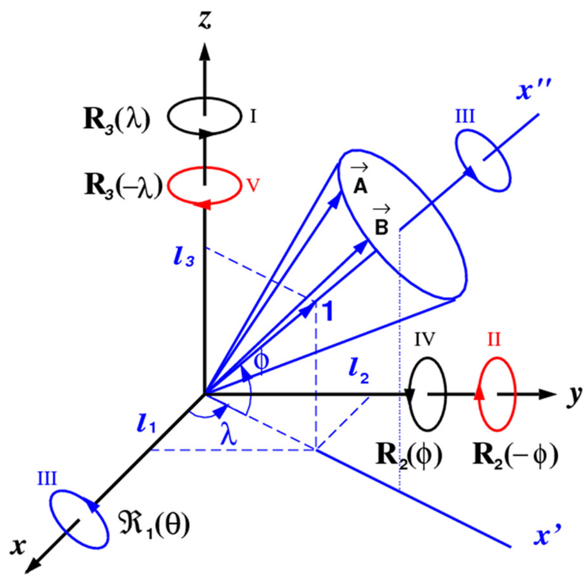

Fig. 3. Rotation of vectors around arbitrary axis determined as a function of rotations around the three axes of the frame

Recall that Eq. (8) was defined assuming positive counterclockwise rotation of vectors; therefore, for the reasons mentioned earlier, the equivalent positive rotation assuming counterclockwise rotation of frames takes the form

$$\mathbf{R}_\ell(\theta) = \Re_\ell(-\theta) = [1] + \sin(-\theta)\ [\ell] + [1 - \cos(-\theta)]\ [\ell]^2 \tag{11}$$

or alternatively

$$\begin{aligned}\mathbf{R}_\ell(\theta) = \Re_\ell^T(\theta) &= [1] - \sin\theta\ [\ell] + (1 - \cos\theta)\ [\ell]^2 \\ &= [1] + \sin\theta\ [\ell]^T + (1 - \cos\theta)\ [\ell]^2\end{aligned} \tag{12}$$

From the previous equation, the three fundamental rotation matrices in Eq. (1) could be easily obtained. For example, the derivation of $\mathbf{R}_1(\theta)$ implies a rotation about the first axis of the frame with known direction cosines $\ell_1 = 1$ and $\ell_2 = \ell_3 = 0$. After substituting these values in Eq. (12) and writing the results explicitly, one has

$$\begin{aligned}\mathbf{R}_1(\theta) = \Re_{\ell_1}^T(\theta) &= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \sin\theta\begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix} \\ &+ (1 - \cos\theta)\begin{bmatrix}0 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \\ &+ \begin{bmatrix}0 & 0 & 0\\0 & 0 & \sin\theta\\0 & -\sin\theta & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix} \\ &+ \begin{bmatrix}0 & 0 & 0\\0 & \cos\theta & 0\\0 & 0 & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & \cos\theta & \sin\theta\\0 & -\sin\theta & \cos\theta\end{bmatrix}\end{aligned} \tag{13}$$

This result was already advanced in Eq. (1). Similarly, the rotation matrices $\mathbf{R}_2(\theta)$ and $\mathbf{R}_3(\theta)$ in Eq. (1) can be obtained. Fig. 4 shows the basic rotation matrices nomenclature as described in this technical note.

Differential Rotation Simplification

Many times in practical applications (e.g., engineering, geodesy, geophysics, photogrammetry, etc.), the assumption that the magnitudes of the rotations are differentially small is introduced. This situation greatly simplifies the mathematical development. For example, although the multiplication of rotation matrices, in general, is not commutative, this property is always fulfilled when the angle of rotation is very small. Consequently, when counterclockwise differential rotation of axes is performed, recalling that $\sin\delta\theta = \delta\theta$ and $\cos\delta\theta = 1$, the following equalities are satisfied:

$$\left.\begin{matrix}\mathbf{R}_3(\delta\theta_3)\mathbf{R}_2(\delta\theta_2)\mathbf{R}_1(\delta\theta_1)\\ \mathbf{R}_2(\delta\theta_2)\mathbf{R}_1(\delta\theta_1)\mathbf{R}_3(\delta\theta_3)\\ \mathbf{R}_1(\delta\theta_1)\mathbf{R}_3(\delta\theta_3)\mathbf{R}_2(\delta\theta_2)\\ \mathbf{R}_1(\delta\theta_1)\mathbf{R}_2(\delta\theta_2)\mathbf{R}_3(\delta\theta_3)\\ \mathbf{R}_2(\delta\theta_2)\mathbf{R}_3(\delta\theta_3)\mathbf{R}_1(\delta\theta_1)\\ \mathbf{R}_3(\delta\theta_3)\mathbf{R}_1(\delta\theta_1)\mathbf{R}_2(\delta\theta_2)\end{matrix}\right\} = \begin{bmatrix}1 & \delta\theta_3 & -\delta\theta_2\\ -\delta\theta_3 & 1 & \delta\theta_1\\ \delta\theta_2 & -\delta\theta_1 & 1\end{bmatrix} = \delta\mathbf{R} \tag{14}$$

The same differential rotation matrix ($\delta\mathbf{R}$) is obtained if one assumes differential rotations in Eq. (12)

$$\begin{aligned}\delta\mathbf{R} = \delta\Re_\ell^T(\theta) &= [1] + \sin\delta\theta\ [\ell]^T + \underbrace{(1 - \cos\delta\theta)}_{\text{zero}}\ [\ell]^2 \\ &= \begin{bmatrix}1 & \delta\theta_3 & -\delta\theta_2\\ -\delta\theta_3 & 1 & \delta\theta_1\\ \delta\theta_2 & -\delta\theta_1 & 1\end{bmatrix}\end{aligned} \tag{15}$$

Similarly, using Eq. (8), when differential counterclockwise rotation of vectors is performed

$$\begin{aligned}\delta\Re &= \begin{bmatrix}1 & -\delta\theta_3 & \delta\theta_2\\ \delta\theta_3 & 1 & -\delta\theta_1\\ -\delta\theta_2 & \delta\theta_1 & 1\end{bmatrix} \\ &= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + \begin{bmatrix}0 & -\delta\theta_3 & \delta\theta_2\\ \delta\theta_3 & 0 & -\delta\theta_1\\ -\delta\theta_2 & \delta\theta_1 & 0\end{bmatrix}\end{aligned} \tag{16}$$

resulting in

$$\delta\mathbf{R} = \delta\Re^T \tag{17}$$

Consequently, as mentioned earlier, counterclockwise rotations of axes and vectors, although both are positive counterclockwise, have opposite signs of the angular argument. This is the most confusing part to comprehend; although both rotations are positive counterclockwise, the angular argument to be plugged into the rotation matrix has opposite signs. However, in one case, one is rotating frames (points remain fixed), and in the other, one is rotating points (frame remains fixed); thus, to avoid any possible confusion, different notation for the rotation matrices is introduced ($\mathbf{R}$ versus $\Re$).

Euler Pole of Rotation

Euler pole of rotation is a term commonly used in geophysics, primarily in the field of plate tectonics, to denote counterclockwise rotation of vectors about a prespecified axis with an orientation defined by two angles (λ, ϕ) as previously noted, and by the angular rotation rate about the axis. Generally, this rotation is assumed differentially small

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$$Positive\left\{\begin{matrix}\textbf{counterclockwise}\\ clockwise\end{matrix}\right\} rotation\ \ of\left\{\begin{matrix}\textbf{axes}\ (frames,\ passive\ rotation)\\ vectors\ (body,\ active\ rotation)\end{matrix}\right\} by\ \theta\ use\left\{\begin{matrix}\mathbf{R}_1(\theta);\mathbf{R}_2(\theta);\mathbf{R}_3(\theta)\\ \Re_\ell^{\mathrm{T}}(\theta)\end{matrix}\right.$$

$$Positive\left\{\begin{matrix}\textbf{counterclockwise}\\ clockwise\end{matrix}\right\} rotation\ \ of\left\{\begin{matrix}\textbf{vectors}\ (body,\ active\ rotation)\\ axes\ (frames,\ passive\ rotation)\end{matrix}\right\} by\ \theta\ use\left\{\begin{matrix}\Re_\ell(\theta)\\ \mathbf{R}_1^{\mathrm{T}}(\theta);\mathbf{R}_2^{\mathrm{T}}(\theta);\mathbf{R}_3^{\mathrm{T}}(\theta)\end{matrix}\right.$$

Fig. 4. Mnemonic sketch to remember the signs of the rotation of frames versus rotation of vectors (coordinates)

and of magnitude $\delta\dot{\Omega} = \sqrt{\delta\dot{\Omega}_1^2 + \delta\dot{\Omega}_2^2 + \delta\dot{\Omega}_3^2}$, where $\delta\dot{\Omega}_1$, $\delta\dot{\Omega}_2$, and $\delta\dot{\Omega}_3$ are the components of the vector $\boldsymbol{\delta\dot{\Omega}}$ along the three Cartesian axes. Conversely, if the three components of the angular rotation rate are known, the angles defining the orientation of the axis in space follow immediately (Soler and Han 2017)

$$\lambda = \arctan\left(\frac{\delta\dot{\Omega}_2}{\delta\dot{\Omega}_1}\right) \tag{18}$$

$$\phi = \arctan\left(\frac{\delta\dot{\Omega}_3}{\sqrt{\delta\dot{\Omega}_1^2 + \delta\dot{\Omega}_2^2}}\right) \tag{19}$$

and the direction cosines of the Euler pole axis can be computed using Eq. (10).

Therefore, if the components of the angular rotation rate ($\delta\dot{\Omega}_1$, $\delta\dot{\Omega}_2$, and $\delta\dot{\Omega}_3$) referring to a 3D frame are initially known, the counterclockwise rotation of any arbitrary vector (point) of components (coordinates) $\{x\ y\ z\}_i^T$ could be determined using Eq. (16) after it is adapted to this particular case by changing differential angular rotations $(\delta\theta_1, \delta\theta_2, \delta\theta_3)$ by differential angular rotation rates $(\delta\dot{\Omega}_1, \delta\dot{\Omega}_2, \delta\dot{\Omega}_3)$ during the time interval $(t' - t)$

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}_i = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_i + (t'-t)\begin{bmatrix} 0 & -\delta\dot{\Omega}_3 & \delta\dot{\Omega}_2 \\ \delta\dot{\Omega}_3 & 0 & -\delta\dot{\Omega}_1 \\ -\delta\dot{\Omega}_2 & \delta\dot{\Omega}_1 & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_i = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_i + \begin{Bmatrix} \delta x \\ \delta y \\ \delta z \end{Bmatrix}_i \tag{20}$$

Under the assumption of active rotation, all coordinates (vector components) refer to the same fixed frame (i). The three elements of the column vector $\{x'\ y'\ z'\}_i^T$ represent the vector of rotated coordinates from the original $\{x\ y\ z\}_i^T$ that were generated by the Euler pole of rotation associated with the three components of the angular rotation rate: $\delta\dot{\Omega}_1$, $\delta\dot{\Omega}_2$, and $\delta\dot{\Omega}_3$. Thus, the matrix vector $\{\delta x\ \ \delta y\ \ \delta z\}_i^T$ contains the differential changes to the coordinates caused by the application of an active angular rotation rate with components $\delta\dot{\Omega}_1$, $\delta\dot{\Omega}_2$, and $\delta\dot{\Omega}_3$ in the interval $(t' - t)$.

If, in contrast, the Euler pole of rotation is explicitly given by the parameters (λ, ϕ) and the magnitude of the total rotation $(\delta\dot{\Omega})$, then the following substitution should be made in Eq. (20):

$$\begin{Bmatrix} \delta\dot{\Omega}_1 \\ \delta\dot{\Omega}_2 \\ \delta\dot{\Omega}_3 \end{Bmatrix} = \begin{Bmatrix} \delta\dot{\Omega}\cos\phi\,\cos\lambda \\ \delta\dot{\Omega}\cos\phi\,\sin\lambda \\ \delta\dot{\Omega}\sin\phi \end{Bmatrix} \tag{21}$$

resulting in the following equation:

$$\begin{Bmatrix} \delta x \\ \delta y \\ \delta z \end{Bmatrix}_i = (t'-t) \times \begin{bmatrix} 0 & -\delta\dot{\Omega}\sin\phi & \delta\dot{\Omega}\cos\phi\,\sin\lambda \\ \delta\dot{\Omega}\sin\phi & 0 & -\delta\dot{\Omega}\cos\phi\,\cos\lambda \\ -\delta\dot{\Omega}\cos\phi\sin\lambda & \delta\dot{\Omega}\cos\phi\,\cos\lambda & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_i \tag{22}$$

Comparing Eqs. (20) and (22) shows that the contribution to the coordinates due to differential rotation of vectors using three rotations around the axes (namely, $\delta\dot{\Omega}_1$, $\delta\dot{\Omega}_2$, $\delta\dot{\Omega}_3$) could be substituted by a unique rotation around the Euler rotation axis pole with orientation angles (λ, ϕ)determined through Eqs. (18) and (19) and the total angular rotation rate $(\delta\dot{\Omega})$. Notice that Eq. (20) is consistent with a positive counterclockwise (active) rotation of vectors, but the same argument could be made for the case of counterclockwise rotation of axes (frames). A recently published article proves numerically, through a rigorous least-squares approach, the application of both alternatives to the theory of plate tectonics (Soler and Han 2017). Similar logic is applied to correct for the effect of the rotation of the plate when the concept of a classical *plate-fixed* geodetic datum is enforced. More about this is discussed later.

Alternative Form of Eq. (8) Using Rotation Matrices Exclusively

Once familiarity with rotation matrices is grasped, a new example that includes all types of rotations described earlier is presented. There is an alternative way to perform a general rotation about an axis other than the direct formulation shown in Eq. (8), and this involves the sequential application of several types of rotations. Recall that rotation matrices always operate from right to left, and this property is determinant in the geometric derivation that follows, which includes counterclockwise and clockwise rotations of frame axes (passive rotations) and counterclockwise rotation of vectors (active rotation). It should also be stressed here that once the sequential product of rotation matrices is established, the algebraic multiplication of a series of concatenated matrices can be done in any arbitrary order.

The idea behind the whole procedure is graphically depicted in Fig. 3. It consists of rotating one of the frame axes into an arbitrary axis with position angles (λ, ϕ) where the rotation of vectors by an angle θ as described by Eq. (8) is performed.

To understand the step-by-step process seen in Fig. 3, the order of all the operations involved are described as follows:

- I. Counterclockwise rotation of the frame about the z-axis by an amount λ. Notice that the x-axis will move to position x' (see Fig. 3).
- II. Clockwise rotation about the y-axis by an amount ϕ. As a consequence of this rotation, the x'-axis will move to x''.

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- III. Counterclockwise rotation of vectors about the x''-axis (the so-called Euler pole axis) by an amount θ (vector **A** moves into position **B**). The frame ($x'' \equiv x$, y'' and z'' not shown in Fig. 3) remains fixed in space during this rotation. Note that, because it is a rotation about an x-axis, the rotation matrix $\Re$ has a subindex of 1. To finalize the procedure, the current frame (x'', y'', z'') is now rotated back to its original position, backtracking the original rotations.
- IV. Opposite rotation to Step 2. Counterclockwise rotation about the y-axis by an amount ϕ. As a consequence of this rotation, the x''-axis will move to x'.
- V. Opposite rotation to Step 1. Clockwise rotation about the z-axis by an amount λ. As a consequence of this rotation, the x'-axis will move to x. At this point, the frame (x, y, z) occupies its original position.

Synoptically, all steps can be written in one single line as follows (only the motions due to rotations of the x-axis are specified):

$$\mathbf{R}_3(\lambda):\ x \to x';\ \mathbf{R}_2(-\phi):\ x' \to x'';\ \Re_1(\theta):\ \mathbf{A} \to \mathbf{B};\ \mathbf{R}_2(\phi):\ x'' \to x';\ \mathbf{R}_3(-\lambda):\ x' \to x \tag{23}$$

The whole procedure can be written mathematically using matrix notation as

$$\begin{aligned}\Re_\ell(\theta) &= \left[\mathbf{R}_2^T(\phi)\mathbf{R}_3(\lambda)\right]^T \Re_1(\theta)\left[\mathbf{R}_2^T(\phi)\mathbf{R}_3(\lambda)\right] \\ &= \mathbf{R}_3(-\lambda)\mathbf{R}_2(\phi)\Re_1(\theta)\mathbf{R}_2(-\phi)\mathbf{R}_3(\lambda)\end{aligned} \tag{24}$$

Notice that the same result could have been obtained if one rotates the y- or z-axis into the Euler pole axis. For example, if one rotates the z-axis into the Euler pole axis, the following matrix expression applies:

$$\Re_\ell(\theta) = \mathbf{R}_3(-\lambda)\mathbf{R}_2\left(\phi - \frac{\pi}{2}\right)\Re_3(\theta)\mathbf{R}_2\left(\frac{\pi}{2} - \phi\right)\mathbf{R}_3(\lambda) \tag{25}$$

But using basic properties of rotation matrices

$$\Re_\ell(\theta) = \mathbf{R}_3(-\lambda)\mathbf{R}_2(\phi)\underbrace{\mathbf{R}_2\left(-\frac{\pi}{2}\right)\Re_3(\theta)\mathbf{R}_2\left(\frac{\pi}{2}\right)}_{\Re_1(\theta)} \times \mathbf{R}_2(-\phi)\mathbf{R}_3(\lambda) = \mathbf{R}_3(-\lambda)\mathbf{R}_2(\phi)\Re_1(\theta)\mathbf{R}_2(-\phi)\mathbf{R}_3(\lambda) \tag{26}$$

In conclusion, it can be stated that, for the selection of orientation angles (λ, ϕ) described in Fig. 3, the following three equalities could be established where the rotation of vectors about the Euler pole axis is performed around the first, second, and third axis of the frame, respectively:

$$\Re_\ell(\theta) = [1] + \sin\theta[\underset{\sim}{\ell}] + (1 - \cos\theta)\ [\underset{\sim}{\ell}]^2 = \begin{cases} \mathbf{R}_3(-\lambda)\mathbf{R}_2(\phi)\Re_1(\theta)\mathbf{R}_2(-\phi)\mathbf{R}_3(\lambda) \\ \mathbf{R}_3\left(\lambda - \frac{\pi}{2}\right)\mathbf{R}_1(\phi)\Re_2(\theta)\mathbf{R}_1(-\phi)\mathbf{R}_3\left(\frac{\pi}{2} - \lambda\right) \\ \mathbf{R}_3(-\lambda)\mathbf{R}_2\left(\phi - \frac{\pi}{2}\right)\Re_3(\theta)\mathbf{R}_2\left(\frac{\pi}{2} - \phi\right)\mathbf{R}_3(\lambda) \end{cases} \tag{27}$$

The proof of any of these three equations requires some algebraic manipulations. Considering that this is a technical note giving practical information with emphasis on didactic concepts and because, to the author's knowledge, the identities of these expressions are not mentioned anywhere in the scientific literature, a concise demonstration of the first case is provided in the appendix. Incidentally, equations similar to the ones on the right-hand side of Eq. (27) were used for the derivation of the corrections due to stellar proper motion in Mueller (1969, p. 115).

Combined Rotations in the Case of a So-Called Geodetic Plate-Fixed Datum

Imagine now that one is interested in a transformation of coordinates between two geodetic Cartesian frames (identified as IGS and D) symbolically represented by the mapping $IGS \to D$. Furthermore, assume that only rotations are involved. In other words, the two frames have the same origin and scale. Hence, the only parameters that one should be concerned with are the three differential rotations $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ about the first, second, and third axes of the rotating frame, respectively, and their variation with respect to time $(\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\varepsilon}_3)$, also known as *rotation rates*. Then, the mathematical general expression describing the transformation of coordinates between the two frames is

$$\begin{Bmatrix} x(t_D) \\ y(t_D) \\ z(t_D) \end{Bmatrix}_D = \begin{bmatrix} 1 & \varepsilon_3(t) & -\varepsilon_2(t) \\ -\varepsilon_3(t) & 1 & \varepsilon_1(t) \\ \varepsilon_2(t) & -\varepsilon_1(t) & 1 \end{bmatrix} \begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix}_{\mathrm{IGS}} + (t_D - t)\begin{bmatrix} 0 & \dot{\varepsilon}_3 & -\dot{\varepsilon}_2 \\ -\dot{\varepsilon}_3 & 0 & \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 & -\dot{\varepsilon}_1 & 0 \end{bmatrix} \begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix}_{\mathrm{IGS}} \tag{28}$$

where $_D$ = geodetic datum; and IGS = a type of international GNSS service frame determined through GNSS observations. The epoch of the coordinates in the IGS frame is t, whereas the epoch of the coordinates in the final geodetic datum frame is t_D. Notice that Eq. (28) is consistent with positive counterclockwise rotation of axes. It is clear that the coordinates of the points referring to Frame D are obtained from the coordinates on Frame IGS after it is rotated to a position of coincidence with Frame D. This is accomplished after rotations are applied to the axes of the IGS frame followed by the addition of the contribution generated by the rotation rates during the time interval $(t_D - t)$.

Now, totally independent of the IGS frame rotations, the coordinates of any point on the surface of the Earth are also affected by the rotation of the plate on which they are located. In other words, the coordinates of the points also change with time because the plate carrying them rotates with respect to the IGS frame as a consequence of the unstoppable geophysical theory of plate tectonics. Recall that this motion of the plate is already implicit in the GNSS observations of the coordinates $\{x(t)\ y(t)\ z(t)\}^T_{\mathrm{IGS}}$ at any time t. Expanding this idea even further, it can be said that the spatial location of the plate at epoch t (the instant of observation) is clearly not the same as at epoch t_D.

The differential changes of the coordinates of an arbitrary point from time t to time t_D due to the rotation of its plate could be written mathematically as [see Eq. (20)]

$$\begin{Bmatrix} \delta x \\ \delta y \\ \delta z \end{Bmatrix}_{\mathrm{IGS}} = (t_D - t)\begin{bmatrix} 0 & -\delta\dot{\Omega}_3 & \delta\dot{\Omega}_2 \\ \delta\dot{\Omega}_3 & 0 & -\delta\dot{\Omega}_1 \\ -\delta\dot{\Omega}_2 & \delta\dot{\Omega}_1 & 0 \end{bmatrix}_p \begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix}_{\mathrm{IGS}} \tag{29}$$

where p = plate to which the point is physically attached. The values of $\left(\delta\dot{\Omega}_1, \delta\dot{\Omega}_2, \delta\dot{\Omega}_3\right)_p$ in Eq. (29) must be expressed in radians $\times$

(time span)$^{-1}$, where the time span is given in units of years, centuries, etc. As previously explained, Eq. (29) is consistent with counterclockwise (active) rotation of vectors (the frame IGS remains fixed and the points move), such as in the particular application of plate tectonics theory.

However, coordinate geodetic databases compiled by the traditional national geodetic organizations provide values at some constant epoch t_D, which, obviously, is not the actual time of observation. Furthermore, the majority of users are generally focused on geographic information systems (GIS) and mapping applications, and the managers in charge of these geographic/land information system (G/LIS) databases prefer that the coordinates of the points that they archive and distribute remain as constant as feasible. Consequently, to achieve such a feat, the coordinates at observation time t (already affected by the motion of the plate) should be brought back to their values at the constant datum epoch t_D by reversing any motion that the plate may have incurred during the interval $(t_D - t)$. Thus, if one wants to correct the final datum coordinates of Eq. (28) by the rotation of the plate, one should add the negative value of Eq. (29) to Eq. (28).

Consequently, the formulation to be used (only rotations are considered) in a plate-fixed datum concept should be

$$\begin{Bmatrix} x(t_D) \\ y(t_D) \\ z(t_D) \end{Bmatrix}_D = \begin{bmatrix} 1 & \varepsilon_3(t) & -\varepsilon_2(t) \\ -\varepsilon_3(t) & 1 & \varepsilon_1(t) \\ \varepsilon_2(t) & -\varepsilon_1(t) & 1 \end{bmatrix} \begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix}_{\mathrm{IGS}} + (t_D - t) \begin{bmatrix} 0 & \dot{\varepsilon}_3 & -\dot{\varepsilon}_2 \\ -\dot{\varepsilon}_3 & 0 & \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 & -\dot{\varepsilon}_1 & 0 \end{bmatrix} \begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix}_{\mathrm{IGS}} + (t_D - t) \begin{bmatrix} 0 & \delta\dot{\Omega}_3 & -\delta\dot{\Omega}_2 \\ -\delta\dot{\Omega}_3 & 0 & \delta\dot{\Omega}_1 \\ \delta\dot{\Omega}_2 & -\delta\dot{\Omega}_1 & 0 \end{bmatrix} \begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix}_{\mathrm{IGS}} \quad (30)$$

and finally

$$\begin{Bmatrix} x(t_D) \\ y(t_D) \\ z(t_D) \end{Bmatrix}_D = \left[\left[\begin{bmatrix} 1 & \varepsilon_3(t) & -\varepsilon_2(t) \\ -\varepsilon_3(t) & 1 & \varepsilon_1(t) \\ \varepsilon_2(t) & -\varepsilon_1(t) & 1 \end{bmatrix} + (t_D - t) \times \begin{bmatrix} 0 & (\dot{\varepsilon}_3 + \delta\dot{\Omega}_3) & -(\dot{\varepsilon}_2 + \delta\dot{\Omega}_2) \\ -(\dot{\varepsilon}_3 + \delta\dot{\Omega}_3) & 0 & (\dot{\varepsilon}_1 + \delta\dot{\Omega}_1) \\ (\dot{\varepsilon}_2 + \delta\dot{\Omega}_2) & -(\dot{\varepsilon}_1 + \delta\dot{\Omega}_1) & 0 \end{bmatrix} \right]\right] \begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix}_{\mathrm{IGS}} \quad (31)$$

If the angular rotation rate of the plate is known as a function of its total value $(\delta\dot{\Omega})$ and the orientation of the Euler pole axis with respect to the IGS frame, the values of Eq. (21) should be plugged into Eq. (31).

The complete transformation equation for the North American Datum of 1983 (NAD 83) with subtle changes in notation was already addressed by Soler et al. (2016), where it is explained that the nomenclature *plate fixed* is a misnomer. In reality, the plates move and carry with them the marks on the ground; however, the coordinates of these marks are *virtually* moved back, as rigorously as possible, to their original positions in space. This is another clear example in which counterclockwise (passive) rotation of frame axes and counterclockwise (active) rotation of vectors are simultaneously applied, although the active rotation has a negative sense to correct the coordinates for the displacement of the plate during the time interval $(t_D - t)$. As it happens, a *negative* counterclockwise rotation of vectors is equal to a *positive* counterclockwise rotation of frame axes, and the mathematical consistency of Eq. (31) is corroborated under the unique assumption that all rotations (active and passive) are positive when counterclockwise. In the opinion of the author, for pure consistency, and because (active) rotation of vectors is always assumed positive counterclockwise in mechanics and geophysics, and to avoid any possible confusion, the (passive) rotation of frames should also be selected as positive counterclockwise (Soler 1997).

Equations similar to Eq. (31) have been used in the transformations implicit in the definition of the NAD 83 since the availability of global positioning system (GPS)–processed results (Soler et al. 2016). Equivalent equations will be extended to the new modernization of the National Spatial Reference System (NSRS) planned to be introduced by the National Geodetic Survey (NGS) around 2022 (Smith et al. 2017).

Conclusions

This technical note intends to clarify the use of rotation matrices around an axis (belonging to a frame or arbitrarily oriented in space) when performing passive (frames rotate, points remain fixed) or active rotations (points or their position vectors move, frame remains fixed). Many times, these two types of rotations are performed sequentially, and it is very important to mathematically differentiate them by initially establishing the corresponding rotation matrix sign convention.

To illuminate the concept, it is shown graphically, and analytically, that counterclockwise passive and active rotations are opposite in sign, and that they rotate frames and vectors (points), respectively. Several theoretical examples are described to facilitate the understanding of the subject matter, and some of them are explicitly proven. It is hoped that this material may be helpful to some readers by streamlining the basic differences between active and passive rotations around axes, and that the clarifications addressed herein may facilitate the correct use of any type of rotation matrices in the future.

Appendix. Proof

With the information provided in the text in conjunction with Fig. 3, the first identity in Eq. (27) could be proved analytically

$$\Re_\ell(\theta) = [1] + \sin\theta\,[\ell] + (1 - \cos\theta)[\ell]^2 = [\mathbf{R}_3(-\lambda)\mathbf{R}_2(\phi)]\Re_1(\theta)[\mathbf{R}_2(-\phi)\mathbf{R}_3(\lambda)] \quad (32)$$

where

$$[\mathbf{R}_2(-\phi)\mathbf{R}_3(\lambda)] = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \end{bmatrix} \quad (33)$$

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$$\Re_1(\theta)\left[\mathbf{R}_2(-\phi)\mathbf{R}_3(\lambda)\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\,\cos\lambda & -\sin\phi\,\sin\lambda & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ -\cos\theta\,\sin\lambda + \sin\theta\,\sin\phi\,\cos\lambda & \cos\theta\,\cos\lambda + \sin\theta\,\sin\phi\,\sin\lambda & -\sin\theta\,\cos\phi \\ -\sin\theta\,\sin\lambda - \cos\theta\,\sin\phi\,\cos\lambda & \sin\theta\,\cos\lambda - \cos\theta\,\sin\phi\,\sin\lambda & \cos\theta\,\cos\phi \end{bmatrix} \tag{34}$$

$$\left[\mathbf{R}_3(-\lambda)\mathbf{R}_2(\phi)\right] = \left[\mathbf{R}_2(-\phi)\mathbf{R}_3(\lambda)\right]^T = \begin{bmatrix} \cos\phi\,\cos\lambda & -\sin\lambda & -\sin\phi\,\cos\lambda \\ \cos\phi\,\sin\lambda & \cos\lambda & -\sin\phi\,\sin\lambda \\ \sin\phi & 0 & \cos\phi \end{bmatrix} = \begin{bmatrix} \ell_1 & -\sin\lambda & -\sin\phi\,\cos\lambda \\ \ell_2 & \cos\lambda & -\sin\phi\,\sin\lambda \\ \ell_3 & 0 & \cos\phi \end{bmatrix} \tag{35}$$

and

$$\Re_\ell(\theta) = \begin{bmatrix} \ell_1 & -\sin\lambda & -\sin\phi\,\cos\lambda \\ \ell_2 & \cos\lambda & -\sin\phi\,\sin\lambda \\ \ell_3 & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ -\cos\theta\,\sin\lambda + \sin\theta\,\sin\phi\,\cos\lambda & \cos\theta\,\cos\lambda + \sin\theta\,\sin\phi\,\sin\lambda & -\sin\theta\,\cos\phi \\ -\sin\theta\,\sin\lambda - \cos\theta\,\sin\phi\,\cos\lambda & \sin\theta\,\cos\lambda - \cos\theta\,\sin\phi\,\sin\lambda & \cos\theta\,\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{bmatrix} \tag{36}$$

and finally

$$\begin{aligned} (1,1) &= \ell_1^2 + \cos\theta\,\sin^2\lambda - \cancel{\sin\theta\,\sin\phi\,\sin\lambda\,\cos\lambda} + \cancel{\sin\theta\,\sin\phi\,\sin\lambda\,\cos\lambda} + \cos\theta\,\sin^2\phi\,\cos^2\lambda \\ &= \ell_1^2 + \cos\theta\,\sin^2\lambda + \cos\theta\,\cos^2\lambda\ell_3^2 = \ell_1^2 + \cos\theta\,\sin^2\lambda + \cos\theta\,\cos^2\lambda\left(1 - \ell_1^2 - \ell_2^2\right) = \ell_1^2 + \cos\theta - \left(\ell_1^2 + \ell_2^2\right)\cos^2\lambda\,\cos\theta \\ &= \ell_1^2 + \cos\theta - \cos^2\phi\,\cos^2\lambda\,\cos\theta = \ell_1^2 + \cos\theta - \ell_1^2\cos\theta = \cos\theta + (1 - \cos\theta)\ell_1^2 \end{aligned} \tag{37}$$

$$\begin{aligned} (1,2) &= \ell_1\ell_2 - \sin\lambda\,\cos\theta\,\cos\lambda - \sin^2\lambda\,\sin\theta\,\sin\phi - \cos^2\lambda\,\sin\theta\,\sin\phi + \sin^2\phi\,\cos\lambda\,\cos\theta\,\sin\lambda \\ &= \ell_1\ell_2 - \sin\lambda\,\cos\theta\,\cos\lambda - \sin\theta\,\sin\phi + \sin^2\phi\,\cos\lambda\,\cos\theta\,\sin\lambda = \ell_1\ell_2 - \sin\theta\,\ell_3 + \cos\theta\left(\sin^2\phi\,\cos\lambda\,\sin\lambda - \sin\lambda\,\cos\lambda\right) \\ &= \ell_1\ell_2 - \sin\theta\,\ell_3 + \cos\theta\left(\left(1 - \cos^2\phi\right)\cos\lambda\,\sin\lambda - \sin\lambda\,\cos\lambda\right) = \ell_1\ell_2 - \sin\theta\,\ell_3 - \cos\theta\,\ell_1\ell_2 = -\sin\theta\,\ell_3 + (1 - \cos\theta)\ell_1\ell_2 \end{aligned} \tag{38}$$

$$(1,3) = \ell_1\ell_3 + \sin\lambda\,\sin\theta\,\cos\phi - \sin\phi\,\cos\lambda\,\cos\theta\,\cos\phi = \ell_1\ell_3 + \sin\theta\,\ell_2 - \ell_1\ell_3\cos\theta = \sin\theta\,\ell_2 + (1 - \cos\theta)\ell_1\ell_3 \tag{39}$$

$$\begin{aligned} (2,1) &= \ell_1\ell_2 - \cos\lambda\,\cos\theta\,\sin\lambda + \cos\lambda\,\sin\theta\,\sin\phi\,\cos\lambda + \sin\phi\,\sin\lambda\,\sin\theta\,\sin\lambda + \sin\phi\,\sin\lambda\,\cos\theta\,\sin\phi\,\cos\lambda \\ &= \ell_1\ell_2 + \cos\lambda\,\cos\theta\,\sin\lambda\left(\ell_3^2 - 1\right) + \sin\theta\,\sin\phi = \ell_1\ell_2 + \cos\lambda\,\cos\theta\,\sin\lambda\left(\cancel{1} - \ell_1^2 - \ell_2^2 - \cancel{1}\right) + \sin\theta\,\ell_3 \\ &= \ell_1\ell_2 - \ell_1\ell_2\cos\theta + \sin\theta\,\ell_3 = (1 - \cos\theta)\ell_1\ell_2 + \sin\theta\,\ell_3 \end{aligned} \tag{40}$$

$$\begin{aligned} (2,2) &= \ell_2^2 + \cos\lambda\,\cos\theta\,\cos\lambda + \cancel{\cos\lambda\,\sin\theta\,\sin\phi\,\sin\lambda} - \cancel{\sin\phi\,\sin\lambda\,\sin\theta\,\cos\lambda} + \sin\phi\,\sin\lambda\,\cos\theta\,\sin\phi\,\sin\lambda \\ &= \ell_2^2 + \cos^2\lambda\,\cos\theta + \sin^2\lambda\,\cos\theta\,\ell_3^2 = \ell_2^2 + \cos^2\lambda\,\cos\theta + \sin^2\lambda\,\cos\theta\left(1 - \ell_1^2 - \ell_2^2\right) = \ell_2^2 + \cos\theta - \left(\ell_1^2 + \ell_2^2\right)\sin^2\lambda\,\cos\theta \end{aligned}$$

$$= \ell_2^2 + \cos\theta - \cos^2\phi \sin^2\lambda \cos\theta = \ell_2^2 + \cos\theta - \ell_2^2 \cos\theta = (1 - \cos\theta)\ell_2^2 + \cos\theta \tag{41}$$

$$(2,3) = \ell_2\ell_3 - \cos\lambda \sin\theta \cos\phi - \sin\phi \sin\lambda \cos\theta \cos\phi = \ell_2\ell_3 - \sin\theta \ell_1 - \cos\theta \ell_2\ell_3 = -\sin\theta \ell_1 + (1 - \cos\theta)\ell_2\ell_3 \tag{42}$$

$$(3,1) = \ell_1\ell_3 - \cos\phi \sin\theta \sin\lambda - \cos\phi \cos\theta \sin\phi \cos\lambda = \ell_1\ell_3 - \sin\theta \ell_2 - \cos\theta \ell_1\ell_3 = -\sin\theta \ell_2 + (1 - \cos\theta)\ell_1\ell_3 \tag{43}$$

$$(3,2) = \ell_2\ell_3 + \cos\phi \sin\theta \cos\lambda - \cos\phi \cos\theta \sin\phi \sin\lambda = \ell_2\ell_3 + \sin\theta \ell_1 - \cos\theta \ell_2\ell_3 = \sin\theta \ell_1 + (1 - \cos\theta)\ell_2\ell_3 \tag{44}$$

$$(3,3) = \ell_3^2 + \cos\phi \cos\theta \cos\phi = \ell_3^2 + \left(1 - \sin^2\phi\right)\cos\theta = \ell_3^2 + \left(1 - \ell_3^2\right)\cos\theta = (1 - \cos\theta)\ell_3^2 + \cos\theta \tag{45}$$

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