Coordinate frames and transformations

Master's Degree in Geomatics Engineering and Geoinformation

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E.T.S.I.Geodesica, Cartografíca y Topográfica

Outline

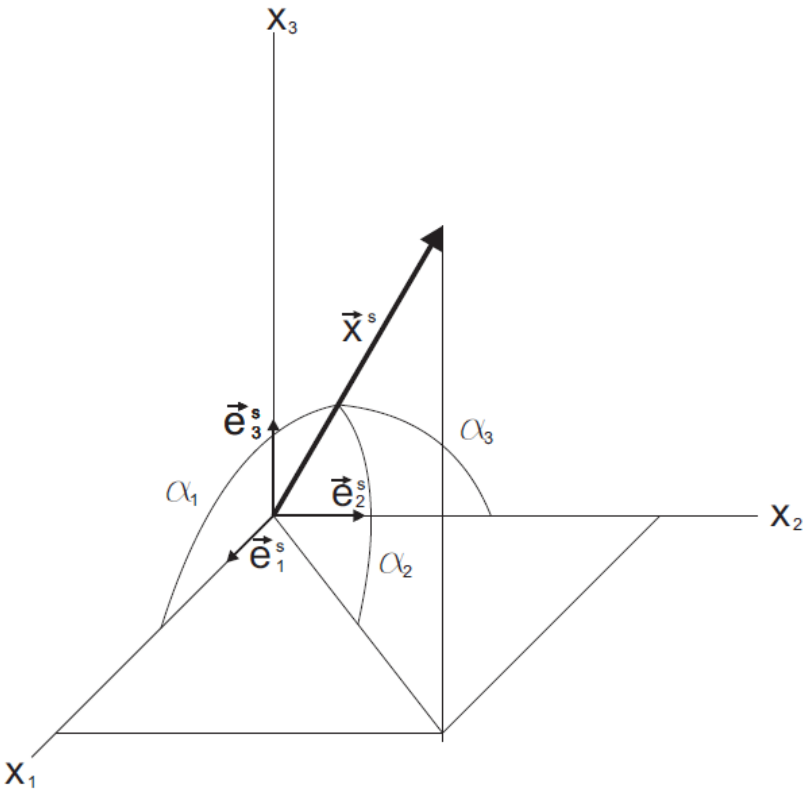
- Cartesian coordinates
- Coordinate frames used in navigation
- Transformations
- Angular rates
- Differential equation of the transformation
- Specific coordinate transformations

Cartesian coordinates

- A coordinate frame may be defined in two interchangeable ways:
  - origin and a set of axes in terms of which the motion of objects may be described
  - position and orientation of an object
- It is equally valid to describe the position and orientation of frame *s* with respect to frame *t* as it is to describe the position and orientation of frame *t* with respect to frame *s*.
- This is a principle of relativity: the laws of physics appear the same for all observers.
- Three frames are usually involved: the reference frame, the object frame and the resolving frame.
- Many navigation problems involve more than one reference frame or even more than one object frame.

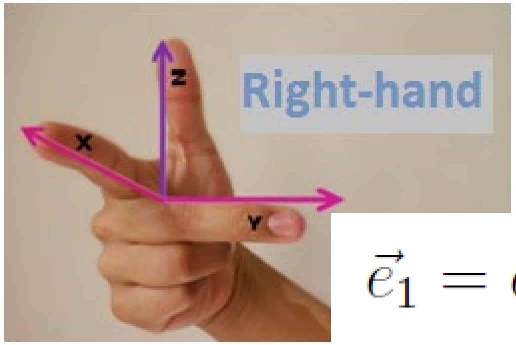
Cartesian coordinates

Cartesian or rectangular coordinates are defined by its origin and three axes that are mutually orthogonal by definition (6 degrees of freedom)



$$O, \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$$

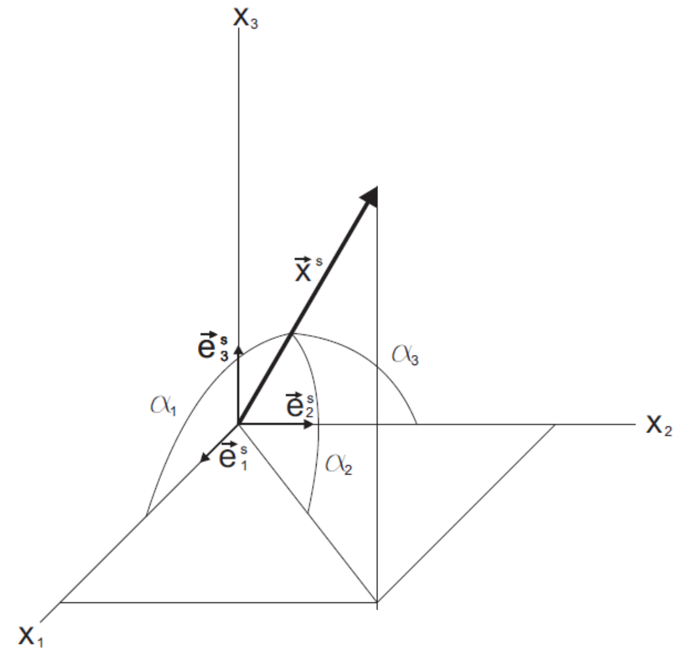
$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$



$$\vec{e}_1 = \vec{e}_2 \times \vec{e}_3 \qquad \vec{e}_2 = \vec{e}_3 \times \vec{e}_1 \qquad \vec{e}_3 = \vec{e}_1 \times \vec{e}_2$$

Cartesian coordinates

Coordinates and direction cosines



$$\vec{x} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1 = \vec{x} \cdot \vec{e}_1 \qquad x_2 = \vec{x} \cdot \vec{e}_2 \qquad x_3 = \vec{x} \cdot \vec{e}_3$$

$$\cos \alpha_1 = \frac{x_1}{\|\vec{x}\|} \qquad \cos \alpha_2 = \frac{x_2}{\|\vec{x}\|} \qquad \cos \alpha_3 = \frac{x_3}{\|\vec{x}\|}$$

Inertial frame (*i-frame*)

Euclidean system in which Newton's laws of motion hold

1. A body in uniform rectilinear motion will remain in uniform rectilinear motion in the absence of applied forces.

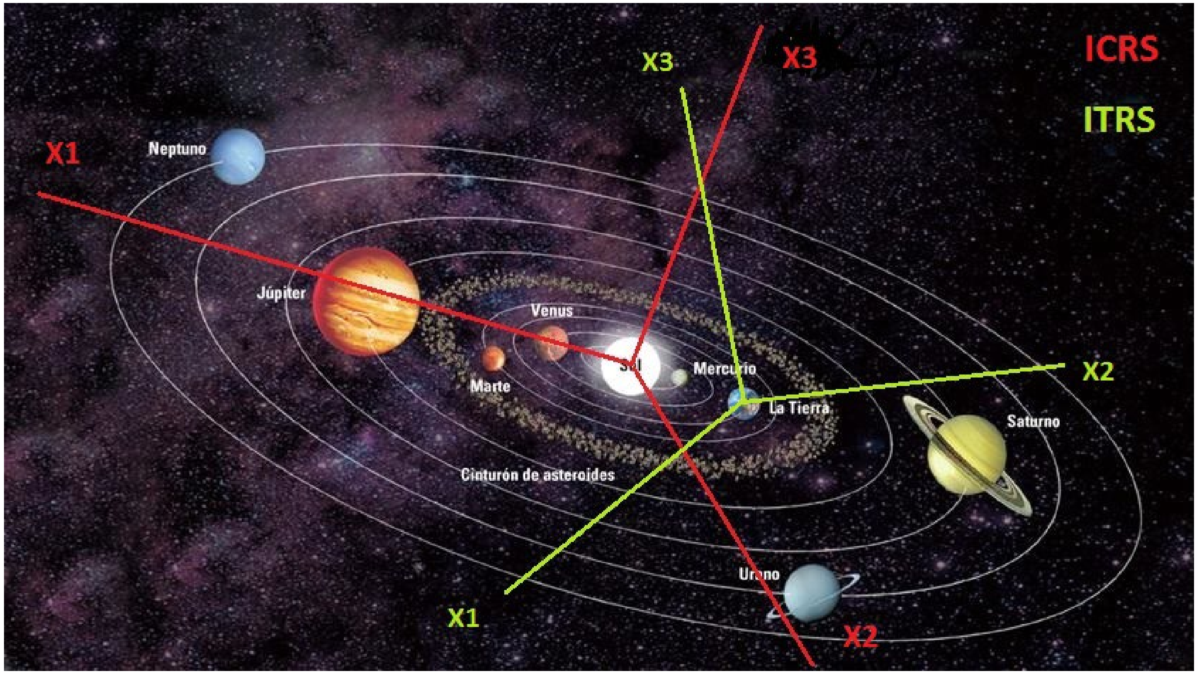
2. The time-rate change of teh linear momentum of a particle equal the sum of applied forces

$$\frac{d}{dt}\left(m_i\dot{\vec{x}}\right) = \vec{F} \qquad \rightarrow \qquad m_i\ddot{\vec{x}} = \vec{F}$$

They are the basis for describing the dynamics of the inertial mesurement units (IMUs)

Inertial frame (*i-frame*)

In our world, a global inertial system is an abstraction, since any frame in the vecinity of the solar system is permeated by a gravitational field that possesses spatially varying gradients.



A body initially at rest will accelerate under the gravitational influence of the sun and planets, thus violating Newton's first law, and therefore, the frame would not be inertial.

Inertial frame (*i-frame*)

It is necessary to modify Newton's Second Law to account for the acceleration due to an ambient gravitational field

$$m_i \ddot{\vec{x}} = \vec{F} + m_g \vec{g}$$

The gravitational acceleration vector $\vec{g}$ can be thought of as the 'proportionality factor' between the gravitational mass m_g and the resultating gravitational force $\vec{F}_g$ as formulated by Newton's Law of Gravitation

$$\vec{F}_g = k \frac{m_g M}{l^2} \vec{e}_l = m_g \vec{g}$$

With $m_i = m_g = m$ (weak Principle of Equivalence)

$$\ddot{\vec{x}} = \vec{a} + \vec{g}$$

Inertial frame (*i-frame*)

Where $\vec{a} = \vec{F}/m$ is known as the specific force o force per unit mass. The specific force is what IMUs measure.

Examples of specific force are atmospheric drag, or the reaction force that Earth surface exerts on us to keep us from falling toward its center.

The vector $\vec{g}$ is assumed to be known from gravity models and planetary ephemeris.

The inertial frame is important in navigation because inertial sensors measure motion with respect to a generic inertial frame, and it enables the simplest form of navigation equations to be used.

Inertial frame (*i-frame*)

ICRS, as realized by the International Earth Rotation Service (IERS), can be use when a precise inertial frame is needed.

$\vec{e}_3$ points at the North Ecliptic Pole and $\vec{e}_1$ points at the direction from the Earth to the Sun at the vernal equinox.

EOP along with precesion-nutation models are needed.

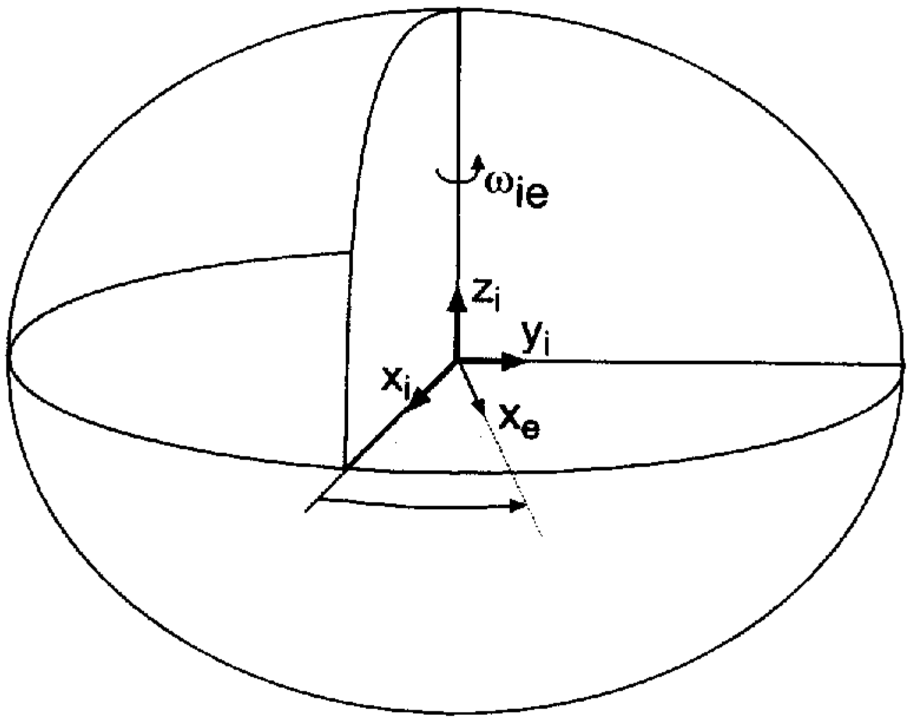
In navigation a simplified version, known as the Earth-centered inertial frame, can be used.

Coordinates in the *i-frame* are denoted

$$\vec{x}^{\,i} = x_1^i \vec{e}_1^{\,i} + x_2^i \vec{e}_2^{\,i} + x_3^i \vec{e}_3^{\,i} = \begin{pmatrix} x_1^i \\ x_2^i \\ x_3^i \end{pmatrix}$$

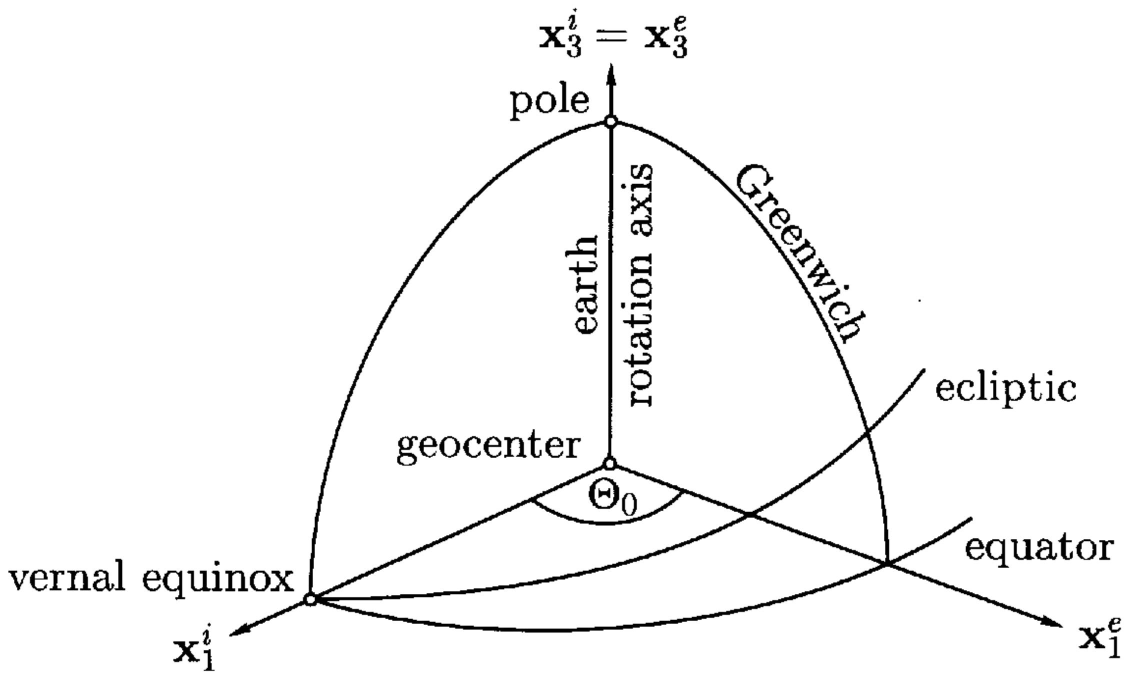
i-frame (Farrell)

Earth-centered inertial frame



i-frame (Hofmann)

Earth-centered inertial frame



Earth frame (*e-frame*)

The Earth frame is important in navigation because the user wants to know their position relative to the Earth, so it is commonly used as both a reference frame and a resolving frame.

The Earth-centered Earth-fixed (ECEF) frame, commonly abbreviated to Earth frame, has its origin at the center of the ellipsoid modelling the Earth's surface , which is roughly at the center of mass.

The z-axis $\vec{e}_3$ always point along the Earth's axis of rotation from the center to the conventional North Pole (IRP).

The x-axis $\vec{e}_1$ points from the center to the intersection of the equator with the conventional meridian (IRM).

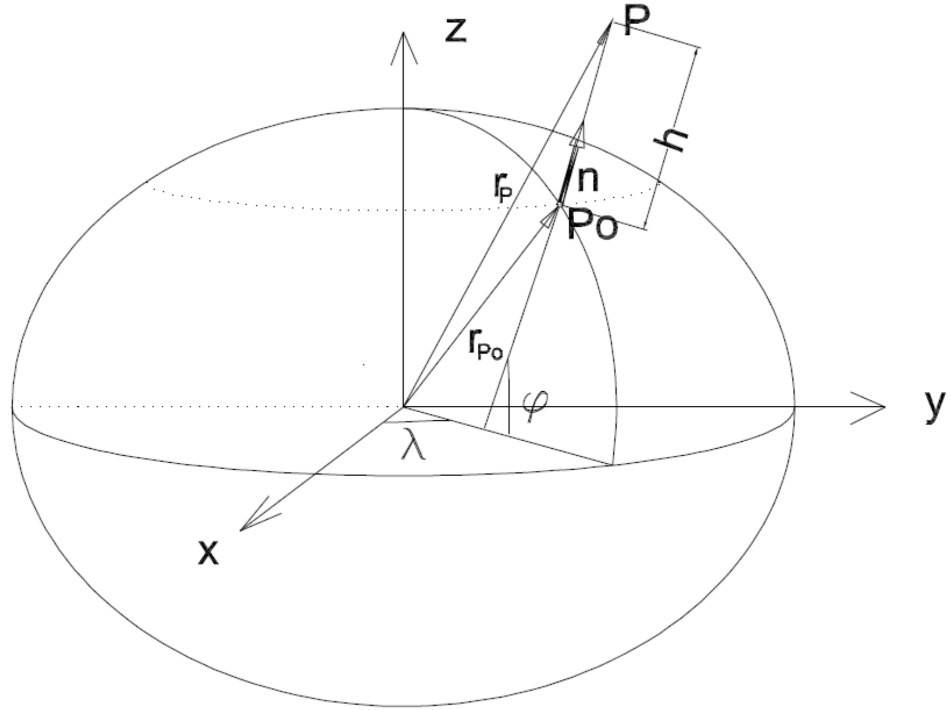
Coordinates in the *e-frame* are denoted

$$\vec{x}^{\,e} = x_1^e \vec{e}_1^{\,e} + x_2^e \vec{e}_2^{\,e} + x_3^e \vec{e}_3^{\,e} = \begin{pmatrix} x_1^e \\ x_2^e \\ x_3^e \end{pmatrix}$$

Earth frame(*e-frame*)

A disadvantage of the ECEF as *e-frame* is that some applications may require horizontal positioning and/or vertical positioning.

The solution is to use a ellipsoid of reference along with geodetic coordinates and ellipsoidal heigth.



$$(\varphi, \lambda, h)$$

Altímeters ⇨ $\begin{cases} h = H + N \\ N = N(\varphi, \lambda) \end{cases}$

Gravitational models ⇨ $\begin{cases} \gamma_N = \gamma_N(\varphi, h) \\ \gamma_E = 0 \\ \gamma_D = \gamma_D(\varphi, h) \end{cases}$

Earth frame (*e-frame*)

$$\begin{aligned} X &= (\nu + h)\cos\varphi\cos\lambda \\ Y &= (\nu + h)\cos\varphi\,\mathrm{sen}\,\lambda \\ Z &= \left[\nu\left(1 - e^2\right) + h\right]\mathrm{sen}\,\varphi \end{aligned}$$

$$\varphi = \mathrm{arc\,tg}\,\frac{Z + e'^2 b\,\mathrm{sen}^3\vartheta}{p - e^2 a\cos^3\vartheta}$$

$$\vartheta = \mathrm{arc\,tg}\,\frac{Za}{pb}$$

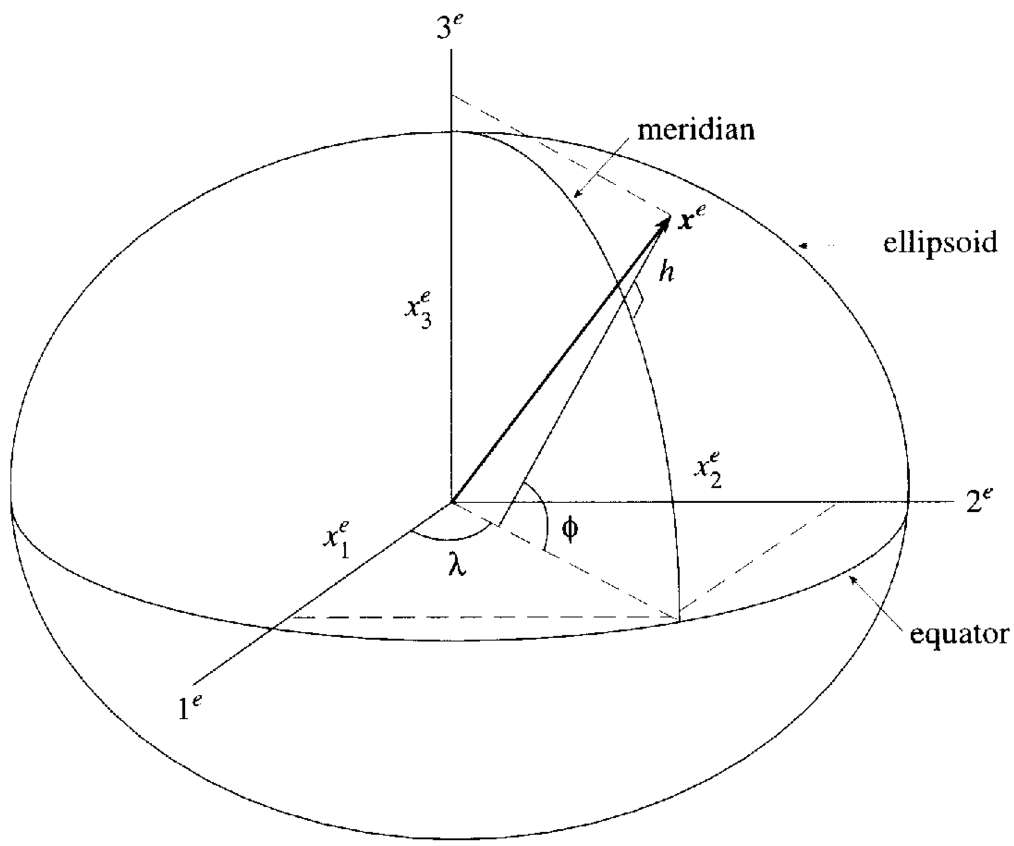
$$\lambda = \mathrm{arc\,tg}\,\frac{Y}{X}$$

$$e'^2 = \frac{a^2 - b^2}{b^2}$$

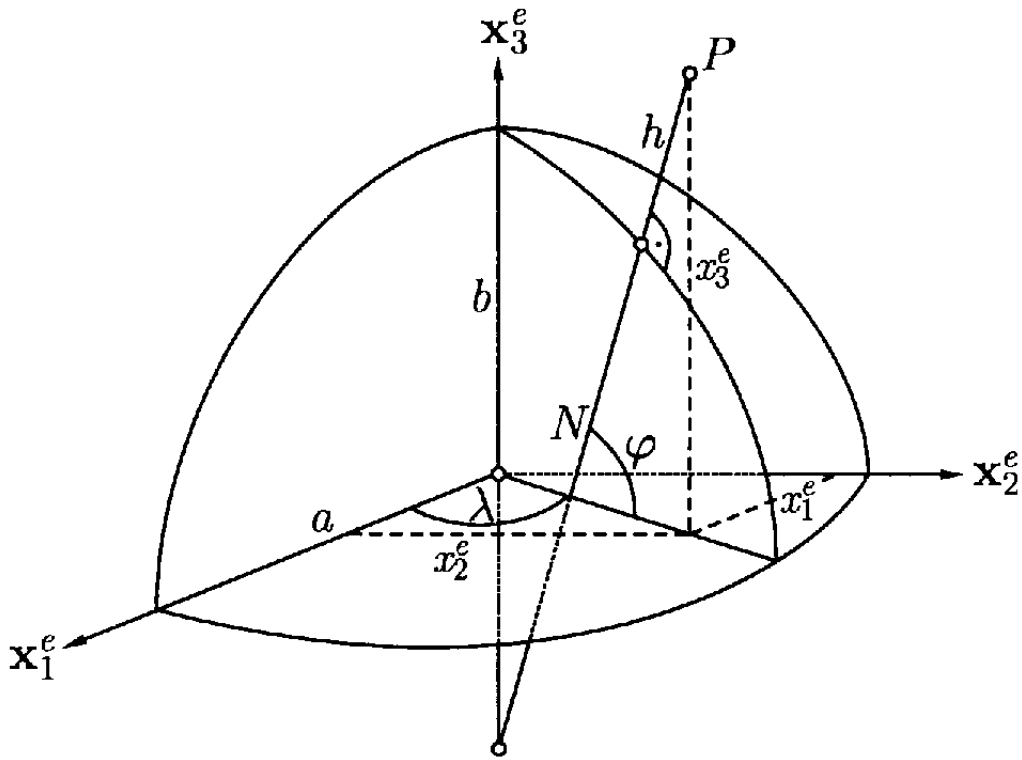
$$h = \frac{p}{\cos\varphi} - \nu$$

$$p = \sqrt{X^2 + Y^2}$$

e-frame (Jekeli)



e-frame (Hofmann)

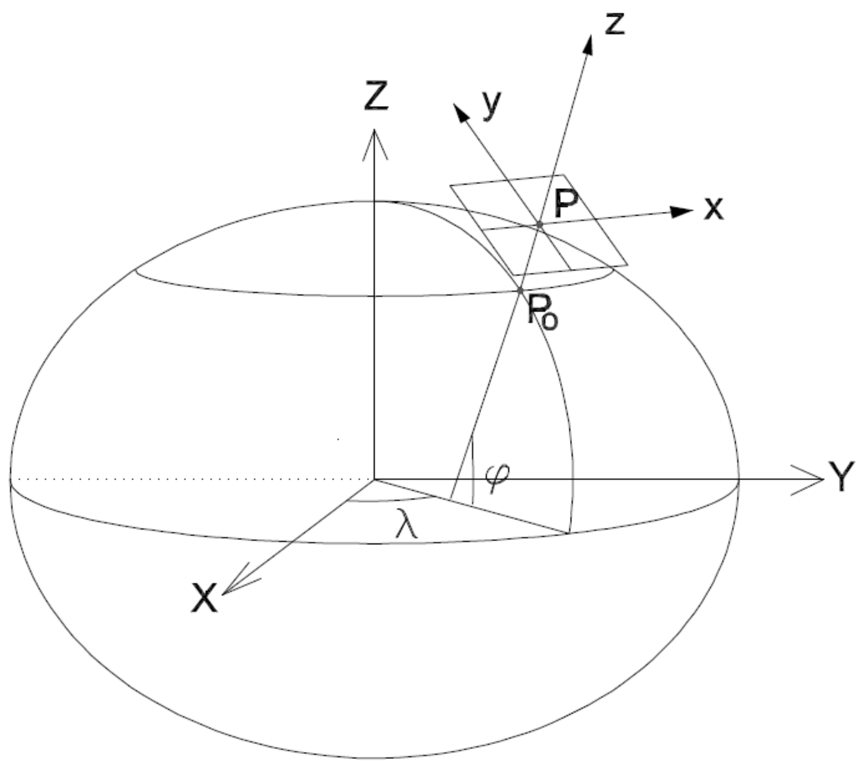


Local navigation frame (*n-frame*)

The local navigation frame, local level navigation frame, geodetic, or geographic frame is denoted by the symbol *n* (some authors use *g, i.e. Xsens*).

Its origin is the point a navigation solution is sought for (i.e., the navigation system, the user, or the host vehicle's center of mass).

Their axes are usually oriented in a different way

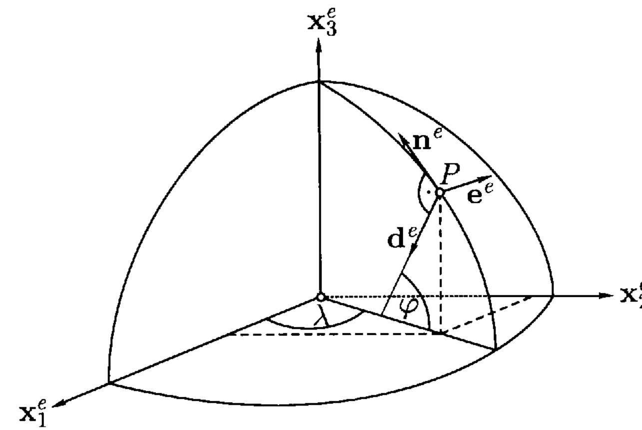


$$\begin{pmatrix} x_{ij} \\ y_{ij} \\ z_{ij} \end{pmatrix} = \begin{pmatrix} -\operatorname{sen}\lambda_i & \cos\lambda_i & 0 \\ -\operatorname{sen}\varphi_i\cos\lambda_i & -\operatorname{sen}\varphi_i\operatorname{sen}\lambda_i & \cos\varphi_i \\ \cos\varphi_i\cos\lambda_i & \cos\varphi_i\operatorname{sen}\lambda_i & \operatorname{sen}\varphi_i \end{pmatrix} \begin{pmatrix} X_{ij} \\ Y_{ij} \\ Z_{ij} \end{pmatrix}$$

Local navigation frame (*n-frame*)

The local navigation frame is important in navigation because the user wants to know their attitude relative to the north, east, and down directions.

For position and velocity, it provides a convenient set of resolving axes, but is not used as a reference frame.



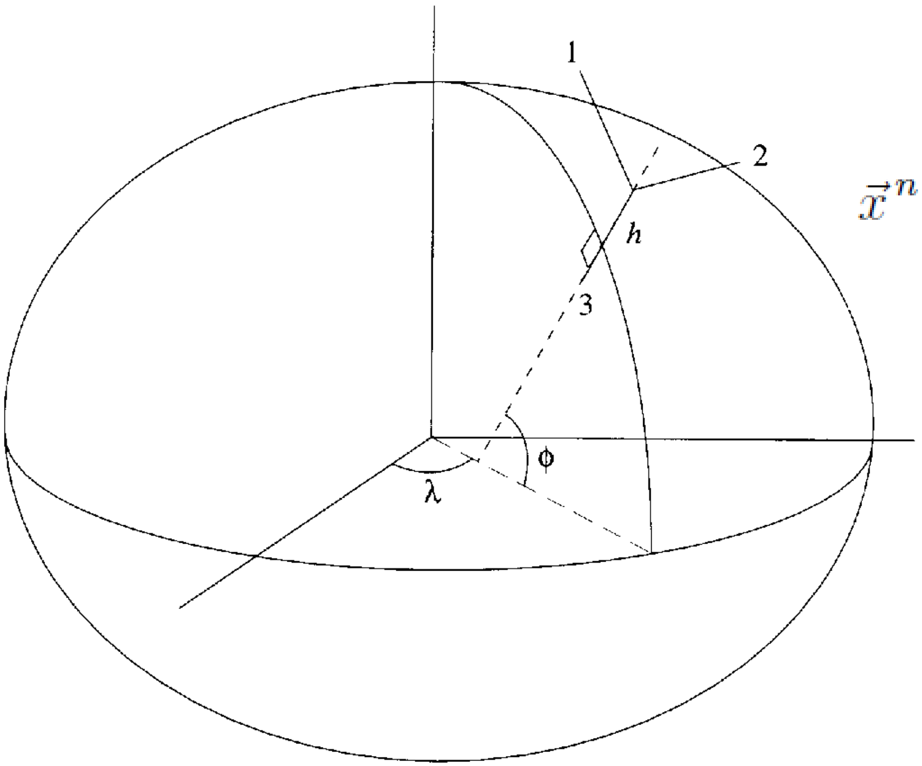
The *z axis, also known* as the down (*D*) *axis, is defined as the normal to the surface of the reference* ellipsoid, pointing roughly toward the center of the Earth.

Simple gravity models assume that the gravity vector is coincident with the *z/D axis of the local navigation frame*

True gravity deviates from this slightly due to local anomalies.

$$\begin{cases} \xi_i = \xi(\varphi_i, \lambda_i) \\ \eta_i = \eta(\varphi_i, \lambda_i) \end{cases}$$

n-frame



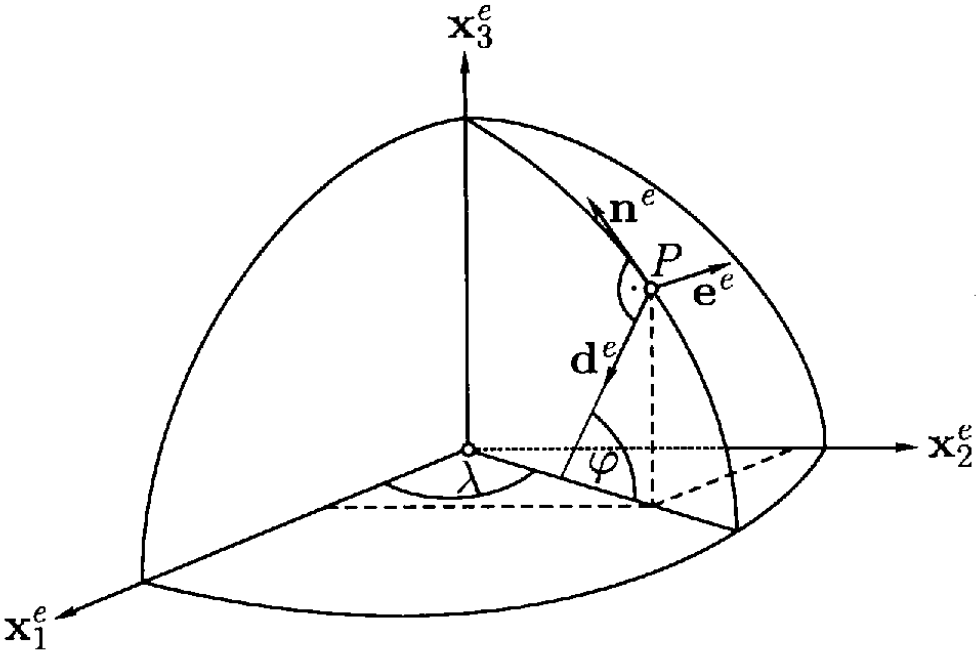
[Jekeli]

$$\vec{x}^{\,n} = x_1^n \vec{e}_1^{\,n} + x_2^n \vec{e}_2^{\,n} + x_3^n \vec{e}_3^{\,n} = \begin{pmatrix} x_1^n \\ x_2^n \\ x_3^n \end{pmatrix}$$

$$C_e^n = R_3(-\lambda) R_2(\varphi + \pi/2) = \begin{pmatrix} -\operatorname{sen}\varphi\cos\lambda & -\operatorname{sen}\lambda & -\cos\varphi\cos\lambda \\ -\operatorname{sen}\varphi\operatorname{sen}\lambda & \cos\lambda & -\cos\varphi\operatorname{sen}\lambda \\ \cos\varphi & 0 & -\sin\varphi \end{pmatrix}$$

n-frame

[Hofmann]



b-frame

The body frame (sometimes known as the vehicle frame) comprises the origin and orientation of the object for which a navigation solution is sought.

The body frame is essential in navigation because it describes the object that is navigating.

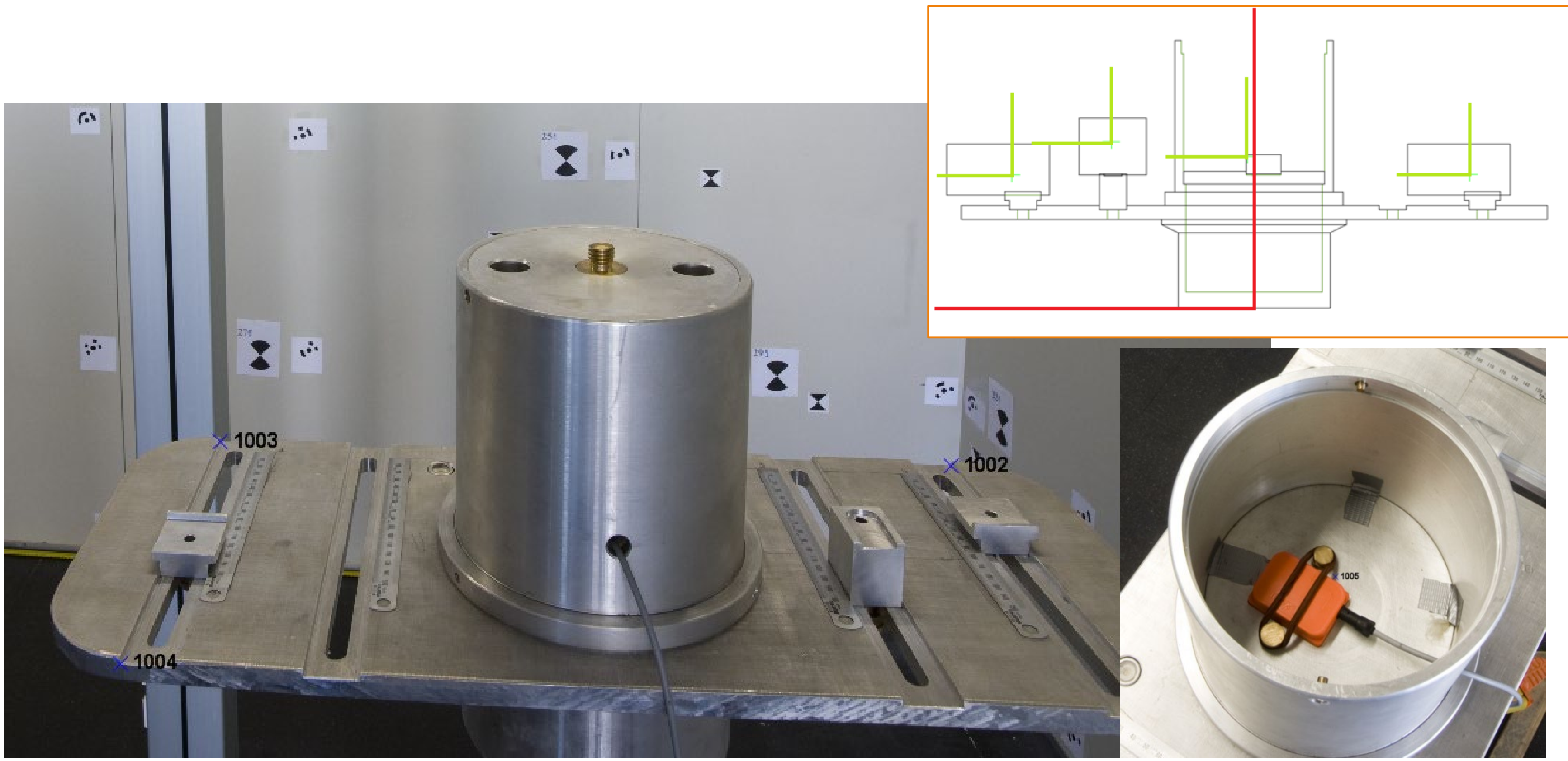
All strapdown inertial sensors measure the motion of the body frame (with respect to a generic generic inertial frame.

The axes are generally defined as x (*forward*), z (*through-the-floor*), and y (*right*), completing the orthogonal set.

$$\vec{x}^{\,b} = x_1^b \vec{e}_1^{\,b} + x_2^b \vec{e}_2^{\,b} + x_3^b \vec{e}_3^{\,b} = \begin{pmatrix} x_1^b \\ x_2^b \\ x_3^b \end{pmatrix}$$

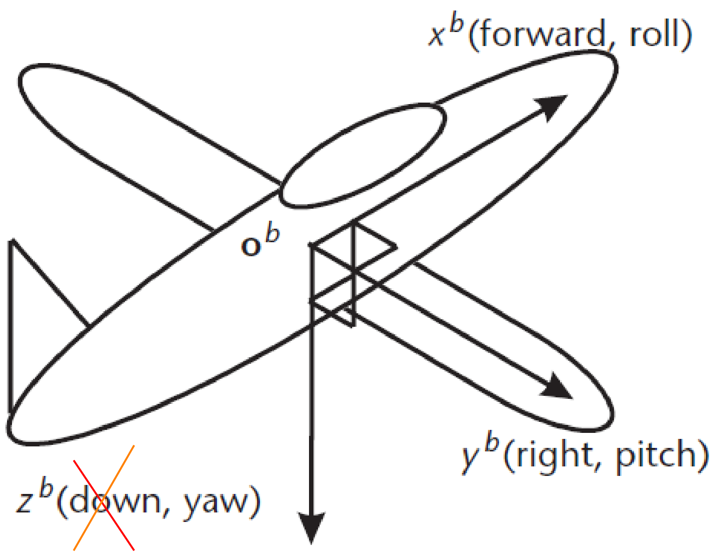
b-frame

Example: *b-frame* can be associated with the system patform where all the sensors to be integrated are installed.



b-frame

b-frame (Roll,Pitch,Yaw)

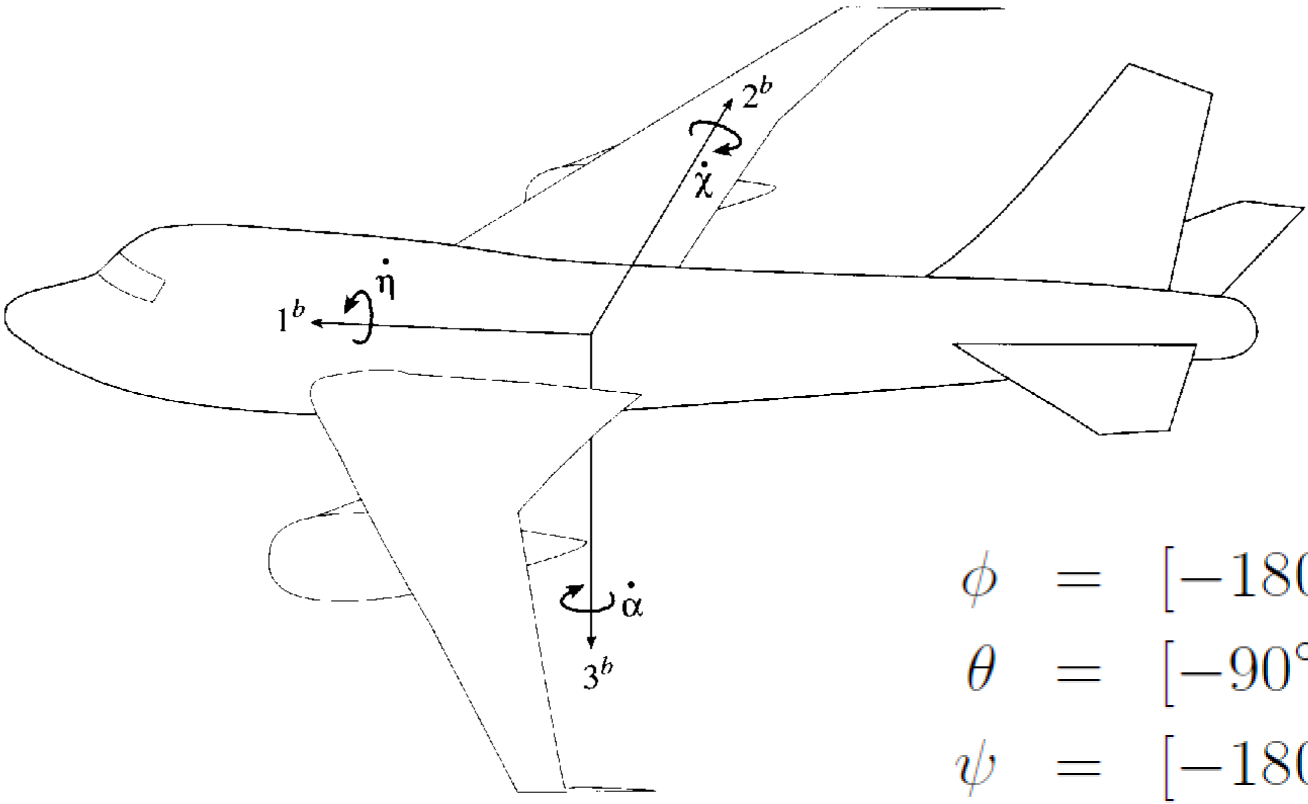


For angular motion, the x-axis is the roll axis, the y-axis is the pitch axis, and the z-axis is the yaw axis.

Hence, the axes of the body frame are sometimes known as roll, pitch, and yaw.

[Groves]

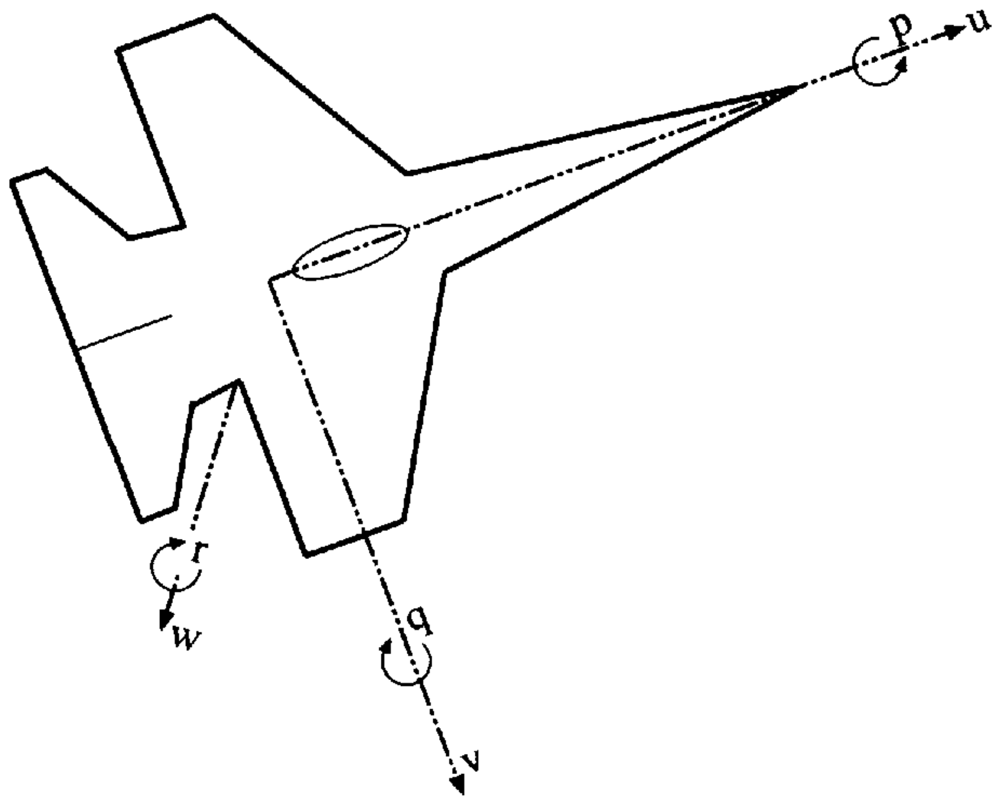
b-frame (Roll,Pitch,Yaw)



$$
\begin{aligned}
\phi &= [-180^\circ, +180^\circ] \\
\theta &= [-90^\circ, +90^\circ] \\
\psi &= [-180^\circ, +180^\circ]
\end{aligned}
$$

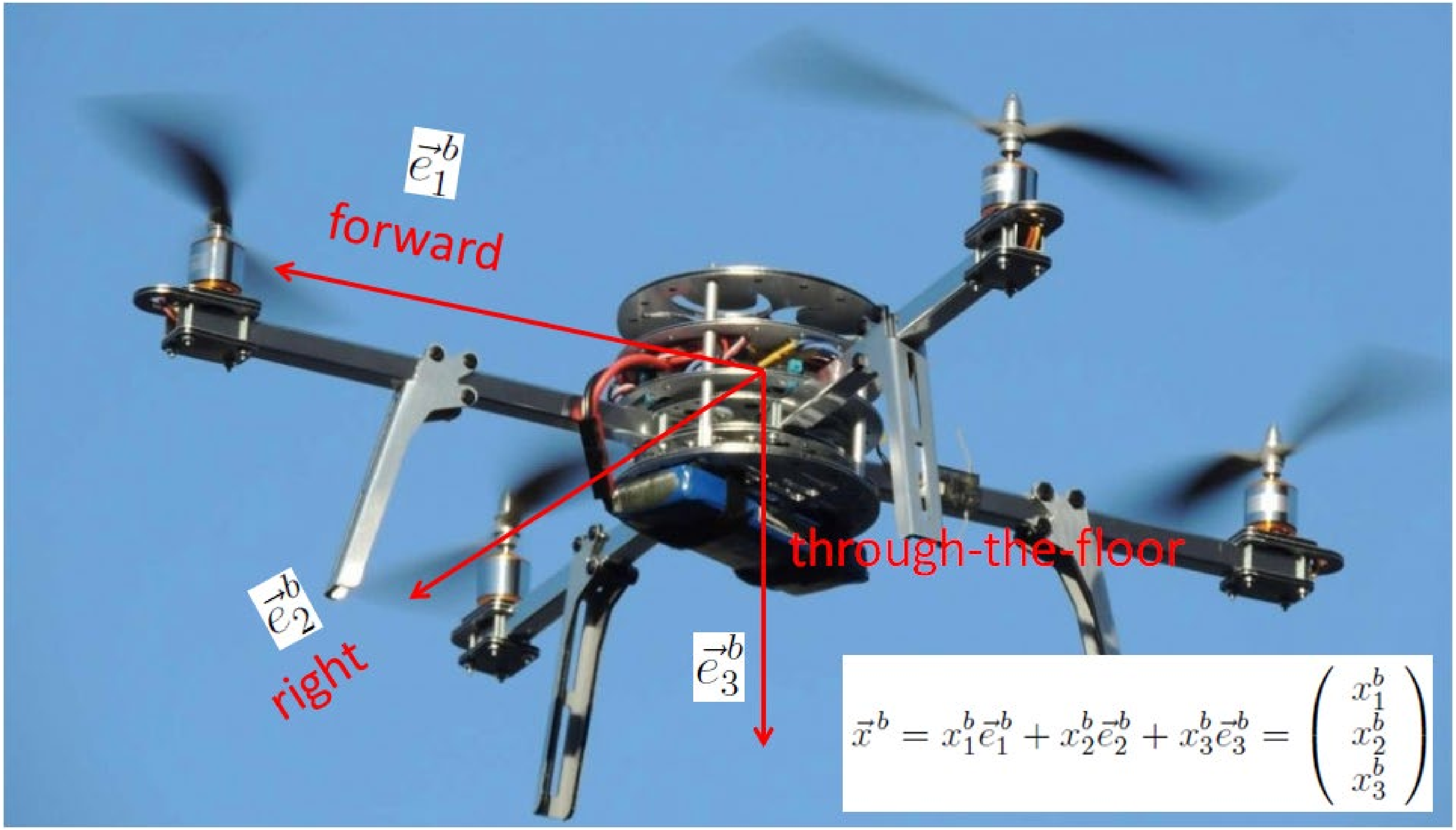
[Jekeli]

b-frame



[Farrell]

b-frame

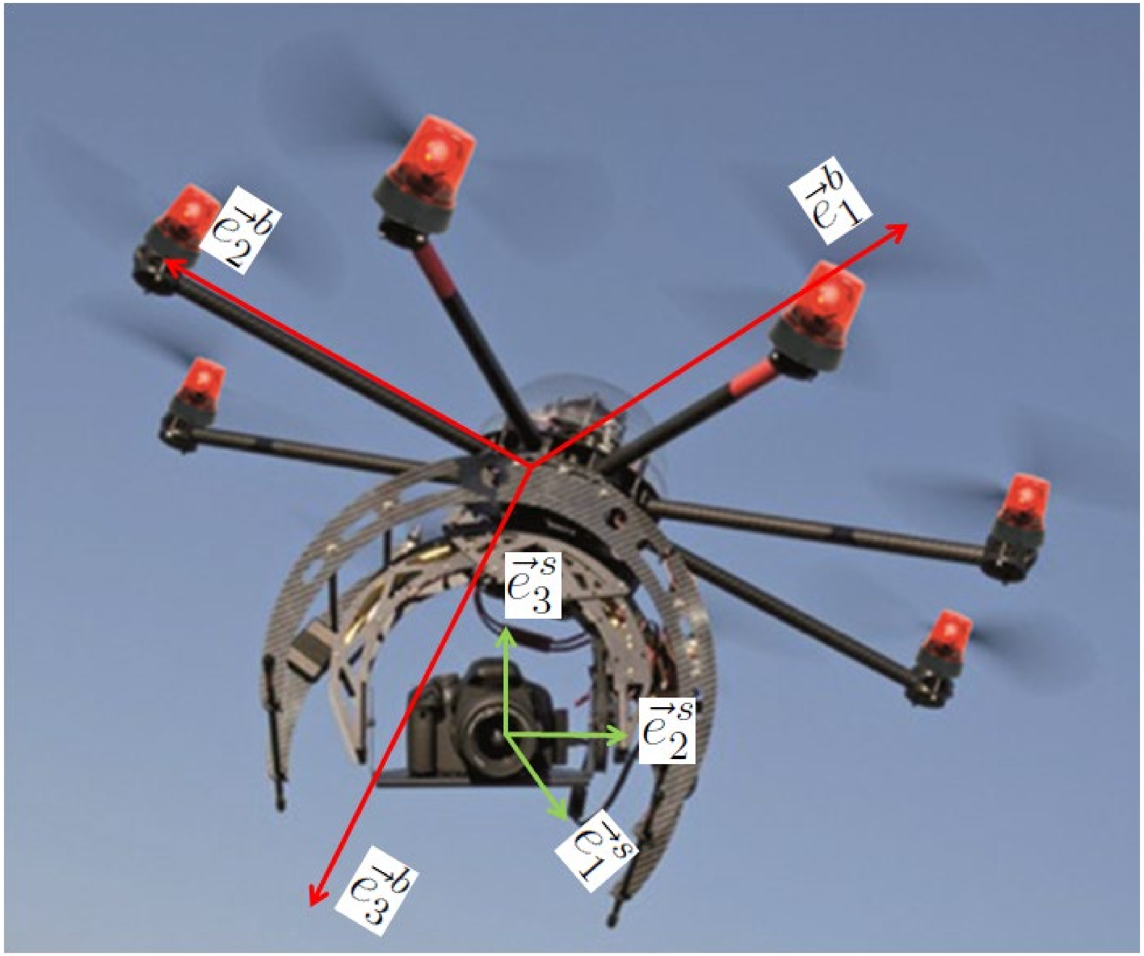


$\vec{e}_1^{\,b}$
forward
$\vec{e}_2^{\,b}$
right
$\vec{e}_3^{\,b}$
through-the-floor
$\vec{x}^{\,b} = x_1^b\vec{e}_1^{\,b} + x_2^b\vec{e}_2^{\,b} + x_3^b\vec{e}_3^{\,b} = \begin{pmatrix} x_1^b \\ x_2^b \\ x_3^b \end{pmatrix}$

Other frames

- The sensor frame (*s-frame*) is a single representative, analytic frame for the navigation system.
  - It is used to model and identify instrument errors in a unified frame for analytical purposes like filtering.
  - In the case of a strapdown system the sensor frame may be identified with the body frame, and in the case of a local-level gimbaled system the sensor frame usually corresponds to the navigation frame.
- Individual instrument frames that serve also to define the input and output axes for each accelerometer and gyroscope.
- The platform frame (or, chassis frame) provides a physical set of fiducial axes that provide a common origin for the instrument cluster.
- To perform inertial navigation, the measurements coordinatized in the instrument frames must be transformed into useful data in an Earth-fixed frame. This requires the development of transformations among coordinate frames, which is the topic of the next sections.

Gimbaled sensor frame



Relationships between frames

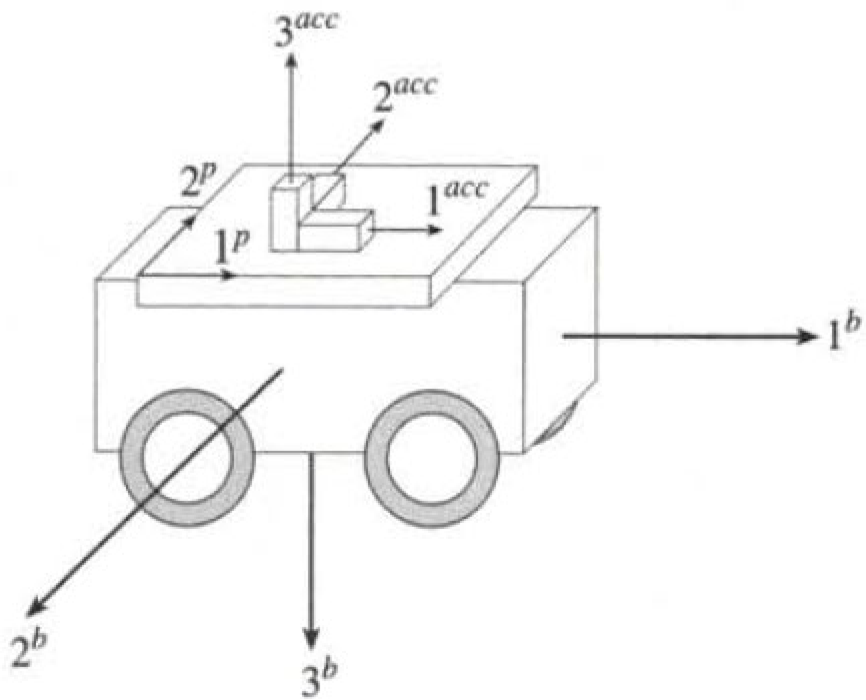


Figure 1.4: Typical relationships between body, platform, and IMU frames.

Coordinate transformations

- Several possibilities exist to define the transformation of coordinates from one frame to another.

- Of primary concern is their relative orientation.

- There may also be a translation between corresponding origins, but this is described simply by a vector of coordinate differences that is applied equally to all points in a frame.

- In general, there will be no need to consider different scales, the scale being universally defined for all systems (International System of Units SI).
  - One sensor of a particular type may yield differently scaled data than another of the same or different type, but rather than endowing each coordinate frame with its own scale, the data themselves are associated with a scale parameter.

Kinematics

- In navigation, the linear and angular motion of one coordinate frame must be described with respect to another.

- Most kinematic quantities, such as position, velocity, acceleration, and angular rate, involve three coordinate frames:

  - Object frame $\rightarrow$ the frame whose motion is described
  - Reference frame $\rightarrow$ the frame with which that motion is respect to
  - Resolving frame $\rightarrow$ the set of axes in which that motion is represented

- The object frame and the reference frame must be different; otherwise, there is no motion.
- The resolving frame may be the object frame, the reference frame, or a third frame.
- To describe these kinematic quantities fully, all three frames must be explicitly stated. Most authors do not do this, potentially causing confusion.

$$\vec{x}^{\gamma}_{\beta\alpha} \quad \vec{v}^{\gamma}_{\beta\alpha} \quad \vec{a}^{\gamma}_{\beta\alpha} \quad \vec{\omega}^{\gamma}_{\beta\alpha}$$

Forms of attitude representation

- Attitude can be represented using a variety of forms such as coordinate transformation matrices (direction cosines), Euler angles (navigation angles), and quaternions.

- All methods of representing attitude fulfill two functions:

- They describe the orientation of one coordinate frame with respect to another (e.g., an object frame with respect to a reference frame).

- They also provide a means of transforming a vector from one set of resolving axes to another.

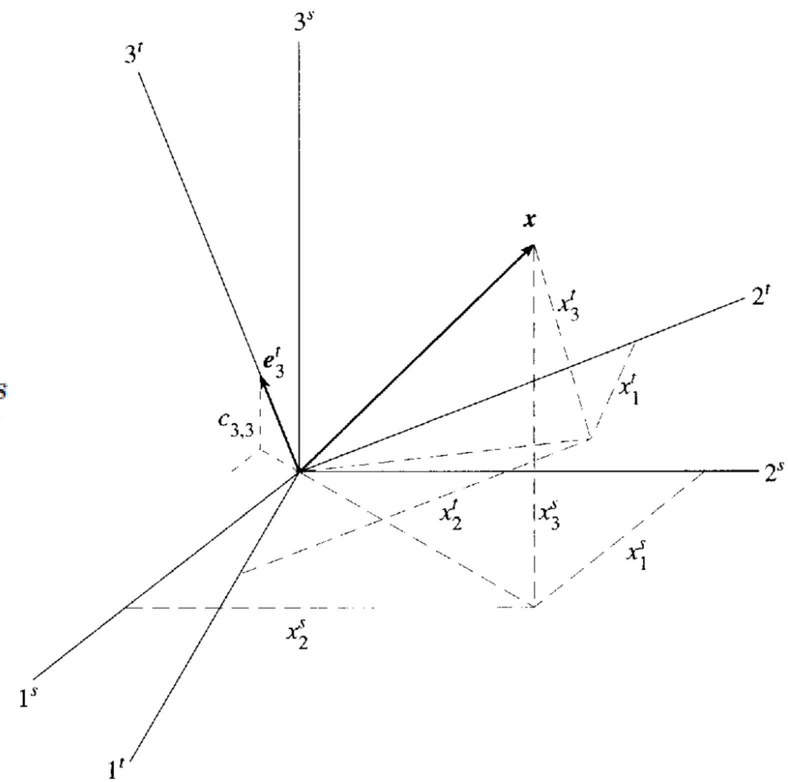
Coordinate transformation matrix

- The coordinate transformation matrix is a 3 × 3 matrix, denoted C (some authors use R or T).

- Where it is used to transform a vector from one set of resolving axes to another, the lower index represents the "from" coordinate frame and the upper index the "to" frame.

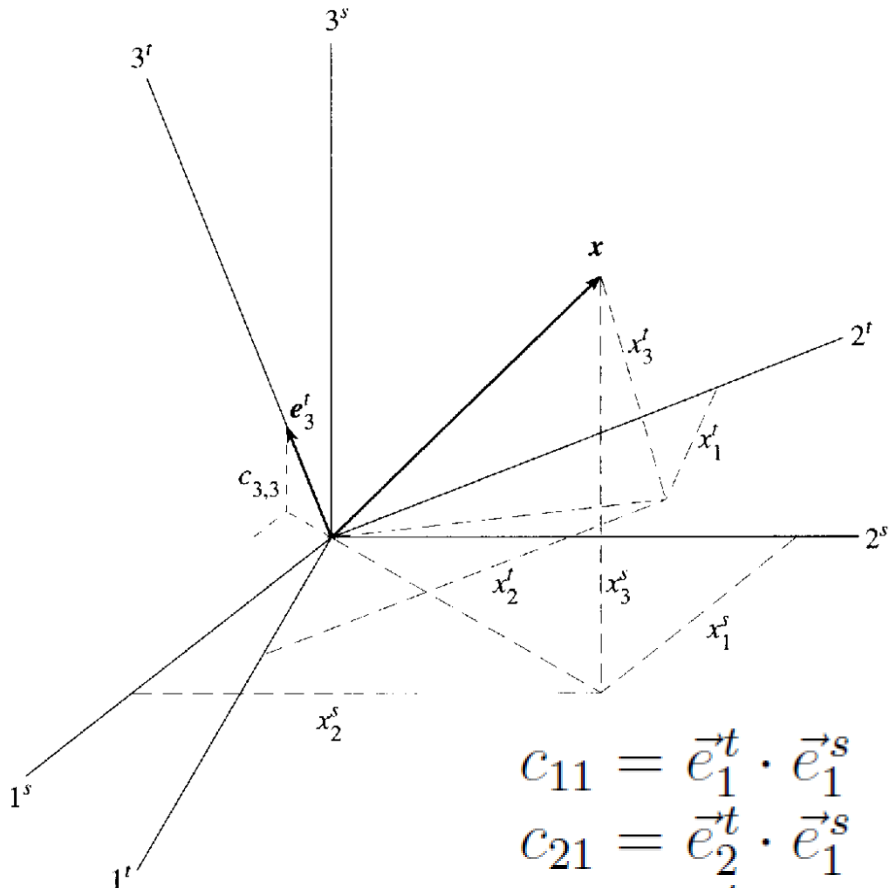
$$C_t^s$$

- The rows of a coordinate transformation matrix are in the "to" frame, whereas the columns are in the "from" frame.



Direction cosines

- Considering two concentric frames



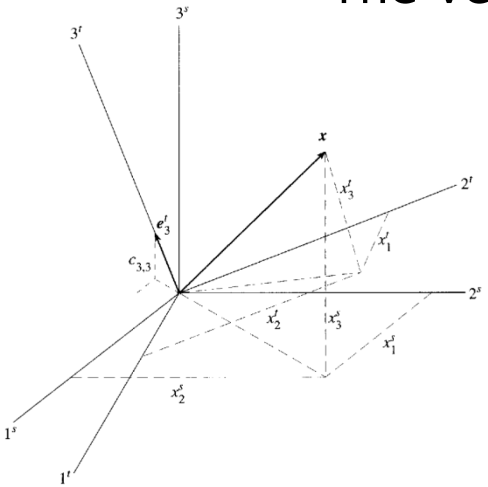
$$\vec{x}^s = x_1^s\vec{e}_1^s + x_2^s\vec{e}_2^s + x_3^s\vec{e}_3^s = \begin{pmatrix} \vec{e}_1^s & \vec{e}_2^s & \vec{e}_3^s \end{pmatrix} \begin{pmatrix} x_1^s \\ x_2^s \\ x_3^s \end{pmatrix}$$

$$\vec{x}^t = x_1^t\vec{e}_1^t + x_2^t\vec{e}_t^t + x_3^t\vec{e}_3^t = \begin{pmatrix} \vec{e}_1^t & \vec{e}_2^t & \vec{e}_3^t \end{pmatrix} \begin{pmatrix} x_1^t \\ x_2^t \\ x_3^t \end{pmatrix}$$

$$\begin{array}{lll} c_{11} = \vec{e}_1^t \cdot \vec{e}_1^s & c_{12} = \vec{e}_1^t \cdot \vec{e}_2^s & c_{13} = \vec{e}_1^t \cdot \vec{e}_3^s \\ c_{21} = \vec{e}_2^t \cdot \vec{e}_1^s & c_{22} = \vec{e}_2^t \cdot \vec{e}_2^s & c_{23} = \vec{e}_2^t \cdot \vec{e}_3^s \\ c_{31} = \vec{e}_3^t \cdot \vec{e}_1^s & c_{32} = \vec{e}_3^t \cdot \vec{e}_2^s & c_{33} = \vec{e}_3^t \cdot \vec{e}_3^s \end{array}$$

Direction cosines

- The vectors $\vec{e}_1^{\,t}, \vec{e}_2^{\,t}, \vec{e}_3^{\,t}$ expressed in the *s-frame* are



$$
\begin{aligned}
c_{11} &= \vec{e}_1^{\,t} \cdot \vec{e}_1^{\,s} & c_{12} &= \vec{e}_1^{\,t} \cdot \vec{e}_2^{\,s} & c_{13} &= \vec{e}_1^{\,t} \cdot \vec{e}_3^{\,s} \\
c_{21} &= \vec{e}_2^{\,t} \cdot \vec{e}_1^{\,s} & c_{22} &= \vec{e}_2^{\,t} \cdot \vec{e}_2^{\,s} & c_{23} &= \vec{e}_2^{\,t} \cdot \vec{e}_3^{\,s} \\
c_{31} &= \vec{e}_3^{\,t} \cdot \vec{e}_1^{\,s} & c_{32} &= \vec{e}_3^{\,t} \cdot \vec{e}_2^{\,s} & c_{33} &= \vec{e}_3^{\,t} \cdot \vec{e}_3^{\,s}
\end{aligned}
$$

- The axis of the *t-frame* expressed in the *s-frame* are

$$
\begin{aligned}
\vec{e}_1^{\,t} &= c_{11}\vec{e}_1^{\,s} + c_{12}\vec{e}_1^{\,s} + c_{13}\vec{e}_1^{\,s} \\
\vec{e}_2^{\,t} &= c_{21}\vec{e}_1^{\,s} + c_{22}\vec{e}_1^{\,s} + c_{23}\vec{e}_1^{\,s} \\
\vec{e}_3^{\,t} &= c_{31}\vec{e}_1^{\,s} + c_{32}\vec{e}_1^{\,s} + c_{33}\vec{e}_1^{\,s}
\end{aligned}
$$

Direction cosine matrix

$$\begin{pmatrix} \vec{e}_1^{\,s} & \vec{e}_2^{\,s} & \vec{e}_3^{\,s} \end{pmatrix} \begin{pmatrix} x_1^s \\ x_2^s \\ x_3^s \end{pmatrix} = \begin{pmatrix} \vec{e}_1^{\,s} & \vec{e}_2^{\,s} & \vec{e}_3^{\,s} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} x_1^t \\ x_2^t \\ x_3^t \end{pmatrix}$$

$$\boxed{\vec{x}^{\,s} = C_t^s \vec{x}^{\,t}} \qquad C_t^s = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

- Only 3 elements are independent $\rightarrow$ 3 degrees of freedom
- Since C_t^s is an orthonormal matrix

$$C_t^s (C_t^s)^T = I \;\Rightarrow\; C_t^s = (C_s^t)^{-1} = (C_s^t)^T$$

Coordinate transformation matrix

C_α^β can also be used to transform ‘transformation matrices’

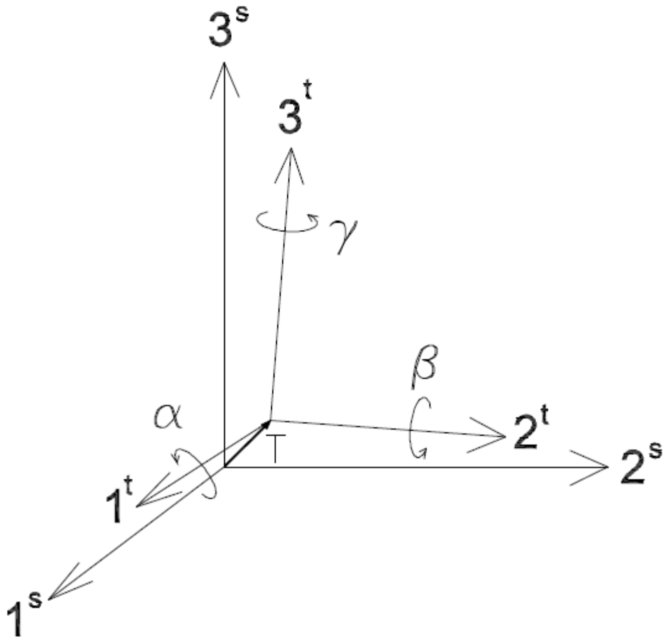
$$\vec{y}^{\,t} = A^t \vec{x}^{\,t}$$

$$\vec{y}^{\,t} = A^t \vec{x}^{\,t} \;\Rightarrow\; C_s^t \vec{y}^{\,s} = A^t C_s^t \vec{x}^{\,s} \;\Rightarrow\; \vec{y}^{\,s} = C_t^s A^t C_s^t \vec{x}^{\,s}$$

$$A^s = C_t^s A^t C_s^t$$

Euler angles (Jekeli notation)

The relative orientation of two frames can also be described by a sequence of rotations. Therefore, an alternative to the direction cosine matrix transformation is the successive application of rotation matrices about specific axes (there are many possibilities)



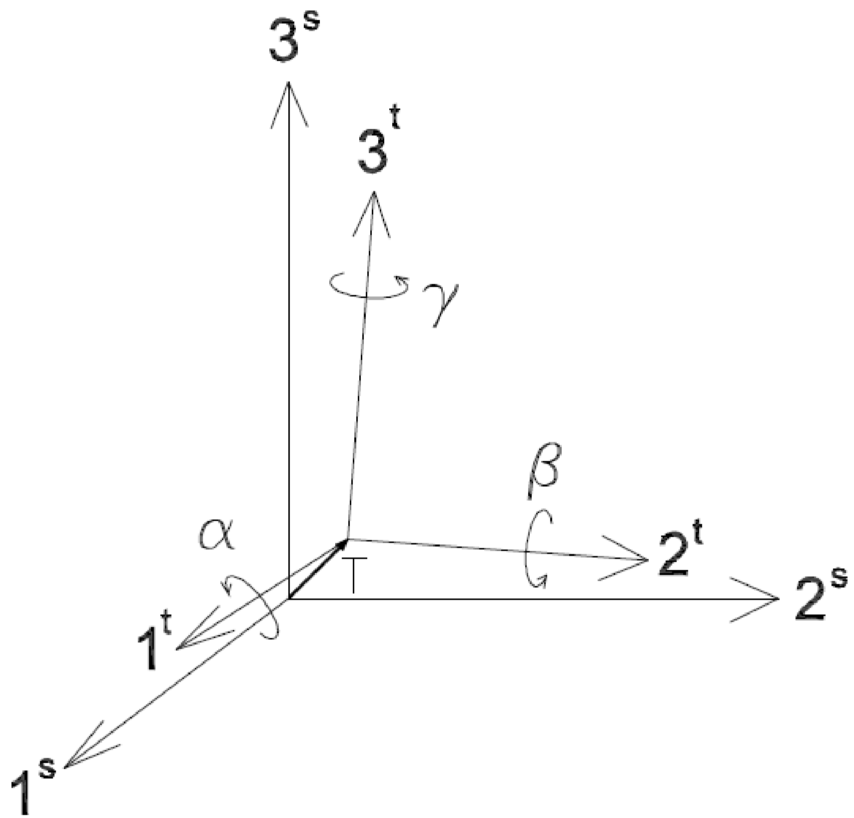
$$\boxed{\vec{x}^{\,s} = R_3(\gamma)R_2(\beta)R_1(\alpha)\vec{x}^{\,t}}$$

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \operatorname{sen}\theta \\ 0 & -\operatorname{sen}\theta & \cos\theta \end{pmatrix} \quad R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\operatorname{sen}\theta \\ 0 & 1 & 0 \\ \operatorname{sen}\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_3(\theta) = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_j^{-1}(\theta) = R_j^T(\theta) \qquad R_j^{-1}(\theta) = R_j(-\theta) \qquad \boldsymbol{R_1(\alpha)R_2(\beta) \neq R_2(\beta)R_1(\alpha)}$$

Euler angles (Jekeli notation)



A rotation is considered **positive** in the counterclockwise sense as viewed along the axis toward the origin (right-hand rule)

Euler angles (Jekeli notation)

$$(R_3(\gamma)R_2(\beta)R_1(\alpha))^{-1} = (R_3(\gamma)R_2(\beta)R_1(\alpha))^T = R_1(-\alpha)R_2(-\beta)R_3(-\gamma)$$

$$C_t^s = R_3(\gamma)R_2(\beta)R_1(\alpha)$$

$$C_t^s = \begin{pmatrix} \cos\gamma\cos\beta & \cos\gamma\,\mathrm{sen}\,\beta\,\mathrm{sen}\,\alpha + \mathrm{sen}\,\gamma\cos\alpha & -\cos\gamma\,\mathrm{sen}\,\beta\cos\alpha + \mathrm{sen}\,\gamma\,\mathrm{sen}\,\alpha \\ -\mathrm{sen}\,\gamma\cos\beta & -\mathrm{sen}\,\gamma\,\mathrm{sen}\,\beta\,\mathrm{sen}\,\alpha + \cos\gamma\cos\alpha & \mathrm{sen}\,\gamma\,\mathrm{sen}\,\beta\cos\alpha + \cos\gamma\,\mathrm{sen}\,\alpha \\ \mathrm{sen}\,\beta & -\cos\beta\,\mathrm{sen}\,\alpha & \cos\beta\cos\alpha \end{pmatrix}$$

Conversely, Euler angles can be obtained from the direction cosine matrix

$$\begin{aligned} \alpha &= \arctan\left(\frac{-c_{32}}{c_{33}}\right) &= \arctan\left(\frac{\cos\beta\,\mathrm{sen}\,\alpha}{\cos\beta\cos\alpha}\right) \\ \beta &= \mathrm{arc\,sen}\,(c_{31}) &= \mathrm{arc\,sen}\,(\mathrm{sen}\,\beta) \\ \gamma &= \arctan\left(\frac{-c_{21}}{c_{11}}\right) &= \arctan\left(\frac{\mathrm{sen}\,\gamma\cos\beta}{\cos\gamma\cos\beta}\right) \end{aligned}$$

atan2 function

Euler angles (Jekeli notation)

For small angles we may approximate $\cos\theta \approx 1$ and $\operatorname{sen}\theta \approx \theta$ and retaining only first-order terms we obtain

$$C_t^s = R_3(\gamma)R_2(\beta)R_1(\alpha) \approx \begin{pmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} = I - \Psi$$

where Ψ is a skew symmetric matrix of the small rotation angles

The order of rotations does not matter!

Euler angles (Jekeli notation)

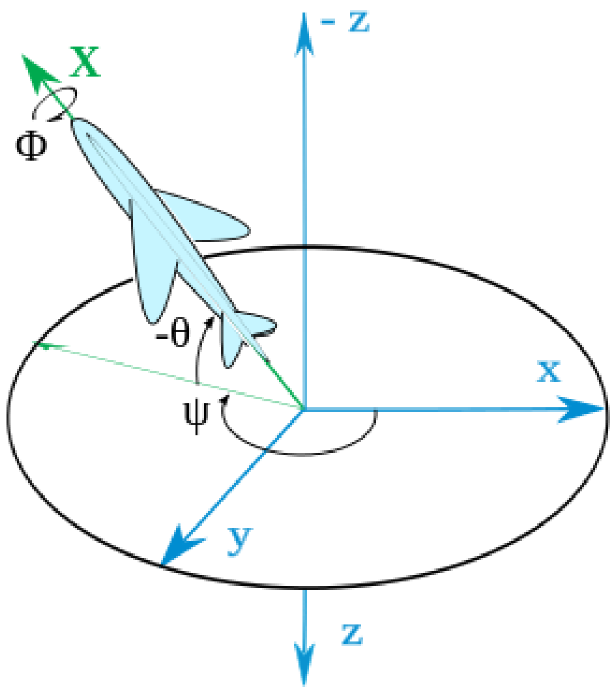
The direct transformation for small angles is given by

$$C_t^s \approx I - \Psi = \begin{pmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{pmatrix}$$

The reverse transformation is given by

$$C_s^t \approx \begin{pmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0 \end{pmatrix} = I - \Psi^T$$

Navigation angles



When the Euler angles relates the ***b-frame*** to the ***n-frame*** (NED) they are usually called navigation angles

- Roll (bank) $\phi = [-180^\circ, +180^\circ]$
- Pitch (elevation) $\theta = [-90^\circ, +90^\circ]$
- Yaw (heading) $\psi = [-180^\circ, +180^\circ]$

because they are used to manoeuvre the vehicle.

Singularity problems when pitch is 90º, for roll and yaw are not uniquely defined!

Navigation angles

$$\boxed{\vec{x}^{\,s} = C_t^s \vec{x}^{\,t}}$$

$$C_t^s = R_3(\psi)R_2(\theta)R_1(\phi)$$

$$C_t^s = \begin{pmatrix} \cos\psi\cos\theta & \cos\psi\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi + \mathrm{sen}\,\psi\cos\phi & -\cos\psi\,\mathrm{sen}\,\theta\cos\phi + \mathrm{sen}\,\psi\,\mathrm{sen}\,\phi \\ -\,\mathrm{sen}\,\psi\cos\theta & -\,\mathrm{sen}\,\psi\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi + \cos\psi\cos\phi & \mathrm{sen}\,\psi\,\mathrm{sen}\,\theta\cos\phi + \cos\psi\,\mathrm{sen}\,\phi \\ \mathrm{sen}\,\theta & -\cos\theta\,\mathrm{sen}\,\phi & \cos\theta\cos\phi \end{pmatrix}$$

Be careful with the rotation matrix in the Sxens MTx User Manual

$$\boxed{\mathbf{x_G} = R_{GS}\mathbf{x_S}}$$

$$\begin{aligned} R_{GS} &= R_\psi^Z R_\theta^Y R_\phi^X \\ &= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \end{aligned}$$

Quaternions

- A rotation may be represented using a quaternion, which is a hyper-complex number with four components

$$q = (q_0, q_1, q_2, q_3) \qquad q_\zeta = (\cos\frac{\zeta}{2}, \vec{n}\,\mathrm{sen}\,\frac{\zeta}{2}) = (q_0, q_1, q_2, q_3)$$

where ζ represents the magnitude of the rotation around a unit vector $\vec{n}$

- Quaternion attitude representation is computationally efficient.

- Manipulation of quaternions is not intuitive.

- Consequently, discussion of quaternions here is limited to their transformation to and from coordinate transformation matrices

Quaternions

The three Euler rotations are substituted by only one ζ rotation around the $\vec{n}$ axis defined by the rotation

$$C_t^s = \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3q_0) & 2(q_1q_3-q_2q_0) \\ 2(q_1q_2-q_3q_0) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_2q_0) & 2(q_2q_3-q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix}$$

$$\boxed{\begin{aligned} \alpha &= \arctan\left(\frac{-2(q_2q_3-q_0q_1)}{q_0^2-q_1^2-q_2^2+q_3^2}\right) \\ \beta &= -\operatorname{arc\,sen}\left(2(q_1q_3+q_2q_0)\right) \\ \gamma &= \arctan\left(\frac{-2(q_1q_2-q_3q_0)}{q_0^2+q_1^2-q_2^2-q_3^2}\right) \end{aligned}}$$

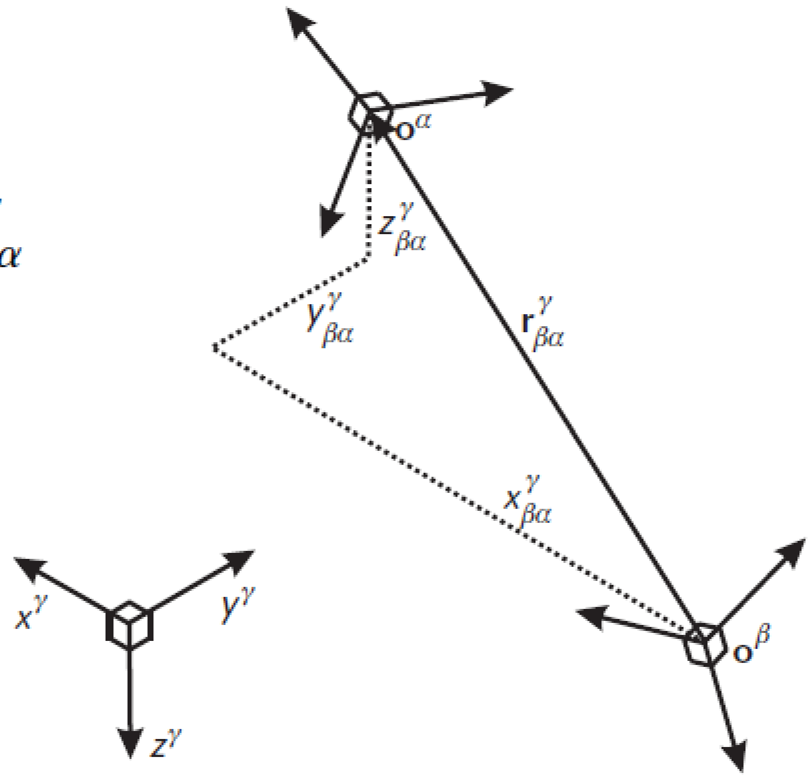
$$\boxed{\begin{aligned} q_0 &= \frac{1}{2}\sqrt{1+c_{11}+c_{22}+c_{33}} \\ q_1 &= \frac{1}{4q_0}(c_{23}+c_{32}) \\ q_2 &= \frac{1}{4q_0}(c_{31}+c_{13}) \\ q_3 &= \frac{1}{4q_0}(c_{12}+c_{21}) \end{aligned}}$$

Translation

- Reference frame, object frame, and resolving frame

$$\mathbf{r}^{\gamma}_{\beta\alpha} = -\mathbf{r}^{\gamma}_{\alpha\beta}$$

$$\mathbf{r}^{\gamma}_{\beta\alpha} = \mathbf{r}^{\gamma}_{\beta\delta} + \mathbf{r}^{\gamma}_{\delta\alpha}$$



$$\mathbf{r}^{\delta}_{\beta\alpha} = \mathbf{C}^{\delta}_{\gamma}\mathbf{r}^{\gamma}_{\beta\alpha}$$

$$\mathbf{r}^{\alpha}_{\alpha\beta} = -\mathbf{C}^{\alpha}_{\beta}\mathbf{r}^{\beta}_{\beta\alpha}$$

Figure 2.9 Position of the origin of frame α with respect to the origin of frame β in frame γ axes.

Axial Vectors

- Because the angular data provided by IMUs are related to angular rates of the instrument or body frame, we need to derive equations that express these sensed rates in terms of rates of angles in, say, the navigation frame.

- An axial vector (also known as pseudo-vector) is the ordered triplet of Eulerian angles (α, β, γ).

- It is not a true vector, except under special circumstances.

- Specifically two such axial "vectors", in general, do not obey the commutativity property of vectors

$$\vec{\psi}_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \qquad \vec{\psi}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \qquad \vec{\psi}_1 + \vec{\psi}_2 \neq \vec{\psi}_2 + \vec{\psi}_1$$

Axial Vectors

This is related to the dependence of the rotational transformation on the order of axis-rotations.

On the other hand, if the angles are small, for instance when they are angular rates, then $\vec{\psi}$ behaves like a vector.

For our purposes, $\vec{\psi} = (\alpha, \beta, \gamma)^T$s always assumed to be a triple of small angles and then

$$\vec{x}^{\,s} = C_t^s \vec{x}^{\,t} = (I - \Psi)\vec{x}^{\,t} = \vec{x}^{\,t} - \Psi \vec{x}^{\,t} = \vec{x}^{\,t} - \vec{\psi} \times \vec{x}^{\,t}$$

$$[\vec{\psi}\times] = \left[\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \times \right] = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} = \Psi$$

Axial vectors

Be $\vec{\psi}^t$ an axial vector that defines a small rotation in the ***t-frame***, then the corresponding differential rotación in the s-frame is obtained by using

$$\vec{\psi}^s = C_t^s \vec{\psi}^t$$

Alternatively, the same operation can be done using matrices

$$\Psi^s = C_t^s \Psi^t C_s^t$$

Angular rates

Here, we consider that the frames are rotating with respect to each other and that the rotations are functions of time.

That is, the angles have velocities (and accelerations) associated with them.

Let $\vec{\omega} = (\omega_1, \omega_2, \omega_3)^T$ be a vector of rotational rates about the three respective, instantaneous axes of a frame.

These rates need not be small because rates, by definition, are infinitesimal angles in the ratio to infinitesimal increments of time.

Angular rates

Notation:

$\vec{\omega}^{\,t}_{st}$ Angular velocity of the ***t-frame*** with respect to the ***s-frame*** with coordinates in the ***t-frame***.

$\vec{\omega}^{\,s}_{ts}$ Angular velocity of the ***s-frame*** with respect to the ***t-frame*** with coordinates in the ***s-frame***.

As vectors, they are related by the corresponding rotational transformation (though its elements also depend on time)

$$\vec{\omega}^{\,t}_{st} = C^t_s \vec{\omega}^{\,s}_{st} = -C^t_s \vec{\omega}^{\,s}_{ts}$$

since

$$\vec{\omega}^{\,s}_{st} = -\vec{\omega}^{\,s}_{ts}$$

Angular rates

Angular velocities can be added or substracted as long as they are in the same frame

$$\vec{\omega}_{st}^{t} = \vec{\omega}_{su}^{t} + \vec{\omega}_{ut}^{t}$$

Alternatively, a skew matrix can be used

$$[\vec{\omega}_{st}^{t}\times] = \Omega_{st}^{t} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

Differential Equation of the Transformation

Usually, we consider two frames that are rotating with respect to each other; that is, their relative orientation changes with time.

$$\dot{C}_t^s$$

It is important to identify the frame in which the time differentiation takes place. Unless otherwise indicated, the time-differentiation is done in the frame designated by the superscript of the variable

$$\dot{C}_t^s = \lim_{\delta\tau \to 0} \frac{C_t^s(\tau + \delta\tau) - C_t^s(\tau)}{\delta\tau}$$

τ initial time

$\tau + \delta\tau$ final time

Differential Equation of the Transformation

The transformation at time $\tau + \delta\tau$ is the result of the transformation up to time τ followed by a small change of the *s-frame* during the interval $\delta\tau$

$$C_t^s(\tau + \delta\tau) = \delta C^s C_t^s(\tau)$$

Where the small-angle transformation can also be written as

$$\delta C^s = I - \Psi^s$$

Differential Equation of the Transformation

$$\dot{C}_t^s = \lim_{\delta\tau \to 0} \frac{(I - \Psi^s)C_t^s(\tau) - C_t^s(\tau)}{\delta\tau} =$$

$$= \lim_{\delta\tau \to 0} \frac{-\Psi^s C_t^s(\tau)}{\delta\tau} = -\lim_{\delta\tau \to 0} \frac{\Psi^s}{\delta\tau} C_t^s(\tau) = -\Omega_{ts}^s C_t^s$$

$$\left.\begin{aligned} \vec{\omega}_{st}^{\,s} &= -\vec{\omega}_{ts}^{\,s} \\ \Omega_{ts}^s &= -\Omega_{st}^s = -C_t^s \Omega_{st}^t C_s^t \end{aligned}\right\} \quad \boxed{\dot{C}_t^s = C_t^s \Omega_{st}^t}$$

Linear velocity transformation

Linear velocities transformation need to use differential equations

$$\dot{\vec{x}}^s = C_t^s \dot{\vec{x}}^t + \dot{C}_t^s \vec{x}^t = C_t^s(\dot{\vec{x}}^t + \Omega_{st}^t \vec{x}^t)$$

Alternatively, (Coriolis' Law)

$$C_s^t \dot{\vec{x}}^s = \dot{\vec{x}}^t + \vec{\omega}_{st}^t \times \vec{x}^t$$

The term in the left is a vector in the ***t-frame*** , but derivation has been done in the ***s-frame***

Specific transformations

Among the usual transformations for sensor integration and navigation they are

- ✓Between the ***e-frame*** and the ***i-frame***
- ✓Between the ***n-frame*** and the ***e-frame***
- ✓Between the ***b-frame*** and the ***n-frame***

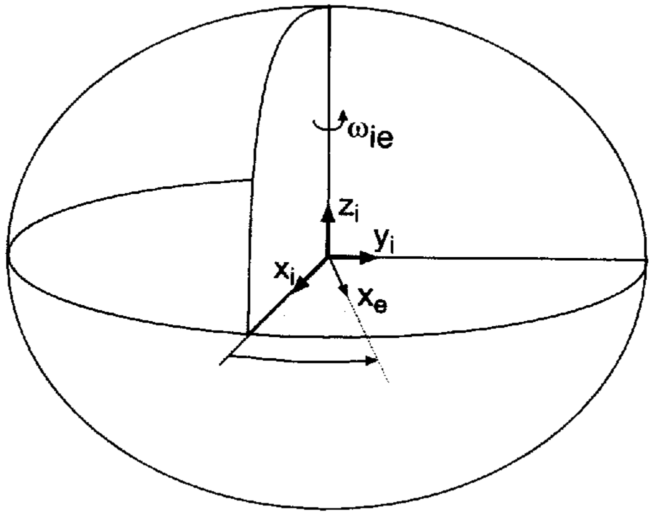
e-frame ↔ *i-frame*

Although EOPs and precesion-nutacion parameters should be included for precise navigation, here we can use a simplified inertial frame that can be safely used in most terrestrial navigation applications.

✓ The *e-frame* and the *i-frame* are assumed to be concentric.

✓ The *e-frame* rotates around the *i-frame* with an uniform angular velocity

✓ ω = 7292115e-11 rad/s (GRS80)



$$\vec{\omega}_{ie}^{e} = (0, 0, \omega_e)^T$$

e-frame ↔ i-frame

The total angle rotated in a time t is $\omega_e t$, therefore the rotation matrix around the third axis is

$$C_i^e = R_3(\omega_e t) = \begin{pmatrix} \cos\omega_e t & \operatorname{sen}\omega_e t & 0 \\ -\operatorname{sen}\omega_e t & \cos\omega_e t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that…

$$\vec{\omega}_{ie}^{\,e} = \vec{\omega}_{ie}^{\,i}$$

e-frame ↔ i-frame

Using ECEF coordinates (X,Y,Z)

$$\begin{pmatrix} x_1^i \\ x_2^i \\ x_3^i \end{pmatrix} = \begin{pmatrix} \cos\omega_e t & -\operatorname{sen}\omega_e t & 0 \\ \operatorname{sen}\omega_e t & \cos\omega_e t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^e \\ x_2^e \\ x_3^e \end{pmatrix}$$

Introducing geodetic coordinates (φ, λ, h)

$$\begin{pmatrix} x_1^i \\ x_2^i \\ x_3^i \end{pmatrix} = \begin{pmatrix} \cos\omega_e t & -\operatorname{sen}\omega_e t & 0 \\ \operatorname{sen}\omega_e t & \cos\omega_e t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (\nu + h)\cos\varphi\cos\lambda \\ (\nu + h)\cos\varphi\operatorname{sen}\lambda \\ [\nu(1 - e^2) + h]\operatorname{sen}\varphi \end{pmatrix}$$

n-frame ↔ e-frame

$(X,Y,Z) \rightarrow (END)$

The transformation consists of two rotations. First, an angle $(\varphi+\pi/2)$ around the local E axis, and then an angle $-\lambda$ around the new D axis as follows

$$C_n^e = R_3(-\lambda)R_2(\varphi+\pi/2) = \begin{pmatrix} -\operatorname{sen}\varphi\cos\lambda & -\operatorname{sen}\lambda & -\cos\varphi\cos\lambda \\ -\operatorname{sen}\varphi\operatorname{sen}\lambda & \cos\lambda & -\cos\varphi\operatorname{sen}\lambda \\ \cos\varphi & 0 & -\sin\varphi \end{pmatrix}$$

n-frame ↔ e-frame

How can the angular rate be obtained?

$$\boxed{\dot{C}_t^s = C_t^s \Omega_{st}^t} \quad \Longrightarrow \quad \Omega_{en}^n = C_e^n \dot{C}_n^e$$

$$\dot{C}_n^e = \begin{pmatrix} -\dot{\varphi}\cos\varphi\cos\lambda + \dot{\lambda}\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\lambda & -\dot{\lambda}\cos\lambda & \dot{\varphi}\,\mathrm{sen}\,\varphi\cos\lambda \\ -\dot{\varphi}\cos\varphi\,\mathrm{sen}\,\lambda - \dot{\lambda}\,\mathrm{sen}\,\varphi\cos\lambda & -\dot{\lambda}\,\mathrm{sen}\,\lambda & \dot{\varphi}\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\lambda \\ -\dot{\varphi}\,\mathrm{sen}\,\varphi & 0 & -\dot{\varphi}\cos\varphi \end{pmatrix}$$

$$\Omega_{en}^n = C_e^n \dot{C}_n^e = \begin{pmatrix} 0 & \dot{\lambda}\,\mathrm{sen}\,\varphi & \dot{\varphi} \\ -\dot{\lambda}\,\mathrm{sen}\,\varphi & 0\,\mathrm{sen}\,\lambda & -\dot{\lambda}\cos\varphi \\ \dot{\varphi} & \dot{\lambda}\cos\varphi & 0 \end{pmatrix}$$

$$\vec{\omega}_{en}^n = (\dot{\lambda}\cos\varphi, -\dot{\varphi}, -\dot{\lambda}\,\mathrm{sen}\,\varphi)^T$$

n-frame ↔ e-frame

Relation between linear velocities

$$\begin{aligned} v_n &= \dot{\varphi}(\rho + h) \\ v_e &= \dot{\lambda}(\nu + h)\cos\varphi \\ v_d &= -\dot{h} \end{aligned}$$

Angular speed of the ***n-frame*** respect to the ***i-frame*** can be easily obtained from

$$\vec{\omega}_{en}^{\,n} = (\dot{\lambda}\cos\varphi, -\dot{\varphi}, -\dot{\lambda}\,\mathrm{sen}\,\varphi)^T$$

$$\vec{\omega}_{in}^{\,n} = ((\dot{\lambda} + \omega_e)\cos\varphi, -\dot{\varphi}, -(\dot{\lambda} + \omega_e)\,\mathrm{sen}\,\varphi)^T$$

b-frame ↔ *n-frame*

Most applications need to transform between the ***b-frame*** and the ***n-frame***

$$(\alpha, \beta, \gamma) \qquad C_b^n = (C_n^b)^T = R_3(-\gamma) R_2(-\beta) R_1(-\alpha)$$

$$(\phi, \theta, \psi) \qquad C_b^n = (C_n^b)^T = R_3(-\psi) R_2(-\theta) R_1(-\phi)$$

$$C_b^n = \begin{pmatrix} \cos\psi\cos\theta & -\operatorname{sen}\psi\cos\theta & \operatorname{sen}\theta \\ \cos\psi\operatorname{sen}\theta\operatorname{sen}\phi + \operatorname{sen}\psi\cos\phi & -\operatorname{sen}\psi\operatorname{sen}\theta\operatorname{sen}\phi + \cos\psi\cos\phi & -\cos\theta\operatorname{sen}\phi \\ -\cos\psi\operatorname{sen}\theta\cos\phi + \operatorname{sen}\psi\operatorname{sen}\phi & \operatorname{sen}\psi\operatorname{sen}\theta\cos\phi + \cos\psi\operatorname{sen}\phi & \cos\theta\cos\phi \end{pmatrix}$$

b-frame ↔ n-frame

Normally, gyros measure angular velocity and rotation angles are subsequently derived from them

$$\boxed{\dot{C}_t^s = C_t^s \Omega_{st}^t}$$

$$\vec{\omega}_{nb}^b = \begin{pmatrix} 1 & 0 & -\operatorname{sen}\psi \\ 0 & \cos\phi & \cos\psi \operatorname{sen}\phi \\ 0 & -\operatorname{sen}\phi & \cos\psi\cos\phi \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -\operatorname{sen}\psi \\ 0 & \cos\phi & \cos\psi \operatorname{sen}\phi \\ 0 & -\operatorname{sen}\phi & \cos\psi\cos\phi \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \operatorname{sen}\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\operatorname{sen}\phi \\ 0 & \operatorname{sen}\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \operatorname{sen}\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\operatorname{sen}\phi \\ 0 & \operatorname{sen}\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \vec{\omega}_{nb}^b$$

b-frame ↔ n-frame

A possible problem when using Euler angles is that there are singularities when pitch is 90º

$$\theta = \pm 90^\circ \Rightarrow \tan\theta \text{ y } \sec\theta$$

Solución: using quaternions

$$\left\{\begin{array}{rcl} \dot{q}_0 & = & \frac{1}{2}(q_1\omega_1 + q_2\omega_2 + q_3\omega_3) \\ \dot{q}_1 & = & \frac{1}{2}(-q_0\omega_1 - q_3\omega_2 + q_2\omega_3) \\ \dot{q}_2 & = & \frac{1}{2}(q_3\omega_1 - q_0\omega_2 - q_1\omega_3) \\ \dot{q}_3 & = & \frac{1}{2}(-q_2\omega_1 + q_1\omega_2 - q_0\omega_3) \end{array}\right.$$