Seminar in Classification of Algebraic Varieties

Talk 2: Curves of Small Genus

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1 Introduction

In the following, X will always denote a curve, this is, a smooth and projective scheme of dimension 1, irreducible $X \to \operatorname{Spec} k$ of finite type over an algebraically closed field k. We will denote by g the genus of X and K a canonical divisor of X.

Theorem. (Riemann-Roch) *Let D be a divisor on X. Then*

$$\dim |D| - \dim |K - D| = \deg D + 1 - g.$$

Definition. A divisor D on X is said to be **special** if $\dim |K - D| \geq 0$. It is thus called **non-special** if $\dim |K - D| = -1$.

Remark. If $\deg D < 0$, then $|D| = \emptyset$, so $\dim |D| = -1$.
If $\deg D > \deg K$, then D is non-special.

Corollary. *Let D be a non-special divisor on X. Then*

$$\dim |D| = \deg D - g.$$

Theorem. (Clifford) *Let D be an effective special divisor on the curve X. Then*

$$\dim |D| \leq \frac{1}{2} \deg D.$$

Remark. If $\mathfrak{d}$ is a base point free linear system of dimension r and degree d, it defines a morphism

$$X \longrightarrow \mathbb{P}^r_k$$

of degree d. This is unique up to an automorphism of $\mathbb{P}^r_k$.
If D is a very ample divisor and $|D|$ is base point free, then the morphism it defines is an immersion and, since X is proper, it is indeed a closed immersion.

Proposition. *Let X be a curve and D a divisor on X.*

a) The complete linear system $|D|$ has no base points if and only if $\forall P \in X$ closed point,

$$\dim |D - P| = \dim |D| - 1.$$

b) D is very ample if and only if $\forall P, Q \in X$ closed points ($P = Q$ allowed),

$$\dim |D - P - Q| = \dim |D| - 2.$$

2 Canonical embedding

Proposition. *If $g \geq 2$, then $\forall n \geq 1$ $|nK|$ has no base points.*

Proof. a) First, let's prove the case $n = 1$. Remember that $\dim |K| = g - 1$. Let $P \in X$ closed. By Riemann-Roch,

$$\dim |K - P| = \dim |P| - \deg P - 1 + g = g - 2 + \dim |P| = \dim |K| - 1 + \dim |P|.$$

Since $P \in |P|$, $|P| \neq \emptyset$, so $\dim |P| \geq 0$. We want to see that $\dim |P| = 0$.

Assume $\dim |P| > 0$. That means $\exists Q \in |P|$ with $P \neq Q$. Then it has no base points, because

$$\operatorname{supp} P \cap \operatorname{supp} Q = \{P\} \cap \{Q\} = \emptyset,$$

so no point can be in the support of all divisors of $|P|$. Hence, $|P|$ defines a morphism

$$|P| : X \longrightarrow \mathbb{P}^1_k$$

of degree 1, which is absurd because X is not rational, as $g > 0$.

Thus, $\dim |P| = 0$, leading to

$$\dim |K - P| = \dim |K| - 1.$$

From this result and the previous proposition, it follows that $|K|$ has no base points.

b) Let $n \geq 2$. Since $g \geq 2$, we have

$$\deg K = 2g - 2 \geq 2,$$

so we get

$$2 \deg K \geq \deg K + 2,$$

and hence

$$\deg nK = n \deg K \geq 2 \deg K \geq \deg K + 2.$$

Now, for all $P \in X$ closed,

$$\deg(nK - P) = \deg nK - 1 > \deg K.$$

By Riemann-Roch,

$$\dim |nK - P| = \deg(nK - P) - g = \deg nK - g - 1 = \dim |nK| - 1.$$

Thus, $|nK|$ has no base points.

□

Definition. For $g \geq 2$ and $n \geq 1$, the base-free linear system $|nK|$ defines a morphism

$$\phi_n : X \longrightarrow \mathbb{P}^N_k$$

of degree $n(2g - 2)$ called the ***n*-th pluricanonical map**, where

$$N = \dim |nK| = \begin{cases} g - 1, & n = 1; \\ (2n - 1)g - 2n, & n \geq 2. \end{cases}$$

Definition. A curve X of genus $g \geq 2$ is said to be **hyperelliptic** if there exists a finite morphism $X \to \mathbb{P}^1_k$ of degree 2.

This definition is equivalent to the curve having a linear system of dimension 1 and degree 2.

Proposition. *If $g \geq 2$, then $|K|$ is very ample if and only if X is not hyperelliptic.*

Proof. By the criterion for very ample, $|K|$ will be very ample if and only if $\forall P, Q \in X$ closed,

$$\dim |K - P - Q| = \dim |K| - 2.$$

Applying Riemann-Roch,

$$\dim |K - P - Q| = \dim |P + Q| - \deg(P + Q) - 1 + g = \dim |P + Q| - 3 + g.$$

Hence, we get the equivalences

$$|K| \text{ is very ample} \iff \forall P, Q \in X \;\; \dim |K - P - Q| = g - 3 \iff \forall P, Q \in X \;\; \dim |P + Q| = 0.$$

Let's now prove the double implication of the statement of the proposition.

- $\Rightarrow$) Assume X is hyperelliptic. Then there is a linear system $\mathfrak{d}$ of dimension 1 and degree 2. Observe that all divisors in $\mathfrak{d}$ are effective of degree 2, i.e., of the form $P + Q$ for $P, Q \in X$. Since $\dim \mathfrak{d} > 0$, $\exists P + Q \in \mathfrak{d}$. Then, $\mathfrak{d} \subseteq |P + Q|$, so $\dim |P + Q| \geq \dim \mathfrak{d} = 1 > 0$.

- $\Leftarrow$) Assume that $\exists P, Q \in X$ closed with $\dim |P + Q| \neq 0$. Since $P + Q \in |P + Q| \neq \emptyset$, then $\dim |P + Q| \geq 1$. Hence, $|P + Q|$ contains a linear divisor $\mathfrak{d}$ of dimension 1 and degree $\deg \mathfrak{d} = \deg(P + Q) = 2$. Thus, X is hyperelliptic.

$\square$

Definition. For a non-hyperelliptic curve X of genus $g \geq 3$, the closed immersion defined by a canonical divisor

$$|K| : X \hookrightarrow \mathbb{P}^{g-1}_k$$

is called the **canonical embedding**. It is unique up to an automorphism of $\mathbb{P}^{g-1}_k$. Its image, which is a curve of degree $2g - 2$, is a **canonical curve**.

Example.
- If X is a non-hyperelliptic curve of genus $g = 3$, then its canonical embedding $|K| : X \to \mathbb{P}^2_k$ has degree 4. This means that it embeds the curve X into $\mathbb{P}^2_k$ as a quartic curve.
- If X is a non-hyperelliptic curve of genus $g = 4$, then its canonical embedding $|K| : X \to \mathbb{P}^3_k$ has degree 6. This means that it embeds the curve X into $\mathbb{P}^3_k$ as a sextic curve.

3 Classification of curves of small genus

We want to study the set $\mathfrak{M}_g$ of all curves of genus g up to isomorphism. One first step to do this is to divide the curves to be classified according to whether they admit linear systems g^r_d of certain dimension r and degree d. For example, if they admit g^1_2, they are hyperelliptic and can be classified apart more easily.

Definition. A curve X is called **trigonal** if it admits a g^1_3.

Proposition. *A curve of genus g has a g^1_d for any $d \geq \frac{1}{2}g + 1$.*
For $d < \frac{1}{2}g + 1$, there exist curves of genus g with no g^1_d.

In particular, for every genus $g \geq 3$, there exist non-hyperelliptic curves.
Curves of genus $g \leq 4$ are always trigonal, but for genus $g \geq 5$ there exist non-trigonal curves.

3.1 Genus 3

The canonical embedding of a curve of genus 3 is a non-singular plane quartic curve. Conversely, every non-singular plane quartic curve has genus 3. Hence, the problem reduces to find all the non-singular quartic curves in $\mathbb{P}^2_k$. These are defined by homogeneous polynomials of degree 4.

Proposition. *$k[x_0, \ldots, x_r]_d = \{f \in k[x_0, \ldots, x_r] \mid f$ homogeneous$, \deg f = d\}$ is a k-vector space of base $\{x_0^{n_0} \cdots x_r^{n_r} \mid n_0 + \cdots + n_r = d\}$. Moreover,*

$$\dim k[x_0, \ldots, x_r]_d = \binom{r+d}{d}.$$

Proof. (Sketch) $\{x_0^{n_0} \cdots x_r^{n_r} \mid n_0 + \cdots + n_r = d\}$ is base of $k[x_0, \ldots, x_r]_d$ by construction of homogeneous polynomials. A typical exercise in combinatorics shows that

$$\#\{x_0^{n_0} \cdots x_r^{n_r} \mid n_0 + \cdots + n_r = d\} = \#\{(n_0, \ldots, n_r) \in \mathbb{N}_0^{r+1} \mid n_0 + \cdots + n_r = d\} = \binom{r+d}{r}.$$

□

In the case $r = 2$, $d = 4$ we get that homogeneous polynomials of degree 4 in 3 variables are determined by $\binom{6}{2} = 15$ coefficients. Hence,

$$\{P \in \mathbb{A}^{15}_k \mid P \text{ closed}\} \longleftrightarrow k[x_0, x_1, x_2]_4.$$

Now, two curves $X, Y \subseteq \mathbb{P}^2_k$ are the same if and only if the defining polynomials f, g satisfy $f = \lambda g$ for some $\lambda \in k$, so

$$\{P \in \mathbb{P}^{14}_k \mid P \text{ closed}\} = \{P \in \mathbb{A}^{15}_k \setminus \{0\} \mid P \text{ closed}\}/k^\times \longleftrightarrow \{X \subseteq \mathbb{P}^2_k \mid X \text{ quartic curve}\}.$$

Now, non-singular quartic curves form an open variety $U \subseteq \mathbb{P}^{14}_k$ and the dimension remains unchanged: $\dim U = 14$.
In our variety of moduli we are considering the isomorphism classes of curves, so $\mathfrak{M}_3^{\text{non-hyp}} = U/\cong$.

Fact. All curves $X, Y \subseteq \mathbb{P}^2_k$ are canonical curves, any isomorphism $X \xrightarrow{\cong} Y$ can be uniquely extended to an automorphism of $\mathbb{P}^2_k$.

Hence,

$$\mathfrak{M}_3^{\text{non-hyp}} = U/\operatorname{PGL}_k(2),$$

where $\operatorname{PGL}_k(2) = \operatorname{Aut}\mathbb{P}^2_k$ is the projective linear group. This group can be represented by the matrices $\operatorname{PGL}_k(n) \cong \operatorname{GL}_{n+1}(k)/k^\times$, so $\dim \operatorname{PGL}_k(2) = (2+1)^2 - 1 = 8$.
Exercise IV.5.2 from Hartshorne states that if X is a curve of genus $g \geq 2$ over a field k with $\operatorname{char} k = 0$, then the group $\operatorname{Aut} X$ is finite. Hence, $\dim \operatorname{Aut} X = 0$, so

$$\dim \mathfrak{M}_3^{\text{non-hyp}} = \dim U - \dim \operatorname{PGL}_k(2) = 14 - 8 = 6.$$

3.2 Genus 4

Fact. Sextic curves in $\mathbb{P}^3_k$ are complete intersections between a unique quadric surface $Q \subseteq \mathbb{P}^3_k$ and a cubic surface F.
Conversely, the complete intersection of a quadric surface and a cubic surface is a sextic curve.
X has exactly one g^1_3 if and only if Q is singular. In that case, Q is a quadric cone.

Complete intersection means that the ideal sheaf of the curve is the sum of the ideal sheaves of the surfaces.
The canonical embedding of a non-hyperelliptic curve of genus 4 is a sextic curve in $\mathbb{P}^3_k$, so we just need to classify sextics of $\mathbb{P}^3_k$.

$$\{P \in \mathbb{P}^9_k \mid P \text{ closed}\} \longleftrightarrow \big(k[x_0, x_1, x_2, x_3]_2 \setminus \{0\}\big)/k^\times \longleftrightarrow \{Q \subseteq \mathbb{P}^3_k \mid Q \text{ quadric surface}\}.$$

Now, the set of non-singular sextic curves in $\mathbb{P}^3_k$ can be described as a projective bundle E over the space of quadric surfaces $\mathbb{P}^9_k$. Considering the projection map $\pi : E \to \mathbb{P}^9_k$, the fibre E_Q of a quadric $Q \in \mathbb{P}^9_k$ is the set of all curves obtained by intersecting Q with cubic surfaces and $H^0(\mathbb{P}^3_k, \mathcal{O}_Q(3))$ is the space of homogeneous cubic polynomials over Q, i.e., the $E_Q = \mathbb{P}(H^0(\mathbb{P}^3_k, \mathcal{O}_Q(3)))$.
Let $Q \subseteq \mathbb{P}^3_k$ be a quadric surface. Twisting by 3 the exact sequence of the closed subscheme, we get

$$0 \longrightarrow \mathcal{L}(-Q) = \mathcal{O}_{\mathbb{P}^3_k}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3_k} \longrightarrow \mathcal{O}_Q \longrightarrow 0,$$

$$0 \longrightarrow H^0\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1)\big) \longrightarrow H^0\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(3)\big) \longrightarrow H^0\big(\mathbb{P}^3_k, \mathcal{O}_Q(3)\big) \longrightarrow 0.$$

$$\dim E_Q = h^0\big(\mathbb{P}^3_k, \mathcal{O}_Q(3)\big) - 1 = h^0\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(3)\big) - h^0\big(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1)\big) - 1 = \binom{6}{3} - \binom{4}{3} - 1 = 15.$$

Thus, the dimension of the projective bundle that parametrises the sextic curves in $\mathbb{P}^3_k$ is

$$\dim E = \dim \mathbb{P}^9_k + \dim E_Q = 9 + 15 = 24.$$

$$\mathfrak{M}_4^{\text{non-hyp}} = E/\operatorname{PGL}_k(3),$$

$$\dim \mathfrak{M}_4^{\text{non-hyp}} = \dim E/\operatorname{PGL}_k(3) = \dim E - \dim \operatorname{PGL}_k(3) = 24 - (4^2 - 1) = 24 - 15 = 9.$$

Now, let's study the curves of genus 4 that only have one g^1_3. Recall that these curves are characterised by their associated quadric surface Q being a cone. Now, all quadric cones – up to isomorphism – can be defined by a homogeneous polynomial of degree 2 with the term x_3^2 missing, so we are left with 9 coefficients. Hence, the variety that parametrises them is $\mathbb{P}^8_k$.
Repeating all the process, the projective bundle F of sextic curves in $\mathbb{P}^3_k$ with one g^1_3 over the quadric cones $\mathbb{P}^8_k$ also has fibres of degree 15, so we get a moduli variety $F/\operatorname{PGL}_k(3)$ of dimension 8.

3.3 Genus 5

Let X be a non-hyperelliptic curve of genus 5.

Fact. X does not have a g_3^1 if and only if its canonical embedding in $\mathbb{P}_k^4$ is the complete intersection of three quadric hypersurfaces.

The curves of genus 5 that don't admit a g_3^1 are the complete intersection of three quadric hypersurfaces, and these can be shown to form a moduli variety of dimension 12.
Let's now explicitly classify the trigonal curves.

Proposition. *X has a g_3^1 if and only if it can be represented as a plane quintic with one node.*

Proof.

- $\Rightarrow$) Let D be a g_3^1. By Riemann-Roch,

$$\dim|K-D| = \dim|D| - \deg D + g - 1 = 1 - 3 + 5 - 1 = 2.$$

  Hence, we have a map $|K-D| : X \to \mathbb{P}_k^2$ of degree 5. Its image is a plane curve, so we can use the formula for the genus:

$$5 = g = \frac{1}{2}(d-1)(d-2) - n = 6 - n,$$

  where n is the number of nodes. This gives $n = 1$.

- $\Leftarrow$) Let $f : X \to \mathbb{P}_k^2$ be a plane quintic curve. Let $D = f^*H$, where $H \subseteq \mathbb{P}_k^2$ is a hyperplane. Then,

$$\deg D = \deg f \cdot \deg H = 5.$$

$$\dim|D| = \dim|f^*H| \geq \dim|H| = h^0\big(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(1)\big) - 1 = \binom{3}{1} - 1 = 2.$$

  By Riemann-Roch,

$$\dim|K-D| = \dim|D| - \deg D + g - 1 = \dim|D| - 1 \geq 1 > 0.$$

  Hence, $K - D$ is a special divisor. We apply Clifford theorem:

$$2 \leq \dim|D| \leq \frac{1}{2}\deg D = \frac{5}{2},$$

  so $\dim|D| = 2$ and therefore, $\dim|K-D| = 1$ and $\deg(K-D) = 8 - 5 = 3$. Thus, $K - D$ is the g_3^1 we were looking for.

$\square$

Now the problem of classifying non-hyperelliptic trigonal curves of genus 5 has turned into classifying quintic plane curves with one node. These are defined by a homogeneous polynomial of degree 5, but – as in the case of quadric cones – one of the terms is missing, so there are $\binom{7}{2} - 1 = 20$ coefficients, so these curves are parametrised by $\mathbb{P}_k^{19}$. Now the classes of equivalence of these curves are represented by $\mathbb{P}_k^{19}/\operatorname{PGL}_k(2)$, which is a variety of dimension $19 - 8 = 11$.