Seminar in Toric Varieties

Talk 3: Affine Toric Varieties

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1 Group ring of a monoid

Definition. A **monoid** $(S, +)$ is a set S together with an associative operation $+ : S \times S \to S$ that has a neutral element $0 \in S$.
If the operation $+$ is commutative, then S is called a commutative monoid.
A monoid homomorphism $x : S_1 \to S_2$ between two monoids $(S_1, +_1), (S_2, +_2)$ is a map such that:

- for all $u, u' \in S_1$, $x(u +_1 u') = x(u) +_2 x(u')$,
- $x(0_1) = 0_2$.

Example. $(\mathbb{N}_0, +)$, $(\mathbb{Z}, +)$, $(\mathbb{C}, \cdot)$.

Definition. Let S be a commutative monoid. Let $\{u_i\}_{i \in I} \subseteq S$ be a family of elements in S.
We say that $\{u_i\}_{i \in I}$ are generators of S if

$$S = \sum_{i \in I} \mathbb{N}_0 u_i,$$

i.e. if $\forall s \in S\ \exists u_{i_1}, \dots, u_{i_n},\ \exists \lambda_1, \dots, \lambda_n \in \mathbb{N}_0$ such that $s = \lambda_1 u_{i_1} + \cdots + \lambda_n u_{i_n}$.
In case there is a finite family of generators, we say that S is finitely generated.

Example. $\mathbb{N}_0^n$ is a monoid with generators $e_1, \dots, e_n$.
Every $(\lambda_1, \dots, \lambda_n) \in \mathbb{N}_0^n$ can be expressed as

$$(\lambda_1, \dots, \lambda_n) = \lambda_1 e_1 + \cdots + \lambda_n e_n.$$

$\mathbb{Z}^n$ is a monoid with generators $e_1, -e_1, \dots, e_n, -e_n$.
Every $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ can be expressed as

$$(\lambda_1, \dots, \lambda_n) = \lambda_1^+ e_1 + \lambda_1^-(-e_1) + \cdots + \lambda_n^+ e_n + \lambda_n^-(-e_n),$$

where

$$\lambda_i^+ = \begin{cases} \lambda_i & \text{if} \quad \lambda_i \geq 0, \\ 0 & \text{if} \quad \lambda_i < 0, \end{cases} \qquad \lambda_i^- = \begin{cases} 0 & \text{if} \quad \lambda_i \geq 0, \\ -\lambda_i & \text{if} \quad \lambda_i < 0. \end{cases}$$

Definition. Given a commutative monoid S, we define its **group ring** $\mathbb{C}[S]$ to be

$$\mathbb{C}[S] := \bigoplus_{u\in S} \mathbb{C}\chi^u,$$

with the multiplication

$$\begin{aligned} \cdot : \mathbb{C}[S] \times \mathbb{C}[S] &\longrightarrow \mathbb{C}[S]. \\ (\chi^u, \chi^{u'}) &\longmapsto \chi^{u+u'} \end{aligned}$$

Note that $\bigoplus_{u\in S} \mathbb{C}\chi^u$ is defined as the $\mathbb{C}$-vector space that has S as a basis.

Proposition. *1) The group ring is in fact a commutative $\mathbb{C}$-algebra with unit χ^0.*

2) If $\{u_i\}_{i\in I}$ is a system of generators of S, then $\{\chi^{u_i}\}_{i\in I}$ is a system of generators of $\mathbb{C}[S]$ as a $\mathbb{C}$-algebra.

3) In particular, if S is finitely generated, then $\mathbb{C}[S]$ is a $\mathbb{C}$-affine algebra, i.e. it is finitely generated.

Proof. 1) By definition, $\mathbb{C}[S]$ is a $\mathbb{C}$-vector space. If we want the multiplication to satisfy associativity and distributivity, it must be defined as

$$\left(\sum_{i=1}^{n} \lambda_i \chi^{u_i}\right) \left(\sum_{j=1}^{m} \mu_j \chi^{u_j}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j \chi^{u_i+u_j}.$$

Check: χ^0 is the unity, the distributivity, the compatibility with scalars, the associativity and the commutativity.

2) Let $\sum_{j=1}^{m} \mu_j \chi^{v_j} \in \mathbb{C}[S]$. We can choose generators $u_1, \dots, u_n$ such that

$$v_j = \sum_{i=1}^{n} \lambda_{ij} u_i,$$

for some $\lambda_{ij} \in \mathbb{N}_0$. Hence,

$$\sum_{j=1}^{m} \mu_j \chi^{v_j} = \sum_{j=1}^{m} \mu_j \chi^{\sum_{i=1}^{n} \lambda_{ij} u_i} = \sum_{j=1}^{m} \mu_j (\chi^{u_1})^{\lambda_{1j}} \cdots (\chi^{u_n})^{\lambda_{nj}}.$$

Thus, $\{\chi^{u_i}\}_{i\in I}$ generate $\mathbb{C}[S]$ as a $\mathbb{C}$-algebra.

□

Proposition. *Given a homomorphism $x : S_1 \to S_2$ of commutative monoids, there exists a unique $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}[S_1] \to \mathbb{C}[S_2]$ such that $\forall u \in S_1$ $\varphi(\chi^u) = \chi^{x(u)}$.*

- *x is injective if and only if φ injective.*
- *x is surjective if and only if φ is surjective.*

- *In particular, if x is a monoid isomorphism, then φ is a $\mathbb{C}$-algebra isomorphism.*

Proof. Let $x : S_1 \to S_2$ be a monoid homomorphism. Define

$$\begin{aligned} \varphi : \mathbb{C}[S_1] &\longrightarrow \mathbb{C}[S_2]. \\ \sum_{i=1}^{n} \lambda_i \chi^{u_i} &\longmapsto \sum_{i=1}^{n} \lambda_i \chi^{x(u_i)} \end{aligned}$$

We already know from linear algebra that a map $x : S_1 \to S_2$ between bases of two vector spaces defines a unique linear map φ between the vector spaces. The only thing remaining to show is that our φ preserves the multiplication.
Check: φ preserves the multiplication.
The points about injectivity and surjectivity of x and φ follow from the already known structure of vector spaces. $\square$

Proposition. *1)* $\mathbb{C}[\mathbb{N}_0^n] \cong \mathbb{C}[X_1, \ldots, X_n]$.

2) $\mathbb{C}[\mathbb{Z}^n] \cong \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$.

Proof. 2) Consider the set of monomials

$$S \coloneqq \{X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mid \alpha_1, \ldots, \alpha_n \in \mathbb{Z}\} \subseteq \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}],$$

with the multiplication as operation. The multiplication is closed in S and $1 = X_1^0 \cdots X_n^0 \in S$, so S is a monoid.

Observe that $\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = \mathbb{C}[S]$, because S is a basis of $\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ because every polynomial can be written as a unique linear combination of monomials. Furthermore, the multiplication is compatible by construction.

Now, consider the map

$$\begin{aligned} x : \mathbb{Z}^n &\longmapsto S, \\ (\alpha_1, \ldots, \alpha_n) &\longmapsto X_1^{\alpha_1} \cdots X_n^{\alpha_n} \end{aligned}$$

which is surjective by definition of S. Let's see that it is a monoid homomorphism. Let $(\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$. Then

$$\begin{aligned} x((\alpha_1, \ldots, \alpha_n) + (\beta_1, \ldots, \beta_n)) &= x(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n) = X_1^{\alpha_1+\beta_1} \cdots X_n^{\alpha_n+\beta_n} \\ &= X_1^{\alpha_1} \cdots X_n^{\alpha_n} X_1^{\beta_1} \cdots X_n^{\beta_n} = x(\alpha_1 \ldots, \alpha_n) x(\beta_1, \ldots, \beta_n). \end{aligned}$$

It is also injective, because for $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, if

$$x(\alpha_1, \ldots, \alpha_n) = X_1^{\alpha_1} \cdots X_n^{\alpha_n} = 1,$$

then $\alpha_1 = \cdots = \alpha_n = 0$, so $(\alpha_1, \ldots, \alpha_n) = 0$.

By the previous proposition, it extends uniquely to a $\mathbb{C}$-algebra isomorphism

$$\varphi : \mathbb{C}[\mathbb{Z}^n] \longrightarrow \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}].$$

Notice that $x(e_i) = X_i$ and $x(-e_i) = X_i^{-1}$. Hence, it will hold also $\varphi(\chi^{e_i}) = X_i$ and $\varphi(\chi^{-e_i}) = X_i^{-1}$.

1) As $\mathbb{N}_0^n$ is a submonoid of $\mathbb{Z}^n$, $\mathbb{C}[\mathbb{N}_0^n]$ is a subalgebra of $\mathbb{C}[\mathbb{Z}^n]$. Hence, the previous $\mathbb{C}$-algebra isomorphism $\varphi : \mathbb{C}[\mathbb{Z}^n] \to \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ restricts to a $\mathbb{C}$-algebra isomorphism $\tilde{\varphi} : \mathbb{C}[\mathbb{N}_0^n] \to \varphi(\mathbb{C}[\mathbb{N}_0^n])$.

   We have said that $e_1, \ldots, e_n$ are generators of the monoid $\mathbb{N}_0^n$, so $\chi^{e_1}, \ldots, \chi^{e_n}$ are generators of the $\mathbb{C}$-algebra $\mathbb{C}[\mathbb{N}_0^n]$, so $\varphi(\chi^{e_1}) = X_1, \ldots, \varphi(\chi^{e_n}) = X_n$ are generators of the $\mathbb{C}$-algebra $\varphi(\mathbb{C}[\mathbb{N}_0^n])$. Hence, $\varphi(\mathbb{C}[\mathbb{N}_0^n]) = \mathbb{C}[X_1, \ldots, X_n]$. Thus,

$$\tilde{\varphi} : \mathbb{C}[\mathbb{N}_0^n] \to \mathbb{C}[X_1, \ldots, X_n]$$

   is a $\mathbb{C}$-algebra isomorphism.

$\square$

2 Geometrical interpretation of the spectrum

Theorem. *(Hilbert's nullstellensatz) Let $R = \mathbb{C}[X_1, \ldots, X_n]$. There is a bijection*

$$\begin{aligned} \mathbb{C}^n &\longleftrightarrow \{\mathfrak{m} \subseteq R \mid \mathfrak{m} \textit{ is a maximal ideal}\} = \operatorname{Spec}_{max} R \\ (\xi_1, \ldots, \xi_n) &\longmapsto (X_1 - \xi_1, \ldots, X_n - \xi_n) \end{aligned}$$

Moreover, for an ideal $I \subseteq R$, this restricts to the bijection

$$\{\xi \in \mathbb{C}^n \mid \forall f \in I \ f(\xi) = 0\} = \widetilde{V}(I) \longleftrightarrow \{\mathfrak{m} \subseteq R \mid I \subseteq \mathfrak{m}, \mathfrak{m} \textit{ is maximal}\} = V(I) \cap \operatorname{Spec}_{max} R$$

This theorem gives an equivalence between the space $\mathbb{C}^n$ and the maximal spectrum $\operatorname{Spec}_{\max} R$ of the ring of polynomials. Furthermore, the affine varieties $\widetilde{V}(I)$ in $\mathbb{C}^n$ correspond to closed subsets of the maximal spectrum with the Zariski topology.
Because of this, every affine variety $\widetilde{V}(I)$ can be seen as a closed subset $V(I)$ of the spectrum.
If $\mathcal{A}$ is an affine $\mathbb{C}$-algebra, if $\{a_1, \ldots, a_n\}$ are generators of $\mathcal{A}$, then the map

$$\begin{aligned} \mathbb{C}[X_1, \ldots, X_n] &\longrightarrow \mathcal{A} \\ X_i &\longmapsto a_i \end{aligned}$$

is a surjective $\mathbb{C}$-algebra homomorphism. Let I be its kernel. Then by the isomorphism theorem, $\mathbb{C}[X_1, \ldots, X_n]/I \cong \mathcal{A}$. Hence,

$$\operatorname{Spec} \mathcal{A} \cong \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n]/I \longleftrightarrow \{\mathfrak{p} \in \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n] \mid I \subseteq \mathfrak{p}\} = V(I) \subseteq \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n],$$

which is just an affine variety in the n-dimensional complex space.
Remember that given a ring R and $f \in R$, we define the localisation $R_f = S^{-1}R$, where $S = \{1, f, f^2, \ldots\}$.

$$\operatorname{Spec} R_f \longleftrightarrow \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \emptyset\} = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\} = D(f) = \operatorname{Spec} R \setminus V(f).$$

Example. Consider the ring of Laurent polynomials

$$R = \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = \mathbb{C}[X_1, \ldots, X_n]_{X_1, \ldots, X_n} = \mathbb{C}[X_1, \ldots, X_n]_{X_1 \cdots X_n}.$$

$$\operatorname{Spec} R = \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n]_{X_1 \cdots X_n} \cong \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n] \setminus V(X_1 \cdots X_n) \cong \mathbb{C}^n \setminus \widetilde{V}(X_1 \cdots X_n) = (\mathbb{C}^*)^n,$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is the multiplicative group of $\mathbb{C}$.

Proposition. *Given a finitely generated commutative monoid S, there exists a 1-1 correspondence*

$$\mathrm{Spec}_{max}\,\mathbb{C}[S] \longleftrightarrow \mathrm{Hom}(S,\mathbb{C}),$$

where $\mathrm{Hom}(S,\mathbb{C})$ *are the monoid homomorphisms from S to* $(\mathbb{C},\,\cdot\,)$.

Proof. Given a monoid homomorphism $x : S \to \mathbb{C}$, there is a unique extension to a $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \varphi : \mathbb{C}[S] &\longrightarrow \mathbb{C}. \\ \chi^u &\longmapsto x(u) \end{aligned}$$

It is clearly surjective, since $\forall \lambda \in \mathbb{C}\ \varphi(\lambda\chi^0) = \lambda$. Hence, $\mathbb{C}[S]/\ker\varphi \cong \mathbb{C}$, which is a field, so $\ker\varphi$ is a maximal ideal of $\mathbb{C}[S]$.
Now, let $\mathfrak{m} \subseteq \mathbb{C}[S]$ be a maximal ideal. We have a map

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook, "i"] \arrow[dr] & \mathbb{C}[S] \arrow[d, two heads, "p"] \\ & \mathbb{C}[S]/\mathfrak{m}. \end{tikzcd}$$

By Hilbert's nullstellensatz, $p \circ i : \mathbb{C} \to \mathbb{C}[S]/\mathfrak{m}$ is a $\mathbb{C}$-algebra isomorphism.
Hence, we have a monoid homomorphism

$$\begin{aligned} S \longrightarrow \mathbb{C}[S] &\longrightarrow \mathbb{C}[S]/\mathfrak{m} \quad \longrightarrow \mathbb{C}. \\ u \longmapsto \chi^u \quad &\longmapsto \overline{\chi^u} = \overline{\lambda\chi^0} \longmapsto \lambda \end{aligned}$$

Let's check that these associations are mutually inverse.
Let $x : S \to \mathbb{C}$ be a monoid homomorphism. We extend it into $\varphi : \mathbb{C}[S] \to \mathbb{C}$. We get the maximal ideal $\ker\varphi$. Notice that $\forall u \in S$

$$\varphi(\chi^u - x(u)\chi^0) = x(u) - x(u)x(0) = x(u) - x(u) = 0,$$

so the monoid homomorphism we finally get

$$\begin{aligned} S \longrightarrow \mathbb{C}[S] &\longrightarrow \mathbb{C}[S]/\ker\varphi \quad \longrightarrow \mathbb{C} \\ u \longmapsto \chi^u \quad &\longmapsto \overline{\chi^u} = \overline{x(u)\chi^0} \longmapsto x(u) \end{aligned}$$

is again x.
Conversely, for maximal ideal $\mathfrak{m} \subseteq \mathbb{C}[S]$, its monoid homomorphism is $S \to \mathbb{C}[S] \xrightarrow{p} \mathbb{C}[S]/\mathfrak{m} \xrightarrow{\cong} \mathbb{C}$, that induces the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[S] \xrightarrow{p} \mathbb{C}[S]/\mathfrak{m} \xrightarrow{\cong} \mathbb{C}$, which has kernel $\mathfrak{m}$. ☐

3 Affine toric varieties

Definition. Given a strongly convex rational polyhedral cone σ, we set $A_\sigma = \mathbb{C}[S_\sigma]$. We call

$$U_\sigma = \mathrm{Spec}\,\mathbb{C}[S_\sigma] = \mathrm{Spec}\,A_\sigma$$

the correspondent **affine toric variety** of σ.

Example. Take $V = \mathbb{R}^n$, $N = \mathbb{Z}^n$, $M = \mathrm{Hom}(N, \mathbb{Z})$.

1) $\sigma = \{0\}$. In this case $S_{\{0\}} = M = (\mathbb{Z}^n)^\vee \cong \mathbb{Z}^n$. The isomorphism between $\mathbb{Z}^n$ and $(\mathbb{Z}^n)^\vee$ is given by the basis: $e_i \mapsto e_i^*$.

   Hence,
$$\mathbb{C}[S_{\{0\}}] = \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}],$$
   where we identify e_i^* with X_i and $-e_i^*$ with X_i^{-1}. Thus, the affine toric variety of $\sigma = \{0\}$ is
$$U_{\{0\}} = \operatorname{Spec} \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = (\mathbb{C}^*)^n.$$

   We call $T = T_N \coloneqq \operatorname{Spec} \mathbb{C}[(\mathbb{Z}^n)^\vee] = (\mathbb{C}^*)^n$ the ***n*-dimensional torus**.

   If τ is a cone contained in σ, then S_σ is a submonoid of S_τ and A_σ is a subalgebra of A_τ, so we have a morphism $U_\tau \to U_\sigma$. Taking $\{0\}$, which is contained in any cone σ, this gives a morphism
$$T_N \longrightarrow U_\sigma$$
   from the torus $T_N = (\mathbb{C}^*)^n$ to the affine toric variety U_σ. This is why these affine varieties are called "toric": because they always contain a torus.

2) Let $\sigma \subseteq \mathbb{R}^2$ be the cone generated by e_2 and $2e_1 - e_2$. We want to calculate the dual cone.

   There are two facets: τ_1 generated by e_2 and τ_2 generated by $2e_1 - e_2$. These are generated by $u_1 = e_1^*$ and $u_2 = e_1^* + 2e_2^*$ respectively, because $u_1, u_2 \in \sigma^\vee$ and $\tau_1 = \sigma \cap u_1^\perp, \tau_2 = \sigma \cap u_2^\perp$.

   Hence, $\{t_1 u_1 + t_2 u_2 \,|\, 0 \leq t_1, t_2 \leq 1\}$ is a system of generators of S_σ, so it is generated by $e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*$. Hence, $A_\sigma = \mathbb{C}[S_\sigma]$ is generated by $X_1, X_1X_2, X_1X_2^2$, i.e.
$$A_\sigma = \mathbb{C}[X_1, X_1X_2, X_1X_2^2] = \mathbb{C}[X, Y, Z]/(Y^2 - XZ),$$
$$U_\sigma = \operatorname{Spec} \mathbb{C}[X, Y, Z]/(Y^2 - XZ) = V(Y^2 - XZ) = \{(X, Y, Z) \in \mathbb{C}^3 \,|\, Y^2 = XZ\}.$$

   The affine toric variety of the cone σ generated by $e_2, 2e_1 - e_2$ is the quadric cone $Y^2 = XZ$.