SOLUTIONS & GRADING SCHEME

Math 105A
Fall 2015
Final Exam
Dec 11 2015
Time Limit: 2 hours

Student's Name (Print): ____________________

Student's ID: ____________________

Print your name and student ID on the top of this page.

This exam contains 15 pages (including this cover page) and 8 problems. **Note that some equations are numbered.** You may *not* use your books, notes, or any calculator in this exam. Do not write in the grading table below.

The following rules apply to the answers you provide in this exam:

- **If you use a theorem, indicate this and explain** why the theorem is being applied.
- **Organize your work**, in a neat and coherent way.
- **Unsupported answers will not receive full credit.** Calculation or verbal explanation is expected.
- **If you need more space, use the back of the pages**; clearly indicate when you have done this.
- **Box your final answer** for full credit.

Question	Points	Score
1	40	
2	20	
3	30	
4	30	
5	35	
6	35	
7	15	
8	20	
Total:	225	

1. Suppose the sequence generated by $p_n = g(p_{n-1})$ converges to a number p.

   (a) (5 points) Given that g is continuous, prove that $p = g(p)$.

   Solution:

$$\begin{aligned} p &= \lim_{n\to\infty} p_n \\ &= \lim_{n\to\infty} g(p_{n-1}) \\ &= g\left(\lim_{n\to\infty} p_{n-1})\right) \\ &= g(p) \end{aligned}$$

   (b) (5 points) What does it mean to say that $\{p_n\}$ "converges quadratically"?

   Solution:

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lambda > 0$$

   (c) (5 points) Under what conditions does $\{p_n\}$ converge quadratically? (You do not need to prove your answer!)

   Solution: The function $g(x)$ should be such that $g'(p) = 0$.

(d) (5 points) Suppose that $g(x) = x - \phi(x)f(x)$, where $\phi(p) \neq 0$. Show that p is a solution to $f(x) = 0$.

Solution: Set $x = p$ in $g(x)$ and use $g(p) = p$ to get

$$0 = \phi(p)f(p),$$

which implies that $f(p) = 0$ since $\phi(p) \neq 0$.

(e) (5 points) Suppose that $\phi(p) = 1/f'(p)$ (and that $|\phi'(p)|$ is finite). Show that $\{p_n\}$ converges quadratically.

Solution: Using $g(x) = x - \phi(x)f(x)$ one finds that

$$g'(p) = 1 - \phi'(p)f(p) - \phi(p)f'(p).$$

Using the relations $\phi(p) = 1/f'(p)$, $f(p) = 0$ and $|\phi'(p)| < \infty$, one finds that

$$g'(p) = 1 - 0 - 1 = 0.$$

(f) (15 points) Suppose further that

$$\phi(x) = \frac{f(x)}{f(x + f(x)) - f(x)}.$$

Taylor expand the numerator and denominator of $\phi(x)$ to first order in the small quantity $x - p$, and use those expansions to evaluate $\lim_{x \to p} \phi(x)$. Use this result to show that $\{p_n\}$ converges quadratically to the solution of $f(x) = 0$.

Solution: Let $x = p + h$. Then $f(x) \approx hf'(p)$, to first order in h (2 points). Using this we find

$$x + f(x) \approx p + h',$$

where $h' = h + hf'(p)$ is of order h (3 points). Therefore

$$f(x + f(x)) \approx f(p + h') \approx h'f'(p)$$

to first order in h (3 points). The denominator is therefore $h\left(f'(p)\right)^2$, to first order in h (2 points). Therefore $\phi(p) = 1/f'(p)$ (3 points). Therefore $\{p_n\}$ converges quadratically by part (e) (2 points).

2. (20 points) Find the PLU decomposition of the matrix

$$A = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 2 \\ -3 & -4 & -1 \end{bmatrix},$$

using Gaussian elimination with scaled partial pivoting.

$$\begin{array}{c c c} |a_{i1}|/s_i & & s_i \\ 1/3 & \multirow{3}{*}{$\begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 2 \\ -3 & -4 & -1 \end{bmatrix}$} & 9 \\ 2/5 & & 5 \\ 3/4 & & 4 \end{array}$$

3 pts.

$$\updownarrow E_1 \leftrightarrow E_3$$

$$\begin{array}{c c} \begin{bmatrix} -3 & -4 & -1 \\ 2 & 5 & 2 \\ 3 & 6 & 9 \end{bmatrix} & \begin{matrix} 4 \\ 5 \\ 9 \end{matrix} \end{array}$$

3pts.

$$\begin{array}{c c c c} & |a_{i2}|/s_i & & \\ & & \multirow{3}{*}{$\begin{bmatrix} -3 & -4 & -1 \\ 0 & 7/3 & 4/3 \\ 0 & 2 & 8 \end{bmatrix}$} & 4 \\ m_{21} = -\frac{2}{3};\ E_2 - \frac{2}{-3}E_1 & 7/15 & & 5 \\ m_{31} = -1;\ E_3 - \frac{3}{-3}E_1 & 2/9 & & 9 \end{array}$$

3 pts.

$$\begin{bmatrix} -3 & -4 & -1 \\ 0 & 7/3 & 4/3 \\ 0 & 0 & 48/7 \end{bmatrix}$$

$$m_{32} = \frac{6}{7};\quad E_3 - \frac{2}{7/3}E_2$$

3pts

4pts $E_1 \Leftrightarrow E_3 \Rightarrow P = P^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$PA = LU$

$\Rightarrow A = P^T LU$

$= PLU.$

2 pts

2pts $L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -1 & 6/7 & 1 \end{bmatrix}$

2pts $U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{bmatrix} = \begin{bmatrix} -3 & -4 & -1 \\ 0 & 7/3 & 4/3 \\ 0 & 0 & 48/7 \end{bmatrix}$

3. Iterative methods to solve the linear system $Ax = b$ take the form

$$Sx_{k+1} = Tx_k + b, \tag{1}$$

where $A = S - T$ decomposes A into a "simpler" matrix S and a remainder matrix T.

(a) (5 points) Suppose the sequence $\{x_k\}$ converges. Show that its limit solves $Ax = b$.

Apply $\lim_{k\to\infty}$ to both sides of Eq (1) to get

$$Sp = Tp + b,$$

where $p = \lim_{k\to\infty} x_k$. Rearrange to get:

$$Ap = b.$$

(b) (5 points) Let $e_k = x^* - x_k$ be the error incurred at the k^{th} iteration relative to a particular solution x^*. Show that $e_k = B^k e_0$. What is B?

Subtract $Sx_{k+1} = Tx_k + b$ from $Sx^* = Tx^* + b$:

$$Se_{k+1} = Te_k$$

$$\Rightarrow \quad e_{k+1} = S^{-1}Te_k = Be_k$$

$$\Rightarrow \quad e_k = B^k e_0.$$

$B = S^{-1}T$

(c) (5 points) Define the spectral radius of a matrix.

The spectral radius of a matrix is it's largest eigenvalue in absolute value:

$$\rho(A) = \max\{|\lambda_i|\},$$

where $\{\lambda_i\}$ ~~are~~ is the set of eigenvalues of A.

(d) (10 points) Suppose that the B you obtained in (b) is symmetric and its spectral radius is less than one. Use these assumptions to show that $\lim_{k\to\infty} e_k = 0$.

3pts. B symm $\Rightarrow$ its eigenvectors v_i form an orthon. basis [Coroll. 9.17, p572] $\Rightarrow$

$$e_0 = c_1 v_1 + \dots + c_n v_n.$$

3pts. $\Rightarrow e_k = c_1 \lambda_1^k v_1 + \dots + c_n \lambda_n^k v_n$, where λ_i are the corresponding evalues of B. Thus each evector grows or decays

4pts. with the powers of λ_i, which go to zero since the largest is < 1. Thus $e_k \to 0$.

Correct sol'ns that don't use assumption of symmetry get 7pts.

(e) (5 points) In fact, any matrix B whose spectral radius is less than one has the property that $B^k \to 0$. Suppose that C is *similar* to such a B. What can you say about the eigenvalues of C? Use this insight to show that $C^k \to 0$.

Similar matrices have the same eigenvalues. [Thm 9.12, p571].

$\therefore \rho(C) = \rho(B) < 1$

$\Rightarrow C^k \to 0$.

4. Consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

(a) (5 points) Referring to Eq. (1), what are S and T for the Jacobi Method?

$$S_J = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad T_J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) (5 points) Use Q3(d) to show that the Jacobi sequence converges to the exact solution.

$$B_J = S_J^{-1} T_J = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

Eigenvalues of B are $\pm \frac{1}{2}$.

$\therefore$ B_J is symm and $\rho(B_J) = 1/2 < 1$

$\xrightarrow[Q3(d)]{}$ $e_k \to 0$, ie. $x_k \to x^*$

observing ~~Showing~~ that the components of ~~...~~ powers of B_J ~~becom~~ appear to approach ϕ gets partial credit.

(c) (5 points) What are S and T for the Gauss-Seidel Method? Substitute them into Eq. (1) to obtain the system of equations satisfied by the components of x_{k+1}.

$$S_{GS} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \quad T_{GS} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Let $x_k = (u_k, v_k)^T$. Then Eq(1) $\Rightarrow$

$$u_{k+1} = \tfrac{1}{2} v_k + 2$$

$$v_{k+1} = \tfrac{1}{2} u_{k+1} - 1$$

(d) (5 points) Compute the spectral radius of $S^{-1}T$ for the Jacobi and Gauss-Seidel Methods. Which method converges fastest? Why?

$\rho_J = \frac{1}{2}$ (see (b)).

$$\rho_{GS} = \rho(S_{GS}^{-1} T_{GS}) = \rho\left(\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}\right) = \max\{0, \tfrac{1}{4}\} = \tfrac{1}{4}.$$

$\rho_{GS} < \rho_J \Rightarrow$ GS converges fastest.

(e) (10 points) Starting from $x_0 = (0, 0)^T$, use the recursion relations you wrote down in (c) to compute x_1 and x_2. What is $\lim_{k\to\infty} x_k$?

5 points.

$$x_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

5 points.

Since these are exact solutions to $Ax=b$, ~~x~~ we must have that:

$$x_\infty = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

5. Let $|\lambda_1| > |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_n|$ be the eigenvalues of an $n \times n$ matrix A. Let $v_1, \cdots, v_n$ be the corresponding eigenvectors, normalized with respect to the *Euclidean norm.*

(a) (10 points) Let

$$x_k = \frac{A^k x_0}{|A^k x_0|}, \tag{2}$$

where x_0 has a component along v_1. Show that $\{x_k\}$ converges to v_1.

5 points

Suppose $x_0 = c_1 v_1 + \cdots + c_n v_n$ $(c_1 \neq 0)$ Then

$A^k x_0 = c_1 \lambda_1^k \left[v_1 + \frac{c_2}{c_1}\left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \cdots + \frac{c_n}{c_1}\left(\frac{\lambda_n}{\lambda_1}\right)^k v_n \right]$

$\to c_1 \lambda_1^k v_1$

5 pts.

$\Rightarrow x_k \to \frac{c_1 \lambda_1^k v_1}{c_1 \lambda_1^k} = v_1.$

(b) (5 points) In terms of eigenvalues, what dictates the rate of convergence?

The size of $|\lambda_2/\lambda_1|$ relative to 1.

(c) (5 points) Let

$$\mu_k = x_k^T A x_k. \tag{3}$$

Show that $\lim_{k\to\infty} \mu_k = \lambda_1$.

$\mu_k \to v_1^T A v_1 = \lambda_1 v_1^T v_1 = \lambda_1$

(d) (10 points) *Hotelling Deflation.* Suppose that v_1 and λ_1 are known, and that the eigenvectors are orthogonal, $v_i^T v_j = \delta_{ij}$. Show that $B = A - \lambda_1 v_1 v_1^T$ has the same eigenvectors and eigenvalues as A except that λ_1 is replaced by 0.

$$(A - \lambda_1 v_1 v_1^T) v_i = \lambda_i v_i - \lambda_1 v_1 (v_1^T v_i)$$ 5 pts

$i=1 \Rightarrow \lambda_1 v_1 - \lambda_1 v_1 (1) = 0$

$i \neq 1 \Rightarrow \lambda_i v_i - \lambda_1 v_1 (0) = \lambda_i v_i$. 5 pts.

Invoking Thm 9.20 p587, and correctly using it, is also acceptable.

(e) (5 points) What is the new limiting value of μ_k when Eqs. (2) and (3) are applied to B instead of A? What constraint needs to be placed on x_0 to obtain this limiting value?

$\mu_k \to \lambda_2$ because $|\lambda_3/\lambda_2| < 1$

x_0 must have a component along v_2.

6. Suppose

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) (5 points) Find the eigenvalues of A.

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(3-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 2, 3.$$

(b) (5 points) Find the corresponding eigenvectors of A, normalized with respect to the *Euclidean norm.*

An eigenvector is a sol^n to the linear system $(A-\lambda I)x = 0$; where λ is the corresponding eigenvalue.

	$i=3$	$i=2$	$i=1$
λ_i	1	2	3
v_i	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

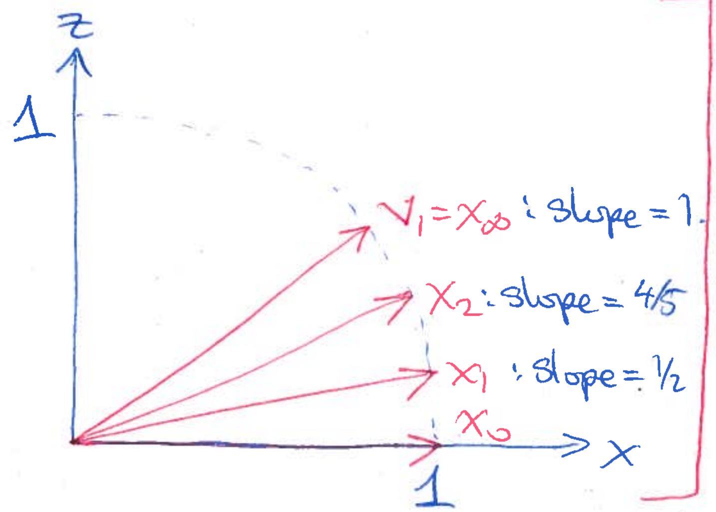
(c) (5 points) Find an orthogonal matrix Q and diagonal matrix D such that $A = QDQ^T$.

A symmetric $\Rightarrow$ $A = QDQ^T$ where

$$Q = [v_1 \quad v_2 \quad v_3]$$

$$D = \cancel{[\lambda_1 \ \lambda_2 \ \lambda_3]} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

(d) (10 points) Starting with $x_0 = (1,0,0)^T$, use Eq. (2) to compute x_1 and x_2. Sketch, in the xz plane, the vectors x_0, x_1, x_2 and x_∞.



2pts

2pts $x_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\0\\1\end{bmatrix}$

2pts $x_2 = \frac{1}{\sqrt{41}}\begin{bmatrix}5\\0\\4\end{bmatrix}$

4pts $x_\infty = v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$

Since v_1 has the largest evalue, λ_1, and $\lambda_1 > \lambda_2$, and x_0 has a ~~comp~~ component along v_1.

(e) (10 points) Use Eq. (3) to compute μ_0, μ_1 and μ_2. What is $\lim_{k\to\infty} \mu_k$?

3pts $\mu_0 = 2$

3pts $\mu_1 = 2\,\tfrac{4}{5}$

3pts $\mu_2 = 2\,\tfrac{40}{41}$

1pt. $\mu_\infty = 3 = \lambda_1$

7. Suppose $A : \mathbb{R}^n \to \mathbb{R}^m$.

(a) (5 points) Define the four fundamental vector spaces of A. For each space, indicate whether it is a subspace of $\mathbb{R}^n$ or $\mathbb{R}^m$.

- row space of $A \subset \mathbb{R}^n$ is the set of all linear combinations of rows of A.
- column space of $A \subset \mathbb{R}^m$ is " " " " " " " columns " ".
- null space of A is $\{x \in \mathbb{R}^n : Ax = 0\} \subset \mathbb{R}^n$.
- " " " A^T is $\{x \in \mathbb{R}^m : A^T x = 0\} \subset \mathbb{R}^m$.

(b) (5 points) Suppose x is in the null space of A. Prove that x is perpendicular to the row space of A.

x in null space of $A \Rightarrow Ax = 0$. Let $r_1, \ldots, r_m$ be the rows of A. Then:

$$\begin{bmatrix} r_1 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow x \perp r_1, \ldots, r_m.$$

$\Rightarrow$ every vector in null space is $\perp$ to " " " row " .

(c) (5 points) Complete the following sentences. The system $Ax = b$ has a solution if b is an element of the

column space of A.

That solution is unique if the dimension of the null space of A is

0.

Note: row space of A = column space of A^T

column space of A = "range" of A.

8. Consider the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) (5 points) Construct an orthonormal basis for the row and column spaces.

The row and column spaces are each 1D, with unit vectors:

$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (row space) $\qquad u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (column space)

(b) (5 points) Do the same for the null spaces of A and A^T.

These spaces are $\perp$ to the row and column spaces, respectively $\Rightarrow$

$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(null space of A) $\qquad$ (null space of A^T)

(c) (5 points) Find the singular values of A.

The singular values are $\sigma_i > 0$ s.t. $Av_i = \sigma_i u_i$

$Av_1 = \sqrt{10}\,u_1 \Rightarrow \sigma_1 = \sqrt{10}$.

$Av_2 = 0$ (as it must be).

Thus the only singular value of A is $\sqrt{10}$.

(d) (5 points) Construct the SVD of A (using *square* matrix factors).

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= [u_1\ u_2]\begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix}\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$

Using the eigenvectors and eigenvalues of A^TA and AA^T to construct correct sols to (c) and (d) also acceptable. [$A^TAv_i = \sigma_i^2 v_i$, $AA^Tu_i = \sigma_i^2 u_i$]