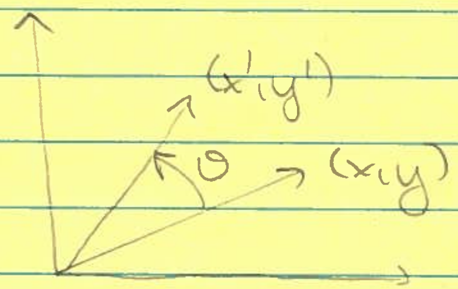
L.23 **QR factorization**

NOTE: PLEASE REVIEW Lecture23.ipynb FIRST.

GIVENS ROTATIONS

Suppose you wanted to rotate a vector thru an angle θ in the plane:

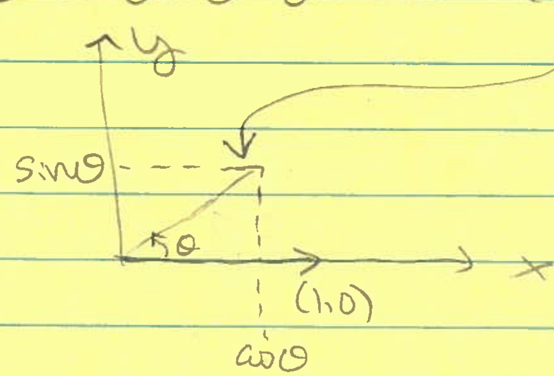


You can do this by matrix multiplication:

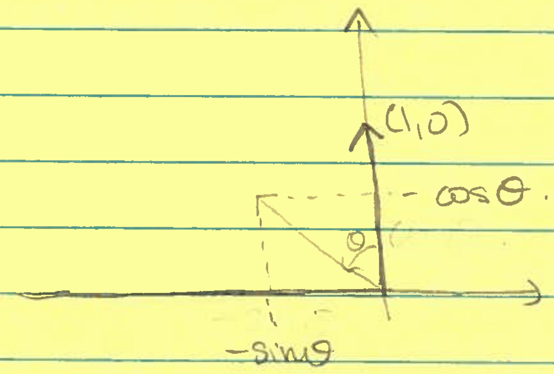
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The way to see this is to remember that the columns of the rotation matrix are obtained by applying a rotation to the unit vectors:

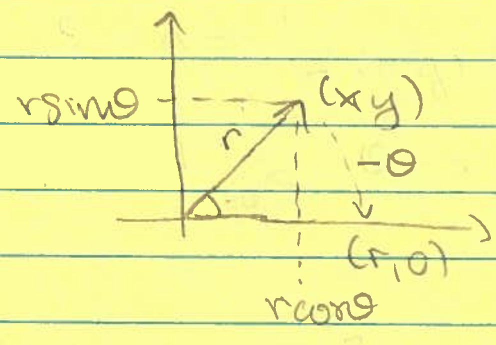
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



and
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



Suppose now that you want to rotate (x,y) so that it lies along x-axis:



$r = \sqrt{x^2+y^2}$

Then

$$x = r\cos(-\theta) = r\cos\theta$$

$$y = r\sin(-\theta) = -r\sin\theta$$

$$\Rightarrow \quad \cos\theta = x/r$$

$$\sin\theta = -y/r$$

Suppose now that you wanted to zero an element in a 2x2 matrix. One way to do that is to multiply by a Givens rotation matrix. To see this, first observe:

$$\underbrace{\begin{bmatrix} c & -s \\ s & c \end{bmatrix}}_{G} \begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \end{matrix}} \end{bmatrix}$$
$$\qquad\qquad\qquad\quad c_1 \quad\; c_2$$

$$= G[c_1, c_2]$$

$$= [Gc_1, Gc_2]$$

So, if we want to change a_{21} to 0, then we'd have to choose G s.t.

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

where
$$\boxed{\begin{aligned} r &= \sqrt{a_{11}^2 + a_{21}^2} \\ c &= a_{11}/r \\ s &= -a_{21}/r. \end{aligned}}$$

In general, a Givens rotation matrix looks like:

$$G = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & c & \cdots & -s \\ & & \vdots & \ddots & \vdots \\ & & s & \cdots & c \\ & & & & \ddots \\ 0 & & & & & 1 \end{bmatrix}$$

Notice that a Givens rotation is orthogonal: For example, for the 2D case:

$$G = [c_1, c_2]$$

and $\quad c_1^T c_2 = (c, s)\begin{pmatrix} -s \\ c \end{pmatrix} = -sc + sc = 0.$

Also:

$$\|c_1\|^2 = \cos^2\theta + \sin^2\theta = 1$$

$$\|c_2\|^2 = (-\sin\theta)^2 + \cos^2\theta = 1.$$

TRIANGULARIZE A MATRIX / We may triangularize a matrix using Givens rotations, repeatedly.

Example $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

Let's zero the (2,1) entry in A:

$$G_1 = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$\left.\begin{aligned} c &= 3/r \\ s &= -1/r \\ r &= \sqrt{3^2+1^2} = \sqrt{10}. \end{aligned}\right\}$$ look at the dashed boxes above

ie. $c = 3/\sqrt{10}$

$s = -1/\sqrt{10}$.

Thus

$$G_1 = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} & 0 \\ -1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$G_1 A = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} & 0 \\ -1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{10} & 6/\sqrt{10} & 1/\sqrt{10} \\ 0 & 8/\sqrt{10} & 3/\sqrt{10} \\ 0 & 1 & 3 \end{bmatrix}$$

Now lets zero the (3,2) entry:

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

Where $c = (8/\sqrt{10})/r$

$s = -1/r$

$r = \sqrt{\left(\frac{8}{\sqrt{10}}\right)^2 + 1^2}$

Turns out that

$c = 0.929$

$s = -0.367$

Then:

$$G_2(G_1 A) = \begin{bmatrix} \sqrt{10} & 6/\sqrt{10} & 1/\sqrt{10} \\ 0 & 2.720 & 1.985 \\ 0 & 0 & 2.441 \end{bmatrix}$$

This upper triangular matrix is usually denoted R and is one factor in the QR factorization of A.