Lec 13

LU Factorization

Forward Substitution

$Ly = b$

$$\Rightarrow \begin{aligned} &l_{11} y_1 &&= b_1 \\ &l_{21} y_1 + l_{22} y_2 &&= b_2 \\ &\vdots \\ &l_{n1} y_1 + l_{n2} y_2 + \cdots + l_{nn} y_n &&= b_n. \end{aligned}$$

$$\Rightarrow \begin{aligned} l_{11} y_1 &= b_1 \\ l_{22} y_2 &= b_2 - l_{21} y_1 \\ &\vdots \\ l_{nn} y_n &= b_n - \left[l_{n1} y_1 + l_{n2} y_2 + \cdots + l_{n,n-1} y_{n-1} \right] \end{aligned}$$

$$\Rightarrow \quad l_{ii} y_i = b_i - \sum_{j=1}^{i-1} l_{ij} y_j$$

or

$$y_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} y_j}{l_{ii}} \quad i = 1, \ldots, n.$$

LU FACTORIZATION (Doolittle's Method)

"Zero" column 1:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[R_3 \to R_3 - m_{31} R_1]{R_2 \to R_2 - m_{21} R_1} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}$$

Elementary row operations such as these constitute a linear transformation on $\mathbb{R}^3$. To see this, consider A as a concatenation of three column vectors:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \in \mathbb{R}^3$$

Clearly, transforming each column vector and concatenating them yields the transformed A.

Let $M^{(1)}$ = matrix of transformation. Since $M^{(1)}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, say, yields the first column of $M^{(1)}$, we may find the 1st column of $M^{(1)}$ by applying the row ops. to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - m_{31} R_1]{R_2 \to R_2 - m_{21} R_1} \begin{bmatrix} 1 \\ -m_{21} \\ -m_{31} \end{bmatrix}$$

Similarly

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\quad '' \quad} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\quad '' \quad} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus

$$M^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}$$

Thus

1st column eliminated

$$\Rightarrow \quad M^{(1)} A = A^{(1)}$$

$$\text{or} \quad A = [M^{(1)}]^{-1} A^{(1)} \qquad (*)$$

It turns out that:

$$[M^{(1)}]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{bmatrix}$$

You can see this by Gaussian elimination or by:

Lemma

Suppose

$$T = I + N$$

where

$$N^2 = 0$$

Then

$$T^{-1} = I - N$$

Pf

$$1 - x^2 = (1-x)(1+x)$$

$$\Rightarrow \quad I - N^2 = (I-N)(I+N)$$

ie. $$I = (I-N)(I+N)$$

$\square$

Thus (*) $\Rightarrow$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}$$

lower triangular

close to upper triangular

Next step: <u>eliminate col^{n} 2</u>:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix} \xrightarrow{R_3 \to R_3 - m_{32}R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}$$

Let $M^{(2)}$ represent this transf$^{\underline{n}}$.

As before the columns of $M^{(2)}$ are results of transforming the basis vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - m_{32}R_2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\quad " \quad} \begin{bmatrix} 0 \\ 1 \\ -m_{32} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus:

$$M^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix}$$

and

$$M^{(2)} A^{(1)} = A^{(2)} \quad \text{2nd col eliminated}$$

or

$$A^{(1)} = [M^{(2)}]^{-1} A^{(2)} \qquad (**)$$

where

$$[M^{(2)}]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{bmatrix}$$

Thus

$$\begin{aligned} A &\overset{(*)}{=} [M^{(1)}]^{-1} A^{(1)} \\ &\overset{(**)}{=} [M^{(1)}]^{-1} [M^{(2)}]^{-1} A^{(2)} \\ &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}}_{U}. \end{aligned}$$

Note: L, U can be combined into: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ m_{21} & a_{22}^{(1)} & a_{23}^{(1)} \\ m_{31} & m_{32} & a_{33}^{(2)} \end{bmatrix}$ w/o loss of information

Example:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{bmatrix}$$

$$\begin{array}{rrrll} 1 & 1 & 0 & & \\ 0 & -1 & -1 & R_2 - 2R_1 & m_{21} = 2 \\ 0 & -4 & -1 & R_3 - 3R_1 & m_{31} = 3 \end{array}$$

$$\begin{array}{rrrll} 1 & 1 & 0 & & \\ 0 & -1 & -1 & & \\ 0 & 0 & 3 & R_3 - 4R_2 & m_{32} = 4 \end{array}$$

Thus

$$A = LU$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$