

Using Gaussian elimination to construct inverse matrices

Consider $Ax=b$. Then $x=A^{-1}b$. Thus one way to solve linear systems is via matrix inversion (though this is (marginally) slower than Gaussian elimination (as mentioned in Lect. 9)).

Given A , want to construct its inverse. Suppose $B=A^{-1}$. Then

$$AB = BA = I$$

We can think of this as solving eqns for elements of B :

$$\text{eg } \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_B = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_I$$

$$\Rightarrow A \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; A \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; A \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Equivalent to 3 linear systems w/ same A -matrix.

\Rightarrow can use a larger augmented matrix:

$$\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}$$

Perform row ops:

$$\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ \rightarrow 0 & 0 & 1 & -1 & 1 & 0 \\ \rightarrow 0 & 2 & 1 & 1 & 0 & 1 \end{array}$$

$$\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}$$

eg $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

- $b_{31} = -1$

- $2b_{21} + b_{31} = 1 \Rightarrow 2b_{21} = 2 \Rightarrow b_{21} = 1$

- $b_{11} + b_{21} - b_{31} = 1 \Rightarrow b_{11} = 1 - 1 - (-1) = 1$

ie. $A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

H/W: find remaining
elts of A^{-1}

DETERMINANTS

Next we discuss the determinant of a matrix:

Defⁿ

- (i) If $A = [a]$ is a 1×1 matrix, then $\det A = a$.
- (ii) Suppose A is an $n \times n$ matrix, $n > 1$. The minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ sub-matrix of A obtained by deleting the i th row and j th column of A .
- (iii) The cofactor A_{ij} is $(-1)^{i+j} M_{ij}$.
- (iv)

$$\begin{aligned} \det(A) &= \sum_{j=1}^n a_{ij} A_{ij} \quad (\text{any } i=1 \dots n) \\ &= \sum_{i=1}^n a_{ij} A_{ij} \quad (\text{any } j=1 \dots n). \end{aligned}$$

Determinants are useful (in theory) because they tell us whether a system is solvable, but they can be useless in practice because of round-off error (see slides).

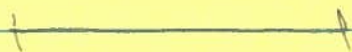
Ex Compute det by cofactor expansion

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= -1 - (-1) + (-1)$$

$$= -1$$



Basic properties of the det.
under Gaussian elimination lead
to another way to compute the det.

$$A \xrightarrow{\text{G.E.}} \tilde{A}$$

$$\Rightarrow \det A = (-1)^{\# \text{ row exchanges}} \prod_{i=1}^n a_{ii}$$

Ex compute det by G.E.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

G.E. \Rightarrow

$$\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{array}$$

$$\rightarrow \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}$$

$$\Rightarrow \det A = (-1)^0 (1)(1)(-1) = -1$$

... as found before.