Math 105A
Fall 2017
Midterm Exam
Fri Nov 3 2017
12.00pm

Student's Name (Print): ____________________

Student's ID: ____________________

Discussion Section Code: ____________________

Print your name and student ID on the top of this page.

This exam contains 6 pages (including this cover page) and 5 problems. You may *not* use your books, notes, or any calculator in this exam. Do not write in the grading table below.

The following rules apply to the answers you provide in this exam:

- **Organize your work**, in a neat and coherent way.
- **Unsupported answers will not receive full credit.** Calculation or verbal explanation is expected.
- **If you need more space, use the back of the pages**; clearly indicate when you have done this.
- **Box your final answer** for full credit.

Question	Points	Score
1	10	
2	10	
3	5	
4	10	
5	20	
Total:	55	

1. (a) (5 points) Consider the sequence $\{p_n\}$ defined by

$$p_n = \sum_{k=1}^{n} \frac{1}{k}.$$

This sequence diverges (to see this intuitively note that $p_n \approx \int_1^\infty dx/x$ for large n). Show that, nevertheless,

$$\lim_{n\to\infty} (p_n - p_{n-1}) = 0.$$

Less: Be mindful of this when testing for convergence in your iterative algorithms!

$$p_n - p_{n-1} = \frac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

(b) (5 points) Show that $p_n = 10^{-2^n}$ converges quadratically.

$$p_n \xrightarrow[n\to\infty]{} 0. \quad \therefore \quad \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{10^{-2^{n+1}}}{(10^{-2^n})^2}$$

$$= \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}}$$

$$= 1.$$

2. (10 points) Find the unique fixed point of $g(x) = 1 + \sqrt{x}$.

$g(x) = x$

$\Rightarrow 1 + \sqrt{x} = x$

Let $y = \sqrt{x}$.
Then
$1 + y = y^2$

$\Rightarrow y^2 - y - 1 = 0$.

$\Rightarrow y = \dfrac{1 \pm \sqrt{1+4}}{2}$

But $y = \sqrt{x} > 0 \Rightarrow y = \dfrac{1+\sqrt{5}}{2} \Rightarrow x = y^2 = \left(\dfrac{1+\sqrt{5}}{2}\right)^2$

3. (5 points) Find all values of α for which the following system has no solutions. Explain your answer.

$$\begin{aligned} 2x - y + 3z &= 5 \\ 4y - 4z &= -4 \\ (5+\alpha)z &= 8 + \alpha \quad (*) \end{aligned}$$

If $\alpha = -5$, then $(*) \Rightarrow 0 \cdot z = 3 \Rightarrow z$ does not exist

If $\alpha \neq -5$, then $(*) \Rightarrow \beta \cdot z = 3$ $(\beta \neq 0) \Rightarrow z = 3/\beta$.

$\Rightarrow$ unique sol^{n} exists.

4. (10 points) Use Gaussian Elimination to compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Reduce given matrix to upper triangular form, U, and use the formula:

$$\det A = (-1)^{\#\text{ row exchanges}} \det U$$

$$= (-1)^{\#\text{row swaps}} \prod_{i=1}^{n} u_{ii}$$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{matrix} \leftarrow \prod_{i=1}^{3} u_{ii}$$

row swaps $= 0$.

$\therefore \det A = (-1)^0 (1)(1)(-1) = -1.$

5. (20 points) Use partial pivoting to compute a PLU factorization of

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

$R_1 \leftrightarrow R_2$ (partial pivoting) $\longrightarrow$
$$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{matrix}$$

elim. 1st col. $\longrightarrow$
$$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{matrix}$$

$m_{21} = 0$
$m_{31} = -1$

$R_2 \leftrightarrow R_3$ (partial pivoting) $\longrightarrow$
$$\begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{matrix}$$

$\longrightarrow$
$$\begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$m_{32} = 1/2$

Two row swaps $\Rightarrow$

$$P = P_{R_2 \leftrightarrow R_3} P_{R_1 \leftrightarrow R_2}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(You may use this page to complete your solution to Q5.)

Now perform G.E. w/ partial pivoting on

$$PA = P_{R_2 \leftrightarrow R_3} P_{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$= P_{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$= \begin{matrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{matrix}$$

$$\xrightarrow[1^{st}\text{ col.}]{\text{elim}} \begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{matrix} \qquad \begin{matrix} m_{21} = -1 \\ m_{31} = 0 \end{matrix}$$

$$\xrightarrow[2^{nd}\text{ col.}]{\text{elim}} \begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad m_{32} = 1/2$$

NOTE: NO ROW SWAPS NECESSARY!

Thus:

$$A = P^T L U$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$