Lec 21

Thm GRAM-SCHMIDT.

Let $\{x_1, x_2, \ldots x_k\}$ be a set of k linearly independent vectors in $\mathbb{R}^n$. Then $\{v_1, v_2, \ldots, v_k\}$ defined by:

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{v_1^T x_2}{v_1^T v_1} v_1$$

$$v_3 = x_3 - \frac{v_1^T x_3}{v_1^T v_1} v_1 - \frac{v_2^T x_3}{v_2^T v_2} v_2$$

$$\vdots$$

$$v_k = x_k - \sum_{i=1}^{k-1} \frac{v_i^T x_k}{v_i^T v_i} v_i ,$$

is a set of k orthogonal vectors in $\mathbb{R}^n$.

Pf
$$v_1^T v_2 = v_1^T \left[x_2 - \frac{v_1^T x_2}{v_1^T v_1} v_1 \right]$$
$$= v_1^T x_2 - v_1^T \cdot \frac{v_1^T x_2}{v_1^T v_1} v_1$$
$$= v_1^T x_2 - \frac{v_1^T x_2}{\cancel{v_1^T v_1}} \cdot \cancel{v_1^T v_1}$$
$$= v_1^T x_2 - v_1^T x_2$$
$$= 0.$$

Thus $v_1 \perp v_2$

$$v_1^T v_3 = v_1^T \left[x_3 - \frac{v_1^T x_3}{v_1^T v_1} v_1 - \frac{v_2^T x_3}{v_2^T v_2} v_2 \right]$$
$$= v_1^T x_3 - v_1^T \cdot \frac{v_1^T x_3}{v_1^T v_1} v_1 - v_1^T \cdot \frac{v_2^T x_3}{v_2^T v_2} v_2$$
$$= v_1^T x_3 - \frac{v_1^T x_3}{\cancel{v_1^T v_1}} \cdot \cancel{v_1^T v_1} - \frac{v_2^T x_3}{v_2^T v_2} \underbrace{v_1^T v_2}_{\substack{=0 \\ \text{(see above)}}}$$
$$= v_1^T x_3 - v_1^T x_3$$
$$= 0$$

Thus $v_1 \perp v_3$

By induction, $v_1 \perp v_i \quad i = 2, \ldots, k$

Similarly:

$$v_2^T v_3 = v_2^T \left[x_3 - \frac{v_1^T x_3}{v_1^T v_1} v_1 - \frac{v_2^T x_3}{v_2^T v_2} \cdot v_2 \right]$$

$$= v_2^T x_3 - 0 - v_2^T x_3 = 0$$

$$v_2^T v_4 = v_2^T \left[x_4 - \frac{v_1^T x_4}{v_1^T v_1} v_1 - \frac{v_2^T x_4}{v_2^T v_2} v_2 - \frac{v_3^T x_4}{v_3^T v_3} v_3 \right]$$

$$= v_2^T x_4 - 0 - v_2^T x_4 - 0$$

$$= 0$$

Again, by induction, $v_2 \perp v_i \quad i = 3 \ldots k$.

Similarly: $v_i \perp v_j \quad j = i+1, \ldots, k$.

$\blacksquare$

ORTHO-NORMAL BASIS

In particular, when the original set of vectors forms a basis for $\mathbb{R}^n$, ie when $k = n$, then the constructed vectors form an orthogonal basis for $\mathbb{R}^n$. From this we can form an orthonormal basis:

$$u_i = \frac{v_i}{v_i^T v_i}.$$

Example $x_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $x_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \frac{v_1^T x_2}{v_1^T v_1} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

where $v_1^T x_2 = (3 \ \ 1) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 8$

$$v_1^T v_1 = (3 \ \ 1) \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 10.$$

Thus

$$v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \frac{8}{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 12/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix}$$

CHK: $v_1^T v_2 = (3 \ \ 1) \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix}$

$$= \frac{1}{5} (3 \ \ 1) \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \frac{1}{5} (-6 + 6) = 0.$$

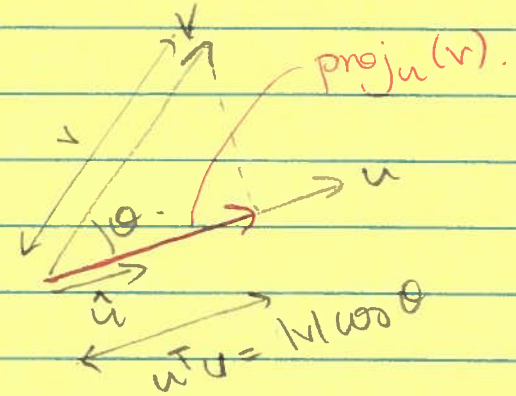
PROJECTION OPERATOR

Define the projection operator by

$$\text{proj}_u v = \frac{u^T v}{u^T u} \cdot u.$$

$$= (\hat{u}^T v)\,\hat{u}$$

where $\hat{u} = u/\sqrt{u^T u}$ = unit vector in direction of u.



In this notation, Gram-Schmidt says:-

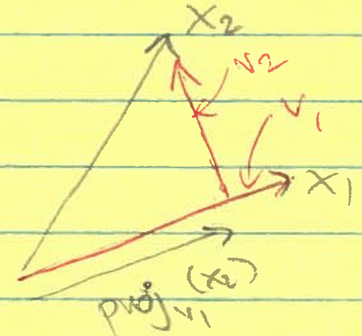
$$v_1 = x_1$$

$$v_2 = x_2 - \text{proj}_{v_1}(x_2)$$

$$v_3 = x_3 - \text{proj}_{v_1}(x_3) - \text{proj}_{v_2}(x_3)$$

$$\vdots$$

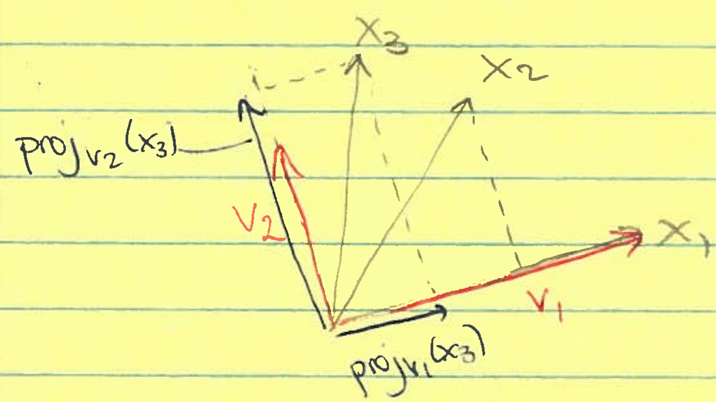
Example



$v_1 \perp v_2$

Geometric constructions like this also help to explain why $\{x_1, \ldots, x_k\}$ is assumed to be a linearly independent set. For, see what happens when they are not:

Example



x_3 = linear combination of x_1 and x_2.

In this case:

$$
\begin{aligned}
v_3 &= x_3 - \text{proj}_{v_1}(x_3) - \text{proj}_{v_2}(x_3) \\
&= x_3 - [\text{proj}_{v_1}(x_3) + \text{proj}_{v_2}(x_3)] \\
&= x_3 - x_3 \\
&= 0 \quad !
\end{aligned}
$$

If, instead, x_3 were pointing out of the page, then $v_3 \neq 0$ (essentially because the projections of x_3 do not change).

This leads to a method to compute the dimension of a space spanned by linearly dependent vectors: use Gram-Schmidt to produce $v_1, v_2, \ldots v_n$, where n labels the first v that is zero. Then $n-1$ is the required dimension.