Lec 16

Eigenvalues, Spectral Radius.

Let A be an $n \times n$ matrix. Then, eigenvectors are vectors that maintain their direction upon application by A, but possibly change their length:

$$Ax = \lambda x \iff (A - \lambda I)x = 0.$$

Now, we recognize this as a linear system of eq$^{\underline{n}}$s. Clearly $x = 0$ is a sol$^{\underline{n}}$. The special values of λ for which there is a non-zero sol$^{\underline{n}}$ are called the eigenvalues of A. The corresponding sol$^{\underline{n}}$s are called the eigenvectors of A.

The only way that $(A - \lambda I)x = 0$ has a non-zero sol$^{\underline{n}}$ is if $(A - \lambda I)$ is singular, ie.

$$\det(A - \lambda I) = p(\lambda) = 0.$$

Once λ is found, we solve $(A - \lambda I)x = 0$ to obtain the corresponding evector.

Ex

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}$$

$p(\lambda) = 0$: $\quad (1-\lambda)(2-\lambda) - 6 = 0.$

$\Rightarrow \quad 2 - \lambda + \lambda^2 = 0.$

$-6 - 2\lambda$

$\Rightarrow \quad \lambda^2 - 3\lambda - 4 = 0.$

$\Rightarrow \quad (\lambda - 4)(\lambda + 1) = 0.$

$\Rightarrow \quad \lambda = 4, -1.$

$\lambda = 4$: $\quad (A - \lambda I)x = 0$: $\quad \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad 3x - 2y = 0.$

$\Rightarrow \quad 3x = 2y$, eg $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\lambda = -1 \quad (A - \lambda I)x = 0$: $\quad \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad 2x + 2y = 0$

$\Rightarrow \quad x = -y$, eg. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Since any scalar multiple of an eigenvector is itself an eigenvector, we might as well scale eigenvector to be of unit length ($\|x\|_2 = 1$):

$$v_1 = \frac{1}{\sqrt{2^2+3^2}}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$v_2 = \frac{1}{\sqrt{1^2+(-1)^2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

ORTHO-NORMAL

A collection of vectors $v^{(i)}$ is called orthonormal if

$$v^{(i)} \cdot v^{(j)} = \delta_{ij}$$

LINEAR INDEPENDENCE

A collection of vectors $v^{(j)}$ is linearly independent if, whenever

$$0 = \sum \alpha_j v^{(j)},$$

then

$$\alpha_j = 0 \quad \forall j.$$

Otherwise, they are called linearly dependent.

Theorem

BASIS VECTORS

Suppose $v^{(1)}, \ldots, v^{(n)}$ is a collection of linearly independent vectors in $\mathbb{R}^n$. Then, for any x, $\exists$ unique $\beta_1, \ldots, \beta_n$ s.t.

$$x = \sum_{i=1}^{n} \beta_i v^{(i)}$$

The vectors $v^{(i)}$ are said to form a basis for $\mathbb{R}^n$.

Theorem

If A is a matrix and $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A w/ associated eigenvectors $v^{(1)}, \ldots, v^{(k)}$, then $\{v^{(i)}\}$ is a linearly independent set.

Pf:

Suppose $\quad 0 = \beta_1 v^{(1)} + \beta_2 v^{(2)} \quad (*)$

Apply A to get:

$$0 = \beta_1 \cdot \lambda_1 v^{(1)} + \beta_2 \lambda_2 v^{(2)}.$$

or

$$0 = \lambda_1 \cdot \beta_1 v^{(1)} + \lambda_2 \cdot \beta_2 v^{(2)}$$

$$= \lambda_1 \cdot (-\beta_2 v^{(2)}) + \lambda_2 \cdot \beta_2 v^{(2)} \quad \text{by} (*)$$

$$= \beta_2 v^{(2)} (\lambda_2 - \lambda_1)$$

Since $\lambda_2 \neq \lambda_1$ and $v^{(2)} \neq 0$, it must be that $\beta_2 = 0$.

Similarly $\beta_1 = 0$.

Thus we have proved that any pair in the set $\{v^{(i)}\}$ is linearly independent, and so they all are. ☒

In particular, if we have n distinct eigenvalues, then the set of eigenvectors forms a basis for $\mathbb{R}^n$.

SPECTRAL RADIUS.

Definition

Spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|$$

where λ is an eigenvetor of A.

Note: If λ is complex, $\lambda = \lambda_R + i\lambda_I$, then

$$|\lambda| = \sqrt{\lambda_R^2 + \lambda_I^2}$$

Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

Earlier we computed: $\lambda = 4, -1$.
Thus $\rho(A) = 4$.

SPECTRAL RADIUS VS MATRIX NORM

Theorem

If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = \sqrt{\rho(A^T A)}$

(ii) $\rho(A) \leq \|A\|$ for any induced norm.

Pf of (ii) Suppose $Ax = \lambda x$, $\|x\| = 1$

Then

$$|\lambda| = |\lambda| \cdot \|x\|$$
$$= \|\lambda x\|$$
$$= \|Ax\|$$
$$\leq \|A\| \qquad \left[\|A\| = \max_{\|x\|=1} \|Ax\| \right]$$

But

$$\rho(A) = \max |\lambda|.$$

Thus:

$$\rho(A) \leq \|A\|.$$

Part (i): Consider special case where A is symmetric.

Let $Ax = \lambda x$.

Then $A^T A x = A^2 x = \lambda A x = \lambda^2 x$.

Thus eigenvalues of $A^T A$ are just squares of evalues of A.

$$\Rightarrow \quad \rho(A^T A) = \max |\lambda^2|$$
$$\Rightarrow \quad \sqrt{\rho(A^T A)} = \max |\lambda| = \rho(A)$$

$$\underset{(11)}{\Longrightarrow} \quad \|A\|_2 = \rho(A).$$