Lec 15

Norms of Vectors & Matrices.

A norm measures size.

Def $\|\cdot\|$ is a vector norm, $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$, if

(i) $\|x\| \geq 0 \ \forall \ x \in \mathbb{R}^n$
(ii) $\|x\| = 0 \iff x = 0$.
(iii) $\|\alpha x\| = |\alpha| \|x\|$.
(iv) $\|x+y\| \leq \|x\| + \|y\|, \quad \forall \ x, y \in \mathbb{R}^n$.

Ex ℓ_2 norm: $\|x\|_2^2 = \sum_{i=1}^{n} x_i^2$

ℓ_2 norm is one member of a family of norms, the ℓ_p-norms:

ℓ_p-norm. ℓ_p norm: $\|x\|_p^p = \sum_{i=1}^{n} |x_i|^p$

Note:

ℓ_∞ norm: $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

To see this, arrange the indices s.t. x_1 is largest in absolute value, ie.

$$|x_1| = \max\{|x_1|, |x_2|, \ldots |x_n|\}.$$

Then

$$\sum_i |x_i|^p = |x_1|^p \sum_i \left[\frac{|x_i|}{|x_1|}\right]^p$$

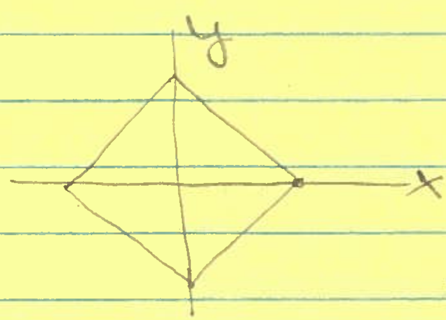
Since each term in the sum is less than or equal to one, we have:

$$\sum_{i=1}^{n} |x_i|^p \xrightarrow[p\to\infty]{} |x_1|^p$$

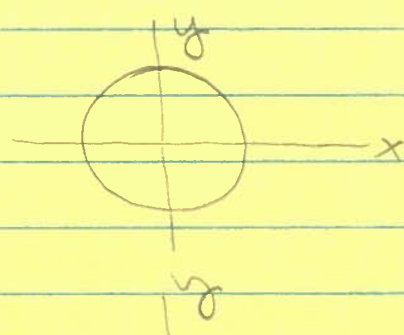
ie. $\|x\|_p \xrightarrow[p\to\infty]{} |x_1|.$

To visualize the l_p-norms, it is helpful to find the vectors in $\mathbb{R}^2$ with a common norm:

l_1



l_2



l_∞

Lemma $\|x\|_2$ is a norm.

Pf

(i) $\|x\|_2 \geq 0$ from definition

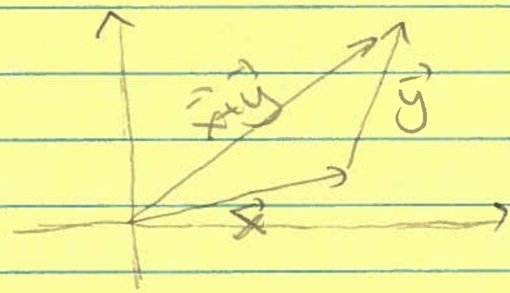
(ii) $x = 0 \Rightarrow \|x\|_2 = 0$

$\|x\|_2 = 0 \Rightarrow \sum |x_i|^2 = 0 \Rightarrow |x_i| = 0 \quad \forall i$

$\Rightarrow x = 0.$

(iii) $\|\alpha x\|_2 = \sqrt{\sum_i |\alpha x_i|^2} = |\alpha| \sqrt{\sum_i |x_i|^2} = |\alpha| \|x\|_2$

(iv) We want to prove the Δ-inequality:



$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|.$

First we need:

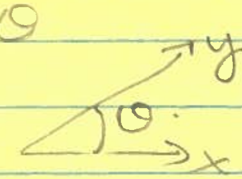
Lemma: Cauchy-Schwarz Inequality.
For $x, y \in \mathbb{R}^n$,

$$x^T y = \sum_i x_i y_i \le \|x\|_2 \|y\|_2$$

Ex: $n=2 \Rightarrow$

$$x^T y = \|x\|_2 \|y\|_2 \cos\theta$$

$$\le \|x_2\|_2 \|y\|_2$$



Pf (general n) In the following I will write $\|\cdot\|$ for $\|\cdot\|_2$. We have:

$$0 \le \|x - \lambda y\|^2 = \sum_i (x_i - \lambda y_i)^2$$

$$= \sum_i x_i^2 + \lambda^2 \sum_i y_i^2 - 2\lambda \sum_i x_i y_i$$

$$\Rightarrow \quad 2\lambda \sum_i x_i y_i \le \sum_i x_i^2 + \lambda^2 \sum_i y_i^2 \qquad (*)$$

Choose $\lambda = \|x\| / \|y\|$. Then $(*) \Rightarrow$

$$2 \frac{\|x\|}{\|y\|} \sum_i x_i y_i \le \|x\|^2 + \|x\|^2 \cdot \boxed{\frac{\sum_i y_i^2}{\|y\|^2}} \to 1$$

$$= 2\|x\|^2$$

$$\Rightarrow \quad \sum_i x_i y_i \le \|x\| \cdot \|y\|$$

▨

Let's return to part (iv) of Lemma p3:
We want to show:

$$\|x+y\| \le \|x\| + \|y\|. \quad \text{("Triangle Inequality")}$$

Consider:

$$\|x+y\|^2 = \sum_i (x_i+y_i)(x_i+y_i)$$

$$= \sum_i x_i^2 + 2\sum_i x_i y_i + \sum y_i^2$$

$$\le \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$$

by Cauchy-Schwarz

$$= \left(\|x\| + \|y\|\right)^2$$

Taking square roots proves the triangle inequality.

$\boxtimes$

Def: Distance between x and $y \in \mathbb{R}^n$, given norm $\|\cdot\|$, is $\|x-y\|$.

Def A sequence $x^{(k)}$ is said to converge to x wrt a norm $\|\cdot\|$ if, given $\varepsilon > 0$, $\exists$ $N(\varepsilon)$ st

$$\|x^{(k)} - x\| < \varepsilon \qquad \forall \ k > N(\varepsilon)$$

THEOREM $x^{(k)} \to x$ in $l_\infty \iff x_i^{(k)} \to x_i$, $i = 1 \cdots n$.

Pf $x^{(k)} \to x \implies \|x^{(k)} - x\|_\infty < \varepsilon$ for $k > N(\varepsilon)$

$$\implies \max_i |x_i^{(k)} - x_i| < \varepsilon \quad \text{— '' —}$$

$$\implies |x_i^{(k)} - x_i| < \varepsilon \quad \text{— '' —} \quad i = 1 \cdots n.$$

$$\implies x_i^{k} \to x_i \qquad i = 1 \cdots n.$$

Conversely: $x_i^{(k)} \to x_i \qquad i = 1 \cdots n.$

$$\implies |x_i^{(k)} - x_i| < \varepsilon \text{ for } k > N_i(\varepsilon), \ i = 1 \cdots n.$$

$$\implies \max |x_i^{(k)} - x_i| < \varepsilon \quad \text{— '' —}$$

$$\implies \|x^{(k)} - x\|_\infty < \varepsilon \text{ for } k > N(\varepsilon) \left[= \max_i N_i(\varepsilon) \right]$$

$$\implies x^{(k)} \to x.$$

$\square$

Matrix Norms We will concentrate on norms for square matrices. A matrix norm $\|\cdot\|$ satisfies:

(i) $\|A\| \geq 0$

(ii) $\|A\| = 0 \iff A = 0$

(iii) $\|\alpha A\| = |\alpha| \|A\|$.

(iv) $\|A+B\| \leq \|A\| + \|B\|$.

(v) $\|AB\| \leq \|A\| \|B\|$.

DISTANCE: The distance between two matrices A and B, wrt $\|\cdot\|$, is $\|A-B\|$.

Natural, or induced, norms are inherited from vector norms:

Thm If $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$, then

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

is a matrix norm.

INTUITION The measure given to a matrix under a natural norm describes how the matrix stretches unit vectors: