Math 105A Sample Midterm
6 questions, 80 points

1. (10 points)

(a). **(5 points)** Define quadratic convergence of a sequence p_n.

$$\lim \frac{|p_{n+1}-p|}{|p_n-p|^2} = \lambda$$

where p = limit of $\{p_n\}$ and $0 < \lambda \leq 1$.

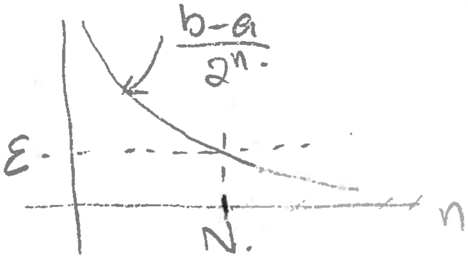
(b). **(5 points)** Given that the sequence generated by $p_n = g(p_{n-1})$ converges to a number p, and that g is continuous, prove that $p = g(p)$. **Hint:** start with $p = \lim_{n\to\infty} p_n$.

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g(\lim_{n\to\infty} p_{n-1}) = g(p).$$

2. (10 points)

Suppose f is continuous on $[a, b]$ and takes values of opposite sign at the interval's endpoints. Using the *bisection method*, determine ~~an upper bound on~~ the *minimum* number of iterations necessary to obtain an approximation, x_*, to the exact zero, x_e, of f with accuracy $|x_* - x_e| < \varepsilon$. Hint: sketch $|x_n - x_e|$, and an upper bound on it, as a function of n, where x_n is the n^{th} iterate of the bisection method.

$$|x_n - x_e| \leq \frac{b-a}{2^n} \longrightarrow$$



$\longleftarrow$

$n \geq N$

$$\Rightarrow \frac{b-a}{2^n} \leq \varepsilon$$

$$\Rightarrow |x_n - x_e| \leq \varepsilon$$

where $\frac{b-a}{2^N} = \varepsilon \Rightarrow 2^N = \frac{b-a}{\varepsilon} \Rightarrow N = \log_2\left(\frac{b-a}{\varepsilon}\right)$

3. (20 points)

(a). **(15 points)** Suppose that $f(x)=(x-p)^m q(x)$ with $\lim_{x\to p} q(x)\neq 0$. Show that a generalized version of Newton's method,

$$p_{n+1}=p_n-m\frac{f(p_n)}{f'(p_n)},$$

converges quadratically. **Hint:** Use the theorem proved in class about conditions under which fixed-point iteration methods converge quadratically.

$p_{n+1} = g(p_n)$ where $g(x) = x - mf(x)/f'(x)$

Thm: $g'(p)=0 \Rightarrow p_n$ converges quadratically.
[Thm also requires that g'' is cts, ie. f''' is cts.]

$$g(x) = x - \frac{m(x-p)^m q}{m(x-p)^{m-1}q + (x-p)^m q'} = x - \frac{m(x-p)q}{mq+(x-p)q'}$$

$$\Rightarrow g'(x) = 1 - \left\{ \frac{mq + m(x-p)q'}{mq+(x-p)q'} - \frac{m(x-p)q}{[mq+(x-p)q']^2}\cdot[mq' + q' + (x-p)q''] \right.$$

$$\Rightarrow g'(p) = 1 - \{1 - 0\} = 0$$

(b). **(5 points)** Consider the following algorithm for solving the equation $f(x)=0$:

$$x_{n+1} = x_n - \phi(x_n) f(x_n).$$

Prove that this scheme is quadratically convergent provided $\lim_{x\to p} \phi(x) = 1/f'(p)$. You may use the theorem proved in class about conditions under which fixed-point iteration methods converge quadratically.

$x_{n+1} = g(x_n)$ where $g(x) = x - \phi(x) f(x)$.

Thm: $g'(p) = 0 \Rightarrow$ quad conv.

$g'(x) = 1 - \phi' f - \phi f'$

$\Rightarrow g'(p) = 1 - \phi(p) f'(p)$ $[f(p)=0]$

Thus, if $\phi(p) = 1/f'(p)$, then $g'(p) = 0$ and seq. converges quadratically.

4. (20 points) Solve the system of equations using Gaussian elimination with scaled partial pivoting:

$$\begin{pmatrix} 1 & 1 & 2\times 10^9 \\ 2 & -1 & 10^9 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2\cdot 10^9 & 1 \\ 2 & -1 & 10^9 & 1 \\ 1 & 2 & 0 & 1 \end{array}\right) \quad \begin{array}{c} s_i \\ 2\cdot 10^9 \\ 10^9 \\ 2 \end{array} \quad \begin{array}{c} |a_{i1}|/s_i \\ \frac{1}{2}\cdot 10^{-9} \\ 2\cdot 10^{-9} \\ 1/2 \end{array}$$

swap.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & -1 & 10^9 & 1 \\ 1 & 1 & 2\cdot 10^9 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 0 & -5 & 10^9 & -1 & 10^9 \\ 0 & -1 & 2\cdot 10^9 & 0 & 2\cdot 10^9 \end{array}\right) \quad \begin{array}{c} |a_{i2}|/s_i \\ 5\cdot 10^{-9} \\ \frac{1}{2}\cdot 10^{-9} \end{array}$$

(no swap)

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -5 & 10^9 & -1 \\ 0 & 0 & \frac{9}{5}\cdot 10^9 & 1/5 \end{array}\right)$$

$\left(2\cdot 10^9 - \frac{1}{5}\cdot 10^9\right)$

x_3: $x_3 = \frac{1}{5}\cdot\frac{5}{9}\cdot 10^{-9} \Rightarrow x_3 = \frac{1}{9}\cdot 10^{-9}$

x_2: $-5x_2 + 10^9 x_3 = -1 \Rightarrow -5x_2 = -1 - \cancel{10^9}\left(\frac{1}{9}\cdot \cancel{10^{-9}}\right) = -\frac{10}{9} \Rightarrow x_2 = \frac{10}{45} = \frac{2}{9}$

x_1: $x_1 + 2x_2 = 1 \Rightarrow x_1 = 1 - 2\left(\frac{2}{9}\right) = 5/9.$

(You may use this page to complete your solution to Q4)

CHK:

$$x_1 + x_2 + 2\cdot 10^9 x_3 = \frac{5}{9} + \frac{2}{9} + 2\cdot\cancel{10^9}\cdot\frac{1}{9}\cdot\cancel{10^{-9}}$$

$$= \frac{7}{9} + \frac{2}{9} = 1 \quad \ldots \text{ as req}^{d}.$$

$$2x_1 - x_2 + 10^9 x_3 = 2\left(\frac{5}{9}\right) - \frac{2}{9} + \cancel{10^9}\cdot\frac{1}{9}\cdot\cancel{10^{-9}}$$

$$= \frac{10}{9} - \frac{2}{9} + \frac{1}{9}$$

$$= 1. \quad \ldots \text{ as req}^{d}.$$

$$x_1 + 2x_2 = \frac{5}{9} + 2\cdot\left(\frac{2}{9}\right) = 1 \quad \ldots \text{ as reqd.}$$

5. (10 points) Determine which of the following matrices are nonsingular and use Gaussian Elimination to determine the inverses of the nonsingular matrices.

(a). $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (b). $\begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}$

(a): $4\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0. \Rightarrow$ singular

(b) $\begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} = 7-4 = 3. \Rightarrow$ non-singular.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right)$$

1st col. of inverse:

y: $3y = -2 \Rightarrow y = -2/3$

x: $x + 2y = 1 \Rightarrow x = 1 - 2(-\frac{2}{3}) = 7/3$

2nd col. of inverse:

y: $3y = 1 \Rightarrow y = 1/3$

x: $x + 2y = 0 \Rightarrow x = -2(\frac{1}{3}) = -2/3$

inverse of (b) is: $\begin{bmatrix} 7/3 & -2/3 \\ -2/3 & 1/3 \end{bmatrix}$

CHK: $\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = I$ .... as reqd.

6. (10 points) Determine the PLU decompositions of the nonsingular matrices in Q5.

$$\begin{matrix} 1 & 2 \\ 2 & 7 \end{matrix}$$

$$\begin{matrix} 1 & 2 \\ 0 & 3 \end{matrix} \qquad m_{21} = 2$$

no now exchanges $\Rightarrow P = I$.

Thus

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}}_{U}.$$