Math 105A
Fall 2015
Final Exam
Dec 11 2015
Time Limit: 2 hours

Student's Name (Print): ____________________

Student's ID: ____________________

Print your name and student ID on the top of this page.

This exam contains 15 pages (including this cover page) and 8 problems. **Note that some equations are numbered.** You may *not* use your books, notes, or any calculator in this exam. Do not write in the grading table below.

The following rules apply to the answers you provide in this exam:

- **If you use a theorem, indicate this and explain** why the theorem is being applied.
- **Organize your work**, in a neat and coherent way.
- **Unsupported answers will not receive full credit**. Calculation or verbal explanation is expected.
- **If you need more space, use the back of the pages**; clearly indicate when you have done this.
- **Box your final answer** for full credit.

Question	Points	Score
1	40	
2	20	
3	30	
4	30	
5	35	
6	35	
7	15	
8	20	
Total:	225	

1. Suppose the sequence generated by $p_n = g(p_{n-1})$ converges to a number p.

   (a) (5 points) Given that g is continuous, prove that $p = g(p)$.

   (b) (5 points) What does it mean to say that $\{p_n\}$ "converges quadratically"?

   (c) (5 points) Under what conditions does $\{p_n\}$ converge quadratically? (You do not need to prove your answer!)

(d) (5 points) Suppose that $g(x) = x - \phi(x)f(x)$, where $\phi(p) \neq 0$. Show that p is a solution to $f(x) = 0$.

(e) (5 points) Suppose that $\phi(p) = 1/f'(p)$ (and that $|\phi'(p)|$ is finite). Show that $\{p_n\}$ converges quadratically.

(f) (15 points) Suppose that

$$\phi(x) = \frac{f(x)}{f(x + f(x)) - f(x)}.$$

Taylor expand the numerator and denominator of $\phi(x)$ to first order in the small quantity $x - p$, and use those expansions to evaluate $\lim_{x \to p} \phi(x)$. Use this result to show that $\{p_n\}$ converges quadratically to the solution of $f(x) = 0$.

2. (20 points) Find the PLU decomposition of the matrix

$$A = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 2 \\ -3 & -4 & -1 \end{bmatrix},$$

using Gaussian elimination with scaled partial pivoting.

3. Iterative methods to solve the linear system $Ax = b$ take the form

$$Sx_{k+1} = Tx_k + b, \tag{1}$$

where $A = S - T$ decomposes A into a "simpler" matrix S and a remainder matrix T.

(a) (5 points) Suppose the sequence $\{x_k\}$ converges. Show that its limit solves $Ax = b$.

(b) (5 points) Let $e_k = x^* - x_k$ be the error incurred at the k^{th} iteration relative to a particular solution x^*. Show that $e_k = B^k e_0$. What is B?

(c) (5 points) Define the spectral radius of a matrix.

(d) (10 points) Suppose that the B you obtained in (b) is symmetric and its spectral radius is less than one. Use these assumptions to show that $\lim_{k\to\infty} e_k = 0$.

(e) (5 points) In fact, any matrix B whose spectral radius is less than one has the property that $B^k \to 0$. Suppose that C is *similar* to such a B. What can you say about the eigenvalues of C? Use this insight to show that $C^k \to 0$.

4. Consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

(a) (5 points) Referring to Eq. (1), what are S and T for the Jacobi Method?

(b) (5 points) Use Q3(d) to show that the Jacobi sequence converges to the exact solution.

(c) (5 points) What are S and T for the Gauss-Seidel Method? Substitute them into Eq. (1) to obtain the system of equations satisfied by the components of x_{k+1}.

(d) (5 points) Compute the spectral radius of $S^{-1}T$ for the Jacobi and Gauss-Seidel Methods. Which method converges fastest? Why?

(e) (10 points) Starting from $x_0 = (0, 0)^T$, use the recursion relations you wrote down in (c) to compute x_1 and x_2. What is $\lim_{k\to\infty} x_k$?

5. Let $|\lambda_1| > |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_n|$ be the eigenvalues of an $n \times n$ matrix A. Let $v_1, \cdots, v_n$ be the corresponding eigenvectors, normalized with respect to the *Euclidean norm*.

   (a) (10 points) Let

   $$x_k = \frac{A^k x_0}{|A^k x_0|}, \tag{2}$$

   where x_0 has a component along v_1. Show that $\{x_k\}$ converges to v_1.

   (b) (5 points) In terms of eigenvalues, what dictates the rate of convergence?

   (c) (5 points) Let

   $$\mu_k = x_k^T A x_k. \tag{3}$$

   Show that $\lim_{k\to\infty} \mu_k = \lambda_1$.

(d) (10 points) *Hotelling Deflation.* Suppose that v_1 and λ_1 are known, and that the eigenvectors are orthogonal, $v_i^T v_j = \delta_{ij}$. Show that $B = A - \lambda_1 v_1 v_1^T$ has the same eigenvectors and eigenvalues as A except that λ_1 is replaced by 0.

(e) (5 points) What is the new limiting value of μ_k when Eqs. (2) and (3) are applied to B instead of A? What constraint needs to be placed on x_0 to obtain this limiting value?

6. Suppose

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) (5 points) Find the eigenvalues of A.

(b) (5 points) Find the corresponding eigenvectors of A, normalized with respect to the *Euclidean norm.*

(c) (5 points) Find an orthogonal matrix Q and diagonal matrix D such that $A = QDQ^T$.

(d) (10 points) Starting with $x_0 = (1, 0, 0)^T$, use Eq. (2) to compute x_1 and x_2. Sketch, in the xz plane, the vectors x_0, x_1, x_2 and x_∞.

(e) (10 points) Use Eq. (3) to compute μ_0, μ_1 and μ_2. What is $\lim_{k\to\infty} \mu_k$?

7. Suppose $A : \mathbb{R}^n \to \mathbb{R}^m$.

   (a) (5 points) Define the four fundamental vector spaces of A. For each space, indicate whether it is a subspace of $\mathbb{R}^n$ or $\mathbb{R}^m$.

   (b) (5 points) Suppose x is in the null space of A. Prove that x is perpendicular to the row space of A.

   (c) (5 points) Complete the following sentences. The system $Ax = b$ has a solution if b is an element of the

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   That solution is unique if the dimension of the null space of A is

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8. Consider the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) (5 points) Construct an orthonormal basis for the row and column spaces.

(b) (5 points) Do the same for the null spaces of A and A^T.

(c) (5 points) Find the singular values of A.

(d) (5 points) Construct the SVD of A (using *square* matrix factors).