PLU factorization

Consider $Ax=b$

If we knew the row exchanges that were required to solve the system by Gaussian elimination, we could arrange the original $eq^{n}s$ in an order that would ensure that no row interchanges were needed.

Let the matrix P represent this a prior sequence of row exchanges then

$$(PA)x = (Pb)$$

can be solved w/o row exchanges.

$\therefore$ The matrix PA can be factored as:

$$PA = LU$$

where L is lower triangular and U is upper triangular.

But how do we compute P?

As before, we may compute the 1^{st} column of P by applying it to the 1^{st} basis vector. Suppose, for concreteness, that our space is $\mathbb{R}^3$, and the only row exchange necessary to solve $Ax=b$ is $R_1 \longleftrightarrow R_2$. Then

$$P\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

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Thus:

$$P = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

Equivalently,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P$$

In words, to compute P we apply row exchanges to the identity matrix.

MEANING OF P^T.

We have:

$$PA = A_{R1 \leftrightarrow R2}$$

$$\Rightarrow \quad (PA)^T = \left(A_{R1 \leftrightarrow R2}\right)^T \qquad (**)$$

We know from linear algebra that

$$LHS = A^T P^T$$

Now consider RHS. First write

$$A = \begin{bmatrix} — R_1 — \\ — R_2 — \\ — R_3 — \end{bmatrix}$$

Then

$$A_{R_1 \leftrightarrow R_2} = \begin{bmatrix} — R_2 — \\ — R_1 — \\ — R_3 — \end{bmatrix}$$

and

$$(A_{R_1 \leftrightarrow R_2})^T = \begin{bmatrix} | & | & | \\ R_2 & R_1 & R_3 \\ | & | & | \end{bmatrix}$$

Thus $(**) \Rightarrow$

$$\begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix} P^T = \begin{bmatrix} | & | & | \\ R_2 & R_1 & R_3 \\ | & | & | \end{bmatrix}$$

Thus, if P represents a seq of row swaps, then P^T represents the corresponding seq of column swaps.

INVERSE OF P. What is the inverse of P?

Any permutation matrix is the product of elementary permutation where by "elementary" I mean a permutation that swaps two rows (or two columns) ONLY.

Ex:

$$P_{R_1 \leftrightarrow R_2} \begin{bmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{bmatrix} = \begin{bmatrix} - R_2 - \\ - R_1 - \\ - R_3 - \end{bmatrix}$$

$$P_{R_2 \leftrightarrow R_3} \begin{bmatrix} - R_2 - \\ - R_1 - \\ - R_3 - \end{bmatrix} = \begin{bmatrix} - R_2 - \\ - R_3 - \\ - R_1 - \end{bmatrix}$$

Thus:

$$P_{R_2 \leftrightarrow R_3} P_{R_1 \leftrightarrow R_2} \begin{bmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{bmatrix} = \begin{bmatrix} - R_2 - \\ - R_3 - \\ - R_1 - \end{bmatrix}$$

Since swapping R_i with R_j is the same as swapping R_j with R_i, each elementary permutation matrix is symmetric and coincides with it's inverse.

Ex
$$P_{R_1 \leftrightarrow R_2} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{symmetric}} = P_{R_2 \leftrightarrow R_1}$$

Also:

$$P_{R_1 \leftrightarrow R_2} \begin{bmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{bmatrix} = \begin{bmatrix} - R_2 - \\ - R_1 - \\ - R_3 - \end{bmatrix}$$

$$\Rightarrow \quad P_{R_2 \leftrightarrow R_1} P_{R_1 \leftrightarrow R_2} \begin{bmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{bmatrix} = \begin{bmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{bmatrix}$$

$\Rightarrow \quad P_{R_2 \leftrightarrow R_1} P_{R_1 \leftrightarrow R_2} = I$ Similarly $P_{R_1 \leftrightarrow R_2} P_{R_2 \leftrightarrow R_1} = I$

$$\Rightarrow \quad \left[P_{R_1 \leftrightarrow R_2} \right]^{-1} = P_{R_2 \leftrightarrow R_1} = P_{R_1 \leftrightarrow R_2}$$

Write a general permutation matrix as the product of elementary permutation matrices:

$$P = P_1 \cdots P_k$$

We have:

$$P^{-1} = (P_1 \cdots P_k)^{-1}$$

$$= P_k^{-1} \cdots P_1^{-1}$$

$$= P_k \cdots P_1$$

$$= P_k^T \cdots P_1^T$$

$$= (P_1 \cdots P_k)^T$$

ie. $\boxed{P^{-1} = P^T}$ $(*)$

Recall: $Ax = b$

$$\Rightarrow PA = LU$$

$$\overset{(*)}{\Rightarrow} \boxed{A = P^T LU.}$$

Ex Find the PLU decomposition of

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

Sol$^{\underline{n}}$

$$\xrightarrow[\text{rows}]{\text{combine}} \begin{matrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{matrix} \qquad \begin{matrix} m_{21} = 1 \\ m_{31} = -1 \end{matrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{matrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{matrix} \qquad m_{32} = 0.$$

Thus one row exchange ($R_2 \leftrightarrow R_3$) was required. The corresponding permutat$^{\underline{n}}$ matrix is:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Now repeat Gaussian elimination, this time doing all the row exchanges up front (which is equivalent to pre-multiplying A by P):

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{} \quad PA = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{rows}]{\text{combine}} \quad \underbrace{\begin{matrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{matrix}}_{\text{same as before}} \qquad \underbrace{\begin{matrix} m_{21} = -1 \\ m_{31} = 1 \\ (m_{32} = 0) \end{matrix}}_{\text{different from before!}}$$

Thus LU factorization of PA is LU, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

CHECK: $LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= PA$$

Thus

$$A = P^T LU = PLU$$

(since P is elementary).

CHECK: We know:

$$LU = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

and P interchanges R_2, R_3. Thus

$$PLU = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \text{ swapped}$$

... which is A, as it should be.

Ex Find PLU factorization of

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

using standard G.E.

Sol^n

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{matrix}$$

$$\xrightarrow[\text{1st col.}]{\text{eliminate}} \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{matrix} \qquad \begin{matrix} m_{21} = 0 \\ m_{31} = -1 \end{matrix}$$

$$\xrightarrow[\text{2nd col.}]{\text{elim.}} \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{matrix} \qquad m_{32} = 2$$

Thus

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P^T.$$

Since the row exchanges already

occured up front, we don't need to redo G.E. and

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} ; \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

CHECK:

$$LU = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$P^T LU = PLU = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \text{ swapped}$$

which is A, as it should be.

Ex Recompute a PLU factorization of

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

this time using partial pivoting.

Sol^{n}

$$\xrightarrow[\text{(PARTIAL PIVOTING)}]{R_1 \leftrightarrow R_2} \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{matrix}$$

$$\xrightarrow[1^{st} \text{ col}^n]{\text{elim}} \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{matrix} \qquad \begin{matrix} m_{21} = 0 \\ m_{31} = -1 \end{matrix}$$

$$\xrightarrow[\text{(PARTIAL PIVOTING)}]{R_2 \leftrightarrow R_3} \begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{matrix}$$

$$\xrightarrow[2^{nd} \text{ col.}]{\text{eliminate}} \begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad m_{32} = 1/2$$

With partial pivoting, we needed 2 row exchanges (instead of 1, previously).

Thus

$$P = \underset{R_2 \leftrightarrow R_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \underset{R_1 \leftrightarrow R_2}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ (notice this is not symmetric - it is not elementary).

Now perform GE w/ partial pivoting on

$$PA = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\xrightarrow{\text{elim } 1^{st} \text{ col.}}$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{matrix} \qquad \begin{matrix} m_{21} = -1 \\ m_{31} = 0 \end{matrix}$$

$\xrightarrow{\text{elim } 2^{nd} \text{ col.}}$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad m_{32} = 1/2$$

Thus

$$A = P^T L U$$

where

$$P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>CHECK</u>:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$P^T L U = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

which is A is required.

These last two examples demonstrate that a PLU factorization is not unique:

$$\overset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}} = \overset{P^T}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \text{w/ partial pivoting}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{w/o partial pivoting}$$

Wow!.