Lec26 Singular Value Decomposition (EXAMPLES)

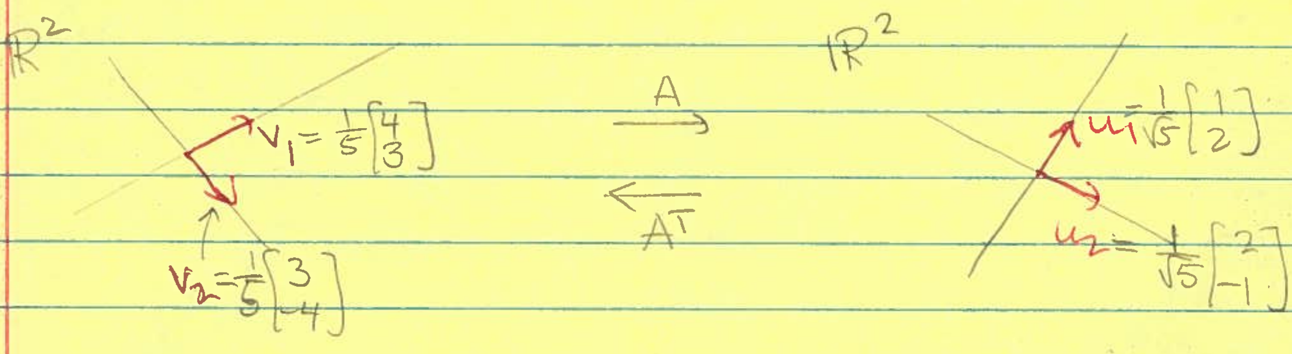
EXAMPLE $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

$\Rightarrow R(A) = \alpha \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\qquad C(A) = \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow N(A) = \gamma \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ $\qquad N(A^T) = \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

CHK: $\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Picture:



Thus

$V = [v_1\ v_2] = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}$ $\qquad U = [u_1\ u_2] = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$

What about Σ?

Recall that eigenvalues of A^TA are $\sigma_1^2, \dots, \sigma_r^2, 0, \dots, 0$. In our case:

$$A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

Since the rows (and the cols) are multiples of each other, we have:

$$\dim R(A^TA) = 1$$

$$\Rightarrow \dim N(A^TA) = 1$$

$\Rightarrow$ one eigenvalue of A^TA must be zero.
Turns out that other evalue is 125.

Thus:

$$\Sigma^T \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 125 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}$$

H/W: CHK that $Av_i = \sigma_i u_i \quad i = 1, 2$
H/W: CHK that $A = U\Sigma V^T$

EXAMPLE $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

$A^T A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (5-\lambda)^2 = 9 \Rightarrow \lambda = 2, 8 \Rightarrow \sigma_1^2 = 8, \sigma_2^2 = 2$

$\Rightarrow \sigma_1 = \sqrt{8}, \sigma_2 = \sqrt{2} \Rightarrow \Sigma = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

$\lambda = 8$: $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 \Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$: $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -x_2 \Rightarrow v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Note: $v_1^T v_2 = 0$, as expected

Thus: $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

What about U? Recall: $Av_i = \sigma_i u_i$

$u_1 = Av_1 / \sigma_1 \propto \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$\Rightarrow u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, after normalization

$$u_2 = Av_2/\sigma_2 \propto \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$\Rightarrow u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, after normalization.

Thus:

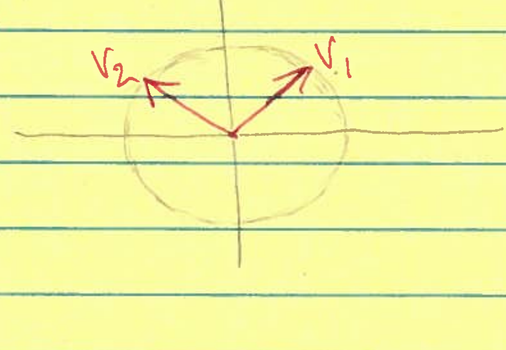
$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

H/W: CHK that $Av_i = \sigma_i u_i$, $A = U\Sigma V^T$.

GEOMETRICAL INTERPRETATION of $A = U\Sigma V^T$

SVD says that the linear transformation represented by A is equivalent to the composition of 3 linear transformations represented by V^T, Σ and U, in that order. Let's see what each of those transformations actually does in the context of the previous example.

Start w/ V^T:

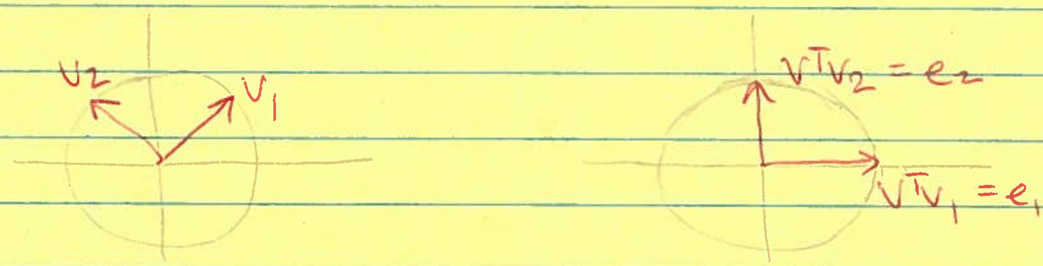


$\xrightarrow{V^T}$?

To fill in the question mark, we need to compute:

$$V^T v_i = \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} v_i = e_i$$

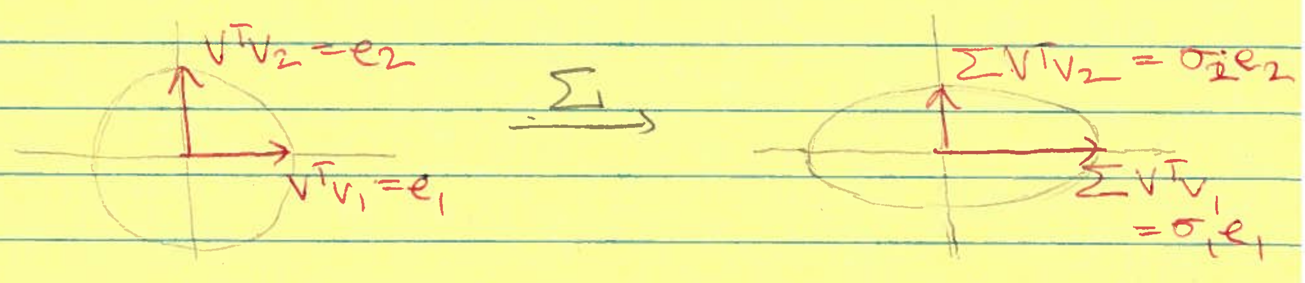
Thus V^T just rotates v_i until it aligns w/ the cardinal axes:



Next apply Σ:

$$\Sigma V^T v_i = \Sigma \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \sigma_i \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \sigma_i e_i$$

$\Rightarrow$ Σ just changes the length of $V^T v_i$ by a factor of σ_i



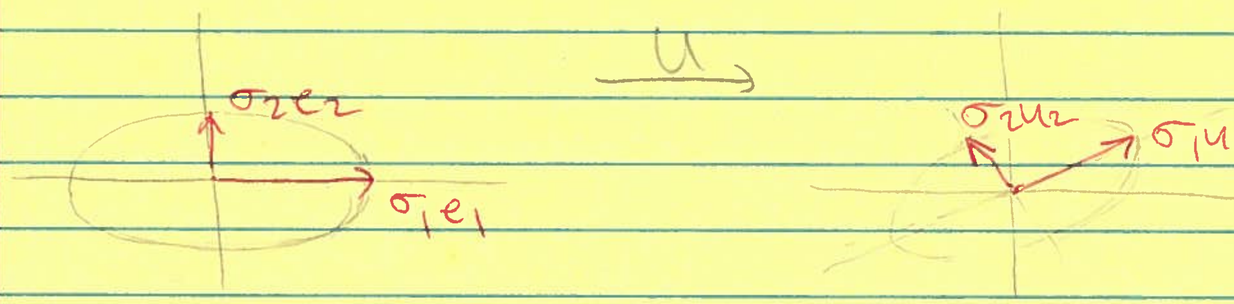
Finally, apply U:

$$U\Sigma V^T v_i = U \cdot \sigma_i \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \sigma_i u_i.$$

ie.

$$U(\sigma_i e_i) = \sigma_i u_i$$

ie. U rotates a vector until it aligns w/ u_i:



Summary

$A = U\Sigma V^T$ transforms the unit circle into an ellipse by rotating (V^T), stretching (Σ), and rotating (U).