QUIZ 3(10 Points)

Choose 2 of the following problems.

(1) Let f be C^{m+1} function, with a taylor series having non zero radius of convergence about p. Suppose p is a root of multiplicity m. Then $f^i(p) = 0$ for $0 \le i \le m-1$, and $f^m(p) \neq 0$. Show that if p is a root of multiplicity m, then the fixed point method

$$g(x) = x - \frac{mf(x)}{f'(x)}$$

has $g'(p) = 0$.
Proof: Expand f in a taylor series about p(noting that $f^i(p) = 0$ for $i < m$), as $f(x) = \frac{f^m(\xi_1(x))}{m!}(x-p)^m$, and $f'(x) = \frac{f^m(\xi_2(x))}{(m-1)!}(x-p)^{m-1}$, where ξ_1 and ξ_2 must lie between x and p. Consider $g(x) = x - \frac{m\frac{f^m(\xi_1(x))}{m!}(x-p)^m}{\frac{f^m(\xi_2(x))}{(m-1)!}(x-p)^{m-1}} = x - \frac{f^m(\xi_1(x))(x-p)}{f^m(\xi_2(x))}$. The functions $f^m(\xi_i(x))$ are differentiable (as f and f' are differentiable). Additionally, $h(x) = f^m(\xi_1(x))/f^m(\xi_2(x))$ is smooth when $f^m(\xi_2(x))$ is not zero, which is the case near p. So we can rewrite this as $g(x) = x - h(x)(x-p)$. Taking derivatives, we obtain $g'(x) = 1 - h'(x)(x-p) + h(x)$. Evaluating at p, the first term is 0. The last term is 1, because ξ_1 and ξ_2 must lie between x and p, so letting $x = p$, we have that ξ_1 and ξ_2 are both equal to $f^m(p)$ (which is not 0), and therefore, the last term is 1. This tells us that $g'(p) = 0$.
This was a little more difficult than I had anticipated, and will not be graded harshly.

(2) Let $p_n = 10^{-2^n}$. Show that p_n converges to 0 quadratically.
Proof; First, p_n converges to 0, because as n goes to ∞, 2^n becomes large, so $\frac{1}{10^{2^n}}$ becomes very small.
To show it converges quadratically we have

$$\frac{|p_{n+1}|}{|p_n|^2} = \frac{10^{2^{n+1}}}{10^{2^{n+1}}} = 1,$$

which tells us the convergence is quadratic.

(3) Use Gaussian elimination to solve

$$\begin{aligned} 2x_1 - 2/3x_2 + 3x_3 &= 1 \\ -x_1 + 2x_3 &= 3 \\ 4x_1 - 9/2x_2 + 5x_3 &= 1 \end{aligned}$$

Start by switching the first two rows, and then eliminate the second two rows to obtain

$$\begin{pmatrix} -1 & 0 & 2 & 3 \\ 0 & -2/3 & 7 & 7 \\ 0 & -9/2 & 13 & 13 \end{pmatrix}$$

Then eliminate the -9/2 element with the second row, via $\frac{-27}{4}R_2 + R_3 \to R_3$. This gives

$$\begin{pmatrix} -1 & 0 & 2 & 3 \\ 0 & -2/3 & 7 & 7 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 13-(27)(13)/4 & 13-(27)(13)/4 \end{pmatrix}$$

Using back substitution, we not obtain $x_3 = 1, x_2 = 0, x_1 = -1$.