Using Gaussian elimination to construct inverse matrices

Consider $Ax=b$. Then $x = A^{-1}b$. Thus one way to solve linear systems is via matrix inversion (though this is (marginally) slower than Gaussian elimination (as mentioned in Lect. 9).

Given A, want to construct its inverse. Suppose $B = A^{-1}$ Then

$$AB = BA = I$$

We can think of this as solving eqns for elements of B:

$$\text{eg} \quad \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$\Rightarrow A\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad A\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Equivalent to 3 linear systems w/ same A-matrix.

$\Rightarrow$ can use a larger augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Perform row ops:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

eg $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

- $b_{31} = -1$
- $2b_{21} + b_{31} = 1 \Rightarrow 2b_{21} = 2 \Rightarrow b_{21} = 1$
- $b_{11} + b_{21} - b_{31} = 1 \Rightarrow b_{11} = 1 - 1 - 1 = -1$

ie. $A^{-1} = \begin{bmatrix} -1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ -1 & \cdot & \cdot \end{bmatrix}$

H/W: find remaining elts of A^{-1}

DETERMINANTS

Next we discuss the determinant of a matrix:

Def^{n}

(i) If $A = [a]$ is a 1×1 matrix, then $\det A = a$.

(ii) Suppose A is an $n\times n$ matrix, $n>1$. The minor M_{ij} is the determinant of the $(n-1)\times(n-1)$ sub-matrix of A obtained by deleting the i^{th} row and j^{th} column of A.

(iii) The cofactor A_{ij} is $(-1)^{i+j}M_{ij}$.

(iv)
$$\det(A) = \sum_{j=1}^{n} a_{ij}A_{ij} \quad (\text{any } i=1..n)$$
$$= \sum_{i=1}^{n} a_{ij}A_{ij} \quad (\text{any } j=1\cdots n).$$

Determinants are useful (in theory) because they tell us whether a system is solvable, but they can be useless in practice because of round-off error (see slides).

Ex Compute det by cofactor expansion

$$\begin{array}{cccc} & + & - & + \\ A = & \multicolumn{3}{c}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}} \end{array}$$

$$\det(A) = 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= -1 - (-1) + (-1)$$

$$= -1$$

Basic properties of the det. under Gaussian elimination lead to another way to compute the det.

$$A \xrightarrow{G.E} \tilde{A}$$

$$\Rightarrow \det A = (-1)^{\#\text{ row exchanges}} \prod_{i=1}^{n} \tilde{a}_{ii}$$

Ex compute det by G.E.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

G.E. $\Rightarrow$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

$$\rightarrow \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{matrix}$$

$$\Rightarrow \det A = (-1)^0 \, (1)(1)(-1) = -1$$

... as found before.