Math 105A Sample Midterm
6 questions, 80 points

1. (10 points)

(a). **(5 points)** Define quadratic convergence of a sequence p_n.

(b). **(5 points)** Given that the sequence generated by $p_n = g(p_{n-1})$ converges to a number *p*, and that *g* is continuous, prove that $p = g(p)$. **Hint**: start with $p = \lim_{n\to\infty} p_n$.

2. (10 points)

Suppose *f* is continuous on [*a, b*] and takes values of opposite sign at the interval's endpoints. Using the *bisection method*, determine *an upper bound* on the *minimum* number of iterations necessary to obtain an approximation, x_*, to the exact zero, x_e, of *f* with accuracy $|x_* - x_e| < \varepsilon$. Hint: sketch $|x_n - x_e|$, and an upper bound on it, as a function of *n*, where x_n is the n^{th} iterate of the bisection method.

3. (20 points)

(a). **(15 points)** Suppose that $f(x) = (x-p)^m q(x)$ with $\lim_{x \to p} q(x) \neq 0$. Show that a generalized version of Newton's method,

$$p_{n+1} = p_n - m\frac{f(p_n)}{f'(p_n)},$$

converges quadratically. **Hint:** Use the theorem proved in class about conditions under which fixed-point iteration methods converge quadratically.

(b). **(5 points)** Consider the following algorithm for solving the equation $f(x)=0$:

$$x_{n+1} = x_n - \phi(x_n) f(x_n).$$

Prove that this scheme is quadratically convergent provided $\lim_{x \to p} \phi(x) = 1/f'(p)$. You may use the theorem proved in class about conditions under which fixed-point iteration methods converge quadratically.

4. (20 points) Solve the system of equations using Gaussian elimination with scaled partial pivoting:

$$\begin{pmatrix} 1 & 1 & 2\times 10^9 \\ 2 & -1 & 10^9 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(You may use this page to complete your solution to Q4)

5. (10 points) Determine which of the following matrices are nonsingular and use Gaussian Elimination to determine the inverses of the nonsingular matrices.

(a). $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (b). $\begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}$

6. (10 points) Determine the PLU decompositions of the nonsingular matrices in Q5.