33-777

Today

I Radiative Transfer
II Local Thermodynamic Equilibrium
III Stellar Atmospheres

In this lecture we will review what we know about radiative transfer and systems in thermodynamic equilibrium + local thermodynamic equilibrium. We will then apply these concepts to stellar atmospheres.

Let's quickly review some of the key concepts from radiative transfer we covered last time.

We defined the concept of <u>specific intensity</u> for a radiation field.

$$dE = I_\nu(\underset{\text{position}}{\vec{r}}, t, \underset{\text{direction}}{\hat{s}}) \cos\theta \, dA \, dt \, d\Omega \, d\nu$$

The specific intensity has units.

$$\frac{\text{erg}}{\text{s} \cdot \text{cm}^2 \cdot \text{Hz} \cdot \text{steradian}}$$

Given the specific intensity, we can calculate a host of useful quantities, including flux, pressure, energy density, etc.

Radiative transfer deals with how the specific intensity of rays change as they travel along a path. We defined two <u>equations of radiative transfer</u>.

① $$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

② $$\frac{dI_\nu}{\tau_\nu} = -I_\nu + S_\nu$$

These two equations state the same thing, but they use different but related quantities

* absorption coefficient, α_ν
* emission coefficient, j_ν
* optical depth, τ_ν

$$\tau_\nu = \int_{s_0}^{s} \alpha_\nu(s')ds'$$

* the source function, S_ν

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu}$$

Using this formalism, a variety of useful concepts can be worked out. One of the most important is <u>Kirchoff's Law</u>. This law states that for matter in thermal equilibrium, the emission and absorption coefficients are related,

$$j_\nu = \alpha_\nu \underbrace{B_\nu(T)}_{\text{Planck's Law}}$$

From here it is useful to define two regimes for emission from material in thermal equilibrium.

① Blackbody Emission ($\tau >> 1$)

$$I_\nu = B_\nu(T)$$

② Thermal Emission ($\tau << 1$)

$$S_\nu = B_\nu(T)$$

At this point it is useful to review the characteristics of material in Thermodynamic Equilibrium (T.E.).

① Planck's Law

For radiation in TE with matter, the spectral energy distribution is given by Planck's Law

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_bT} - 1}$$

② Maxwell-Boltzmann Distribution

For an ideal gas, the velocity distribution of particles is defined as

$$dn_v = 4\pi n \left(\frac{m}{2\pi K_b T}\right)^{3/2} e^{\frac{-mv^2}{2K_bT}} v^2 dv$$

③ Boltzmann Equation

In general the particles in a gas may gain or lose internal energy, e.g. H atoms may transition between different electron states. For a gas in TE, the ratio of particles in energy state E_b to E_a is given by

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b - E_a)/K_bT}$$

where g_b, g_a are statistical weights to account for degeneracy amongst states.

④ Saha Equation

For a gas in TE, the relative number of atoms in different ionization states is given by

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i}\left(\frac{2\pi m_e k_b T}{h^2}\right)^{3/2} e^{-\chi_i/k_b T}$$

where Z_i is the partition function for an atom in ionization state i

$$Z_i = \sum_{j=1}^{\infty} g_j e^{-(E_j - E_1)/k_b T}$$

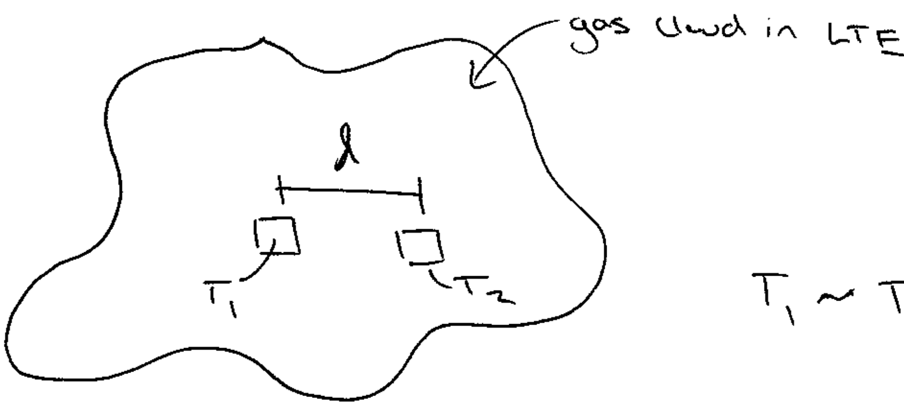
ground state energy

and χ_i is the ionization energy needed needed to remove an electron from the ground state of an atom in ionization state i. For example for hydrogen in the ground state (HI), $i=1$

$$\chi_1 = 13.6 \text{ eV}$$

Unfortunatly (or fortunatly to keep things interesting) astrophysical objects | sources are rarely perfect blackbodies. For example, Stars contain temperature gradients, i.e. they are much hotter in their cores than their surface. Given this, to what degree can we leverage our tools developed for systems in thermal equilibrium?

Here it is useful to define the concept of <u>local thermodynamic equilibrium</u> (LTE).



$T_1 \sim T_2$

Given enough time, a gas cloud will reach LTE if over scales comparable to the mean free path of photons / gas particles the temperature is approximatly constant. If this is the case, then locally in the cloud

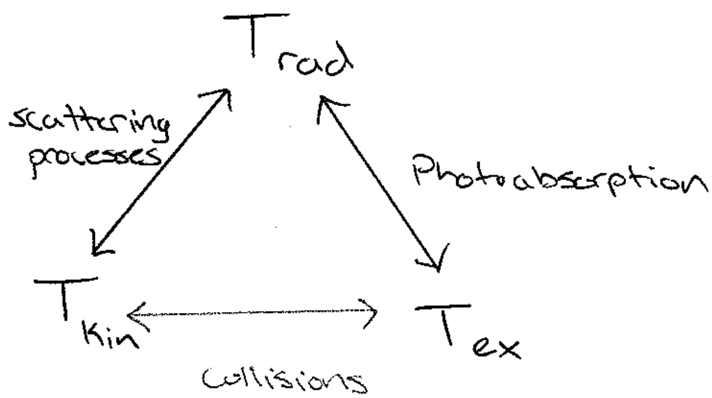
$$\frac{\partial T(\vec{r})}{\partial t} = 0 \quad [\text{const. temp}]$$

Here it is important to note there are multiple ways to define temperature

- radiation Temperature
- Kinetic Temperature
- Excitation Temperature

Depending on whether we are talking about photons, Kinetic energy of gas particles, or quantum state of atoms / Molecules in the gas.

In LTE,



$$T_{rad} = T_{kin} = T_{ex} = T_{rad}$$

In addition, because locally the system looks like one in TE,

$$S_\nu = B_\nu(T)$$ } but note that T may be a function of s, τ

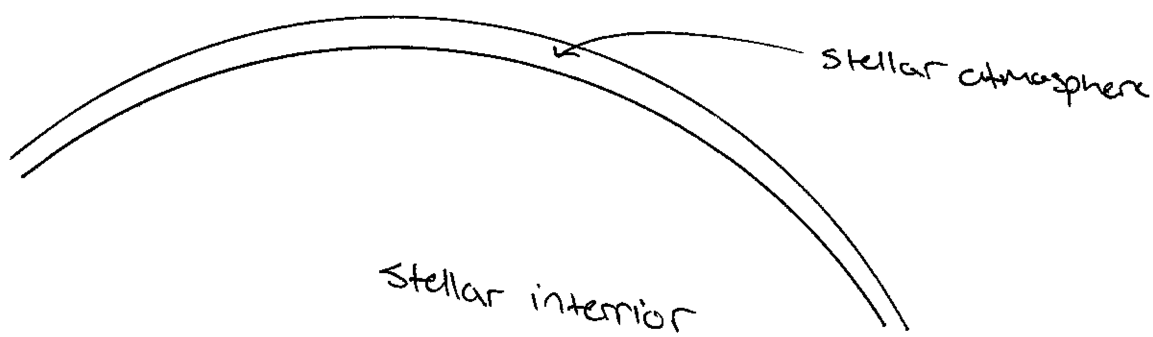
Thus the equation of radiative transfer is given by

$$\frac{dI_\nu}{d\tau} = -I_\nu + B_\nu$$ } For system in LTE

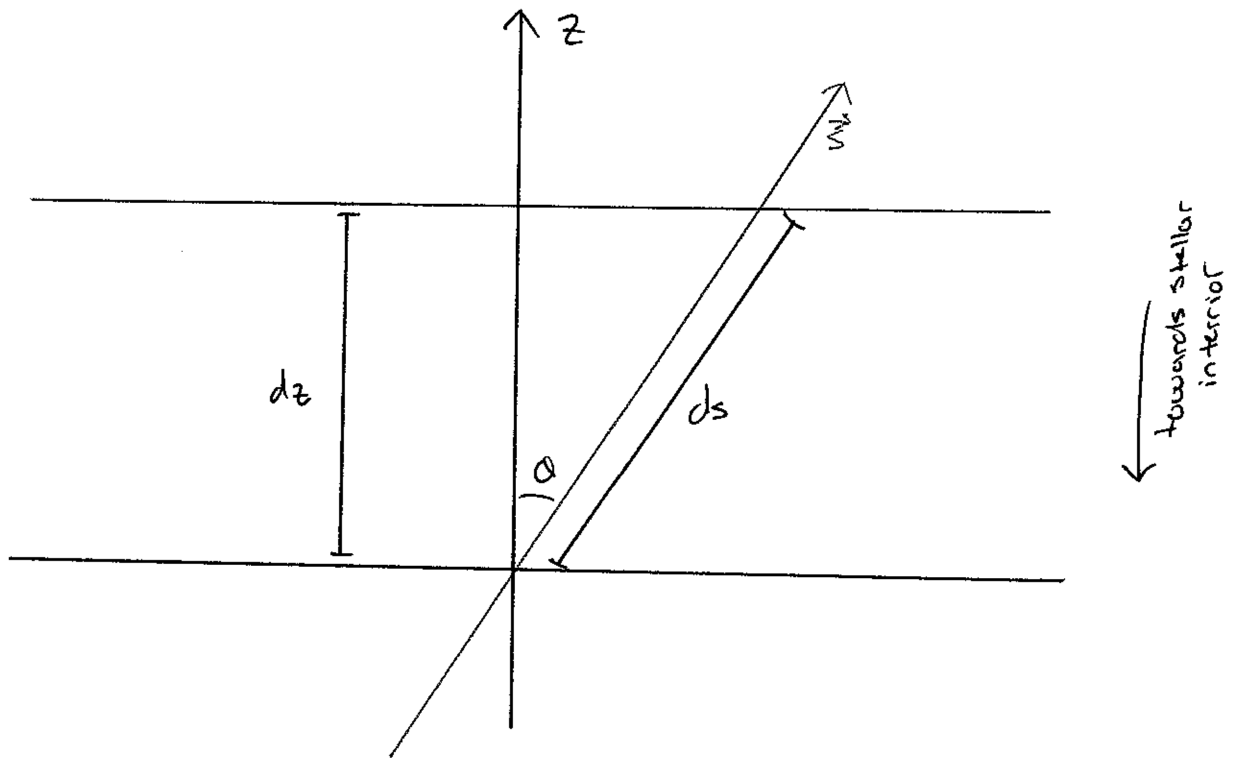
Stellar Atmospheres

Here, let's apply our understanding of radiative transfer to stellar atmospheres. This is our first opportunity to see how we can learn about the interrior of stars by examing the light we receive from stars.

It is usually the case that the atmosphere of a star is thin compared to the size of the star.



Given this it is safe to approximate the geometry as plane parallel.



Here the $\hat{z}$ direction is outwards towards the stellar surface. Here the differential path length along a ray, ds, that makes an angle, θ, wrt to $\hat{z}$ is given by,

$$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu} \qquad \boxed{\mu \equiv \cos\theta}$$

Let's define the <u>vertical optical depth</u>,

$$d\tau_\nu = -\alpha_\nu \underbrace{dz}_{\text{note } dz \neq ds}$$

Note that this is different from our usual definition for optical depth τ_ν'. The two are related via,

$$\tau_\nu' = \frac{\tau_\nu}{\cos\theta} = \tau_\nu \sec\theta$$

For material in LTE, and where the plane parallel approximation is valid,

$$I_\nu = I_\nu(z, \mu)$$

* note specific intensity is not a function of any other coordinate (t, ϕ).

Thus, we can rewrite the equation of radiative transfer as,

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

$$\Rightarrow \mu \frac{dI_\nu(z,\mu)}{dz} = j_\nu - \alpha_\nu I_\nu$$

multiply each side by $\frac{1}{\alpha_\nu}$ gives

$$\mu \frac{dI_\nu(z,\mu)}{dz\,\alpha_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu$$

$$-\mu \frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$\Rightarrow \mu \frac{dI_\nu(\tau_\nu,\mu)}{d\tau_\nu} = I_\nu - S_\nu$$

Recall that for material in LTE, $S_\nu = B_\nu(T)$. So, in order to solve this, we need to know the temperature as a function of radius inside a star. In general, this would also require us to know the frequency dependence of α_ν.

To side-step these problems, consider a toy version of this problem where α_ν is independent of ν.

This is called a <u>grey atmosphere</u>.

In this case, we can drop the frequency dependence on the optical depth and use the bolometric specific intensity and source function.

$$I = \int I_\nu \, d\nu$$

$$S = \int S_\nu \, d\nu$$

In this case the transfer function for a plane parallel grey atmosphere becomes,

$$\mu \frac{dI}{d\tau} = I - S$$

Integrating over solid angle,

$$\int_\Omega \mu \frac{dI}{d\tau} \, d\Omega = \int_\Omega (I - S) \, d\Omega$$

Applying Leibniz's rule,

$$\frac{d}{d\tau}\int I \mu \, d\Omega = \int I \, d\Omega - \int S \, d\Omega$$

$$\frac{d}{d\tau}\underbrace{\int I \cos\theta \, d\Omega}_{\text{radiative Flux}} = \int I \, d\Omega - \underbrace{S\int d\Omega}_{\text{note } S \text{ is independent of the direction}}$$

$$\Rightarrow \frac{d}{d\tau} F = \underbrace{4\pi \langle I \rangle}_{\text{mean specific intensity}} - 4\pi S$$

$$\Rightarrow \frac{d}{d\tau} F = 4\pi\left(\langle I \rangle - S\right)$$

This relation tells us how flux depends on optical depth. Similarly, we can derive the relation for radiative pressure by multiplying by $\cos\theta$ and integrating over solid angle to get,

$$\boxed{\frac{d}{d\tau} P = \frac{1}{c} F}$$

This is pretty interesting! Let's consider the consequences of this relation.

First note that in an equilibrium stellar atmosphere, absorption and emission processes must balance out, i.e. there are no sources or sinks of energy. Any energy generated in the core must pass out as constant flux through the outer layers. In this case,

$$\frac{d}{d\tau} F = 0 = 4\pi \left(\langle I \rangle - S \right)$$

$$\Rightarrow \quad \langle I \rangle = S$$

This condition defines radiative equilibrium.

Since the radiative flux is constant under the condition of radiative equilibrium, we can solve for the radiative pressure.

$$\frac{dP}{d\tau} = \frac{1}{c} F$$

$$\Rightarrow \; P = \frac{1}{c} F \tau + q$$

integration const.

Recall that for an isotropic radiation field,

radiative energy density

$$P = \frac{1}{3} u$$

for isotropic radiation field

Not proven here, but at sufficiently large optical depth inside a star where $T \gg T_{eff}$, the radiation field is very close to isotropic.

If we make the assumption this holds throughout the star, this is called the Eddington Approximation. Let's run with this...

For an isotropic radiation field,

$$u = \frac{4\pi}{c} \langle I \rangle$$

From this we can relate pressure and intensity

$$P = \frac{1}{3} u = \frac{4\pi}{3c} \langle I \rangle$$

Under radiative equilibrium $\langle I \rangle = S$

$$\Rightarrow \quad P = \frac{4}{3} \frac{\pi}{c} S$$

Then from our equation for pressure, we can solve for the source function inside the star.

$$P = \frac{F}{c}(\tau + q)$$

$$\Rightarrow \quad S = \frac{3F}{4\pi}(\tau + q)$$

By setting $\tau = 0$ at the surface, it can be shown that $q = \frac{2}{3}$

$$\Rightarrow \quad 4\pi S = 3F\left(\tau + \frac{2}{3}\right)$$

$$\Rightarrow \quad 4\pi \langle I \rangle = 3F\left(\tau + \frac{2}{3}\right)$$

$$\Rightarrow \quad cU = 3F\left(\tau + \frac{2}{3}\right)$$

Noting that for blackbody radiation,

① $u = aT^4$

radiation constant

② $F = \sigma T_{eff}^4$

surface flux

③ $\sigma = \dfrac{ca}{4}$ } relation between radiation const. and σ

$$\Rightarrow \quad caT^4 = 3\sigma T_{eff}^4 \left(\tau + \frac{2}{3}\right)$$

$$\Rightarrow \quad T^4 = \frac{3}{4} T_{eff}^4 \left(\tau + \frac{2}{3}\right)$$

This tells us how temperature varies as a function of optical depth inside a grey stellar atmosphere.

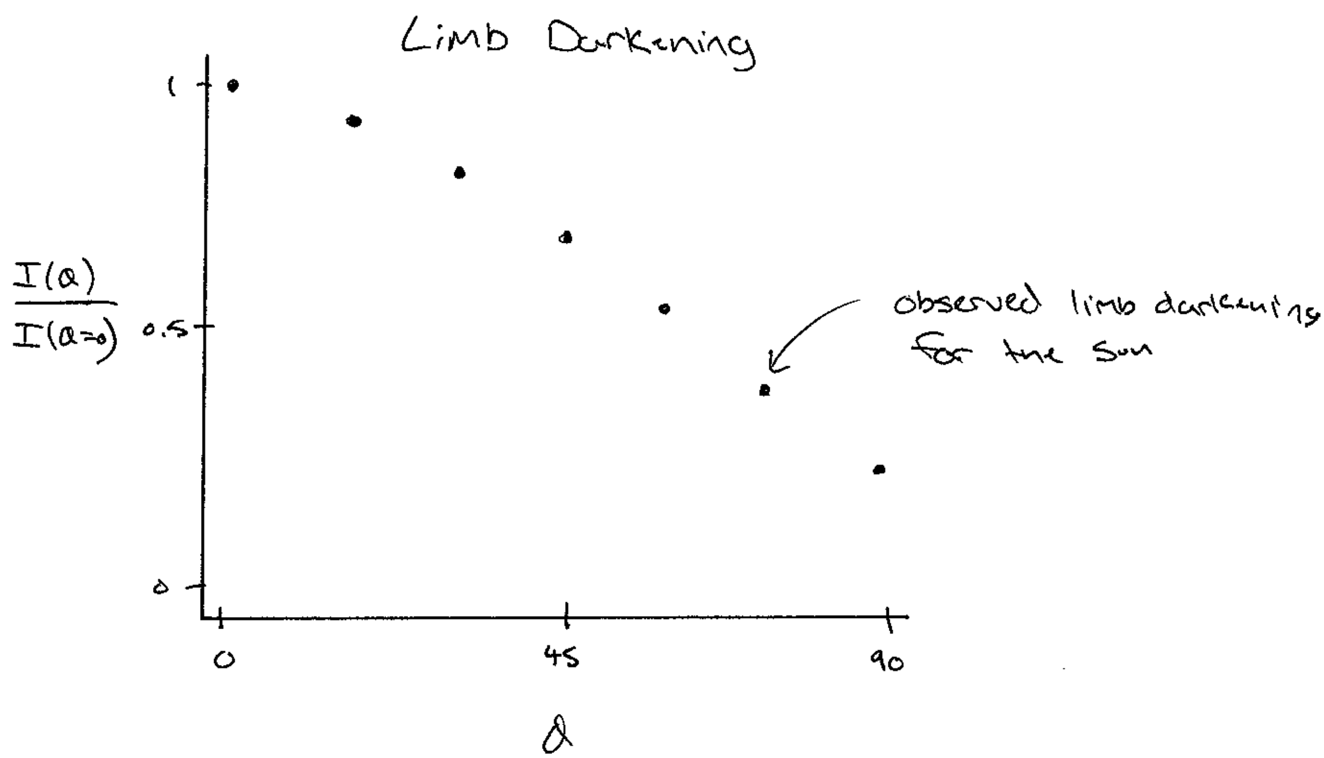
This is an interesting result! We can solve for the optical depth where $T = T_{eff}$.

$$T_{eff}^4 = \frac{3}{4} T_{eff}^4 \left(\tau + \frac{2}{3}\right)$$

$$\Rightarrow \quad \tau = \frac{2}{3}$$

This result can be interpreted as the optical depth we "see" when we look at the Sun.

Let's look at an application of this model to explain an observable property of the Sun. Limb darkening is the observed projected radial dependence of intensity over the disk of the Sun.



Given what we know, we can now make a theoretical prediction for the solar limb darkening.

Starting with the transfer equation,

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

$$\left(\mu \frac{dI_\nu}{d\tau_\nu}\right)\left(e^{-\frac{\tau_\nu}{\mu}}\right) = (I_\nu - S_\nu) e^{-\frac{\tau_\nu}{\mu}}$$

$$\frac{dI_\nu}{d\tau_\nu} \mu e^{-\frac{\tau}{\mu}} - I_\nu e^{-\tau/\mu} = -S_\nu e^{-\tau_\nu/\mu}$$

$$\mu \frac{d}{d\tau_\nu}\left(I_\nu e^{-\tau_\nu/\mu}\right) = -S_\nu e^{-\tau_\nu/\mu}$$

Integrating both sides gives

$$I_\nu e^{-\tau_\nu'/\mu} \Big|_{\tau_{\nu,0}}^{\tau_\nu} = -\int_{\tau_{\nu,0}}^{\tau_\nu} \frac{S_\nu}{\mu} e^{-\tau_\nu'/\mu} \, d\tau_\nu'$$

Let's consider radiation moving from deep in the interior ($\tau_{\nu,0} = \infty$) outwords ($\mu \geq 0$).

$$I_\nu e^{-\frac{\tau_\nu'}{\mu}} \Big|_{\infty}^{\tau_\nu} = - \int_{\infty}^{\tau_\nu} \frac{S_\nu}{\mu} e^{-\frac{\tau_\nu'}{\mu}} d\tau'$$

$$I_\nu e^{-\tau_\nu/\mu} = \int_{\tau_\nu}^{\infty} \frac{S_\nu}{\mu} e^{-\tau_\nu'/\mu} d\tau_\nu'$$

For radiation coming at of the surface $\tau = 0$,

$$I_\nu(\tau = 0, \mu) = \int_0^{\infty} \frac{S_\nu}{\mu} e^{-\tau_\nu'/\mu} d\tau_\nu'$$

Now, let's apply our grey atmosphere model

$$I_\nu = I$$
$$S_\nu = S = \frac{3F}{4\pi}(\tau + q)$$

Now the emergent intensity is given by,

$$I(\tau=0,\mu) = \frac{3F}{4\pi}\int_0^\infty (\tau'+q)e^{-\tau'/\mu}\,\frac{d\tau'}{\mu}$$

$$= \frac{3F}{4\pi}(\mu+q)$$

derivation of $q=\frac{2}{3}$

The emergent flux is then found by integrating over $\mu = 0 \rightarrow 1$.

$$F = 2\pi\int_0^1 I\mu\,d\mu$$

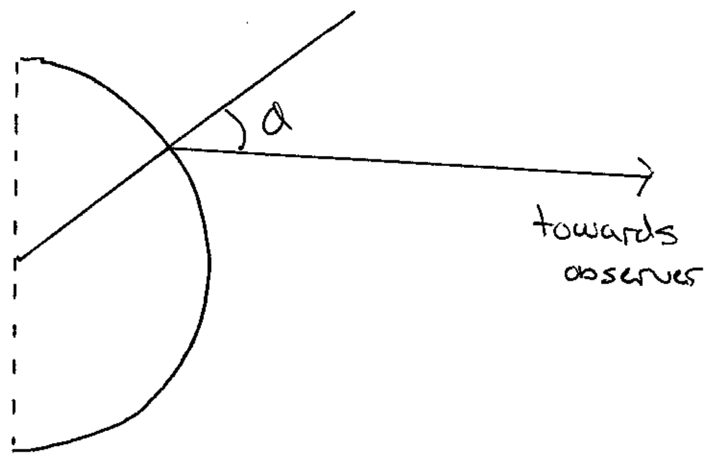
$$F = \frac{3F}{2}\int_0^1 (\mu+q)\mu\,d\mu$$

$$F = \frac{3F}{2}\left(\frac{1}{3}+\frac{q}{2}\right)$$

$$\Rightarrow q = \frac{2}{3}$$

The intensity variation as a function of μ is

$$I(\tau=0,\mu) = \frac{3F}{4\pi}\left(\mu + \frac{2}{3}\right)$$

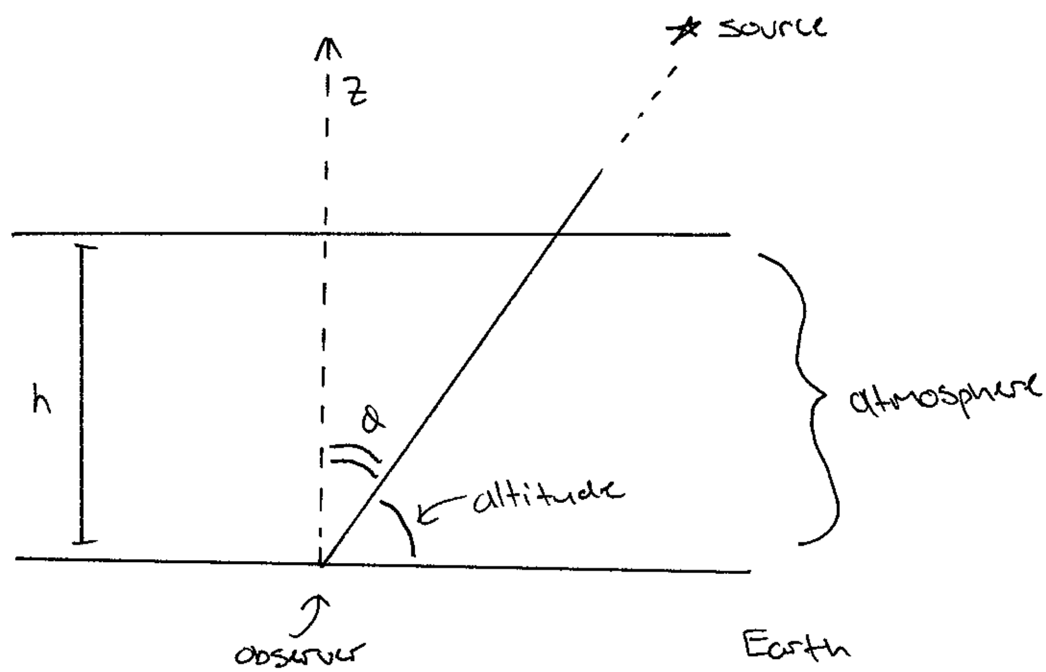


Rays from the center of the solar disk are normal to the surface, $\alpha = 0$. Rays from near the limbs, $\alpha > 0$

$$\Rightarrow \quad \frac{I(\tau=0,\mu)}{I(\tau=0,1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right)$$

$$= \frac{3}{5}\mu + \frac{2}{5}$$

Let's look at another example that is important for observational astronomers. When measuring the magnitude of a star, it is desirable to obtain a quantity that is independent of time and the location on Earth of the observation. Consider that light is scattered by the Earths atmosphere. The observed intensity of an object then depends on the altitude of the object.



The optical depth to the source is related to the vertical optical depth

$$\tau'_\nu = \tau_\nu \sec\theta$$

standard optical depth

vertical optical depth

Ignoring any emission in the Earth's atmosphere

unknown

$$I_\nu(\theta) = I_\nu(\theta=0)\, e^{-\tau_\nu \sec\theta}$$

observable

unknown

angle from zenith