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Today

I Stellar Interriors

II Hydrostatic Equilibrium

In this short lecture, we will begin our study of stellar interriors.

Stellar Interriors

First, let's quickly summarize what we know about the Sun.

- distance from Earth

  $1\ \text{AU} = 1.5 \times 10^{8}\ \text{km}$

- mass

  $1\ M_\odot = 2 \times 10^{33}\ \text{g}$

- radius

  $1\ R_\odot = 7 \times 10^{10}\ \text{cm}$

- luminosity

  $1\ L_\odot = 3.8 \times 10^{33}\ \text{erg/s}$

- effective surface temperature

  $T_{eff\odot} = 5800\ \text{K}$

- age of the solar system

  $t \sim 4.5 \times 10^{9}\ \text{yr}$

Given the mass and size of the Sun, we can calculate the mean density inside the Sun.

$$\bar{\rho}_\odot = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3} = 1.4 \frac{g}{cm^3}$$

This is nearly the density of liquid water at $\sim 1 \frac{g}{cm^3}$

From the effective temperature, we can use Wein's law to get the energy of the typical photon that escapes the Sun.

$$h\nu_{max} = 2.8\,KT = 1.4\,eV$$

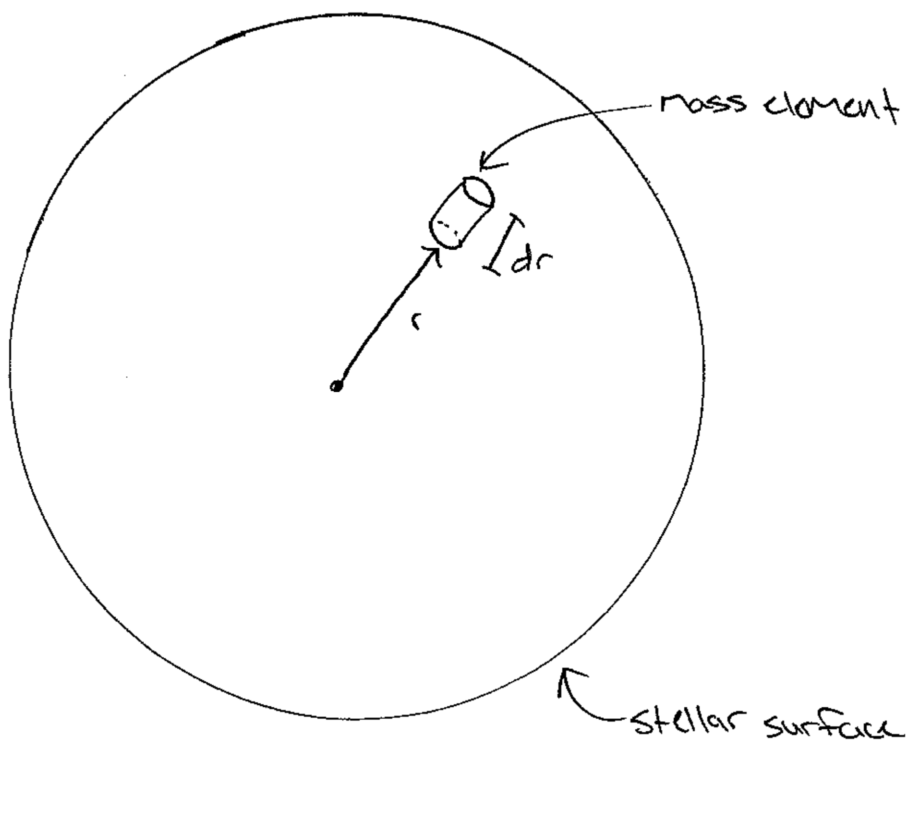
* note that you get a slightly different answer if you use the other form of Wein's law

$$\lambda_{max} = \frac{0.29\ cm \cdot K}{T} = 500\ nm$$

$\Rightarrow$ Emission peaks in the visible spectrum.

It turns out that stars are just big spherical (to good approximation) balls of gas held together by gravity and supported against collapse by pressure gradients. We can see this by considering the free-fall time, the time scale of collapse if there were no pressure support.

Consider a mass element inside a star



The gravitational potential energy of this element is simply,

$$dU = -\frac{GM(r)\,dm}{r}$$

where the notation $M(r)$ is the mass interior to r,

$$M(r) = \int_0^r 4\pi (r')^2 \rho(r')\,dr'$$

From conservation of energy, we can calculate the velocity of this mass element as it falls inwards,

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{GM(r')}{r} - \frac{GM(r')}{r'}$$

where r' is the initial radial position, and the mass interior to the element is constant, $M(r')$.

With some algebra, we can set up an integral to calculate the free fall time, τ_{ff}.

$$\tau_{ff} = \int_0^{\tau_{ff}} dt = -\int_{r'}^{0} \left\{ 2GM(r') \left[\frac{1}{r} - \frac{1}{r'} \right] \right\}^{-1/2} dr$$

$$= \left(\frac{(r')^3}{2GM(r')} \right)^{1/2} \underbrace{\int_0^1 \left(\frac{x}{1-x} \right)^{1/2} dx}_{\text{evaluates to } \frac{\pi}{2}}$$

$$\tau_{ff} = \left(\frac{3\pi}{32G\bar{\rho}} \right)^{1/2}$$

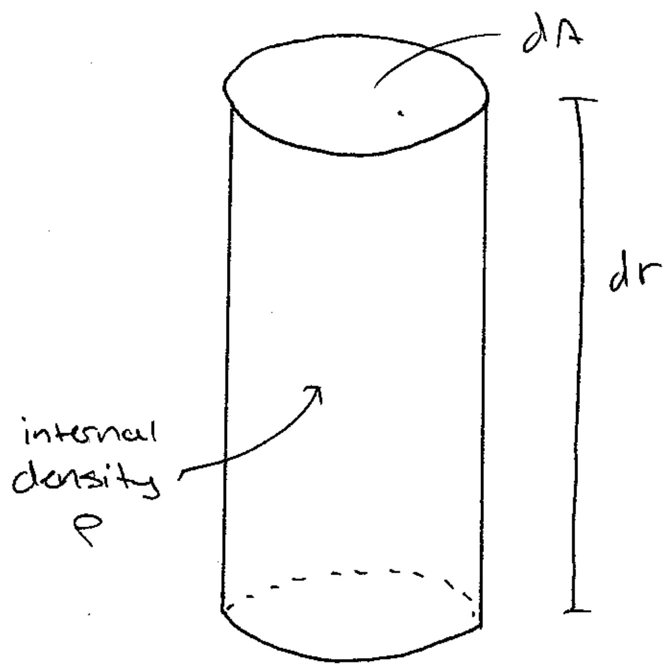
Using the mean density of the Sun, the free fall time of the Sun is

$$\tau_{ff\odot} = 1800 \text{ s}$$

Given that this time scale is so short, this implies there must be some form of pressure support in the Sun.

The condition that the internal pressure balance out the inward pull of gravity is called <u>hydrostatic equilibrium</u>.

Consider again our mass element inside the star. This time we will define it to be a cylinder.



The mass inside the cylinder is

$$dm = \rho \, dV$$
$$= \rho \, dr \, dA$$

The forces acting on our cylinderical mass element are gravity pulling inwards and pressure pushing on all sides.

Let's assume that the horizontal forces cancel out. From Newton's second law,

$$dm \frac{d^2\vec{r}}{dt^2} = \vec{F}_{net} = \vec{F}_{top} + \vec{F}_{bottom} + \vec{F}_{grav}$$

Given that this is an equilibrium state,

$$\vec{F}_{net} = 0$$

Dropping the vector notation,

$$-F_{top} + F_{bottom} - F_{grav} = 0$$

$$\Rightarrow \underbrace{F_{top} - F_{bottom}}_{\equiv dF} = -F_{grav}$$

Given the definition of pressure $\left(P = \frac{F}{A}\right)$

$$dP = \frac{dF}{dA} = -\frac{F_{grav}}{dA}$$

Recalling that the gravitational force is defined as,

$$F_{grav} = \frac{GM_1M_2}{r^2}$$

The force acting on ar mass element is given by,

$$F_{grav} = \frac{GM(r)\,dM}{r^2}$$

$$\Rightarrow \quad dP = -\frac{GM(r)\,dm}{r^2\,dA}$$

$$\Rightarrow \quad dP = -\frac{GM(r)}{r^2\,dA}\rho\,dr\,dA$$

$$\boxed{dP = -\frac{GM(r)}{r^2}\rho\,dr}$$

This is an important and widely applicable equation in astrophysics.

Let's examine one consequence of hydro-static equilibrium. Taking our equation we just derived, we can do some manipulation.

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\int_0^{R_*} 4\pi r^3 \frac{dP}{dr}\,dr = \int_0^{R_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r}$$

integrating over entire star

Notice that $dM(r) = \rho(r)4\pi r^2\,dr$ is the mass of a shell. As a result, this is just the gravitational potential energy of the star.

The left hand side can be integrated by parts to give,

$$P(r)\,4\pi r^3 \Big|_0^{R_*} - 3\int_0^{R_*} P(r)\,4\pi r^2\,dr$$

By definition we take $P(R_*) = 0$. As a result, this first term is zero. The second term is,

$$-3\int_0^{R_*} P(r)\,4\pi r^2\,dr = -3\,\bar{P}\,V$$

volume of star

Volume averaged pressure

As a result, putting this together, the hydrostatic equilibrium implies,

$$\bar{P} = -\frac{1}{3}\frac{E_{grav}}{V}$$

We can rewrite the virial theorem for a star made of a monoatomic, non-relativistic, ideal gas.

pressure is given by the equation of state,

$$PV = NKT$$

and energy

$$E_{thermal} = \frac{3}{2} NKT$$

$$\Rightarrow P = \frac{2}{3} \frac{E_{thermal}}{V}$$

Again, we can integrate over the entire star

$$\int_0^{R_*} P(r) 4\pi r^2 dr = \int_0^{R_*} \frac{2}{3} \frac{E_{thermal}}{V} 4\pi r^2 dr$$

$$\Rightarrow \quad \bar{P}V = \frac{2}{3} E^{tot}_{thermal}$$

total thermal energy in star

Substituting this into our first statement of the virial theorem,

$$\bar{P} = -\frac{1}{3} \frac{E_{grav}}{V}$$

$$\Rightarrow \quad E^{tot}_{thermal} = -\frac{E_{grav}}{2}$$

This is pretty interesting. This implies that stars have negative heat capacity! Stars heat up as they lose energy!

This is one form of the virial theorem valid for gravitationally bound systems.

Hydrostatic equilibrium is one of a few equations of stellar structure.

Another simple addition is a mass conservation equation,

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

which we have already used to define the total mass interior to r

$$M(r) = \int_0^r 4\pi (r')^2 \rho \, dr$$

Note that in the special case $\rho = \text{const.}$

$$M(r) = \frac{4}{3}\pi r^3 \rho.$$

Using these two equations of stellar structure, we can estimate the pressure at the center of the Sun to within an order-of-magnitude.

$$\frac{dP}{dr} = - \frac{GM(r)}{r^2} \rho(r)$$

Let's estimate $\rho(r) = \rho$ const.

$$\Rightarrow \int_{P_c}^{0} dP = - \int_{0}^{R_*} \frac{GM(r)}{r^2} \rho \, dr$$

$$\Rightarrow -P_c = - G\rho \int_{0}^{R_*} \frac{M(r)}{r^2} dr$$

$R_* \equiv$ stellar radius
$P_c \equiv$ central pressure
$P_c \equiv P(r=0)$

$$= - \frac{2\pi}{3} G \rho^2 R_*^2$$

Recall that for constant ρ, $\rho = \frac{M_*}{\frac{4}{3}\pi R_*^3}$

$$P_c = \frac{3G}{8\pi} \frac{M_*^2}{R_*^4}$$

for solar values $(M_\odot, R_\odot)$

$$P_{c,\odot} \simeq 10^{14} \frac{N}{m^2} \simeq 10^9 \text{ atm}$$

Let's now estimate the central temperature in the Sun. Making the reasonable assumption that the material in the central regions is an ideal gas,

$$P = nKT$$

The number density

$$n = \frac{\rho}{\bar{m}}$$

mean mass

Let's define the mean molecular weight

$$\mu \equiv \frac{\bar{m}}{m_H}$$

Then we can write the ideal gas law as,

$$P = \frac{\rho}{\mu m_H} kT$$

Of course the total pressure is actually a combination of thermal and radiation.

$$P_{tot} = \underbrace{\frac{\rho}{\mu m_H} kT}_{\text{thermal}} + \underbrace{\frac{1}{3} a T^4}_{\text{radiation}}$$

Let's (safely) ignore radiation for now. If we assume the Sun is pure ionized hydrogen,

$$\bar{m} = \frac{1}{2}(m_p + m_e) \simeq \frac{1}{2} m_H$$

$$\Rightarrow \mu \simeq \frac{1}{2}$$

From the ideal gas law

$$T = \frac{P \mu m_H}{\rho k}$$

Again, let's take ρ to be constant and use the pressure, P_c, from our last estimate.

$$T_\odot(r=0) \simeq 4 \times 10^7 \text{ K}$$

It turns out these conditions are sufficient for Hydrogen → Helium fusion.