33-777

Today

I Equations of stellar Structure

II Stellar Scaling Relations

In this lecture we continue to derive the remaining equations of stellar Structure.

In the last lecture we derived the first two equations important to stellar structure,

(1) Hydrostatic Equilibrium

$$\frac{dP(r)}{dr} = -\frac{GM(r)}{r^2}\rho(r)$$

(2) Mass conservation

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

While the second equation is rather trivial, the first is fundamental. Hydrostatic equilibrium tells us that stars are supported against collapse by self-gravity by an internal pressure gradient. A closely related statement are various forms of the <u>virial theorem</u>,

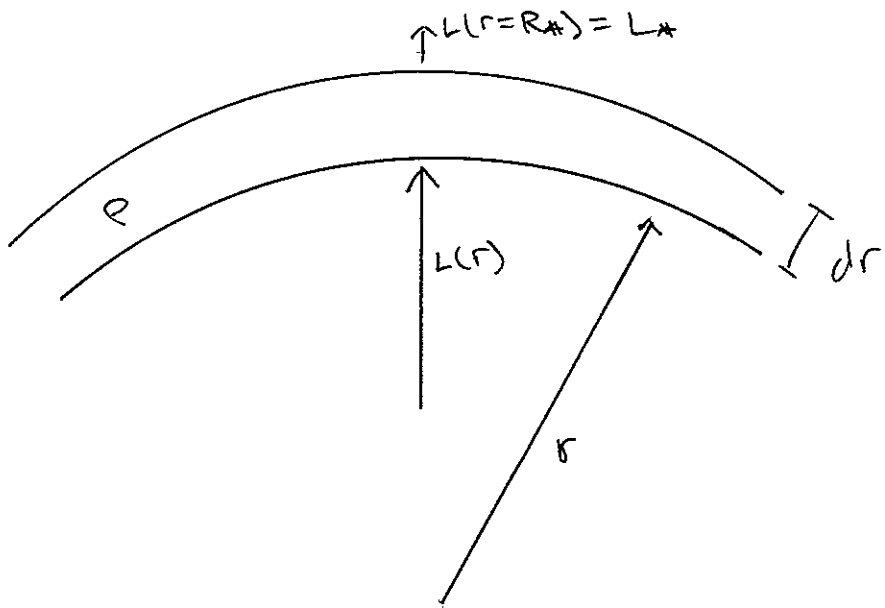
$$E_{thermal} = -\frac{1}{2} E_{grav}$$

In equilibrium, there is a balance between a star's thermal and gravitational energy.

The only plausible source of thermal energy in a star (over most of its life) is nuclear fusion. From order of magnitude calculations, we saw that in the core of the Sun,

$$T_c \simeq 10^7 \text{ K}$$
$$P_c \simeq 10^9 \text{ atm}$$

This turns out to be sufficient to fuse hydrogen into helium (more on this later). Let's now focus on how this energy is transported from the stellar interior out.



Consider $L(r)$ be the total energy flux per unit time which passes through a spherical shell of radius r within the star. At the surface, this must be equal to the total luminosity of the star.

$$L(r = R_*) = L_*$$

Within a star, the amount of energy added to luminosity, $L(r)$, by a shell of thickness dr, density, $\rho(r)$, with a rate of energy generation per unit mass, ε, is,

$$dL(r) = 4\pi r^2 dr\, \rho\, \varepsilon$$

which gives the third equation of stellar structure, the <u>energy flux</u>,

$$\frac{dL}{dr} = 4\pi r^2 \rho\, \varepsilon$$

This energy flux is sourced by a temperature gradient. As a result, our description of stellar structure is incomplete until we specify how heat is transported inside a star. First, we will consider <u>radiative heat transfer</u>.

Recall that stars with interriors in LTE, a state of <u>radiative equilibrium</u> is achieved. This told us that radiative energy flux is driven by a radiative pressure gradient (think "photon wind").

$$\frac{dP}{d\tau} = \frac{F}{c}$$

Recall that starting from here we were able to show that in a grey atmosphere, there must be a temperature gradient.

without making the "grey" assumption, we can derive a similar result. Adding back the frequency dependence,

$$\frac{dP_\nu}{d\tau_\nu} = \frac{F_\nu}{c}$$

From the definition of optical depth $d\tau_\nu = -\alpha_\nu dz$, we can rewrite this as,

$$F_\nu = -\frac{c}{\alpha_\nu}\frac{dP_\nu}{dz}$$

To get the total flux, we just need to integrate over all frequencies,

$$F = \int F_\nu d\nu = -c\int \frac{1}{\alpha_\nu}\frac{dP_\nu}{dz} d\nu$$

This may seem hopeless, but luckily we defined the very convenient Rosseland mean opacity, $\bar{\kappa}$, for just this purpose!

We defined the Rosseland mean as a weighted average over the blackbody spectrum.

$$\frac{1}{\bar{K}} = \frac{\int_0^\infty \frac{1}{K_\nu} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}$$

If we rearrange this to get,

$$\int_0^\infty \frac{1}{K_\nu} \frac{\partial B_\nu}{\partial T} d\nu = \frac{1}{\bar{K}} \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu$$

Now recall that,

$$P_\nu = \frac{4\pi}{3c} B_\nu(T)$$

Chain rule

$$\Rightarrow \frac{dP_\nu}{dz} = \frac{4\pi}{3c} \frac{dB_\nu}{dT} \frac{dT}{dz}$$

And that $K_\nu = \frac{\alpha_\nu}{\rho}$.

We can then rewrite our equation for the flux as,

$$F = -\frac{c}{\rho} \int \frac{1}{\kappa_\nu} \frac{4\pi}{3c} \frac{\partial B_\nu}{\partial T} \frac{dT}{dz} d\nu$$

$$= -\frac{4\pi}{3} \frac{1}{\rho} \frac{dT}{dz} \int \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T}$$

From the Rosseland mean opacity this is just

$$F = -\frac{4\pi}{3} \frac{1}{\rho} \frac{dT}{dz} \frac{1}{\bar{\kappa}} \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu$$

$$\frac{\partial B_\nu}{\partial T} = \frac{dP_\nu}{dz} \frac{3c}{4\pi} \frac{1}{(dT/dz)})$$

$$\Rightarrow F = -\frac{c}{\bar{\kappa}\rho} \int_0^\infty \frac{d}{dz} P_\nu \, d\nu$$

$$= -\frac{c}{\bar{\kappa}\rho} \frac{d}{dz} P$$

Finally, we note that for blackbody radiation,

$$u = aT^4 \qquad \text{(energy density)}$$

and

$$P = \frac{1}{3}u$$

$$\Rightarrow \quad P = \frac{a}{3}T^4$$

The energy flux inside a star is then

$$\boxed{F = \frac{-c}{\kappa\rho}\frac{d}{dz}\left(\frac{a}{3}T^4\right)}$$

We can use our third equation of stellar structure to relate this to the luminosity.

Finally, we can notice that for blackbody radiation

$$u = aT^4 \quad \text{and} \quad P = \frac{1}{3}u$$

$$\Rightarrow \quad P = \frac{a}{3}T^4$$

We can then write the radiative enery flux as

$$F = -\frac{c}{\bar{\kappa}\rho}\frac{d}{dz}\left(\frac{a}{3}T^4\right)$$

Replacing z (height in stellar atmosphere) with radius inside star, the luminosity is then

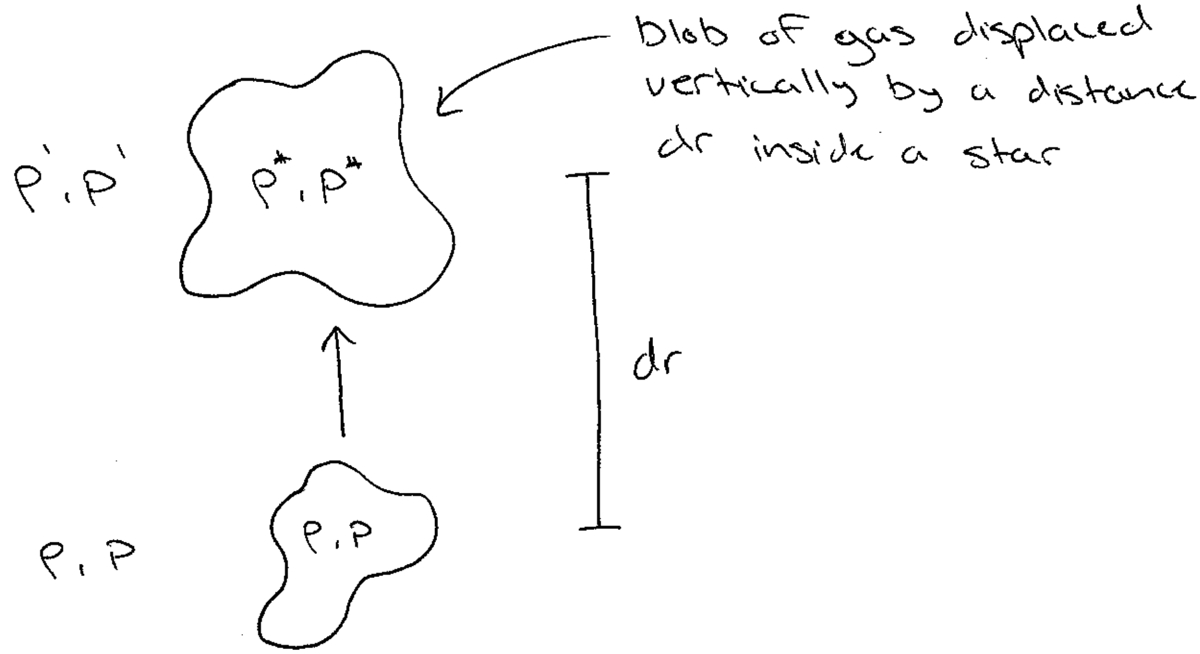
$$L(r) = 4\pi r^2 F = -4\pi r^2 \frac{c}{\bar{\kappa}\rho}\frac{d}{dr}\left(\frac{a}{3}T^4\right)$$

$$\Rightarrow \frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L(r)}{4\pi r^2}$$

This is the fourth equation of stellar structure, the <u>radiative heat transfer equation</u>. Needless to say, this is only valid if convection and conduction are not important. Let's ignore conduction for now as it turns out to be unimportant for stars.

We cannot afford to ignore <u>convection</u>. For radiative heat transfer, matter does not need to move. By contrast however, convection occurs when blobs of hot gas rise (and cool blobs sink) within the star physically transporting the heat.

Let's work out the physical conditions when we expect this to occur.



Before displacement, our gas blob has density and pressure that match its surroundings, ρ & P. After a vertical displacement from radius r to $r + dr$, the blob's surroundings will have a new density and pressure, ρ' & P'.

If we assume that pressure equilizes quickly, we can treat the displacement <u>adiabatically</u>.

$$\Rightarrow \quad P' = P^*$$

Now, if

- $\rho^* < \rho' \Rightarrow$ blob continues to rise due to buoyancy.

By assuming adiabatic displacement

$$P V^{\gamma} = \text{const}$$

γ adiabatic index

$$\Rightarrow \quad \rho^* = \rho \left(\frac{P^*}{P} \right)^{1/\gamma} \qquad \gamma = \frac{C_p}{C_v}$$

for monoatomic gas

$$\Rightarrow \quad \rho^* = \rho \left(\frac{P'}{P} \right) \qquad \gamma = \frac{5}{3}$$

For a given pressure gradient inside the star at the location of the blob dP/dr,

$$P' = P + \frac{dP}{dr} \underbrace{\Delta r}_{\text{vertical displacement}}$$

Substituting thin in for P', expanding and keeping only linear terms in Δr gives

$$\rho^* \simeq \rho + \frac{\rho}{\partial P} \frac{dP}{dr} \Delta r \quad \Big\} \text{ density of blob.}$$

The density outside the displaced blob is given by,

$$\rho' = \rho + \frac{d\rho}{dr}\Delta r$$

Recall that for an ideal gas $P = \rho R_{specific} T$,

$$\Rightarrow \rho' = \rho + \frac{\rho}{P}\frac{dP}{dr}\Delta r - \frac{\rho}{T}\frac{dT}{dr}\Delta r$$

The difference between the blob's density and the surrounding density is then,

$$\begin{aligned}\rho^* - \rho' &= \left(\rho + \frac{\rho}{\gamma P}\frac{dP}{dr}\Delta r\right) - \left(\rho + \frac{\rho}{P}\frac{dP}{dr}\Delta r - \frac{\rho}{T}\frac{dT}{dr}\Delta r\right) \\ &= \left[-\left(1 - \frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}\right]\Delta r\end{aligned}$$

Since the pressure and temperature must both decrease with radius, the condition for stability against convection becomes,

$$\rho^* - \rho' > 0 \Rightarrow \left|\frac{dT}{dr}\right| < \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|$$

This is the Schwarzschild stability condition against convection. If the temperature gradient in the stellar interior becomes larger than a critical value, then convection will kick-in.

Convection is a very efficient heat transport mechanism. Because of this, convection tends to maintain the temperature gradient at or near the critical value. As a result, we can take this to be the convective heat transfer equation.

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr}$$

This completes our derivation of the

Equations of Stellar Structure

① $$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$ } Hydrostatic Equilibrium

② $$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$ } mass conservation

③ $$\frac{dT(r)}{dr} = \frac{-3}{4ac}\frac{\bar{\kappa}\rho(r)}{T(r)^3}\frac{L(r)}{4\pi r^2}$$

or

$$\frac{dT(r)}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T(r)}{P(r)}\frac{dP(r)}{dr}$$

} heat transfer

④ $$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\,\varepsilon$$ } energy conservation

In order to solve this set of equations, we must specify a few more relations.

Additions

(1) equation of state

$$P(\rho, T, X_i)$$

(2) opacity

$$\bar{\kappa}(\rho, T, X_i)$$

(3) energy generation rate

$$\varepsilon(\rho, T, X_i)$$

(4) chemical composition

- Mass fraction of element i

$$X_i = \frac{\rho_i}{\rho}$$

- ionization state of element i

After specifying these additions, we have four independent functions of radius, ρ, T, M, L and four independent equations.

Note that there are some boundary conditions.

At the center of the star,

* $M(r=0) = 0$
* $L(r=0) = 0$

At the surface of the star,

* $\rho(r=R_*) = 0$
* $M(r=R_*) = M_*$
* $L(r=R_*) = L_*$

Let's take a moment to discuss the "additions" to our basic equations of stellar structure.

First, let's discuss the equation of state.

The equation of state is a function of the form,

$$P(\rho, T, x_i)$$

$\nwarrow$ composition.

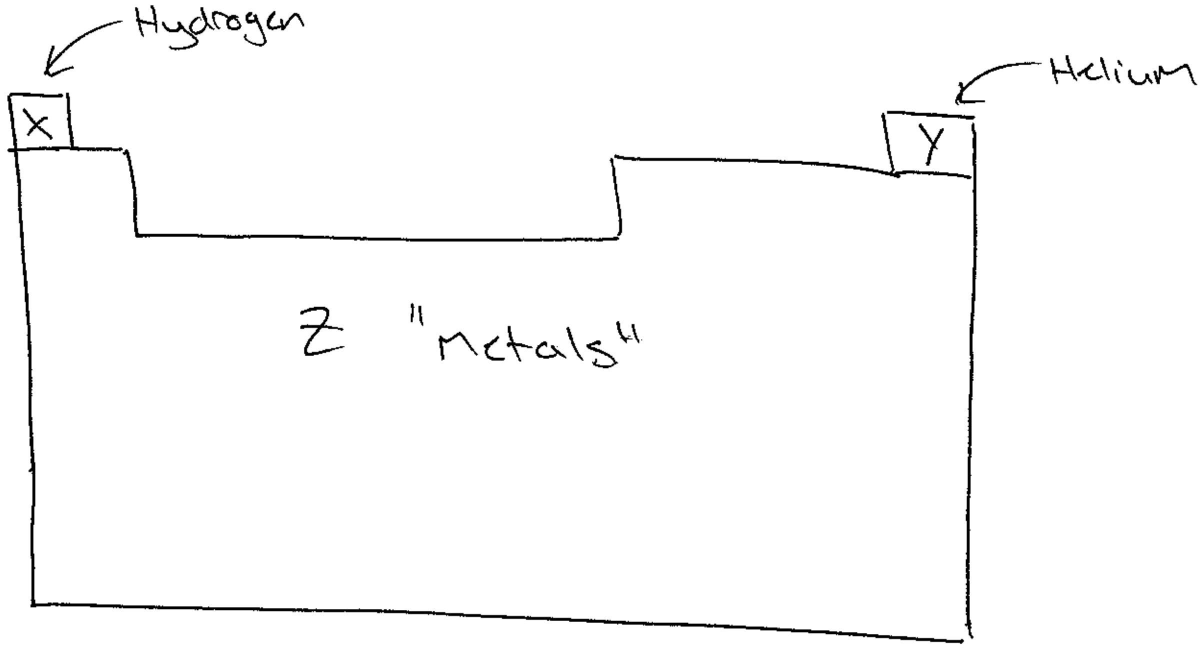
In general the equation of state will depend on the temperature and density in addition to the chemical composition.

As stated before (but not proven) in normal, main sequence stars, a classical, non-relativistic ideal gas is an excellent description. In this case,

$$P = nKT = \frac{\rho}{\bar{m}} KT$$

$\nwarrow$ mean particle mass

At this point it is necessary to mention the astronomer's view of the periodic table.



On a cosmic scale less than 1% by mass of baryons are found outside of hydrogen and helium atoms.

If we let X, Y, Z be the mass fractions of hydrogen, helium, and metals respectively, then the number density of each element is given by,

$$n_H = \frac{X\rho}{m_H}, \quad n_{He} = \frac{Y\rho}{4m_H}, \quad n_A = \frac{Z_A \rho}{A m_H}$$

atomic mass number

For a completely ionized gas, each element different number of particles (electrons plus nuclei).

$$n = 2n_H + 3n_{He} + \underbrace{\sum \frac{A}{2} n_A}_{\substack{\text{actually this} \\ \text{is } Z+1 \simeq \frac{A}{2} \\ \uparrow \\ \text{atomic \#}}}$$

$$\Rightarrow \quad n = \frac{\rho}{m_H}\left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)$$

$$= \frac{\rho}{2m_H}\left(3X + \frac{Y}{2} + 1\right) \qquad X + Y + Z \equiv 1$$

Thus, the mean molecular weight is given by,

$$\mu = \frac{\bar{m}}{m_H} = \frac{\rho}{n m_H} = \left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)^{-1}$$

note this is a good approximation in the stellar interior, but this breaks down in the outer cooler regions where ionization may be low and molecules can form.

For the Sun,

$$X = 0.71$$
$$Y = 0.27 \quad \Rightarrow \quad \mu_\odot = 0.61$$
$$Z = 0.02$$

However, this is a function of position. In the solar core, a significant fraction of the available hydrogen has been burnt into He, such that $\mu = 0.85$.

In some situations we must take into account radiation pressure in the equation of state such that

$$P = \underbrace{\frac{\rho K T}{\bar{\mu}}}_{\text{ideal gas}} + \underbrace{\frac{1}{3} a T^4}_{\text{radiation pressure}}$$

In even more exotic cases (white dwarfs, neutron stars, evolved stellar cores) Fermi-Dirac statistics are more relavent.

We have already discussed calculations of opacity in some detail, and let's put off a detailed look at nuclear energy generation mechanisms and rates till next time.

Let's pause and consider what we can learn about stars with what we have so far. Unfortunatly, these equations of stellar structure can not be analytically solved simultaneously. In general numerical methods must be employed to find solutions. Generally, the parameter space that is explored is for different initial masses and chemical composition.

Let's make some crude estimates to solve these equations for some interesting quantities. Let's estimate the hydrostatic equilibrium condition

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho$$

$$\sim \frac{P}{R} \propto \frac{M}{R^2}\rho$$

using $\rho \sim \frac{M}{R^3}$

$$\Rightarrow \quad P \propto \frac{M\rho}{R} \sim \frac{M^2}{R^4}$$

From the equation of state

$$P \propto \rho T \quad \Rightarrow P \propto \frac{M}{R^3}T$$

From these two relations

$$\frac{M^2}{R^4} \propto \frac{M}{R^3}T \quad \Rightarrow \quad T \propto \frac{M}{R}$$

This tells us that the typical temperature inside a star is proportional to $\frac{M}{R}$.

Let's do a similar thing to the radiative transfer equation. Here we assume heat transfer is radiative everywhere and $\bar{\kappa} = \text{const}$ inside a star.

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L(r)}{4\pi r^2}$$

$$\sim \frac{T}{R} \propto \left(\frac{M}{R^3}\right)\left(\frac{1}{T^3}\right)\left(\frac{L}{R^2}\right)$$

$$\Rightarrow L \propto \frac{(TR)^4}{M}$$

Earlier we saw that $T \propto \frac{M}{R}$

$$\Rightarrow L \propto M^3$$

From this we may expect more massive stars to be significantly more luminous than lower mass stars!

We have already seen that stars can be treated as approximate black-bodies. In this case,

$$L = 4\pi R^2 \sigma T_{eff}^4$$

If we make the rough approximation that $T_{eff} \propto T$ ← typical internal temp.

$$\Rightarrow \quad L \propto R^2 T^4 = (RT)^2 T^2$$

We saw that $L \propto M^3$ and $RT \propto M$

$$\Rightarrow \quad M^3 \propto M^2 T^2$$

$$\Rightarrow \quad M \propto T^2$$

Substituting luminosity back in,

$$L \propto M^3 \Rightarrow M \propto (L)^{1/3}$$

$$\Rightarrow \quad L^{1/3} \propto T^2$$

$$\Rightarrow L \propto T^6$$

and again using $T_{eff} \propto T$

$$L \propto T_{eff}^{\,6}$$

compare this to what we saw on the H-R diagram

Finally, if we assume that stars have a finite amount of "fuel", we can estimate their lifetime.

$$\tau_{life} \propto \frac{M}{L}$$

$$\Rightarrow \quad \tau_{life} \propto \frac{1}{M^2}$$

More massive stars have shorter lives!