33-777

Feb 30, 2020

Today

I. Introduction to Stars

II. Radiative Transfer

In this lecture, we will review some basics about stars before reviewing radiative transfer theory. This sets the base for a multi-lecture overview of stellar astrophysics.

I have utilized multiple sources for this material including,

- "Radiative Processes in Astrophysics" by G.B. Rybicki and A.P. Lightman
- "Astrophysics for Physicists" by A.R. Choudhuri

Introduction to Stars

Let's begin our study of stars with a brief review of the anatomy and taxonomy of stars. First, for the purposes of this class we will define stars as gravitationally bound balls of gas supported against gravitational collapse by nuclear fusion processes. With this in mind, we can make a distintion between stars and related (but distinct) classes of objects such as:

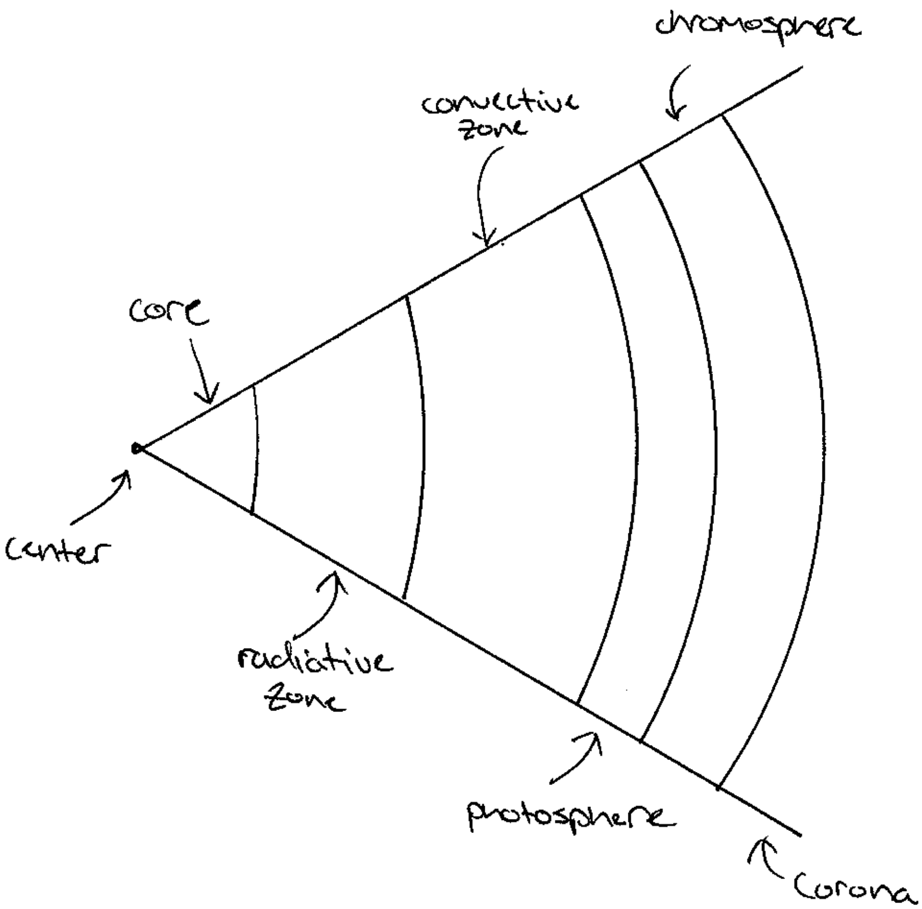
- planets
- brown dwarfs

$\Big\}$ "failed stars"

- stellar remanents
  + white dwarfs
  + neutron stars
  + black holes

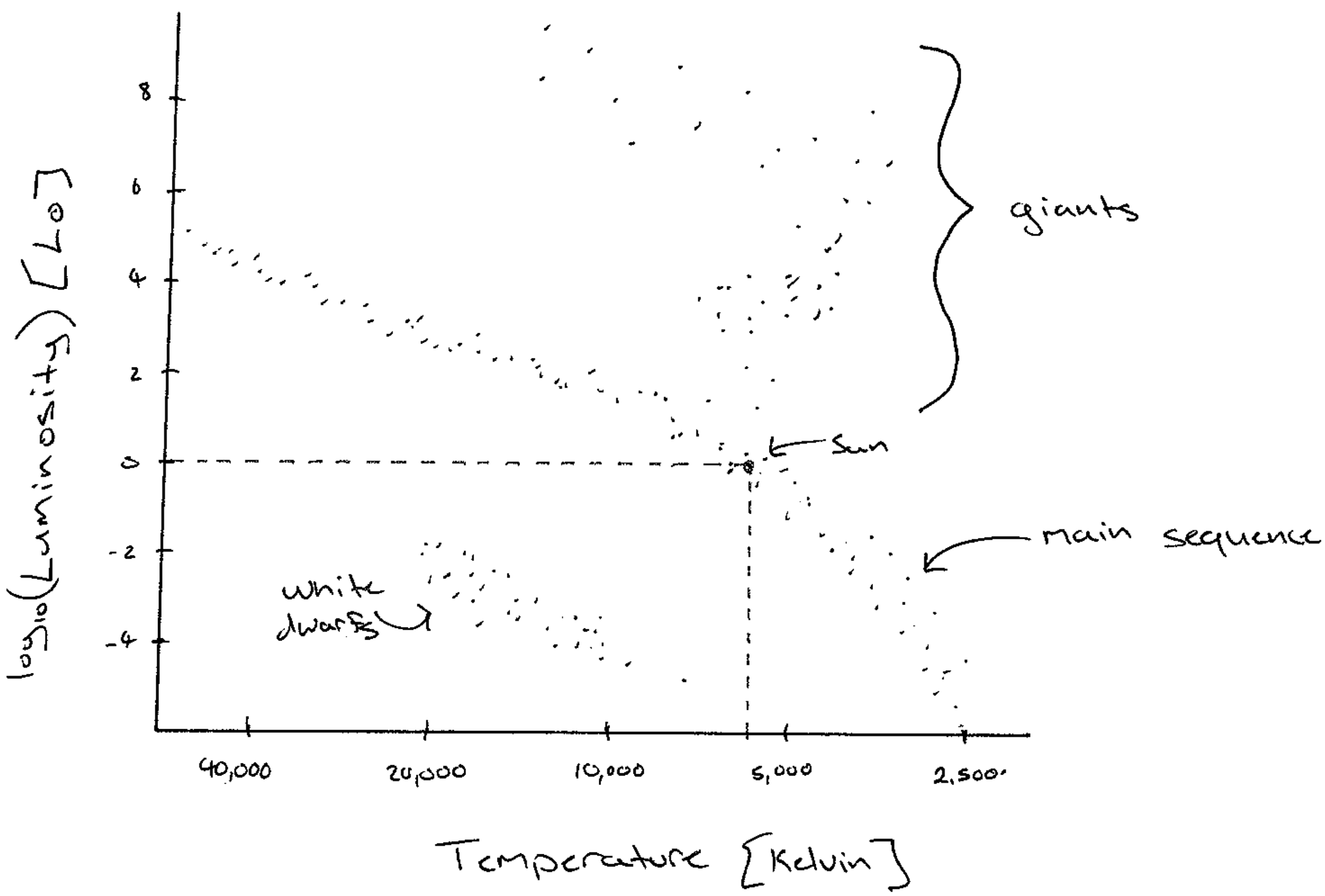
$\Big\}$ end products of stellar evolution

It is also convenient to talk about different "parts" of stars. These parts are generally distinguished as regions where different physical processes are more/less important



- stellar core
    - energy production region
- radiative zone
    - dominated by radiative energy transport
- convective zone
    - dominated by convection
- photosphere
    - "surface of star"
    - origin of emitted photons

Perhaps the most ubiquitous plot | figure in astrophysics is the Hertzsprung-Russel Diagram (H-R Diagram), also known as a color-magnitude diagram (CMD) depending on the form.



We will spend quite a bit of effort understanding why stars fall where they do on this diagram, and how stars evolve on the H-R diagram over the course of their lives.

An examination of stellar spectra show that stars are well approximated as blackbodies. From this fact, we can immediately learn something about the range of stellar sizes. Recall that for a blackbody, the flux at the surface is given by the stefan-boltzmann law.

$$f = \sigma T^4$$

σ stefan-boltzmann const.

Thus, for a star (assuming it is spherical), the luminosity is given by,

$$L_* = 4\pi R_*^2 \sigma T_{eff}^4$$

For a constant size, the luminosity of a star scales as,

$$L_* \propto T_e^4 \quad \Big\}\ \text{const. } R_*$$

The majority of stars are found along the <u>main sequence</u>. For these stars,

$$L_* \propto T_{eff}^8 \quad \Big\}\ \text{main sequence stars}$$

This tells us that hotter main sequence stars are larger,

$$R_* \propto T_{eff}^2 \quad \Big\}\ \text{main sequence stars.}$$

The typical sizes are between

$$R_* \sim 0.25 - 25 \ R_\odot$$

solar radius

Astronomers also identify other classes of stars on the H-R diagram, e.g.

- Red Giants

  ~5,000 K stars that are significantly larger than the Sun.

- White Dwarfs

  Hot objects that are much smaller than main sequence stars.

The spectra of stars are not perfect blackbodies. Stars show various atomic and molecular lines (generally absorption lines) ontop of their underlying blackbody spectrum. Stars are classified by their spectral features. These spectral types are

O B A F G K M

This classification scheme is known as the "Harvard system". Originally, this system was based on the strength of the hydrogen Balmer (absorption) lines. Recall that the Balmer series are the atomic lines corresponding to the $n = 2 \rightarrow 3$ electron transitions in the hydrogen atom. In this original (now out-dated) system, stellar types $A \rightarrow Q$ were ordered such that type A had the strongest Balmer lines. The current ordering was proposed by Annie Jump Cannon and corresponds to a temperature sequence (some types were dropped/combined when this was done).

Type	T_{eff}
O	> 28,000 K
B	10,000 - 28,000 K
A	7,500 - 10,000 K
F	6,000 - 7,500 K
G	5,000 - 6,000 K
K	3,500 - 5,000 K
M	< 3,500 K

This classification scheme is refined by adding an integer, 0-9, after the letter type. In this case A0 would be the hottest A-type, while A9 would be the coolest, etc.

Various atomic and molecular lines are more or less prominent in the various stellar types, including: H, Ca, Fe, K, TiO, ...
To understand why various lines are more or less apparant in stars of various temperatures, we need to dig into how EM radiation interacts with the gas in stellar atmospheres. This is our next topic.

It's worth quickly noting here that stars are also catagorized into luminosity classes. These are typically indicated with roman numerals I - VI, I being the most luminious. The Sun is a G2V star.

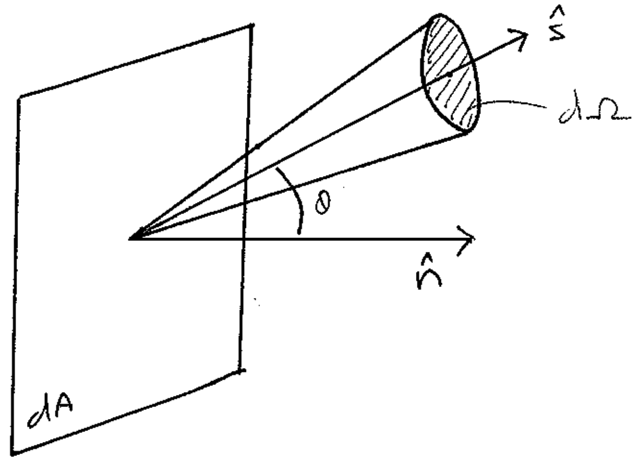
Radiative Transfer

As we have discussed, nearly all we know about astrophysical phenomina is based on the analysis of electromagnetic radiation we receive from astronomical sources. We will start our discussion of stars with a detailed look at how radiation travels through the outer layers of a star. Before considering the specifics of stars, it is necessary review the theory of radiative transfer, the macroscopic study of how radiation interacts with matter.

A detailed description of radiation can be built up by considering individual rays of radiation. The key here is that rays have a well defined direction, allowing us to describe how radiation flows from one place (e.g. a star) to another (us on earth).

However, since a single ray carries infinitesimal energy, we will generally consider bundles of rays. This allows us to define the specific intensity of radiation as follows. Consider all rays passing through a small area, dA, within a solid angle, $d\Omega$, of a specific ray traveling in a direction $\hat{s}$. The total energy passing through dA in this direction within the frequency range $\nu, \nu + d\nu$ is given by,

$$dE = I_\nu(\vec{r}, t, \hat{s}) \cos\theta \, dA \, dt \, d\Omega \, d\nu$$



$$F_\nu = \int I_\nu \cos\theta \, d\Omega$$ — Flux

and

$$P_\nu = \int I_\nu \cos^2\theta \, d\Omega$$ — Pressure

Recall that by integrating over frequency we can obtain the total (also called bolometric) quantities.

$$F = \int F_\nu \, d\nu$$

$$P = \int P_\nu \, d\nu$$

$$I = \int I_\nu \, d\nu$$

Another quantity of obvious interest is the energy density of the radiation field.

Here α is the angle between the direction of the specific ray and the normal to the surface, $\hat{n}$.

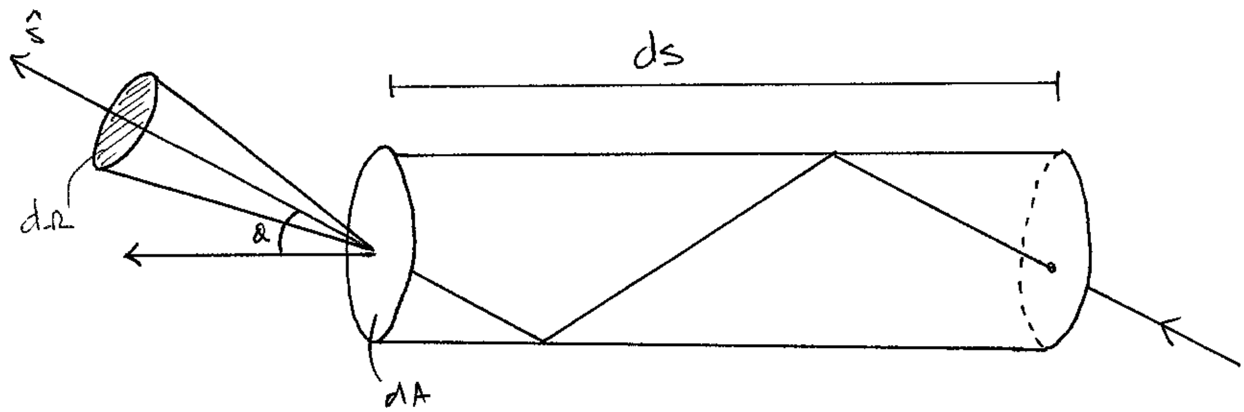
This defines the specific intensity (also called the "brightness") I_ν. Note that the specific intensity has units,

$$\frac{\text{erg}}{\text{s} \cdot \text{cm}^2 \cdot \text{Hz} \cdot \text{steradian}}$$

Also note that I_ν is generally a function of location, $\vec{r}$, time, t, and direction, $\hat{s}$. The specific intensity is a complete description of a radiation field.

The first two moments* of the specific intensity are the radiative <u>flux</u> and <u>pressure</u>.

* Multiplication by powers of $\cos\alpha$ and integration over $d\Omega$



The specific energy density, u_ν, is defined as the radiation energy per unit volume per unit frequency interval. This quantity can be calculated by considering how much energy flows through a cylinderical trap open at both ends. A ray that enters the cylinder at an angle α will be in the trap for a time

$$dt = \frac{ds}{c \cos\alpha}$$

where c is the speed of light.

The energy inside the trap due to rays entering at an angle α is then,

$$dE_\nu = I_\nu \cos\alpha \, dA\, dt\, d\Omega$$

$$= I_\nu \cos\alpha \, dA \left(\frac{ds}{c\cos\alpha}\right) d\Omega$$

$$= \frac{1}{c} \underbrace{dA\, ds}_{\text{volume of cylinder}} d\Omega \, I_\nu$$

$$\Rightarrow \frac{dE_\nu}{dA\, ds} = u_\nu(\alpha) = \frac{1}{c} I_\nu d\Omega$$

The specific energy density is then found by integrating over solid angle (α, ϕ).

$$u_\nu = \frac{1}{c} \int_\Omega I_\nu \, d\Omega$$

$$= \frac{4\pi}{c} \langle I_\nu \rangle$$

where $\langle I_\nu \rangle$ is the mean intensity (sometimes called J_ν):

$$\langle I_\nu \rangle = \frac{1}{4\pi} \int I_\nu \, d\Omega$$

Note that if the radiation field is isotropic, then,

$$\langle I_\nu \rangle = I_\nu$$

and we can write,

$$u_\nu = \frac{4\pi}{c} I_\nu$$

A special case is blackbody radiation which is isotropic. If we define the specific intensity of blackbody radiation to be $B_\nu(T)$:

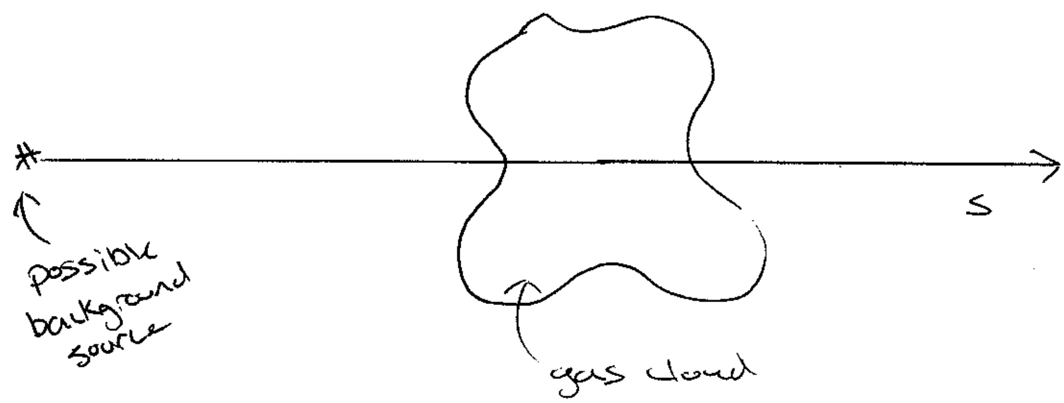
temperature

$$u_\nu = \frac{4\pi}{c} B_\nu(T)$$

which lets us give the energy density of blackbody radiation:

$$u_\nu \, d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$$

Now, let's consider how specific intensity changes as we move along a ray path through space.



As a ray transits a small path of length ds, which may be through a cloud of gas, the specific intensity changes

$$dI_\nu = \underbrace{dI_{\nu,\text{loss}}}_{\substack{\text{combined effect of scattering and absorption which}\\ \text{removes photons from the ray direction } \hat{s}.}} + \underbrace{dI_{\nu,\text{gain}}}_{\substack{\text{combined effect of all processes}\\ \text{which add photons in the direction } \hat{s}.}}$$

It should be noted that as a ray travels through empty space, the specific intensity remains constant because there are no scattering/absorption processes (or emission) in a vacuum.

$$\left.\frac{dI_\nu}{ds} = 0 \quad \right\} \text{empty space}$$

In the presence of matter, to quantify absorption processes, we define the <u>absorption coefficient</u>, α_ν, which accounts for the attenuation of specific intensity. We also define the <u>emission coefficient</u>, j_ν, which describes how specific intensity is added to along the ray path as a result of emission processes. With this we can write down the <u>equation of radiative transfer</u>.

$$\underbrace{\frac{dI_\nu}{ds}}_{\substack{\text{change in specific}\\ \text{intensity along}\\ \vec{s}.}} = \underbrace{-\alpha_\nu I_\nu}_{\substack{\text{attenuation}\\ \text{of } I_\nu \text{ due to}\\ \text{scattering/absorption}}} + \underbrace{j_\nu}_{\substack{\text{addition of specific}\\ \text{intensity due to}\\ \text{emission.}}}$$

In addition to the empty space scenario discussed earlier ($\alpha_\nu = 0$, $j_\nu = 0$), there are two other special cases which are easy to solve for.

① Absorption Only ($j_\nu = 0$)

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

$$\Rightarrow I_\nu(s) = I_\nu(s_0) \exp\left[-\int_{s_0}^{s} \alpha_\nu(s')\,ds'\right]$$

② Emission only ($\alpha_\nu = 0$)

$$\frac{dI_\nu}{ds} = j_\nu$$

$$\Rightarrow I_\nu(s) = I_\nu(s_0) + \int_{0}^{s} j_\nu(s')\,ds'$$

At this point it is useful to define the optical depth, τ_ν.

$$\tau_\nu = \int_{s_0}^{s} \alpha_\nu(s')ds'$$

The optical depth is a measure of the total absorption along the path of a ray. In the absorption only case, the specific intensity along the ray path is then,

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu}$$

We can then define two limiting cases.

① optically thick $(\tau_\nu \gg 1)$

The intensity is decreased by more than one e-folding

② optically thin $(\tau_\nu \ll 1)$

$$I_\nu \sim I_{\nu,0}$$

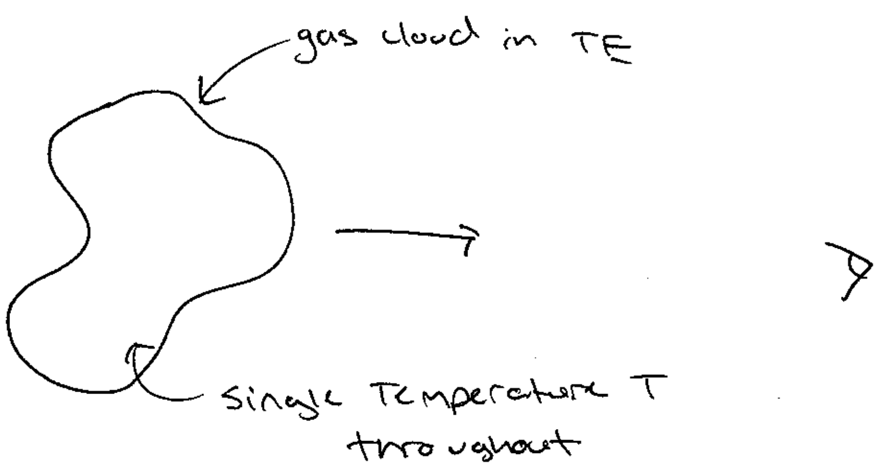
Finally, we can define a new quantity, the source function.

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu}$$

Note that S_ν has units of specific intensity. This quantity allows us to write down the other common form of the radiative transfer equation.

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

Before looking at the general solution, let's consider a cloud of gas in thermodynamic equilibrium with no background source.



Because the system is in equilibrium, there is no net transport of energy accross the Cloud.

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu = 0$$

$$\Rightarrow \quad I_\nu = \frac{j_\nu}{\alpha_\nu} = S_\nu$$

For systems in thermodynamic equilibrium, the source function is the blackbody intensity function, $B_\nu(T)$.

$$\Rightarrow \quad I_\nu = S_\nu = B_\nu(T)$$

$$\Rightarrow \quad j_\nu = \alpha_\nu B_\nu(T)$$

This last result is known as Kirchoff's law. This just states that for a blackbody, there must be a balance between emission and absorption. To see the significance of this result, Let's come back to this after looking at the general solution.

Let's go back to the general radiative transfer equation,

$$\frac{dI_\nu}{d\tau_\nu} = - I_\nu + S_\nu$$

Solving for the specific intensity,

$$\frac{dI_\nu}{d\tau_\nu} e^{\tau_\nu} = - I_\nu e^{\tau_\nu} + S_\nu e^{\tau_\nu}$$

$$\Rightarrow \quad \frac{dI_\nu}{d\tau_\nu} e^{\tau_\nu} + I_\nu e^{\tau_\nu} = S_\nu e^{\tau_\nu}$$

Applying the product rule $\left[(f \cdot g)' = f' \cdot g + f \cdot g' \right.$

$$\frac{d}{d\tau_\nu}\left(I_\nu e^{\tau_\nu}\right) = S_\nu e^{\tau_\nu}$$

By integrating over optical depth from $0 \to \tau_\nu$, we come to the general solution.

$$\underbrace{I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}}_{\substack{\text{intensity of} \\ \text{background source}}} + \underbrace{\int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu)\, d\tau_\nu}_{\substack{\text{integrated contribution} \\ \text{of emission}}}$$

One special case is one where the source function is constant along the ray path. In this case, we can simplify this solution to,

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)$$

Now again, we can consider the optically thin and thick cases again when the source function is constant.

1. optically thin ($\tau_\nu \ll 1$)

using $e^{-\tau_\nu} \simeq 1 - \tau_\nu$ (taylor series approx.)

we can write the transfer equation solution as

$$I_\nu(\tau_\nu) = I_\nu(0)(1-\tau_\nu) + S_\nu\left[1-(1-\tau_\nu)\right]$$

$$\Rightarrow I_\nu(\tau_\nu) = \underbrace{I_\nu(0)(1-\tau_\nu)}_{\text{attenuated background source}} + \underbrace{S_\nu \tau_\nu}_{\text{contribution from cloud}}$$

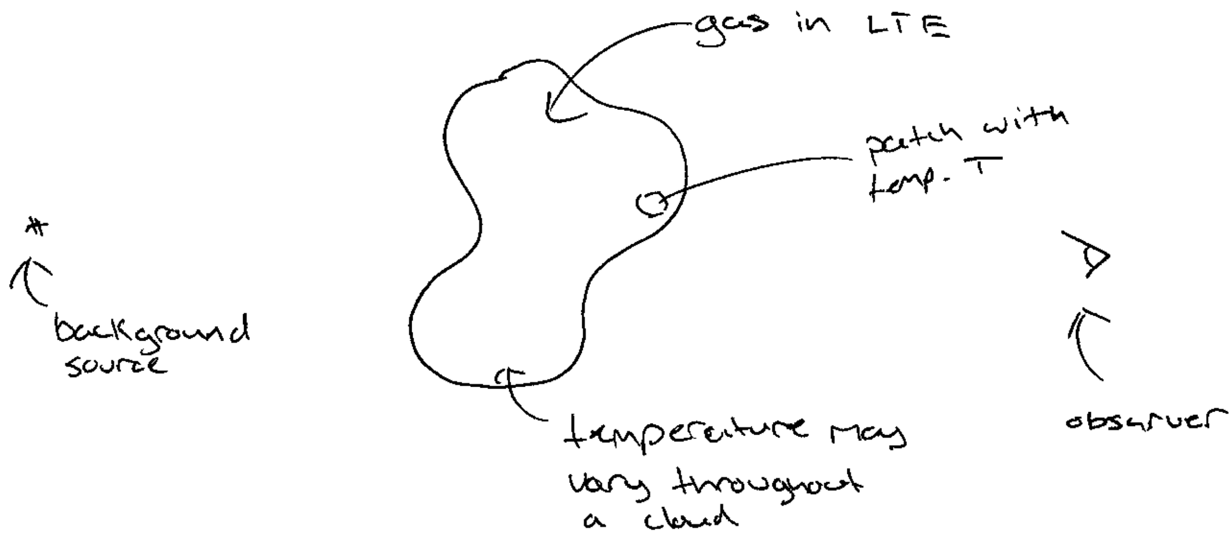
2. optically thick ($\tau_\nu \gg 1$)

$$I_\nu(\tau_\nu) = S_\nu$$

We can relate this back to Kirchoff's law; For an optically thick blackbody, the specific intensity is the Planck spectrum, ie,

$$I_\nu = B_\nu(T) \quad \Big\} \text{optically thick blackbody w/ temperature } T$$

Let's consider a special case, a cloud of gas in local thermodynamic equilibrium (LTE).



For material in LTE, the source function is simply the blackbody intensity spectrum.

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu} = B_\nu(T)$$

Kinetic temperature of gas cloud particles

Now the solution* is given by,

$$I_\nu = I_\nu(s_0)e^{-\tau_\nu} + B_\nu(T)(1-e^{-\tau_\nu})$$

* note that we have taken some liberties here since this solution assumes S_ν = const.

Now, again we can examine our two extreme cases,

① $\tau_\nu >> 1$

$$\Rightarrow \quad I_\nu = B_\nu(T)$$

② $\tau_\nu << 1$

$$\Rightarrow \quad I_\nu = I_\nu(s_0)(1-\tau_\nu) + B_\nu(T)\tau_\nu$$

Here it is important to note that for a cloud in LTE, the source function does not need to be constant inside the cloud. If it is, the cloud is in thermal equilibrium (const. temperature throughout). Stars are examples of optically thick gas clouds in LTE. We don't see a blackbody spectrum of a single temperature because I_ν is a function of τ_ν and stars contain temperature gradients!