

THESIS

Fully chaotic conservative models for some torus homeomorphisms

Alejo García-Sassi

August 2nd, 2024

Advisors

Dr. Alejandro Passeggi
Dr. Fábio Armando Tal



UNIVERSIDAD
DE LA REPÚBLICA
URUGUAY

Doctorado en Matemática
PEDECIBA
Universidad de la República
Montevideo, Uruguay



Doutorado em
Matemática Aplicada
Universidade de São Paulo
São Paulo, Brasil

*Reconquistar el olvidado balbuceo
que hacía juego en el origen con las cosas
y dejar que los pedazos se peguen después solos
como se sueldan los huesos y las ruinas.*

ROBERTO JUARROZ

Agradecimientos

La meritocracia es una fábula. Este trabajo no existiría de no ser por la red que me sostiene, todos los días, en lo simbólico y en lo concreto.

A Fábio y al Rata. Por su paciencia, su calidez, sus propuestas, sus consejos. Por sus ideas. Por mantener la calma todas las veces que yo la perdí. Por dejarme ser y dejarme hacer. No me alcanzarían tres vidas para saldar esa entrega.

A Juliana. Al Leva. Al Gordo. Por su cariño. Por mostrarme que se puede hacer matemática con los pies sobre la tierra. Por el rol decisivo que tuvieron y tienen en el matemático que soy.

A Pierre-Antoine. Por su trabajo incansable. Por sus correcciones y recomendaciones, determinantes en la calidad y claridad del trabajo. A André y a Salvador, por haberse tomado el tiempo para ser parte del proceso de revisión y evaluación.

A quienes me escucharon describir los mismos objetos, explicar los mismos argumentos, cometer los mismos errores. Una vez, y otra vez, y otra más. A Bellati, a Elena, a Favio, a Nelson, a Sofía. A Patrice. Por el interés, las sugerencias, los aportes, las referencias.

A Lydia. A Analú, a Claudia, a Natalia. A Paola. Por sus gestiones, su plasticidad, su diligencia. Por la confianza vertida.

A mi familia. A mi padre, por enseñarme que pensar es una de las formas del placer. A mi madre, por ser el pilar fundamental de mi crecimiento. A mi hermana, por apoyarme aún yendo en contra de sí misma. A mis primxs, por ser mi refugio. A mis tíos y mis tíos, por ayudarme a canalizar mi energía.

A mis amigxs, por enseñarme que pensar es solamente una, de las formas del placer. A Anara, a Chuchi, a Gero, a León, a Lily, a Natia. A Bruno, a Ernesto, a Facundo, a Nico. Por aliviar mi dolor. Por compartir mi felicidad.

A mis compañerxs del IMERL, por hacer que se parezca un poco más a mi casa. Al Alongo, a Clarita, a Ernesto, a Florencia, a Fiori, a Javier, a Jose, al Jimmy, a Marcelo, a Matilde, a Meli, a Nancy, a Nacho, a Pablito, a Paula, a Tabaré.

A Carpeta. Por ser como la brisa.

Abstract

We study homotopic-to-the-identity torus homeomorphisms, whose rotation set has nonempty interior. We prove that any such map is monotonically semiconjugate to a homeomorphism that preserves the Lebesgue measure, and that has the same rotation set. Furthermore, the dynamics of the quotient map has several interesting chaotic traits: for instance, it is topologically mixing, it has a dense set of periodic points and it is continuum-wise expansive. In particular, this shows that a convex compact set of \mathbb{R}^2 with nonempty interior is the rotation set of the lift of a homeomorphism of \mathbb{T}^2 if and only if it is the rotation set of the lift of a conservative homeomorphism.

Resumen

Estudiamos homeomorfismos del toro homotópicos a la identidad, cuyo conjunto de rotación tiene interior no vacío. Probamos que cualquier mapa de ese tipo es monótonamente semiconjugado a un homeomorfismo que preserva la medida de Lebesgue, y que tiene el mismo conjunto de rotación. Más aún, la dinámica del mapa cociente tiene varios aspectos interesantes: por ejemplo, es topológicamente mixing, tiene un conjunto denso de puntos periódicos y es continuum-wise expansiva. En particular, esto muestra que un conjunto compacto convexo de \mathbb{R}^2 con interior no vacío, es el conjunto de rotación del levantamiento de un homeomorfismo de \mathbb{T}^2 si y solamente si es el conjunto de rotación del levantamiento de un homeomorfismo conservativo.

Resumo

Estudamos homeomorfismos do toro homotópicos à identidade, cujo conjunto de rotação tem interior não vazio. Provamos que qualquer mapa desse tipo é monotonamente semiconjugado a um homeomorfismo que preserva a medida de Lebesgue e que tem o mesmo conjunto de rotação. Além disso, a dinâmica do mapa quociente tem várias qualidades caóticas interessantes: por exemplo, ela é topologicamente mixing, tem um conjunto denso de pontos periódicos e é continuum-wise expansiva. Em particular, isso mostra que um conjunto compacto convexo de \mathbb{R}^2 com interior não vazio é o conjunto de rotação do levantamento de um homeomorfismo de \mathbb{T}^2 , se e somente se for o conjunto de rotação do levantamento de um homeomorfismo conservativo.

Contents

Chapter 1. Introduction	17
1. Motivation	17
2. Torus Rotation Theory	18
3. Statement of results	19
4. Organization of the article	22
5. Some open questions	23
Chapter 2. Preliminaries	25
1. Basic notation	25
2. Torus topological dynamics	27
3. Strictly Toral Dynamics	29
4. Equivariant Brouwer Theory	30
5. Forcing Theory	31
6. Topological and Markovian horseshoes	32
7. Prime End Theory	35
8. Higher genus Rotation Theory	36
9. The Fine Curve Graph	37
Chapter 3. Canonically foliated strips	39
1. Construction	40
2. Crossing continua	45
Chapter 4. Anchoring techniques	51
1. Semianchors	51
2. The Anchoring Lemma	53
3. Proofs of Theorems C and E	56
Chapter 5. The essential factor	61
1. Dynamically bounded splitting	61
2. Monotone quotient	63
3. Quick dynamical consequences	64
Chapter 6. Stable and unstable sets	67
1. Weak stability	67
2. Shape and size	67
3. Dynamical properties	69
Chapter 7. The essential factor is rotationally mixing	75
1. Total anchoring	76

2. Mixing pairs of continua	80
3. Proof of Theorem D	82
Chapter 8. Heteroclinic pseudo-rectangles	85
1. Topological structure	85
2. Dense family of fitted heteroclinic pseudo-rectangles	90
Chapter 9. The essential factor has a dense family of horseshoes	99
1. Topological horseshoes	99
2. Markovian horseshoes	104
Chapter 10. Proof of Theorem A	113
1. The essential factor is conservative	113
2. The proof	114
Bibliography	115

CHAPTER 1

Introduction

1. Motivation

The study of arbitrary dynamical systems on a manifold is a very hard task, as there can be too much information and features for one to get the grasp of. One of the most well established ways to parse all that information is to try to relate a given dynamical system to some simpler model dynamics which still encompass central attributes from the original system, but where our understanding of the phenomena is more robust. This is typically done by finding semiconjugacies between the original system or some restriction of it, to certain factor dynamics. Information may be lost in this process, but several aspects of the original system become clearer in the study of these factors.

This strategy has so far been very successfully used in the study of one-dimensional dynamics. For instance, the Poincaré Rotation Theory provides simple models for homeomorphisms of the circle, and the Kneading Theory of Milnor and Thurston showed the existence of semiconjugacies from positive entropy multimodal maps of the interval, to simple piecewise-linear constant-slope models.

The use of models by factors is also a prevalent strategy in the study of surface homeomorphisms. It started with the study of Smale's Horseshoes and their semiconjugacy with the shift in the symbolic space, one of the most well understood models in dynamics. Katok in [Kat80] showed that, if a $C^{1+\alpha}$ diffeomorphism f of a surface has positive topological entropy, then there exists some invariant subset for a power f^k of f such that the restriction of f^k to this invariant set is semiconjugate to the Bernoulli shift.

Other deep interesting models are obtained when one considers the isotopy classes of maps in surfaces. Nielsen-Thurston Theory has classified all the possible isotopy classes, and it has been shown that maps which are isotopic to Linear Anosov diffeomorphisms of \mathbb{T}^2 , are extensions of these ([Fra69]). For closed surfaces it is also known that if a map is isotopic to a pseudo-Anosov map, then there exists a semiconjugacy between some invariant closed set of the original dynamics to the pseudo-Anosov homeomorphism ([Han85]). One therefore has very good models to understand the dynamics when the induced action in the homotopy group is sufficiently complex.

But none of those models help in understanding the rotational behaviour of a dynamical system on a surface if it lies in the isotopy class of the identity. In this paper we attempt to provide models that preserve this specific feature, which has been increasingly relevant in Surface Dynamics. We will concentrate our work in the study of homeomorphisms of \mathbb{T}^2 which are homotopic to the identity, a set we

denote by $\text{Homeo}_0(\mathbb{T}^2)$, but we imagine most of what is done here will be able to be extended in the correct circumstances to other closed surfaces. For this study, we will make use of Rotation Theory and the modern Equivariant Brouwer Theory techniques. We will postpone the description of the latter to the body of the text, let us say a few words on Rotation Theory.

2. Torus Rotation Theory

Given a surface S with a non-trivial fundamental group and an isotopy $(f_t)_{t \in [0,1]}$ connecting the identity f_0 to a homeomorphism $f = f_1$ of S , one can ask how the trajectories of points by the isotopy *loop* around the surface. For the torus case, the classical notion of rotation set was given birth in [MZ89].

DEFINITION 1.1 (Rotation set). Given f in $\text{Homeo}_0(\mathbb{T}^2)$ and a lift \widehat{f} of f to \mathbb{R}^2 , one defines the *rotation set* of \widehat{f} as

$$\rho(\widehat{f}) := \left\{ v \in \mathbb{R}^2 : \exists \widehat{z}_k \in \mathbb{R}^2, n_k \rightarrow +\infty \text{ such that } \lim_{k \rightarrow \infty} \frac{\widehat{f}^{n_k}(\widehat{z}_k) - \widehat{z}_k}{n_k} = v \right\}.$$

This can be proved to be equivalent to the set of mean rotations of f -invariant measures, that is,

$$\rho_{\text{inv}}(\widehat{f}) := \left\{ v \in \mathbb{R}^2 : \exists \mu \text{ which is } f - \text{invariant s.t. } \int \widehat{f}(\widehat{z}) - \widehat{z} d\mu = v \right\}$$

In [MZ89] it is proven that $\rho(\widehat{f})$ is a compact convex subset of \mathbb{R}^2 (in particular it is connected), which is then either a point, a segment or a set with nonempty interior. Given every lift \widehat{f} commutes with the deck transformations, the rotation set of two different lifts will be the same up to integer translations. We will then say that the rotation set of f has non-empty interior if this is true for the rotation set of a lift \widehat{f} of f . One notes that the notion of rotation sets was already introduced by Schwartzmann using invariant measures in [Sch57], and a topological version was later given by Fried in [Fri82].

Rotation sets have been a very effective tool in the study of torus homeomorphisms, as they can in some cases encode several dynamical properties. For instance, when the rotation set has nonempty interior, it is known that the homeomorphism has positive topological entropy, as proved in [LM91], using Nielsen-Thurston classification of surface homeomorphisms from [Thu88], after having punctured the surface on an f -invariant set and restricting the dynamics to the resulting surface. Franks shows the abundance of periodic orbits in [Fra89]), and ergodic measures realizing each rotation vector in the interior of the rotation set are found in [MZ91]. New proofs of these results and several improvements are recently obtained using Forcing Theory by Le Calvez and Tal, found in [LCT18, LCT22], using Brouwer-Le Calvez's dynamically transverse decomposition built in [LC05a].

Also, assuming some extra regularity, inside the class of $\mathcal{C}^{1+\alpha}$ diffeomorphisms, if f has a lift \widehat{f} such that $\rho(\widehat{f})$ has the origin in its interior, then Addas-Zanata in [AZ14] and [AZ15] uses Pesin Theory (see for example [Pol93, Part I]) to find

saddle periodic points with different rotation vectors such that, for any lift of that point, its unstable manifold intersects the stable manifolds of each of its integer translates in a topologically transverse way. If, furthermore, one assumes that f is transitive, then one obtains that \widehat{f} (and therefore f itself) must be topologically mixing.

Given all this applications, the study of the *realization* of rotation sets, i.e. understanding which subsets of \mathbb{R}^2 are rotation sets of lifts of elements of $\text{Homeo}_0(\mathbb{T}^2)$, became also a popular topic. One says that a subset D of \mathbb{R}^2 is realized as a rotation set of a homeomorphism f if f has a lift whose rotation set is D . For the case in which it has nonempty interior, Kwapisz proved in [Kwa92] that every rational polygon (i.e. its vertices have rational coordinates) can be realized as the rotation set of a torus homeomorphism: Pasiegi proves that rational polygons are yielded as rotation sets for a generic set of homeomorphisms ([Pas13]), and Guih  neuf and Koropecki prove ([GK17]) that this is a necessary condition for the rotation set to be stable under small perturbations.

Kwapisz found in 1995 the first example of a rotation set with infinitely many extremal points ([Kwa95]), and in 2017 Boyland, De Carvalho and Hall find a one-parameter family of examples, many of them also having this property ([BdCH15]). They also describe explicitly how the rotation set changes as the parameter changes, and obtain examples for which there exists an extremal point which is not the rotation of any ergodic measure for the homeomorphism. Also, a result by B  guin, Crovisier and Le Roux proves that every irrational vector can appear as the extremal point of a rotation set with nonempty interior ([B  6]).

There are examples of compact convex sets that have been proven to never be a rotation set of a torus homeomorphism, but only in a very specific case ([LCT18]). But it is not known if there exists a rotation set with uncountable many extremal points. In particular, one does not know if for instance the closed unit ball is a realized as a rotation set.

3. Statement of results

Our main result shows that, whenever f has a rotation set with nonempty interior, then it is always an extension of another homeomorphism with much better understood dynamics, and with the same rotation set. Let us first introduce some notions for the sake of precision.

Factors and extensions. Given two metric spaces X and Y , and two continuous maps $f : X \rightarrow X$, $g : Y \rightarrow Y$, we will say that g is a *factor of f* (and equivalently, we will also say that f is an *extension of g*) if there exists a semi-conjugacy h from f to g , that is, a continuous surjection $h : X \rightarrow Y$ such that $h \circ f = g \circ h$. A map $h : X \rightarrow Y$ is *monotone* if the preimage of every point is connected.

Diameter of continua. Let $\widehat{\pi} : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ be the classical covering projection.

DEFINITION 1.2. Given an open set $U \subset \mathbb{T}^2$, we will say that U is *inessential* if every loop in U is homotopically trivial in \mathbb{T}^2 . Otherwise, we will say it is *essential*.

We will say an arbitrary set $X \subset \mathbb{T}^2$ is *inessential* if it has an inessential neighbourhood. Otherwise we will say it is essential.

Note that a connected set $D \subset \mathbb{T}^2$, holds that every lift of D (which is a connected component of $\widehat{\pi}^{-1}(D)$), is homeomorphic to D . D is then said to be inessential if a lift of \widehat{D} of D satisfies that $\widehat{D} \cap \widehat{D} + v = \emptyset$ for every non-null $v \in \mathbb{Z}^2$.

DEFINITION 1.3. Given a continuum $K \subset \mathbb{T}^2$, we will define its lifted diameter $\text{diam}(K)$ as the euclidean diameter of any connected component of its lifts \widehat{K} to the universal covering \mathbb{R}^2 of \mathbb{T}^2 .

Note that the diameter of K in this case is finite if, and only if, it is inessential. Thus, for the context of inessential continua we will abuse notation and write *diameter* for its lifted diameter.

DEFINITION 1.4 (Dynamical diameter). Let $K \subset \mathbb{T}^2$ be a continuum, and $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ a homeomorphism. We will define the *dynamical diameter* $\mathcal{D}_f(K)$ as

$$\mathcal{D}_f(K) := \sup_{j \in \mathbb{Z}} \{\text{diam}(f^j(K))\}$$

A continuum K will be *dynamically bounded* if $\mathcal{D}_f(K) < \infty$, and it will be *dynamically unbounded* otherwise.

A homeomorphism h of a metric space is *continuum-wise expansive* if there exists $\varepsilon > 0$ such that, for any non-trivial continuum K , the supremum of the diameters of $h^j(K)$ with $j \in \mathbb{Z}$ is greater than ε . Continuum-wise expansive maps have several strong properties, see for instance [Her04]. For f in $f \in \text{Homeo}_0(\mathbb{T}^2)$ we say that it is *infinitely continuum-wise expansive* if, for any continuum K , we have that $\mathcal{D}_f(K) = \infty$.

Topological entropy. Let us recall a definition of topological entropy by Bowen in [Bow71]. Let f be a uniformly continuous map of a metric space (X, d) . For every $n \in \mathbb{Z}^+$, we define the distances

$$\begin{aligned} d_n(x, x') &= \sup\{d(f^j(x), f^j(x')) : 0 \leq j \leq n\}, \\ d_{\pm n}(x, x') &= \sup\{d(f^j(x), f^j(x')) : -n \leq j \leq n\} \end{aligned}$$

We will say a subset $Y \subset X$ is (n, ε) -separated (similarly $(\pm n, \varepsilon)$ -separated) with respect to f , if for every pair $y, y' \in Y$ we have that $d_n(y, y') > \varepsilon$ (similarly $d_{\pm n}(y, y') > \varepsilon$).

Define $s_n(\varepsilon, K)$ as the maximal number of elements of an (n, ε) -separated subset of K , with respect to f , and define $s_{\pm n}(\varepsilon, K)$ in the same fashion. Let us define

$$(1.1) \quad h(f, K) = \lim_{\varepsilon \rightarrow 0} \left(\limsup_{n \rightarrow +\infty} \frac{1}{n} \log(s_n(\varepsilon, K)) \right)$$

We define the *(topological) entropy of f carried by K* as

$$h_{\pm}(f, K) = \lim_{\varepsilon \rightarrow 0} \left(\limsup_{n \rightarrow +\infty} \frac{1}{n} \log(s_{\pm n}(\varepsilon, K)) \right)$$

We will say that K carries positive entropy whenever $h(f, K)$ is positive. We then define the topological entropy $h(f)$ of f as

$$h(f) = \sup_{K \text{ compact}} h(f, K).$$

We will say f is *tight* if for every non-trivial continuum K , we have that $h_{\pm}(f, K)$ is positive. Note that this notion has been of interest during these last years: De Carvalho and Paternain define the 0-entropy quotient to recover tight dynamics in a more complex topological space in [dCP02], this result has been retaken for example in [BdCH23] by Boyland, De Carvalho and Hall.

THEOREM A. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ have a lift \hat{f} such that $\rho(\hat{f})$ has nonempty interior. Then, there exists a factor map $g \in \text{Homeo}_0(\mathbb{T}^2)$ by a monotone semi-conjugacy, having a lift \hat{g} with the following properties:*

- $\rho(\hat{g}) = \rho(\hat{f})$.
- g is area preserving.
- g is topologically mixing, and if $\vec{0} \in \text{int}(\rho(\hat{g}))$, \hat{g} is topologically mixing.
- g is infinitely continuum-wise expansive.
- g is tight: for each non-trivial continuum $K \subset \mathbb{T}^2$, we have $h_{\pm}(g, K) > 0$.
- For each open set $U \subset \mathbb{T}^2$, g has a Markovian horseshoe $X \subset U$ (see Definition 2.28). In particular, $\overline{\text{Per}(g)} = \mathbb{T}^2$.

Note that Theorem A, in addition to presenting dynamical consequences for homeomorphisms of \mathbb{T}^2 , has a direct consequence for the study of the realization of rotation sets in the form of the following corollary:

COROLLARY B. *A convex compact subset of \mathbb{R}^2 with nonempty interior is realized as the rotation set of an homeomorphism of \mathbb{T}^2 if and only if it is also realized as the rotation set of an area-preserving homeomorphism of \mathbb{T}^2 .*

In particular, this holds for the closed unit disk.

We remark that Theorem A for the case where f is a $\mathcal{C}^{1+\alpha}$ diffeomorphism was obtained in a recent work by De Carvalho, Koropecki and Tal, in an yet unfinished pre-print. That work relied on Pesin's theory and the work of Addas-Zanata in [AZ14], and the main structure of the proof is similar to ours. There are some technical difficulties that appear in the proof since we are working with no regularity, particularly in showing the existence of dense horseshoes, but the most crucial point is the proof of the Theorem C below. In any case, we outline here every step so that the work is self-contained.

THEOREM C. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$, such that $\rho(f)$ has nonempty interior. Then there exists $M > 0$ such that, if K is a continuum and $\text{diam}(K) > M$, then $\mathcal{D}_f(K) = \infty$.*

The main improvement here is that we showed this result using equivariant Brouwer techniques and Forcing Theory, bypassing Pesin's Theory and the need for regularity.

Note that if f belongs to the isotopy class of an linear Anosov automorphism A , then any lift \widehat{f} to \mathbb{R}^2 has that $d(\widehat{f}(\widehat{z}) - A(\widehat{z}))$ is uniformly bounded, and therefore we obtain that

REMARK 1.5. If we take $f \in \text{Homeo}(\mathbb{T}^2)$ to be in the isotopy class of an Anosov linear automorphism, the thesis from Theorem C still holds.

This allows us to get a monotone semiconjugacy for this isotopy class using analogous anchoring arguments, and obtain a factor g which is again infinitely continuum-wise expansive, is area-preserving and has a dense set of topological horseshoes.

Apart from its use in the proof of Theorem A, Theorem C has also a few other applications. For instance, one can now obtain for homeomorphisms a result known before only for diffeomorphisms (see [AZ14]):

THEOREM D. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ be a transitive homeomorphism.*

- (1) *If $\rho(f)$ has nonempty interior, then f is topologically mixing.*
- (2) *Furthermore, if f has a lift \widehat{f} with $\vec{0} \in \text{int}(\rho(\widehat{f}))$, then \widehat{f} is also topologically mixing.*

Finally, Theorem C also solves a relevant question (Question G from [KT13]). It was known that, if f is a non-wandering isotopic-to-the-identity homeomorphism whose rotation set has nonempty interior, then the lifted diameter of any periodic open topological disk (usually thought as elliptic islands) is bounded, and the bound only depended on the period of the disk. But it was not known if the bound could be independent of the period, except in the $\mathcal{C}^{1+\alpha}$ setting where it was proven in [AZ14].

THEOREM E. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ be a non-wandering homeomorphism whose rotation set has nonempty interior. Then, there exists $M > 0$ such that the lifted diameter of any periodic topological disk is less than or equal to M .*

We will say a homeomorphism f is under the General Hypothesis if it satisfies the hypothesis of Theorem A.

4. Organization of the article

Looking at the big picture, the proof of Theorem A, can be split into two big parts. In the first one we will build a factor g , by collapsing dynamically bounded continua for f (assuming that even makes sense). This is done in Sections 3 through 5. The second part (Sections 6 through 9) is dedicated to fully understanding the dynamics of g and recovering every piece for the proof of Theorem A. Let us now comment on the role of each section.

As usual, Chapter 2 is devoted to settle notation and explain some preliminary results and techniques we will use later.

In Chapter 3 we introduce the notion of *canonically foliated strip*, which is the notion of the article and will aid us throughout the whole work. In Chapter 4 we already use it to develop *anchoring* techniques for inessential continua, to prove that there is a uniform bound for the diameter of dynamically bounded

continua for f , which is Theorem C (we also obtain Theorem E as a result). The whole existence of the factor g relies on this theorem: this is showed in Chapter 5, where we build the mentioned factor and immediately obtain some basic dynamical properties for it:

- There exists a lift \hat{g} of g such that $\rho(\hat{g}) = \rho(\hat{f})$,
- g is infinitely continuum-wise expansive,
- g is tight.

In Chapter 6 we prove some topological and dynamical properties for stable and unstable sets of points, which we use in the following three sections. In Chapter 7 we make a refinement of the anchoring arguments from Chapter 4 which we call *total anchoring*, and use it to prove that g is rotationally mixing.

Sections 8 and 9 are a little bit technical, and dedicated to proving the density of Markovian horseshoes for g : in Chapter 8 we introduce the notion of *heteroclinic pseudo-rectangle* using stable and unstable sets, and prove their density in 8.9; in Chapter 9 we show that inside any heteroclinic pseudo-rectangle there exists a Markovian horseshoe, using the total anchoring we previously developed.

The article finishes in Chapter 10, where we show that up to choosing a topological model for the quotient torus, g is area-preserving. This allows us to prove Theorem A, which we also do in this section.

5. Some open questions

Here are some questions which appeared with the development of the techniques used in this article.

- (1) Can the uniform bound M for the diameter of dynamically bounded continua, be in turn bounded by only using some dynamical invariants of f ? Our construction heavily depends on the choice of an isotopy I from the identity to f , and a transverse foliation \mathcal{F} (see Definition 2.20 for details).
- (2) Can we give a lower bound for the period for f , of the realizing points built in Proposition 3.5, using the size of the rotation set? This could yield lower bounds for the topological entropy of f , and (more subtly) for the topological entropy carried by continua.
- (3) Which conditions on the realizability of rotation sets, imposes the fact that g is area-preserving?
- (4) Consider the action of f in the fine curve graph of the torus (see Definition 2.44). Can we use it to fully describe the action of g ?
- (5) Suppose on top of the General Hypotheses that f is nonwandering: in which case it is also strictly toral. Is the set of periodic points dense in $\text{Ess}(f)$? (see Definitions 2.14 and 2.15).
- (6) Even simpler: when f is transitive (and therefore $\text{Ess}(f) = \mathbb{T}^2$), is it true that f has a dense set of periodic points? (Note that the answer is indeed positive for the factor g). This was the very first motivating question we faced in the production of this article, and there are no proofs known to us, even for the \mathcal{C}^∞ , area-preserving case.

- (7) Let us go to the context of cloed orientable hyperbolic surfaces S_g , of genus $g \geq 2$. It seems reasonable to search for an analogous semiconjugacy to the one in Theorem A. Which conditions on the homological or homotopical rotation sets (See Definitions 2.42 and 2.43), would allow us to replicate the anchoring techniques from Chapter 4 and get a uniform bound for the diameter of dynamically bounded continua, as in Theorem C? We imagine that a condition as the one in the main result in [?] is sufficient, but this can probably be improved.
- (8) A very recent pre-print by Militon [?] provides, for a homeomophism g of a surface S of genus ≥ 2 homotopic to a pseudo-Anosov (pA) map, a pártition of the S into stable sets as well as a partition of the surface into unstable sets, which immitates the well known foliations of a pA map. It appears that, in that context, a continuum K is dinamically bounded if and only if it is contained in a connected component of the intersection of a stable set and an unstable set. Is this sufficient to obtain a result as Theorem C in this case? And does this imply that a factor with similar properties exists?

CHAPTER 2

Preliminaries

1. Basic notation

Surfaces. Throughout this paper, a *surface* S will be a two dimensional orientable topological manifold. We will say that S is respectively a *plane*, *annulus*, *sphere* or *torus* if it is homeomorphic to \mathbb{R}^2 , $\mathbb{R}^2 \setminus \{0\}$, \mathbb{S}^2 or \mathbb{T}^2 . A set $U \subset S$ will be a *disk* if it is homeomorphic to $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

Homeomorphisms. All maps considered in this work are continuous. Moreover, we will say that a map $f : S \rightarrow S$ is a *homeomorphism* if f is bijective, and both f and its inverse are continuous.

DEFINITION 2.1. Given two surfaces S, S' and two endomorphisms $f : S \rightarrow S$ and $f' : S' \rightarrow S'$ we will say that f and f' are topologically conjugate if there exists a homeomorphism $h : S \rightarrow S'$ such that $hf = gh$.

Covering spaces. Given two surfaces S and \tilde{S} , we will say that $\tilde{\pi} : \tilde{S} \rightarrow S$ is a *covering projection* if every point $z \in S$ has a neighbourhood U , such that $\tilde{\pi}^{-1}(U)$ is a disjoint union of disks \tilde{D}_i in \tilde{S} , such that for every value of i , we have that $\tilde{\pi}|_{\tilde{D}_i}$ is a homeomorphism.

Under these conditions, we will say that \tilde{S} is a *covering space* of S , and it will be the *universal covering* if it is simply connected. It is very well known that the universal covering of a surface is unique up to homeomorphisms, and it is homeomorphic to \mathbb{R}^2 whenever $S \neq \mathbb{S}^2$. When there is no place for ambiguity, we will lighten notation and simply call $\tilde{\pi}$ and \tilde{S} a *covering*, indistinctively.

For the particular case when $S = \mathbb{A} \simeq \mathbb{R}^2 \setminus \{0\}$ is an annulus, we will take the universal covering \mathbb{A} which is a plane, and the covering projection will be $\check{\pi}(x, y) = (e^{-y}\cos(2\pi x), e^{-y}\sin(2\pi x))$. We will then have the deck transformation $T \in \text{Homeo}^+(\mathbb{A})$ defined by $T(x, y) = (x+1, y)$. Similarly, for the case of the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$, simply take the universal covering \mathbb{R}^2 with the natural projection, and we will have that the group of deck transformations will be generated by $T_x(x, y) = (x+1, y)$ and $T_y(x, y) = (x, y+1)$.

Given $f : S \rightarrow S$, and considering \tilde{S} the universal covering of S , we may take $\tilde{f} : \tilde{S} \rightarrow \tilde{S}$ a *lift* of f , as a map such that $f\tilde{\pi} = \tilde{\pi}\tilde{f}$ (we know that \tilde{f} is determined up to postcomposition with a deck transformation). Notice that, for the case in which S is hyperbolic, there exists a unique lift F of f , which commutes with the deck transformations.

Curves and foliations.

DEFINITION 2.2. Given two points $p, p' \in S$, a *path* or *curve* from p to p' will be a continuous function $\gamma : [0, 1] \rightarrow S$ such that $\gamma(0) = p$, $\gamma(1) = p'$. We will say the path is *simple* when γ is injective, and *closed* when $p = p'$.

We will say γ is respectively a *segment*, *line*, *circle* if it is homeomorphic to $[0, 1], \mathbb{R}, \mathbb{S}^1$ by a proper application h to \mathbb{R}^2 .

DEFINITION 2.3. Given a surface S , an *oriented topological foliation* \mathcal{F} is a partition of S in one-dimensional manifolds such that for each $p \in S$, there exists a neighbourhood U_p and a homeomorphism $h : U_p \rightarrow (-1, 1) \times (-1, 1)$ which preserves orientation and sends \mathcal{F} into the foliation by vertical lines, oriented from top to bottom.

Given $z \in S$, we define ϕ_z as the leaf going through z , similarly we define ϕ_z^+, ϕ_z^- as the semi-leaves starting at z , which respect the orientation.

Given a leaf ϕ of a singular oriented topological foliation in a surface S , we define $\alpha(\phi), \omega(\phi)$ as the usual α and ω limits of curves in surfaces.

Topological notions.

DEFINITION 2.4. Let $X \subset \mathbb{T}^2$ be a set, $U \subset \mathbb{T}^2$ an open set, and $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ a homeomorphism.

- An open set $U \subset \mathbb{T}^2$ is said to be *inessential* if every closed curve in U is homotopically trivial. X is said to be *inessential* when it has an inessential neighbourhood. For the case of connected sets, this is equivalent to Definition 1.2.
- X is *essential* when it is not inessential. Moreover, it is *fully essential* when $\mathbb{T}^2 \setminus X$ inessential.
- U is said to be *annular* if it is homeomorphic to an open annulus.
- We define the *filling* $\text{Fill}(X)$ of X , as the union of X with every inessential component of its complement. We will say that X is *filled* whenever $X = \text{Fill}(X)$.
- We will define the *orbit* of X as the union of the past and future iterates of X by the homeomorphism f .

DEFINITION 2.5. Let f be a uniformly continuous homeomorphism of the surface S . We will say that f is *mixing* if given two open sets $U, V \subset S$, there exists n_0 such that

$$f^j(U) \cap V \neq \emptyset, \quad \text{for every } n \geq n_0.$$

LEMMA 2.6. If f has a power f^j which is topologically mixing, then f is also topologically mixing.

PROOF. Let $U, V \subset S$ be two open sets. And let us write $V_k = f^{-k}(V)$, where $k > 0$. By hypothesis, there exists m_0 such that

$$(f^j)^m(U) \cap V_k \neq \emptyset, \quad \text{for every } 0 \leq k \leq j-1, m \geq m_0.$$

This means that we may take $n_0 = jm_0$ and obtain that

$$f^n(U) \cap V \neq \emptyset, \quad n \geq n_0,$$

which concludes the proof. \square

DEFINITION 2.7 (Hausdorff distance). Let (X, d) be a metric space, and let $K_1, K_2 \subset X$ be two nonempty subsets. We define the *Hausdorff distance* $d_H(K_1, K_2)$ between K_1 and K_2 as

$$d_H(K_1, K_2) = \inf\{r \geq 0 : K_2 \subset B(K_1, r), K_1 \subset B(K_2, r)\},$$

where

$$B(K, r) = \{x \in X : d(x, K) < r\}.$$

We recall that whenever X is compact, the metric space $(\mathcal{K}(X), d_H)$ of continua of X with the Hausdorff distance, is also compact.

The following definition precises when a curve crosses another one from left to right, or viceversa.

DEFINITION 2.8 (Intersection number). Given a surface S , and two curves $\gamma, \gamma' : [0, 1] \rightarrow S$ with a simple isolated intersection point $\gamma(t_0) = \gamma'(t'_0)$ in their interior, we will say that $\gamma \wedge \gamma' = 1$ whenever there exists a homeomorphism h from a neighbourhood of z to $(0, 1)^2$, such that $h(\gamma(t)) = (t, \frac{1}{2})$ and such that the vertical coordinate of $h \circ \gamma'$ is strictly increasing in a neighbourhood of t'_0 .

Note that this notion extends naturally for pairs of curves with finitely many transversal intersections, is bilinear and behaves as expected when concatenating curves.

DEFINITION 2.9 (Lines' relative position). Let $\lambda_0, \lambda_1, \lambda_2 \subset \mathbb{R}^2$ be oriented lines. We will say that λ_2 is above λ_1 relative to λ_0 (or equivalently, λ_1 is below λ_2 relative to λ_0), if

- The three lines are pairwise disjoint
- None of the lines separate the other two,
- If η_1, η_2 are two disjoint paths that respectively join $z_1 = \lambda_0(t_1)$ to $z'_1 \in \lambda_1$ and $z_2 = \lambda_0(t_2)$ to $z'_2 \in \lambda_2$, and the paths do not meet the three lines except at their endpoints, then $t_2 > t_1$.

Note that this definition only depends on the orientation of λ_0 .

2. Torus topological dynamics

DEFINITION 2.10. Let $f \in \text{Homeo}(\mathbb{T}^2)$. We will say that f is rotationally mixing if there exists a lift \widehat{f}^j of a power f^j to \mathbb{R}^2 , such that \widehat{f}^j is topologically mixing.

Let \mathbb{T}^2 be 2-dimensional torus, and let $f \in \text{Homeo}(\mathbb{T}^2)$. Then, the class $[f] \in \text{MCG}(\mathbb{T}^2)$ of homeomorphisms isotopic to f is determined by the action on the homology group, which for the torus is equivalent to the fundamental group and equal to \mathbb{Z}^2 . There is an extensive explanation on surface mapping class groups in [FM12].

This action is determined by the action of the homeomorphism on a fixed base of the fundamental group (we may very well take the canonical basis for this),

which in turn is defined by a 2×2 integral matrix. We then have a natural association

$$[f] \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The idea is obtain dynamical results from algebraic properties of this matrix. We know that we have two eigenvalues λ, λ^{-1} (its product must be one because if we take a lift \hat{f} to \mathbb{R}^2 , the image of a fundamental domain must have area equal to 1).

Take a look to the trace of this matrix. We know that $\text{tr}(M) = a+d = \lambda+\lambda^{-1}$. If $|\text{tr}(M)| < 2$, then both λ and λ^{-1} must belong to the unit complex ball (so that their product is equal to 1). Given that the trace is an integer, it can only be $-1, 0$ or 1 , each of these cases yielding exactly one possibility (up to permutation) for the pair λ, λ^{-1} . Easy computations for each of these cases we have that $[f^{12}] = \text{Id}$. We will say in this situation that f has *finite order*.

Note that we have covered every case with non-real eigenvalues. For the rest of the analysis, we may have that the trace is equal to or greater than 2. If $|\text{tr}(M)| = 2$, then either $\lambda = 1 = \lambda^{-1}$ or $\lambda = -1 = \lambda^{-1}$. For the first case, we have that

$$\text{either } [f] = \text{Id} \text{ or } [f] \text{ is conjugate to } \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

. Note that for this case we have $(1, 0)$ as an eigenvector. The second case is analogous,

$$\text{either } [f] = \text{Id} \text{ or } [f] \text{ is conjugate to } \begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}.$$

We again that $(1, 0)$ is an eigenvector, which is equivalent to saying that (up to isotopy), the closed curve associated to this element of the fundamental group is preserved by the homeomorphism. For these two cases we will say that f is *reducible*.

We finish the analysis with $|\text{tr}(M)| > 2$. We may assume that $|\lambda| > 1 > |\lambda^{-1}|$. We will write $\lambda^+ = \lambda$, $\lambda^- = \lambda^{-1}$ and note that our associated matrix has a diagonal form such that

$$[f] \mapsto \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix}.$$

By the Spectral Theorem, we know that this linear representative has two perpendicular eigenvectors v^+, v^- such that the foliations in these directions are respectively λ -expanded and λ -contracted. For this case we will say that $[f]$ is *Anosov*.

We are now able to state the central result from Nielsen-Thurston Theory, which is the surface homeomorphism classification for the much simpler case of the torus:

THEOREM 2.11 (Torus Homeomorphisms Classification). *Let f be a torus homeomorphism. Then one of the following holds:*

- *f has finite order: there exists a power f^n of f such that $f^n \sim \text{Id}$.*
- *f is reducible: there exists a closed curve class which is invariant by f .*

- f is isotopic to a linear Anosov homeomorphism.

3. Strictly Toral Dynamics

The results for this section come from [KT13]. The following notion tries to encode torus homeomorphisms whose dynamics can not be reduced to simpler subsurfaces. Throughout this section we will assume that f is a *non-wandering, isotopic-to-the identity* torus homeomorphism.

DEFINITION 2.12. We will say f is *irrotational* if there exists a lift \hat{f} in \mathbb{R}^2 such that $\rho(\hat{f}) = (0, 0)$.

DEFINITION 2.13. We will say that f is *annular* if there exists a lift \hat{f} of f , a vector $v \in \mathbb{Z}_*^2$ and some $M \in \mathbb{R}$, such that for every $\hat{z} \in \mathbb{R}^2$ and every $j \in \mathbb{Z}$, we have that

$$-M < \langle \hat{f}^j(\hat{z}) - \hat{z}, v \rangle < M,$$

that is, the displacement of any orbit in one particular rational direction is uniformly bounded.

It can be proved in this case that there exists a finite covering of \mathbb{T}^2 by another torus, such that the corresponding lift of f acting in this torus has an invariant annular set (see for example [Jä08][Remark 3.10]).

DEFINITION 2.14 (Strictly toral, [KT13]). Let $f \in \text{Homeo}_0(\mathbb{T}^2)$, and assume that f non-wandering. We will say that f is *strictly toral* if neither of the following properties hold:

- (1) There exists a power f^k of f such that $\text{Fix}(f^k)$ is fully essential, and such that f^k is irrotational,
- (2) There exists a power f^k of f such that f^k is annular

Note that the first item is essentially saying that f is *planar*, in the sense that, up to taking a power of f , understanding its dynamics is as difficult as understanding plane dynamics (i.e. the return maps for each disk in the complement of the set of fixed points). For the annular case, up to a finite covering we have an invariant annular set, whose complement is also annular, and therefore it suffices to understand annulus dynamics.

Let us now explain how the dynamics in the complement of these two cases is rightfully called strictly toral.

DEFINITION 2.15. We define the set $\text{Ine}(f)$ of *inessential points* (or simply *inessential set*) of f , as

$$\text{Ine}(f) = \{z \in \mathbb{T}^2 : \exists \text{ a neighbourhood } U \text{ of } z \text{ such that the orbit of } U \text{ is inessential}\}.$$

We define $\text{Ess}(f)$ the *set of essential points* of f as

$$\text{Ess}(f) = \mathbb{T}^2 \setminus \text{Ine}(f)$$

Note that both of these sets are invariant. Moreover, we have by [KT13, Theorems A and B] that, if f is strictly toral, then $\text{Ine}(f)$ is an open inessential set, which is in turn the union of bounded periodic disks, whose diameter is uniformly bounded if we look at the set of disks with a fixed period q . Therefore, $\text{Ess}(f)$ is a fully essential closed set. We also know from [KT13] that

REMARK 2.16. If f is strictly toral, then f is transitive $\iff \text{Ess}(f) = \mathbb{T}^2$.

Note that if the rotation set $\rho(f)$ has nonempty interior, then f is strictly toral.

A topological model for the Chirikov-Taylor Standard map. Strictly toral maps have been extensively studied by mathematicians and physicists with different approaches, as they appear naturally from several physical dynamical systems, the most well-known being the Chirikov-Taylor Standard map (see for example [Chi79]), defined as $S(x, y) = (\bar{x}, \bar{y})$, where

$$\begin{aligned}\bar{y} &= y + k \sin(x), \\ \bar{x} &= x + y + k \sin(x),\end{aligned}$$

which is obtained as the return map for the quantum periodically kicked rotator. In the context of KAM theory, the set $\text{Ine}(f)$ is called the set of *elliptic islands*, and $\text{Ess}(f)$ is known as the *chaotic sea*. Very little is understood for the properties of $\text{Ess}(f)$, appearing for large values of k . Another examples of physical relevance can be found in [LKFA90].

4. Equivariant Brouwer Theory

This theory was developed by Le Calvez, and generalizes the foliations by Brouwer lines for fixed-point-free orientation-preserving homeomorphisms of the plane.

DEFINITION 2.17. Given a closed surface and an isotopy $I : [0, 1] \times S \rightarrow S$, we will define the set $\text{Fix}(I)$ of fixed points of I by

$$\text{Fix}(I) = \{z \in S : I(t, z) = z \text{ for every } t \in [0, 1]\}.$$

We will also define $\text{Dom}(I) = S \setminus \text{Fix}(I)$.

DEFINITION 2.18. Given $f : S \rightarrow S$ and an isotopy $I : [0, 1] \times S \rightarrow S$ from the identity to f , we will say that I is *maximal* if for every $z \in \text{Fix}(f) \cap \text{Dom}(I)$, we have that the path $I_z(t)$ is not homotopically trivial in $\text{Dom}(I)$.

The following result is due to Béguin, Crovisier and Le Roux, and can be found in [BCLR20].

THEOREM 2.19 ([BCLR20]). *Given an isotopy I' from the identity to a homeomorphism f of a closed surface S , there exists a new isotopy I which is homotopic to I' with fixed endpoints (seen as paths in the space of homeomorphisms of S) such that I is a maximal isotopy.*

This allows us to use the equivariant version of the Brouwer-Le Calvez transverse foliation built in [LC05b] and work with the following object:

DEFINITION 2.20 (Maximal dynamically transverse decomposition). Let f be a homeomorphism of the surface S which is isotopic to the identity, by the maximal isotopy I . We will say that a singular oriented foliation \mathcal{F} of S is dynamically transverse to I if

- $\text{Sing}(\mathcal{F}) = \text{Fix}(I)$
- Every path by the isotopy $\gamma_z(t) : [0, 1] \rightarrow \text{Dom}(I)$, $\gamma_z(t) = I(t, z)$ has a representative $\gamma'_z : [0, 1] \rightarrow \text{Dom}(I)$ which is homotopic to γ with fixed endpoints in $\text{Dom}(I)$, such that \mathcal{F} is positively transverse to γ'_z (i.e. the new path only crosses leaves from right to left).

In this context, we will say that (f, I, \mathcal{F}) is a maximal dynamically transverse decomposition (MDTD) for f . The desired foliation appears as the projection of an equivariant version of the plane foliation by Le Calvez for the universal covering \tilde{M} of $M = \text{Dom}(I)$, that is, we first obtain a maximal dynamically transverse decomposition $(\tilde{f}, \tilde{I}, \tilde{\mathcal{F}})$ in the universal covering of $\text{Dom}(I)$ (where $\tilde{\mathcal{F}}$ is a non-singular equivariant foliation), and then we project it to obtain (f, I, \mathcal{F}) .

We will define $I_{\mathcal{F}}^t(z) = \gamma'_z(t)$ the transverse path for z , which is defined for times in $[0, 1]$, and we will simply write $I_{\mathcal{F}}(z)$ for said path. We will extend this map in the usual way for other values of t , by concatenating respective transverse paths of iterates by f :

$$I_{\mathcal{F}}^n(z) = \prod_{0 \leq j \leq n-1} I_{\mathcal{F}}(f^j(z)), \quad I_{\mathcal{F}}^{\mathbb{Z}}(z) = \prod_{j \in \mathbb{Z}} I_{\mathcal{F}}(f^j(z))$$

We can also lift these transverse paths to the universal covering of the surfaces we are working in.

We will write ϕ to denote leaves in \mathcal{F} , and similarly $\tilde{\phi}$ for their lifts in $\tilde{\mathcal{F}}$. Given a transverse path $\gamma : [a, b] \rightarrow \text{Dom}(I)$ and $t \in [a, b]$, we will write $\phi_{\gamma(t)}$ to denote the leaf such that $\gamma(t) \in \phi_{\gamma(t)}$ (and similarly for their respective lifts).

5. Forcing Theory

The Theory of Forcing for transverse trajectories has less than ten years of existence, and has been developed in [LCT18], and [LCT22], the latter having been published in 2022. It has already yielded several strong results in the description of surface topological dynamics, and uses the elements from Definition 2.20.

We say that two transverse paths $\tilde{\gamma}, \tilde{\gamma}'$ whose images are in $\text{Dom}(\tilde{I})$ are equivalent if they have lifts to \tilde{M} that start and finish at the same leaf.

We say a path $\tilde{\gamma} : [a, b] \rightarrow \tilde{M}$ from $\tilde{\phi}_{\tilde{\gamma}(a)}$ to $\tilde{\phi}_{\tilde{\gamma}(b)}$ is *admissible of order n* if it is equivalent to a transverse path $\tilde{I}_{\tilde{\mathcal{F}}}^n(\tilde{\gamma})$, that is, if

$$\tilde{f}^n(\tilde{\phi}_{\tilde{\gamma}(a)}) \cap \tilde{\phi}_{\tilde{\gamma}(b)} \neq \emptyset.$$

We say a path $\gamma : [a, b] \rightarrow \text{Dom}(I)$ is *admissible of order n* if it has a lift that is admissible of order n .

The following definition comes from [LCT18, Section 3].

DEFINITION 2.21 ($\tilde{\mathcal{F}}$ -transverse intersection, [LCT18]). Let $\tilde{\mathcal{F}}$ be an oriented foliation of the plane and let $(\tilde{f}, \tilde{I}, \tilde{\mathcal{F}})$ be a dynamically transverse decomposition. Let $\tilde{\gamma}_1 : J_1 \rightarrow \mathbb{R}^2$, $\tilde{\gamma}_2 : J_2 \rightarrow \mathbb{R}^2$ be two transverse paths such that $\tilde{\phi}_{\gamma_1(t_1)} = \tilde{\phi}_{\gamma_2(t_2)} = \tilde{\phi}$.

We will say that $\tilde{\gamma}_1$ intersects $\tilde{\gamma}_2$ $\tilde{\mathcal{F}}$ -transversely and positively (or equivalently, that $\tilde{\gamma}_2$ intersects $\tilde{\gamma}_1$ $\tilde{\mathcal{F}}$ -transversely and negatively) if there exist $a_1, b_1 \in J_1$ with $a_1 < t_1 < b_1$, and $a_2, b_2 \in J_2$ with $a_2 < t_2 < b_2$, such that

- $\tilde{\phi}_{\tilde{\gamma}_2(a_2)}$ is below $\tilde{\phi}_{\tilde{\gamma}_1(a_1)}$ relative to $\tilde{\phi}$,
- $\tilde{\phi}_{\tilde{\gamma}_2(b_2)}$ is above $\tilde{\phi}_{\tilde{\gamma}_1(b_1)}$ relative to $\tilde{\phi}$.

In this context we can also say that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have an $\tilde{\mathcal{F}}$ -transverse intersection

We will say that two transverse paths γ_1, γ_2 intersect \mathcal{F} -transversally whenever they have lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ that intersect $\tilde{\mathcal{F}}$ -transversally.

If two lifts of γ_1 have a $\tilde{\mathcal{F}}$ -transverse intersection, we will say that γ_1 has a transverse self-intersection.

The fundamental result of [LCT18] is without a doubt Proposition 20, which roughly says we may concatenate pieces of transverse trajectories, whenever these paths intersect $\tilde{\mathcal{F}}$ -transversally:

PROPOSITION 2.22 (Forcing, [LCT18]). *Let $\gamma_1 : [a_1, b_1] \rightarrow S$, $\gamma_2 : [a_2, b_2] \rightarrow S$ be two admissible transverse paths, which are respectively admissible of order n_1 and n_2 , and intersect \mathcal{F} -transversally at $\gamma_1(t_1) = \gamma_2(t_2)$. Then,*

- $\gamma_1|_{[a_1, t_1]} \gamma_2|_{[t_2, b_2]}$ and $\gamma_2|_{[a_2, t_2]} \gamma_1|_{[t_1, b_1]}$ are both admissible of order $n_1 + n_2$.
- Furthermore, either one of these paths is admissible of order $\min(n_1, n_2)$, or both are admissible of order $\max(n_1, n_2)$.

We will use [LCT18, Propositions 7 and 26], and [LCT22, Theorem M] to build the canonically foliated strips in Chapter 3. For the reader who wants some insight on how these techniques work, going through [LCT18, Sections 3 and 4], could be a useful practice.

6. Topological and Markovian horseshoes

Here we introduce two objects with horseshoe-like dynamics, the former is easier to detect, the latter yields stronger consequences.

Topological horseshoes. We start by defining a purely topological notion of horseshoe, introduced by Kennedy and Yorke in [KY01].

DEFINITION 2.23. Let C be a compact set of a surface S , and let C_0, C_1 be two fixed disjoint compact subsets of C , each of them intersecting each connected component of C . We will say that a continuum K of C is a *connection* if it intersects both C_0 and C_1 .

We will say that a continuum K' of C is a *preconnection* (for the homeomorphism f) when $f(K')$ is a connection.

DEFINITION 2.24. We will say that the homeomorphism f has *crossing number* equal to m for the compact set C , if m is the greatest integer for which there exist $C_0, C_1 \in C$ such that each connection contains m disjoint preconnections.

THEOREM 2.25 ([KY01]). *Suppose that a power f^j of a homeomorphism f of the surface S has crossing number $m > 1$ for the locally connected, compact set $C \subset S$. Then, there exists a compact subset $Q \subset C$, such that $f^j|_Q$ is semiconjugate to the Bernoulli shift in m symbols.*

In particular, $h_{\text{top}}(f)$ is positive.

In this case, we will say that Q is a topological horseshoe in m symbols for f , with period j .

This notion generalizes the classical Smale's Horseshoe, which satisfies the hypothesis of Theorem 2.25, see Figure 1 for a brief explanation.

Note that although we obtain positive topological entropy, we do not recover the existence periodic points for f as we can not make sure that the preimages by the semiconjugacy of periodic points for the Bernoulli shift, contain periodic points for f .

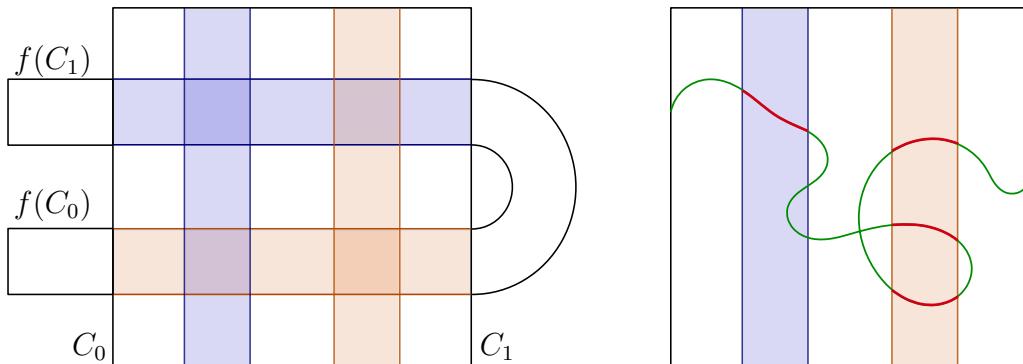


FIGURE 1. For the classical Smale Horseshoe, every connection (green) has at least two respective subcontinua (red) connecting the vertical sides of the blue and orange regions, which implies it has at least two disjoint preconnections.

Markovian horseshoes. Stronger notions of horseshoe than the one in [KY01] are described for example in [PPS18], [LCT22] and [GM22, Section 9.2]. The notion used in [GM22] is slightly stronger and adapted to the particular context of hyperbolic surface homeomorphisms.

This will be our case for the surface $S = \mathbb{R}^2 \setminus \text{Dom}(\widehat{I})$, whenever $\text{Dom}(\widehat{I})$ is nonempty. Furthermore, be aware that although this notion is called *rotational horseshoe* when defined, in our context the horseshoes will rotate around singularities of $\widehat{\mathcal{F}}$, which is not itself rotation in our original torus.

DEFINITION 2.26. We will say a compact connected set $R \subset S$ is a *rectangle* if it is homeomorphic to $[0, 1]^2$ by a homeomorphism $h : [0, 1]^2 \rightarrow h([0, 1]^2) \subset S$. We will call *sides* of R , the image of the sides of $[0, 1]^2$ by h , the *horizontal sides* being $R^- = h([0, 1] \times \{0\})$ and $R^+ = h([0, 1] \times \{1\})$.

DEFINITION 2.27. Let $R_1, R_2 \subset S$ be two rectangles. We will say that $R_1 \cap R_2$ is a *Markovian intersection* if there exists a homeomorphism h from a neighbourhood of $R_1 \cup R_2$ to an open subset of \mathbb{R}^2 , such that

- $h(R_2) = [0, 1]^2$;
- Either $h(R_1^+) \subset \{(x, y) \in \mathbb{R}^2 : y > 1\}$ and $h(R_1^-) \subset \{(x, y) \in \mathbb{R}^2 : y < 0\}$;
or $h(R_1^+) \subset \{(x, y) \in \mathbb{R}^2 : y < 0\}$ and $h(R_1^-) \subset \{(x, y) \in \mathbb{R}^2 : y > 1\}$;
- $h(R_1) \subset \{(x, y) \in \mathbb{R}^2 : y > 1\} \cup [0, 1]^2 \cup \{(x, y) \in \mathbb{R}^2 : y < 0\}$.

DEFINITION 2.28 (Markovian horseshoe). [GM22]. Let f be a homeomorphism of a hyperbolic surface S , let \tilde{f} its lift to the universal covering \tilde{S} , and let $m > 1$ be an integer.

We will say that f has a *Markovian horseshoe* in m symbols with deck transformations W_1, \dots, W_m , if there exists a rectangle $\tilde{R} \subset \tilde{S}$, and some $j \geq 1$ such that

for every $1 \leq i \leq m$, the intersections $W_i(\tilde{R}) \cap \tilde{f}^j(\tilde{R})$ are Markovian.

In this context, we will say that j is the period of the horseshoe. Whenever \tilde{R} projects injectively by the covering projection, we may also say that R contains a Markovian horseshoe in m symbols for f . In this case, the set

$$\bigcap_{n \in \mathbb{Z}} f^{nj}(R)$$

is the Markovian horseshoe, and j is again its period.

For the purpose of this work, we will adapt this notion and say that a homeomorphism $f \in \text{Homeo}_0(\mathbb{T}^2)$ has a Markovian horseshoe if that happens for the restriction $\widehat{f^k}|_{\text{Dom}(\widehat{I})}$ of a lift of some power f^k , where \widehat{I} is taken as in 2.20.

The proof of the following result can be found in [GM22].

PROPOSITION 2.29. *Let $f \in \text{Homeo}(S)$. Suppose that f has a Markovian horseshoe X of period j contained in the rectangle R , and with associated deck transformations W_1, \dots, W_m ($m \geq 2$), which generate a free group. Then, there exists a compact subset $\tilde{Y} \subset \tilde{S}$ such that*

- *The map $f^j|_X$ has an extension $\tilde{g} : \tilde{Y} \rightarrow \tilde{Y}$, which in turn is an extension of the Bernoulli shift $\sigma : \{1, \dots, m\}^{\mathbb{Z}} \rightarrow \{1, \dots, m\}^{\mathbb{Z}}$ in m symbols by the semiconjugacy h .*

In particular, the following diagram commutes:

$$\begin{array}{ccc} \{1, \dots, m\}^{\mathbb{Z}} & \xrightarrow{\sigma} & \{1, \dots, m\}^{\mathbb{Z}} \\ h \uparrow & \tilde{g} & \uparrow h \\ \tilde{Y} & \xrightarrow{\quad} & \tilde{Y} \\ \pi \downarrow & f^j & \downarrow \pi \\ X & \xrightarrow{\quad} & X \end{array}$$

- *the preimage by h of every periodic sequence by the shift σ , contains a periodic point of g (which then projects to a periodic point of f^j).*

REMARK 2.30. If $f \in \text{Homeo}(S)$ has a Markovian horseshoe X of period j in m symbols, then $h_{\text{top}}(f) \geq \frac{\log(m)}{j}$. In particular, the existence of a Markovian horseshoe for f implies positiveness of topological entropy of f .

Note that the existence of a Markovian horseshoe implies the existence of a topological one. The Markovian ones behave better, in particular

REMARK 2.31. Markovian horseshoes coming from deck transformations forming a free group yield exponential growth of the amount of periodic orbits, as the period goes to infinity. Topological horseshoes do not ensure the existence of periodic points.

7. Prime End Theory

We will use classical Prime End Theory by Carathéodory in order to prove the density of horseshoes, in Chapter 9. An exhaustive construction of the definitions and results we exhibit can be found for example in [Mil06, Chapter 17] or [CL66, Chapter 9], we include the results we will use later for the sake of completeness.

Let us take $\mathcal{O} \subset \mathbb{S}^2$ a simply connected open set, whose complement has more than one point. By the Riemann mapping theorem, it is holomorphic to the canonical disk $\mathbb{D} = B(0, 1) \subset \mathbb{R}^2$.

DEFINITION 2.32. We will say $\gamma : (0, 1) \rightarrow \mathcal{O}$ is a *crosscut* if it is homeomorphic to $(0, 1)$, and its closure is homeomorphic to $[0, 1]$, with its two endpoints in $\partial\mathcal{O}$.

Using the Jordan Curve Theorem, we know that every crosscut γ separates \mathcal{O} into two connected components. Each component N of $\mathcal{O} \setminus \gamma$ will be called a *cross-section*, which holds $\gamma = U \cap \partial N$.

DEFINITION 2.33. A *fundamental chain* $\mathcal{N} = \{N_j\}_{j \in \mathbb{N}}$ will be a decreasing sequence ($N_{j+1} \subset N_j$ for every j) of cross-sections $N_i \subset \mathcal{O}$, such that the diameter of the corresponding crosscuts tends to 0 as j tends to infinity.

Two fundamental chains $\mathcal{N} = \{N_j\}_{j \in \mathbb{N}}$ and $\mathcal{N}' = \{N'_j\}_{j \in \mathbb{N}}$ will be equivalent if every N_j contains some N'_k and conversely every N'_j contains some $N_{k'}$.

DEFINITION 2.34. A *prime end* \check{z} for the pair $(\text{cl}(\mathcal{O}), \partial\mathcal{O})$ is an equivalence class of fundamental chains.

DEFINITION 2.35. We define the *impression* $\text{Imp}(\check{z}) \subset \text{cl}(\mathcal{O})$ of a prime end \check{z} as

$$\bigcap_{j \in \mathbb{N}} \text{cl}(N_j),$$

where $\{N_j\}$ is any fundamental chain which defines \check{z} .

REMARK 2.36. For every prime end \check{z} , we have that $\text{Imp}(\check{z})$ is a compact connected subset of $\partial\mathcal{O}$.

DEFINITION 2.37. We define the *Caratheodory compactification* $\check{\mathcal{O}}$ of \mathcal{O} as the disjoint union of \mathcal{O} and the set of prime ends of \mathcal{O} , with the topology having $\mathcal{O} \cup \mathcal{N}$ as a base, where

- \mathcal{O} is the set of open subsets of \mathcal{O} ,
- $\mathcal{N} = \bigcup \underline{N}$, where \underline{N} is the union of any cross-section N with the set of prime ends which are given by a fundamental chain $\{N_j\}$ with $N_1 \subset N$.

THEOREM 2.38 (Caratheodory compactification). *Any conformal isomorphism $h : \mathbb{D} \rightarrow \mathcal{O}$ extends uniquely to an isomorphism from $\check{\mathbb{D}} = \text{cl}(\mathbb{D})$ to $\check{\mathcal{O}}$.*

We will write \check{z} for points in $\check{\mathbb{D}}$, note that the set of prime ends is then homeomorphic to $\partial\check{\mathbb{D}} \simeq \mathbb{S}^1$.

We will say a curve $\eta : [0, 1] \rightarrow \mathcal{O}$ lands at $z \in \partial\mathcal{O}$ if we can extend η continuously in $\text{cl}(\mathcal{O})$ with $\gamma(1) = z$.

We will say $z \in \partial\mathcal{O}$ is accessible if there exists a curve in \mathcal{O} which lands at z . Note that any neighbourhood U of a point z in \mathcal{O} intersects both \mathcal{O} and its complement. By drawing a curve inside U between points in each of these components, and which does not go through z , we obtain the following result.

REMARK 2.39. The set of accessible points is dense in $\partial\mathcal{O}$.

For this paper, we will say a prime end $\check{z} = e^{i\theta}$ is accessible, if there exists a simple curve $\check{\gamma} : [0, 1] \rightarrow \mathbb{D}$ with $\check{\gamma}|_{[0,1]} \subset \mathbb{D}$, $\check{\gamma}(1) = e^{i\theta}$, which holds that

$$h(\check{\gamma}) \text{ lands at some } z \in \partial\mathcal{O}$$

THEOREM 2.40 (17.4, [Mil06]). *The set of accessible prime ends has full Lebesgue measure in \mathbb{S}^1 . In particular, it is dense.*

8. Higher genus Rotation Theory

We have studied the torus case in Chapter 1. For closed orientable hyperbolic surfaces $S = S_g$, of genus $g \geq 2$, the study of rotation sets for homeomorphisms is very much different: the geometry is now hyperbolic so geodesics drift exponentially, and the fundamental group of the surface is not abelian anymore. Thus, we get two different rotation notions: a homological and a homotopical one.

Homological rotation. Fix a bounded fundamental domain D for the action of the fundamental group Γ by deck transformations in the universal covering \tilde{S} . Let \tilde{f} be the only lift of f which commutes with the deck transformations. For each $z \in S$, let \tilde{z} is the lift of z in D , and define $a_z \in \Gamma$ as the only deck transformation such that $a_z^{-1}\tilde{f}(\tilde{z}) \in D$. For $a \in \Gamma$, we will denote $[a] \in H_1(S, \mathbb{R})$ its homology class.

DEFINITION 2.41. Given an f -ergodic probability measure μ , the *homological rotation vector* of μ is

$$\rho_{H_1}(\mu) := \int_S [a_y] d\mu(y) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} [a_{f^i(x)}], \quad \text{for } \mu - \text{almost every } x \in S.$$

DEFINITION 2.42. The *homological rotation set* $\rho_{H_1}(f)$ of f is the set of vectors $r \in H_1(S, \mathbb{R})$ such that there exists $(x_k)_k \in S^{\mathbb{N}}$ and $(n_k)_k \in \mathbb{N}^{\mathbb{N}}$ with $\lim_{k \rightarrow +\infty} n_k =$

$+\infty$ and such that

$$\lim_{k \rightarrow +\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} [a_{f^i(x_k)}] = r.$$

The *ergodic (homological) rotation set* $\rho_{H_1}^{\text{erg}}(f)$ of f is

$$\rho_{H_1}^{\text{erg}}(f) = \{\rho_{H_1}(\mu) \mid \mu \text{ is } f\text{-ergodic}\}.$$

Sufficient homological-rotation conditions for positive topological entropy are found by Pollicott in [Pol92] and [Mat97]. Katok proves that for the case of flows the homological rotation set is totally isotropic. ([KH95]). Lellouch gets several results for this rotation set in his PhD work ([Lel19]), and Alonso, Brum and Passegi describe this rotation set for a generic set of homeomorphisms in [ABP20].

Homotopical rotation. Using the classical model, define the set of geodesics os \tilde{S} as $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$, and given a geodesic (α, ω) given by its ends, we define $\pi_{\alpha, \omega}(\tilde{x})$ as the orthogonal projection of \tilde{x} on $\pi_{\alpha, \omega}$.

DEFINITION 2.43. The *homotopical rotation set* $\rho(f)$ consists of the triples (α, ω, v) with $\alpha, \omega \in \partial \tilde{S} \simeq \mathbb{S}^1$, $v > 0$ such that there exist sequences $\{\tilde{x}_k\}_k \in \tilde{S}$ and $n_k \rightarrow +\infty$ such that

$$\lim_{k \rightarrow +\infty} \left(\tilde{x}_k, \tilde{f}^{n_k}(\tilde{x}_k), \frac{d(\pi_{\alpha, \omega}(\tilde{x}_k), \pi_{\alpha, \omega}(\tilde{f}^{n_k}(x_k)))}{n_k} \right) \rightarrow (\alpha, \omega, v).$$

Guilhéneuf and Militon obtain in [GM22] sufficient homotopical-rotation conditions for the existence of topological horseshoes. A slightly stronger notion called geodesic tracking is introduced by Lessa in [Les11]. Using both homological and homotopical arguments, Lessa, Militon and the author partially describe the shape of ergodic rotation sets in [GSGL23].

9. The Fine Curve Graph

A brand new object which yielded intensive research for surface homeomorphisms during the last five years is the fine curve graph, which was introduced by Bowden, Hensel and Webb in [BHW22], published in 2022.

DEFINITION 2.44 (Fine Curve graph, [BHW22]). Let S be a closed orientable surface of genus $g \geq 1$. Let the *fine curve graph* $\mathcal{C}^\dagger(S)$ be the graph whose vertices correspond to essential simple closed curves in S . Two such vertices are joined by an edge when the corresponding curves are disjoint. We denote by d^\dagger the distance in $\mathcal{C}^\dagger(S)$, where the length of each edge is equal to 1.

When S is the 2-torus we change the definition of the edges of $\mathcal{C}^\dagger(S)$: two vertices are joined by an edge when the corresponding curves intersect at most once.

In this work [BHW22] it is also proven that $\mathcal{C}^\dagger(S)$ is a Gromov hyperbolic space, taking the distance d^\dagger which is induced by the graph structure. A homeomorphism f of S acts naturally as an automorphism of this graph, in fact

$\text{Homeo}(S)$ is isomorphic to $\text{Aut}(\mathcal{C}^\dagger(S))$, combining the proofs by Long, Margalit, Pham, Verberne, Yao in [LMP⁺21], and by Le Roux and Wolff in [RW22]. Such an automorphism induces a well defined translation length

$$|f| = \lim_{k \rightarrow +\infty} \frac{1}{k} d^\dagger(f^k(x), x),$$

which lets us classify these automorphisms as either hyperbolic ($|f| > 0$), parabolic ($|f| = 0$ but no finite diameter orbits) or elliptic (f has finite diameter orbits). For the case of $\text{Homeo}_0(\mathbb{T}^2)$, results by Bowden, Hensel, Mann, Militon and Webb prove in [BHM⁺22] and by Guihéneuf and Militon in [GM23] uniquely determine this class for the action, only by looking at the rotation set and at sublinear deviations from it.

CHAPTER 3

Canonically foliated strips

The purpose of this chapter is to build the main tool for the proof of Theorem C, which is called a *canonically foliated strip*. Its construction uses some notions of Le Calvez-Tal Forcing Theory for surface homeomorphisms. Given our homeomorphism f under the hypothesis of Theorem C, we may assume up to taking a power and an adequate lift, that the origin $\vec{0} \in \rho(\hat{f})$ for some lift \hat{f} .

Let us settle notation for the remainder of the current and the following chapters. We will take a maximal dynamically transverse decomposition (f, I, \mathcal{F}) for f , where \mathcal{F} is a singular foliation with $\text{Sing}(\mathcal{F}) = \text{Fix}(I)$, and such that I lifts to an isotopy \hat{I} from the identity to \hat{f} . We may then take a lift of this decomposition to the universal covering \mathbb{R}^2 of \mathbb{T}^2 , obtaining $(\hat{f}, \hat{I}, \hat{\mathcal{F}})$. Restricting to our domain $\text{Dom}(\hat{I}) = \mathbb{R}^2 \setminus \text{Fix}(\hat{I})$, we may then lift again to the universal covering $\tilde{\mathbb{D}}$ of $\text{Dom}(\hat{I})$, and get the corresponding $(\tilde{f}, \tilde{I}, \tilde{\mathcal{F}})$, where \tilde{f} acts as a Brouwer homeomorphism on the topological plane $\tilde{\mathbb{D}}$.

Unless explicitly stated, we shall use for the remainder of the work the diacritic $\hat{}$ (i.e. \hat{f}) to denote objects in the universal covering \mathbb{R}^2 of \mathbb{T}^2 , and we shall use the diacritic $\tilde{}$ (i.e. \tilde{f}) to denote objects in the universal covering $\tilde{\mathbb{D}}$ of $\text{Dom}(\hat{I})$.

Be aware that we will develop the proofs without going to the -more frequent-universal covering of the punctured torus $\text{Dom}(I)$. In our context, for instance, the group of deck transformations acting in $\tilde{\mathbb{D}}$ is necessarily not finitely generated, as $\text{Dom}(\hat{I})$ has infinitely many punctures for each puncture of $\text{Dom}(I)$.

For the particular case of Proposition 3.5 we will use maximal dynamically transverse decompositions in two other auxiliary spaces, which will be explained in its proof.

DEFINITION 3.1 (Canonically foliated strip). Let \hat{f} be a lift to \mathbb{R}^2 of a homeomorphism $f \in \text{Homeo}_0(\mathbb{T}^2)$, and let $(\hat{f}, \hat{I}, \hat{\mathcal{F}})$ be an MDTD for \hat{f} . We will say that $\hat{A} \subset \mathbb{R}^2$ is a *canonically foliated strip* (CFS) for \hat{f} if

- (1) There exists $v \in \mathbb{Z}^2$, $\hat{z} \in \mathbb{R}^2$ and some $j > 0$, such that $\hat{f}^j(\hat{z}) = \hat{z} + v$, and such that \hat{A} is the saturated set of $\hat{A} \cap \hat{I}_{\hat{\mathcal{F}}}^{\mathbb{Z}}(\hat{z})$ by leaves of $\hat{\mathcal{F}}$.
- (2) There exists a homeomorphism $h : \hat{A} \rightarrow \mathbb{R}^2$, such that $h(\hat{f}|_{\hat{A}})$ is the foliation by vertical lines, oriented from top to bottom.

In this context, we will say that \hat{z} is a *realizing point* for \hat{A} .

REMARK 3.2. If \hat{A} is a CFS, there exists a non-null vector $v \in \mathbb{Z}^2$ such that $\hat{A} + v = \hat{A}$.

The goal is then to prove the following result.

PROPOSITION 3.3. *Let \widehat{f} be a lift to \mathbb{R}^2 of $f \in \text{Homeo}(\mathbb{R}^2)$, with $\vec{0} \in \text{int}(\rho(\widehat{f}))$. Then, there exist $j, p \in \mathbb{Z}^+$, and four families*

$$\widehat{\mathcal{A}}^\rightarrow = \{\widehat{A}_k^\rightarrow\}_{k \in \mathbb{Z}}, \quad \widehat{\mathcal{A}}^\uparrow = \{\widehat{A}_k^\uparrow\}_{k \in \mathbb{Z}}, \quad \widehat{\mathcal{A}}^\leftarrow = \{\widehat{A}_k^\leftarrow\}_{k \in \mathbb{Z}}, \quad \widehat{\mathcal{A}}^\downarrow = \{\widehat{A}_k^\downarrow\}_{k \in \mathbb{Z}}$$

of canonically foliated strips $\widehat{A}_k^\rightarrow, \widehat{A}_k^\uparrow, \widehat{A}_k^\leftarrow, \widehat{A}_k^\downarrow$, with respective realizing points $\widehat{z}_k^\rightarrow, \widehat{z}_k^\uparrow, \widehat{z}_k^\leftarrow, \widehat{z}_k^\downarrow$, such that

(1) For every $k \in \mathbb{Z}$, we have that

$$\widehat{f}^j(\widehat{z}_k^\rightarrow) = \widehat{z}_k^\rightarrow + (p, 0), \quad \widehat{f}^j(\widehat{z}_k^\uparrow) = \widehat{z}_k^\uparrow + (0, p),$$

$$\widehat{f}^j(\widehat{z}_k^\leftarrow) = \widehat{z}_k^\leftarrow + (-p, 0), \quad \widehat{f}^j(\widehat{z}_k^\downarrow) = \widehat{z}_k^\downarrow + (0, -p),$$

(2) For every pair of values $k, k' \in \mathbb{Z}$ with $k \neq k'$, we have that

$$\text{cl}(\widehat{A}_k^\rightarrow) \cap \text{cl}(\widehat{A}_{k'}^\rightarrow) = \emptyset, \quad \text{cl}(\widehat{A}_k^\uparrow) \cap \text{cl}(\widehat{A}_{k'}^\uparrow) = \emptyset,$$

$$\text{cl}(\widehat{A}_k^\leftarrow) \cap \text{cl}(\widehat{A}_{k'}^\leftarrow) = \emptyset, \quad \text{cl}(\widehat{A}_k^\downarrow) \cap \text{cl}(\widehat{A}_{k'}^\downarrow) = \emptyset.$$

Let us start with an elementary result regarding transverse foliations:

LEMMA 3.4. *If $\vec{0} \in \rho(\widehat{f})$, the diameter of leaves $\widehat{\phi} \in \widehat{\mathcal{F}}$ is uniformly bounded.*

PROOF. By Franks realization result in [Fra89], let us take $\widehat{z}_1, \widehat{z}_2, \widehat{z}_3$ three periodic points realizing different rotation directions and such that $\vec{0}$ belongs to the convex hull of the set containing the respective three rotation vectors. Let $\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3$ be their respective transverse paths for their whole orbits,

$$\widehat{\gamma}_j = \widehat{I}_{\widehat{\mathcal{F}}}^{\mathbb{Z}}(\widehat{z}_j), \quad 1 \leq j \leq 3.$$

which we will build as periodic by respective integer translations, given that they come from periodic orbits. Furthermore, note that these paths have a well defined slope, again because they come from periodic orbits (and the three slopes are different from each other). We may then take three sufficiently long pieces $\widehat{\gamma}'_1, \widehat{\gamma}'_2, \widehat{\gamma}'_3$ such that their concatenation defines a region \widehat{R} on its right (i.e. we obtain a *triangle* with its sides being positively and cyclically oriented), such that \widehat{R} contains a fundamental domain. Given that the used paths are positively transverse with respect to the foliation, we obtain that the future half-leaf of any point inside the region \widehat{R} , must also stay inside \widehat{R} , which proves that the diameter of these half-leaves is uniformly bounded since \widehat{R} contains a fundamental domain. The same argument reiterated implies the diameter of past half-leaves is also uniformly bounded, which completes the proof. \square

1. Construction

We will use this section to prove Proposition 3.3, by constructing the respective families. The following result is key for this construction.

PROPOSITION 3.5. *If $\vec{0} \in \text{int}(\rho(\widehat{f}))$, then*

(1) There exists $\widehat{z}^\rightarrow \in \mathbb{R}^2$ and two integers $p^\rightarrow, q^\rightarrow > 0$, holding $\widehat{f}^{q^\rightarrow}(\widehat{z}^\rightarrow) = \widehat{z}^\rightarrow + (p^\rightarrow, 0)$, and such that the transverse path $\widehat{I}_{\widehat{\mathcal{F}}}^{\mathbb{Z}}(\widehat{z}^\rightarrow)$ has no $\widehat{\mathcal{F}}$ -transverse self-intersections.

(2) Moreover, for every $q'^\rightarrow \geq q^\rightarrow$, there exists $\widehat{z}'^\rightarrow \in \mathbb{R}^2$ such that

$$\widehat{f}^{q'^\rightarrow}(\widehat{z}'^\rightarrow) = \widehat{z}'^\rightarrow + (p^\rightarrow, 0)$$

and such that the transverse path $\widehat{I}_{\widehat{\mathcal{F}}}^{\mathbb{Z}}(\widehat{z}'^\rightarrow)$ has no $\widehat{\mathcal{F}}$ -transverse self-intersections.

PROOF. (1). By the realization result in [Fra89], there exists \widehat{z}_0 such that $\widehat{f}^{q_0}(\widehat{z}_0) = \widehat{z}_0 + (1, 0)$. Let us take a dynamically transverse decomposition $(\widehat{f}, \widehat{I}, \widehat{\mathcal{F}})$.

Given the foliation $\widehat{\mathcal{F}}$ is equivariant by integer translations, we can project our decomposition to the annulus $\breve{\mathbb{A}} = \mathbb{S}^1 \times \mathbb{R}$, by taking $(\widehat{x}, \widehat{y}) \sim (\widehat{x}', \widehat{y}')$ whenever $\widehat{x}' - \widehat{x} \in \mathbb{Z}$, and obtain the corresponding $(\breve{f}, \breve{I}, \breve{\mathcal{F}})$. Let us then take \breve{z} as the projection to the annulus of a point $\widehat{z} \in \mathbb{R}^2$.

Let us define $\breve{\gamma}_0 : \mathbb{R} \rightarrow \breve{\mathbb{A}}$ by

$$\breve{\gamma}_0(t) = \breve{I}_{\breve{\mathcal{F}}}^{q_0 t}(\breve{z}_0)$$

Note that $\breve{\gamma}_0(0) = \breve{\gamma}_0(1)$, we may then think it as a loop in the annulus (with the same name), with domain equal to \mathbb{S}^1 . Note that it suffices to prove that $\breve{I}_{\breve{\mathcal{F}}}^{\mathbb{Z}}(\breve{z}^\rightarrow)$ has no $\breve{\mathcal{F}}$ -transverse self-intersection: this fact will then imply that $\widehat{I}_{\widehat{\mathcal{F}}}^{\mathbb{Z}}(\widehat{z}^\rightarrow)$ has no $\widehat{\mathcal{F}}$ -transverse intersection with any of its horizontal integer translates, which is on its turn stronger than our conclusion.

Compactifying the annulus in the classical way to get a sphere lets us apply Proposition 7 in [LCT18], from which we know that $\breve{\gamma}_0$ has an $\breve{\mathcal{F}}$ -transverse self-intersection if and only if $\breve{\gamma}_0|_{[0,2]}$ has an $\breve{\mathcal{F}}$ -transverse self-intersection.

Given the path $\breve{\gamma}_0$ is at a positive distance from the set $\text{Sing}(\breve{\mathcal{F}})$, we can assume up to taking an equivalent path, that $\breve{\gamma}_0|_{[0,1]}$ has a finite number ℓ_0 of self-intersections counted with multiplicity (without looking at $\breve{\mathcal{F}}$ -transversality). If $\breve{\gamma}_0|_{[0,2]}$ has no $\breve{\mathcal{F}}$ -transverse self-intersection, the proof is finished.

Let us then suppose that there exists an $\breve{\mathcal{F}}$ -transverse self-intersection at $\breve{\gamma}_0(s) = \breve{\gamma}_0(t)$, with $s, t \in [0, 2]$, $s < t$. In this case, we will build a process which finds $\breve{z}_1 \in \mathbb{R}^2$ (its projection to the annulus being \breve{z}_1), holding the following properties:

- There exists $q_1 > 0$ such that $\widehat{f}^{q_1}(\widehat{z}_1) = \widehat{z}_1 + (p_1, 0)$, where $p_1 \in \mathbb{Z}^+$,
- The transverse trajectory $\breve{I}_{\breve{\mathcal{F}}}^{q_1}(\breve{z}_1)$ is a subloop of $\breve{\gamma}_0$, which has at most $\ell_1 \leq \ell_0 - 1$ self-intersections counted with multiplicity (without looking at $\breve{\mathcal{F}}$ -transversality).

We shall use $\rho(\breve{\gamma}_0|_{[s,t]})$ to denote the integer such that, when taking lifts to \mathbb{R}^2 , we obtain

$$\widehat{\gamma}_0(t) = \widehat{\gamma}_0(s) + (\rho(\breve{\gamma}_0|_{[s,t]}), 0).$$

We define $\rho(\breve{\gamma}_0|_{[t,s+1]})$ in the same fashion. Note that one of these two numbers is positive, given their sum is equal to 1.

Let us write $\pitchfork_{\check{\mathcal{F}}}$ to denote $\check{\mathcal{F}}$ -transverse intersection between curves. Note that from

$$\check{\gamma}_0|_{[0,2]}(s) \pitchfork_{\check{\mathcal{F}}} \check{\gamma}_0|_{[0,2]}(t)$$

we derive that

$$\begin{aligned} \check{\gamma}_0|_{[1,3]}(s+1) &\pitchfork_{\check{\mathcal{F}}} \check{\gamma}_0|_{[0,2]}(t), \quad \check{\gamma}_0|_{[0,2]}(s) \pitchfork_{\check{\mathcal{F}}} \check{\gamma}_0|_{[-1,1]}(t-1), \\ \check{\gamma}_0|_{[1,3]}(s+1) &\pitchfork_{\check{\mathcal{F}}} \check{\gamma}_0|_{[-1,1]}(t-1) \end{aligned}$$

This means that the closed curve $\check{\gamma}_0|_{[-1,3]}$ which contains all the previous ones, has transverse intersections at the pairs of times $(s, t), (s+1, t), (s, t-1)$ and $(s+1, t-1)$.

Let us go to the universal covering $\check{\mathbb{D}}$ of $\text{Dom}(\check{I})$, and name $(\check{f}, \check{I}, \check{\mathcal{F}})$ to the corresponding dynamically transverse decomposition. Let $\check{\gamma}_0$ be a lift of $\check{\gamma}_0$ to this covering. By hypothesis on $\check{\gamma}_0$, we know there exists a deck transformation \check{T} such that there exists an $\check{\mathcal{F}}$ -transverse intersection at $\check{T}^j(\check{\gamma}_0(t)) = \check{T}^{j+1}(\check{\gamma}_0(s))$ for every $j \in \mathbb{Z}$.

Let us first assume that $t-s < 1$, the other case is analogous. Given that $\check{\gamma}_0|_{[-1,3]}$ is admissible of order $4q_0$, we know by [LCT22, Theorem M] the path $\check{\gamma}_0|_{[s,t]}$ is realized by a periodic point \check{z}_1 of period $q_1 = 4q_0$, and therefore by definition we have that its projection $\check{\gamma}_0|_{[s,t]}$ is realized by a periodic point $\check{z}_1 = \pi(\check{z}_1)$ of period $q_1 = 4q_0$.

If $\rho(\check{\gamma}_0|_{[s,t]})$ is positive, then we define \hat{z}_1 as any lift of \check{z}_1 to \mathbb{R}^2 . Note that

- $\hat{f}^{q_1}(\hat{z}_1) = \hat{z}_1 + \rho(\check{\gamma}_0|_{[s,t]})$
- $\check{I}_{\check{\mathcal{F}}}^{q_1}(\check{z}_1) = \check{\gamma}_0|_{[s,t]}$ is a subloop of $\check{\gamma}_0$, and therefore the number ℓ_1 of self-intersections counted with multiplicity, is strictly lower than ℓ_0 .

If $\rho(\check{\gamma}_0|_{[s,t]})$ is negative, we have that $\rho(\check{\gamma}_0|_{[t,s+1]})$ is positive, and the solution is analogous, because again by [LCT22, Theorem M] we get that the path $\check{\gamma}_0|_{[t,s+1]}$ is realized by a periodic point of period $q_1 = 4q_0$.

We are left with the case where $1 < t-s < 2$. In this case, note that $(t-1)-s < 1$, and again we will look at the two subpaths $\check{\gamma}_0|_{[s,t-1]}$ and $\check{\gamma}_0|_{[t-1,s+1]}$. Both are realized by periodic points of period $4q_0$, and one of them rotates positively in the annulus \check{A} , which will be our desired \check{z}_1 .

If \check{z}_1 has an $\check{\mathcal{F}}$ -transverse self intersection, this process can be iterated to obtain \hat{z}_2, q_2, ℓ_2 in the same fashion as before. Note that this process ends in at most $k \leq \ell_0$ iterations. This means that the point $\check{z}^\rightarrow = \hat{z}_k$ obtained after the k -th and last iteration of the process, must have a transverse path with no $\check{\mathcal{F}}$ -transverse self-intersections (otherwise we would be able to reiterate the process), and also holds

$$\hat{f}^{q_k}(\hat{z}_k) = \hat{z}_k + (p_k, 0), \text{ for some } q_k > 0,$$

which concludes the proof of the first item.

(2). Let $\check{z}^\rightarrow \in \mathbb{R}^2$ as obtained in the previous item, and let us take its projection \check{z}^\rightarrow to the annulus. Define its transverse path $\check{\gamma} : \mathbb{R} \rightarrow \check{A}$ by

$$\check{\gamma}(t) = \check{I}_{\check{\mathcal{F}}}^{q^\rightarrow t}(\check{z}^\rightarrow),$$

and note that for $n \in \mathbb{Z}^+$, $\check{\gamma}|_{[0,n]}$ is an admissible path of order nq^\rightarrow because \check{z}^\rightarrow has period q^\rightarrow .

By [LCT18, Proposition 26], it suffices to prove that there exists an admissible path which intersects $\check{\gamma}$ $\check{\mathcal{F}}$ -transversely and positively, and an admissible path which intersects $\check{\gamma}$ $\check{\mathcal{F}}$ -transversely and negatively: in that case we will obtain that for every $q'^\rightarrow \geq q^\rightarrow$, the path $\check{\gamma}|_{[0,1]}$ is realized up to equivalence, as the transverse path of a periodic point of period q' , whose transverse trajectory will have no $\check{\mathcal{F}}$ -transverse self-intersection, and the proof will be then finished.

Let us find these two paths. Let \check{A}^\rightarrow be the saturated set by leaves of $\check{\mathcal{F}}$ of $\check{\gamma}|_{[0,1]}$. Using the result from the first item, take \check{z}^\uparrow and \check{z}^\downarrow lifts of periodic points of f which respectively rotate $(0, p^\uparrow/q^\uparrow)$ and $(0, -p^\downarrow/q^\downarrow)$ ($p^\uparrow, p^\downarrow > 0$), and take their respective projections $\check{z}^\uparrow, \check{z}^\downarrow$ to the annulus \check{A} . Using that

- The diameter of leaves of $\check{\mathcal{F}}$ is uniformly bounded because $\check{\mathcal{F}}$ is the projection of $\widehat{\mathcal{F}}$ (see Lemma 3.4),
- The transverse paths

$$\check{\gamma}^\uparrow(t) = \check{I}_{\check{\mathcal{F}}}^{q^\uparrow t}(\check{z}^\uparrow), \quad \check{\gamma}^\downarrow(t) = \check{I}_{\check{\mathcal{F}}}^{q^\downarrow t}(\check{z}^\downarrow),$$

both go through \check{A}^\rightarrow in opposite directions,

we obtain that the following sets

$$J^\uparrow = \{t \in \mathbb{R} : \check{\phi}_{\check{\gamma}(t)} \cap \check{\gamma}^\uparrow \neq \emptyset\}, \quad J^\downarrow = \{t \in \mathbb{R} : \check{\phi}_{\check{\gamma}(t)} \cap \check{\gamma}^\downarrow \neq \emptyset\}$$

are both bounded. Finally, by [ST21, Proposition 15], we obtain that for sufficiently long pieces $\check{\gamma}^\uparrow|_{[s^\uparrow, t^\uparrow]}, \check{\gamma}^\downarrow|_{[s^\downarrow, t^\downarrow]}$, we have that

$$(3.1) \quad \check{\gamma}^\uparrow|_{[s^\uparrow, t^\uparrow]} \text{ intersects } \check{\gamma}^\rightarrow \text{ } \check{\mathcal{F}}\text{-transversally and positively,}$$

$$(3.2) \quad \check{\gamma}^\downarrow|_{[s^\downarrow, t^\downarrow]} \text{ intersects } \check{\gamma}^\rightarrow \text{ } \check{\mathcal{F}}\text{-transversally and negatively,}$$

which concludes the proof. □

We emphasize that the saturated set X of $\widehat{I}_{\widehat{\mathcal{F}}}^{\mathbb{Z}}(\widehat{z}^\rightarrow)$ by leaves of \mathcal{F} is a topological trivially foliated plane: given that $\widehat{I}_{\widehat{\mathcal{F}}}^{\mathbb{Z}}(\widehat{z}^\rightarrow)$ has no $\widehat{\mathcal{F}}$ -transverse self intersection, by the Poincaré-Bendixson Theorem, we obtain that every leaf $\widehat{\phi} \subset X$ intersects $\widehat{I}_{\widehat{\mathcal{F}}}^{\mathbb{Z}}(\widehat{z}^\rightarrow)$ in exactly one point. This, together with the fact that can parametrize each leaf varying continuously, gives us coordinates (s, t) for every point in X , where we have the homeomorphism $\mathbb{R}^2 \rightarrow X$ given by

$$(s, t) \mapsto \widehat{\phi}_t(\widehat{I}_{\widehat{\mathcal{F}}}^s(\widehat{z}^\rightarrow))$$

We now build the four families of closure-disjoint CFS.

PROOF OF PROPOSITION 3.3. Using the first item in Proposition 3.5, we find the existence of four periodic points $\widehat{z}^\rightarrow, \widehat{z}^\uparrow, \widehat{z}^\leftarrow$ and \widehat{z}^\downarrow , with respective periods $q^\rightarrow, q^\uparrow, q^\leftarrow, q^\downarrow$, which respectively rotate $(p^\rightarrow/q^\rightarrow, 0)$, $(0, p^\uparrow/q^\uparrow)$, $(-p^\leftarrow/q^\leftarrow, 0)$ and

$(0, -p^\downarrow/q^\downarrow)$, (where $p^\rightarrow, p^\uparrow, p^\leftarrow, p^\downarrow > 0$), and whose transverse trajectories by the isotopy \widehat{I} have no $\widehat{\mathcal{F}}$ -transverse self-intersection.

Using the second item from Proposition 3.5, let us take $q'^\rightarrow = (q^\rightarrow q^\uparrow q^\leftarrow q^\downarrow) p^\rightarrow \geq q^\rightarrow$ to obtain a new periodic point which we will still call \widehat{z}^\rightarrow for the sake of simplicity, such that its transverse trajectory has no $\widehat{\mathcal{F}}$ -transverse self-intersection, and moreover

$$\widehat{f}^{q'\rightarrow}(\widehat{z}^\rightarrow) + (p^\rightarrow, 0).$$

Therefore, \widehat{z}^\rightarrow has rotation vector $(1/(q^\rightarrow q^\uparrow q^\leftarrow q^\downarrow), 0)$. In the same fashion we obtain \widehat{z}^\uparrow , \widehat{z}^\leftarrow and \widehat{z}^\downarrow with the same rotation speed $1/(q^\rightarrow q^\uparrow q^\leftarrow q^\downarrow)$ (the speed being the norm of the rotation vector).

Note that if we take a power f^j of f , then we may take a new isotopy \widehat{I}^j from the identity to \widehat{f}^j defined as $\widehat{I}^j(t, \widehat{z}) = \widehat{I}(jt, \widehat{z})$, and we then have that $(\widehat{f}^j, \widehat{I}^j, \widehat{\mathcal{F}})$ is a maximal dinamically transverse decomposition for \widehat{f}^j (notice that we have used the same singular foliation). We then also have the following equality for the transverse paths

$$(\widehat{I}^j)_\widehat{\mathcal{F}}^n(\widehat{z}) = \widehat{I}_\widehat{\mathcal{F}}^{jn}(\widehat{z}), \text{ where } n \in \mathbb{Z}$$

This implies that if $\widehat{I}_\widehat{\mathcal{F}}^{\mathbb{Z}}(\widehat{z}^\rightarrow)$ has no $\widehat{\mathcal{F}}$ -transverse self-intersection, then $(\widehat{I}^j)_\widehat{\mathcal{F}}^{\mathbb{Z}}(\widehat{z}^\rightarrow)$ also has no $\widehat{\mathcal{F}}$ -transverse self-intersection.

This on its turn shows that if we take

$$q = q^\rightarrow q^\uparrow q^\leftarrow q^\downarrow, \underline{p} = |\text{lcm}(p^\rightarrow, p^\uparrow, p^\leftarrow, p^\downarrow)|,$$

we have that

$$\begin{aligned} \widehat{f}^{qp}(\widehat{z}^\rightarrow) &= \widehat{z}^\rightarrow + (\underline{p}, 0), & \widehat{f}^{qp}(\widehat{z}^\uparrow) &= \widehat{z}^\uparrow + (0, \underline{p}), \\ \widehat{f}^{qp}(\widehat{z}^\leftarrow) &= \widehat{z}^\leftarrow + (-\underline{p}, 0), & \widehat{f}^{qp}(\widehat{z}^\downarrow) &= \widehat{z}^\downarrow + (0, -\underline{p}). \end{aligned}$$

For this \widehat{z}^\rightarrow , we can define \widehat{A}^\rightarrow as the saturated set by leaves of $\widehat{\mathcal{F}}$, of the transverse path $\widehat{I}_\widehat{\mathcal{F}}(\widehat{z}^\rightarrow)$. By construction, we obtain that \widehat{A}^\rightarrow is a trivially foliated *horizontal strip* (*i.e.* homeomorphic to a plane, and invariant by integer horizontal translations). In this context, we will say that \widehat{A}^\rightarrow is a *canonically foliated strip* (CFS), and that \widehat{z}^\rightarrow is a *realizing point* for \widehat{A}^\rightarrow .

Given that the diameter of leaves of $\widehat{\mathcal{F}}$ is uniformly bounded (see Lemma 3.4) and that \widehat{A}^\rightarrow is invariant by integer horizontal translations, there exists $d^\rightarrow \in \mathbb{Z}^+$ such that

$$(3.3) \quad \text{cl}(\widehat{A}^\rightarrow) \cap \text{cl}(\widehat{A}^\rightarrow + (0, d')) = \emptyset, \text{ for every } d' \geq d^\rightarrow$$

In the same way as before, we get three analogous canonically foliated strips, which we will call A^\uparrow, A^\leftarrow and A^\downarrow , with their respective realizing points $\widehat{z}^\uparrow, \widehat{z}^\leftarrow$ and \widehat{z}^\downarrow , and integers $d^\uparrow, d^\leftarrow, d^\downarrow$ as in Equation 3.3.

Let us take p the smallest multiple of \underline{p} such that

$$p \geq \max\{d^\rightarrow, d^\uparrow, d^\leftarrow, d^\downarrow\} := d$$

. Again, we obtain that

$$(3.4) \quad \widehat{f}^{qp}(\widehat{z}^\rightarrow) = \widehat{z}^\rightarrow + (p, 0),$$

and similarly for the other three realizing points.

Let us then take $\widehat{z}^\rightarrow = (\widehat{x}^\rightarrow, \widehat{y}^\rightarrow)$, such that $0 \leq \widehat{y}^\rightarrow < 1$, and build its corresponding CFS \widehat{A}^\rightarrow . Now, consider the family $\widehat{\mathcal{A}}^\rightarrow = \{\widehat{A}_k^\rightarrow\}_{k \in \mathbb{Z}}$ of closure-disjoint horizontal CFSs, defined by

$$(3.5) \quad \widehat{A}_k^\rightarrow = \widehat{A}^\rightarrow + (0, pk),$$

and note that $(\widehat{x}_k^\rightarrow, \widehat{y}_k^\rightarrow) = \widehat{z}_k^\rightarrow = \widehat{z}^\rightarrow + (0, pk)$ is a realizing point for $\widehat{A}_k^\rightarrow$. We then have that

$$k = \lfloor \frac{\widehat{y}_k^\rightarrow}{p} \rfloor.$$

We define in the same fashion the families

$$(3.6) \quad \widehat{\mathcal{A}}^\uparrow = \{\widehat{A}_k^\uparrow\}_{k \in \mathbb{Z}}, \quad \widehat{\mathcal{A}}^\leftarrow = \{\widehat{A}_k^\leftarrow\}_{k \in \mathbb{Z}} \text{ and } \widehat{\mathcal{A}}^\downarrow = \{\widehat{A}_k^\downarrow\}_{k \in \mathbb{Z}},$$

each made of closure-disjoint CFSs, with the desired realizing points, which concludes the proof. \square

We will work with this four families in the following sections. We shall write

$$(3.7) \quad \widehat{\mathcal{A}} = \widehat{\mathcal{A}}^\rightarrow \cup \widehat{\mathcal{A}}^\uparrow \cup \widehat{\mathcal{A}}^\leftarrow \cup \widehat{\mathcal{A}}^\downarrow.$$

We shall write \widehat{A} for an arbitrary canonically foliated strip in $\widehat{\mathcal{A}}$.

REMARK 3.6. We will sometimes work with the closure of our canonically foliated strips and use the same notation $\widehat{A}^{\rightarrow, \uparrow, \leftarrow, \downarrow}$ for the sake of simplicity. In this context, we will refer to this closure as a closed CFS. However, we will refer to the open ones unless explicitly stated.

2. Crossing continua

We will introduce the notion of crossing continuum for a closed CFS, which is central for the remainder of our work, and describe the topology of the strips.

Boundary components: Top and Bottom. Note that by the Uniformization Theorem, we know that the complement of a CFS \widehat{A} has no bounded connected components. Moreover, there are exactly two connected components on this complement, because \widehat{A} is invariant by the integer translation given by their realizing point, but bounded in the orthogonal direction (recall that the diameter of leaves of $\widehat{\mathcal{F}}$ is uniformly bounded by Lemma 3.4). With this uniform bound we also obtain that the boundary $\partial\widehat{A}$ has exactly two connected components.

DEFINITION 3.7. Given a CFS \widehat{A} with a realizing point \widehat{z} , a point \widehat{w} on its complement will be *on its top* (and we will write $\widehat{w} \in T(\widehat{A})$) if it belongs to the connected component of its complement included in $R(\widehat{I}_F^\mathbb{Z}(\widehat{z}))$, and it will be *on its bottom* ($\widehat{w} \in B(\widehat{A})$) otherwise.

In addition, we will write $\partial\widehat{A} = \partial_T\widehat{A} \cup \partial_B\widehat{A}$, where

$$\partial_T\widehat{A} = \partial(T(\widehat{A})), \text{ and } \partial_B\widehat{A} = \partial(B(\widehat{A})).$$

Note that both $\partial_T\widehat{A}^\rightarrow$ and $\partial_B\widehat{A}^\rightarrow$ are connected, since \widehat{A}^\rightarrow is bounded in the vertical direction.

Once again, given the diameter of leaves in $\widehat{\mathcal{F}}$ is uniformly bounded, we get that each boundary component $\partial_T\widehat{A}$ and $\partial_B\widehat{A}$ of a canonically foliated strip, contains infinitely many singularities of the foliation. This means that if we take a lift $\tilde{A} \subset \tilde{\mathbb{D}}$ of $\widehat{A} \subset \mathbb{R}^2$ (which we will also call CFS), then its boundary and its complement will both have infinitely many connected components, each one determined by one leaf of $\partial\widehat{A}$. We will now settle notation for these components.

DEFINITION 3.8. Let \widehat{A} be a canonically foliated strip, \tilde{A} a lift of \widehat{A} to $\tilde{\mathbb{D}}$. We will write

$$\partial_T\tilde{A} := \partial\tilde{A} \cap \{\tilde{\pi}^{-1}(\widehat{\phi}_T) : \widehat{\phi}_T \in \partial_T\widehat{A}\}, \quad \partial_B\tilde{A} := \partial\tilde{A} \cap \{\tilde{\pi}^{-1}(\widehat{\phi}_B) : \widehat{\phi}_B \in \partial_B\widehat{A}\}.$$

Moreover, we will write $T(\tilde{A})$ for the (closed) connected components of the complement of \tilde{A} which are bounded by leaves $\tilde{\phi}_T \subset \partial_T\tilde{A}$, and similarly we will write $B(\tilde{A})$ for the (closed) connected components of the complement of \tilde{A} which are bounded by leaves $\tilde{\phi}_B \subset \partial_B\tilde{A}$.

DEFINITION 3.9. Given a lift \tilde{A} of a canonically foliated strip, and $\tilde{\phi}_T \subset \partial_T\tilde{A}$, we will write $T_{\tilde{\phi}_T}(\tilde{A})$ for the closed connected component of \tilde{A}^C bounded by $\tilde{\phi}_T$ (similar fashion for $\tilde{\phi}_B \subset \partial_B\tilde{A}$).

Crossing continua and diameter measurement.

DEFINITION 3.10 (Crossing continuum). Given a closed CFS \widehat{A} and a continuum $\widehat{K} \in \mathbb{R}^2$, we will say that \widehat{K} is a *crossing continuum* (relative to \widehat{A}) if it intersects both connected components of $\partial\widehat{A}$. In addition, a crossing continuum will be *minimal* if it does not strictly contain any crossing subcontinua.

REMARK 3.11. Given \widehat{A} a closed CFS, any crossing continuum \widehat{K} has a minimal crossing subcontinuum $\widehat{K}' \subset \widehat{A}$. In particular, \widehat{K}' intersects one leaf (or singularity of the foliation) on each connected component of $\partial\widehat{A}$.

DEFINITION 3.12 (Vertical diameter). Given a continuum $\widehat{K} \in \mathbb{R}^2$, we will define its *vertical diameter* $V\text{diam}(\widehat{K})$ as the supremum of the difference of the y -coordinates in pairs of points belonging to \widehat{K} .

The *horizontal diameter* of a continuum, $H\text{diam}(\widehat{K})$ is defined analogously, looking at the x -coordinates. For inessential continua $K \in \mathbb{T}^2$ we recall that we take their lifted diameter as $\text{diam}(K) = \text{diam}(\widehat{K})$, where \widehat{K} is any of its lifts to \mathbb{R}^2 . Accordingly, if we go to the universal covering $\tilde{\mathbb{D}}$ of $\text{Dom}(\widehat{I})$ and take $\tilde{K} \in \tilde{\mathbb{D}}$, we will write $\text{diam}(\tilde{K}) = \text{diam}(\widehat{K}) = \text{diam}(\tilde{\pi}(\tilde{K}))$. The same holds for horizontal and vertical diameter.

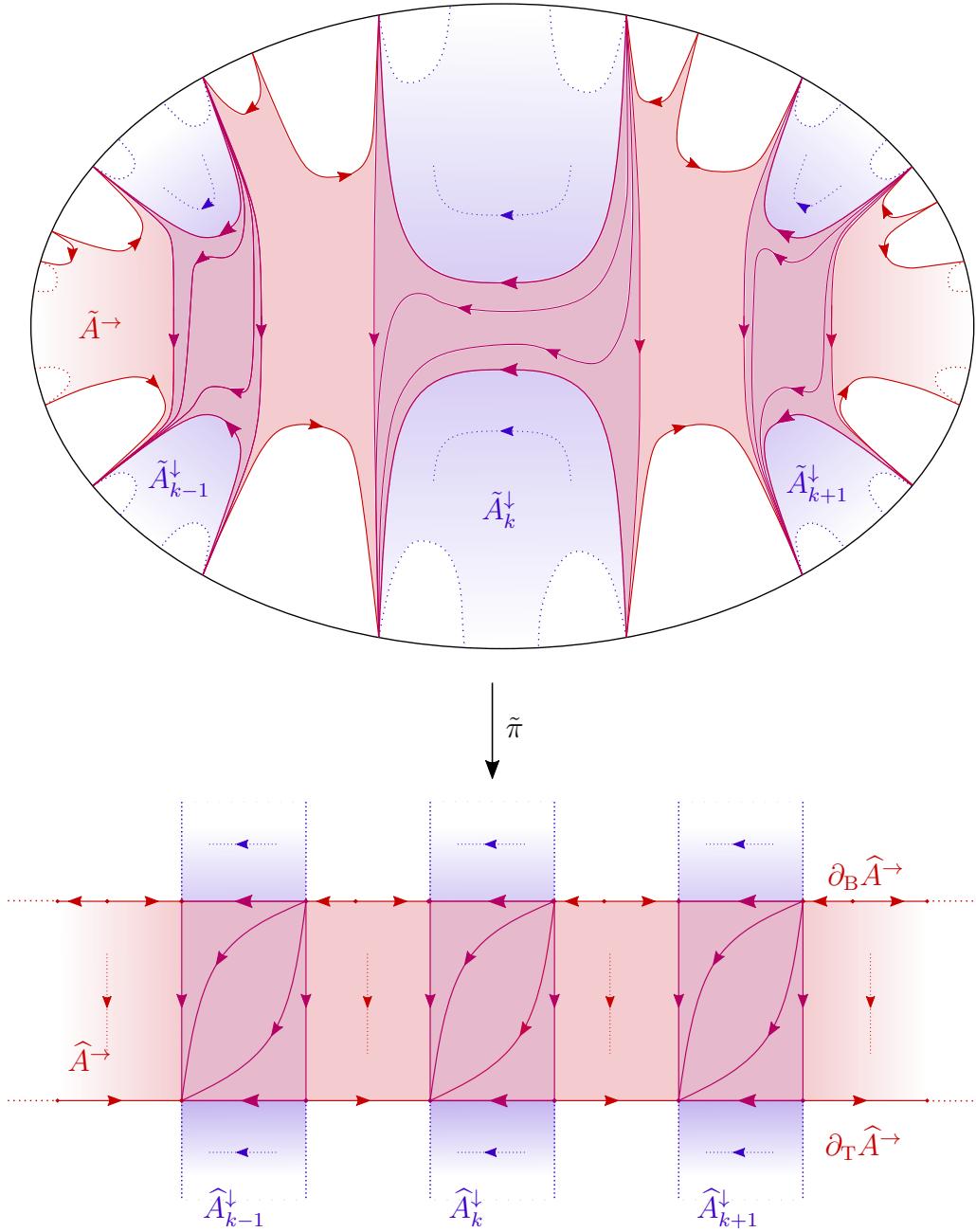


FIGURE 2. A canonically foliated strip \widehat{A}^\rightarrow , with two boundary connected components, and one of its lifts, with infinitely many of them.

We emphasize that the (lifted) diameter of any continuum (either included in \mathbb{T}^2 or in $\widehat{\mathbb{D}}$) is always measured in \mathbb{R}^2 , respectively taking a lift or projecting.

Inside the strip: Left and Right.

DEFINITION 3.13. Given a canonically foliated strip \widehat{A} with a realizing point \widehat{z} and a crossing continuum \widehat{K} , we say a point $\widehat{w} \in \widehat{A} \setminus \widehat{K}$ is *on its left* ($\widehat{w} \in L_{\widehat{A}}(\widehat{K})$) if

it belongs to the connected component of $\widehat{A} \setminus \widehat{K}$ which contains $\widehat{f}^j(\widehat{z})$ for arbitrarily large values of j , and it is *on its right* ($\widehat{w} \in R_{\widehat{A}}(\widehat{K})$) if it belongs to the connected component of $\widehat{A} \setminus \widehat{K}$ which contains $\widehat{f}^j(\widehat{z})$ for arbitrarily large negative values of j .

These notions lift naturally to the lifts $\tilde{A} \subset \tilde{\mathbb{D}}$ of our canonically foliated strips. Moreover, note that each of the two ends of the strip belong to one of this connected components (different from each other). For the case of CFS \tilde{A} in the universal covering $\tilde{\mathbb{D}}$, we may identify each of these ends $l_{\tilde{A}}$ (the future) and $r_{\tilde{A}}$ (the past) with a point in the boundary of $\tilde{\mathbb{D}}$.

Note that the closure of a leaf $\widehat{\phi} \subset \widehat{A}$ is a crossing continuum for \widehat{A} . We may then apply the *left-right* notion to leaves $\widehat{\phi} \subset \widehat{A}$ by taking their closure: we will say that $\widehat{z} \in L_{\widehat{A}}(\widehat{\phi})$ whenever $\widehat{z} \in L_{\widehat{A}}(\text{cl}(\widehat{\phi}))$. Analogously, $\widehat{z} \in R_{\widehat{A}}(\widehat{\phi})$ if $\widehat{z} \in R_{\widehat{A}}(\text{cl}(\widehat{\phi}))$.

This leads us to the following:

REMARK 3.14. Let \widehat{A} be a CFS and $\widehat{\phi} \subset \widehat{A}$ a leaf. Then, if $\tilde{A} \subset \tilde{\mathbb{D}}$ is a lift of \widehat{A} , we obtain that

$$L_{\widehat{A}}(\widehat{\phi}) = \tilde{\pi}(L(\tilde{\phi}) \cap \tilde{A}), \text{ and } R_{\widehat{A}}(\widehat{\phi}) = \tilde{\pi}(R(\tilde{\phi}) \cap \tilde{A})$$

REMARK 3.15. Given the relative orientation between the transverse foliation and the paths by the isotopy, the notions of Top, Bottom, Left and Right we just defined coincide with the visual concepts, for canonically foliated strips \widehat{A}^\leftarrow , where the realizing point goes *from right to left*.

Fundamental domains. Recall the definition of $\widehat{\mathcal{A}}$ from Equations 3.5 through 3.7, and let us explain the notion of fundamental domains for a CFS $\widehat{A}^\rightarrow \subset \widehat{\mathcal{A}}^\rightarrow$. Recall that every $\widehat{A}_k^\downarrow \in \widehat{\mathcal{A}}^\downarrow$ must cross \widehat{A}^\rightarrow from bottom to top at least once.

Then, we may take $\widehat{\phi}_{T,0}^\downarrow \subset \widehat{A}^\rightarrow \cap \partial_T \widehat{A}_0^\downarrow$, and then define

$$\widehat{\phi}_{T,k}^\downarrow = \widehat{\phi}_{T,0}^\downarrow + (pk, 0),$$

where p is as in Equation 3.4.

These leaves define the fundamental domains

$$\widehat{D}_k^\rightarrow = L_{\widehat{A}^\rightarrow}(\widehat{\phi}_{T,k}^\downarrow) \cap R_{\widehat{A}^\rightarrow}(\widehat{\phi}_{T,k+1}^\downarrow).$$

Note that all of these fundamental domains behave nicely when taking natural lifts, therefore obtaining that any \tilde{A}^\rightarrow is divided in the following domains:

$$\tilde{D}_k^\rightarrow = L_{\tilde{A}^\rightarrow}(\tilde{\phi}_{T,k}^\downarrow) \cap R_{\tilde{A}^\rightarrow}(\tilde{\phi}_{T,k+1}^\downarrow).$$

This construction can be accordingly made with any other canonically foliated strip $\widehat{A}^\uparrow, \widehat{A}^\leftarrow, \widehat{A}^\downarrow$ and their respective lifts to the universal covering. We emphasize that each of these fundamental domains has *length* p , more precisely, we have that $\widehat{\phi}_{T,k}^\downarrow \in \partial_T \widehat{A}_k^\downarrow$.

Let us finish the Section with a result on how points travel through fundamental domains.

LEMMA 3.16. *Let $\widehat{A} \in \widehat{\mathcal{A}}$ be a CFS, $\widehat{\phi} \subset \widehat{A}$ a leaf in the fundamental domain \widehat{D}_k , and $n > 0$ an integer.*

Then, there exists k' such that for any leaf $\widehat{\phi}' \subset \widehat{D}_{k'} \subset \widehat{A}$, when we take a set of natural lifts $\tilde{A}, \tilde{\phi}, \tilde{\phi}'$ to $\tilde{\mathbb{D}}$, we have that

$$\tilde{f}^j(\tilde{\phi}') \subset R(\tilde{\phi}), \text{ for every } 0 \leq j \leq n.$$

Let us write the proof for a horizontal CFS \widehat{A}^\rightarrow , as the other cases are analogous.

PROOF. Fix $n > 0$, a leaf $\phi \subset \widehat{A}^\rightarrow \in \widehat{\mathcal{A}}^\rightarrow$, and let \widehat{D}_k be the fundamental domain of \widehat{A} such that $\phi \subset \widehat{D}_k^\rightarrow$ (see the construction above). Given that \widehat{f} is the lift to \mathbb{R}^2 of an isotopic-to-the-identity torus homeomorphism (which is uniformly continuous), we have that there exists $M > 0$ such that

$$(3.8) \quad \sup_{\widehat{z} \in \mathbb{R}^2} d(\widehat{f}(\widehat{z}), \widehat{z}) \leq M, \text{ which implies that } \sup_{\widehat{z} \in \mathbb{R}^2} d(\widehat{f}^n(\widehat{z}), \widehat{z}) \leq nM.$$

Now, recall that by Lemma 3.4, the diameter of leaves is uniformly bounded, which implies that the closure $\text{cl}(\widehat{D}^\rightarrow)$ of every fundamental domain is compact. Given fundamental domains are obtained by taking horizontal translations of one another, we recover that there exists $k' < k$, such that

$$d(\tilde{z}', \widehat{D}_k^\rightarrow) > 2nM, \text{ for every } \tilde{z}' \in \widehat{D}_{k'}^\rightarrow.$$

This, together with Equation 3.8, implies that taking any leaf $\widehat{\phi}' \subset \widehat{D}_{k'}^\rightarrow$, and a natural set of lifts $\tilde{\phi}', \tilde{\phi}$ of $\widehat{\phi}'$ and $\widehat{\phi}$ to $\tilde{\mathbb{D}}$, we have that

$$\tilde{f}^j(\tilde{\phi}') \subset R(\tilde{\phi}), \text{ for every } 0 \leq j \leq n,$$

which concludes the proof. \square

CHAPTER 4

Anchoring techniques

The goal of this chapter is to prove Theorem C, whose statement we now recall (see Definition 1.4 for details),

Theorem C. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$, such that $\rho(f)$ has nonempty interior. Then, there exists $M > 0$ such that for any continuum $K \in \mathbb{T}^2$, $\mathcal{D}_f(K) > M \implies \mathcal{D}_f(K) = \infty$.*

The idea of the proof resides in the canonically foliated strips we built in the previous section, and heavily uses Brouwer-Le Calvez's dynamically transverse decomposition. We will use the construction and notation of Chapter 3, and recall that up to taking a power, we may assume that $\vec{0} = \text{int}(\rho(\hat{f}))$.

REMARK 4.1. Let $f \in \text{Homeo}_0(\mathbb{T}^2)$, a continuum $K \in \mathbb{T}^2$ and $n > 0$. Then $\mathcal{D}_f(K) = \infty \iff \mathcal{D}_{f^n}(K) = \infty$

This remark shows it is enough to prove Theorem C for any power of f .

REMARK 4.2. Given each of the properties from Theorem A can be checked up to taking a power of f , we will assume that f has a lift \hat{f} with $\vec{0}$ in $\text{int}(\rho(\hat{f}))$, and we will work with such a lift from now on.

We will now prove Theorem C for f^{qp} , where q and p are as in Equation 3.4. Let us then rename $f := f^{qp}$, $\hat{f} = (\hat{f})^{qp}$ (which still has the origin in the interior of its rotation set). This implies that

$$(4.1) \quad \hat{f}(\hat{z}^\rightarrow) = \hat{z}^\rightarrow + (p, 0),$$

and analogously for any other realizing point in any direction. The key fact here is

REMARK 4.3. Each realizing point from a CFS advances one fundamental domain per iteration of \hat{f} .

1. Semianchors

Note that if $\text{diam}(\hat{K}) > M$, then we either have $\text{Vdiam}(\hat{K}) > \frac{\sqrt{2}M}{2}$ or $\text{Hdiam}(\hat{K}) > \frac{\sqrt{2}M}{2}$. This, together with the symmetry of this section's hypothesis, shows that in order to prove Theorem C, it is enough to prove that there exists $M > 0$ such that for every continuum $K \in \mathbb{T}^2$,

$$\text{Vdiam}(K) > M \implies \mathcal{D}_f(K) = \infty.$$

In particular, note that continua with large vertical diameter will be crossing for some horizontal canonically foliated strip. Let us assume that $\text{Vdiam}(\widehat{K})$ is sufficiently large so that there exists $k \in \mathbb{Z}$ such that \widehat{K} goes through $\widehat{A}_{k_0}^\rightarrow$. Whenever there is no ambiguity for this value of k_0 , we will fix this horizontal strip and denote it as \widehat{A}^\rightarrow .

For the remainder of this section and the following one, let us assume that $K \subset \text{Dom}(I)$, that is, K does not intersect any singularity.

LEMMA 4.4. *Let $n \in \mathbb{Z}^+$. Then, there exists $M_n > 0$ such that:*

For any continuum \widehat{K} with $\text{Vdiam}(\widehat{K}) \geq M_n$, there exists a horizontal CFS $\widehat{A}^\rightarrow \in \widehat{\mathcal{A}}^\rightarrow$, and two points $\widehat{z}_{\text{TT}}, \widehat{z}_{\text{BB}} \in \widehat{K}$ such that

$$(4.2) \quad \widehat{I}^t(\widehat{\phi}_{\widehat{z}_{\text{TT}}}) \in T(\widehat{A}^\rightarrow) \text{ and } \widehat{I}^t(\widehat{\phi}_{\widehat{z}_{\text{BB}}}) \in B(\widehat{A}^\rightarrow), \text{ for every } -n \leq t \leq n.$$

PROOF. Fix $n > 0$. Let us then start by stating the three following facts:

- The vertical diameter of any CFS $\widehat{A}^\rightarrow \in \widehat{\mathcal{A}}^\rightarrow$ is uniformly upper-bounded. Let us then take d_1 an upper bound for that quantity.
- As proven in Lemma 3.4, the diameter of leaves is also uniformly bounded. Let us then take d_2 a bound for this diameter.
- The length of the vectors $\{\widehat{I}^t(\widehat{z}) - \widehat{z} : \widehat{z} \in \mathbb{R}^2, |t| < n\}$ is also uniformly bounded. This is because \widehat{I} is \mathbb{Z}^2 -periodic, given it is defined as the lift of an isotopy in the torus, which is on its turn compact. Let us then take d_3 an upper bound for this family of lengths.

Finally, let us take

$$M_n = 2(d_1 + d_2 + d_3) + p,$$

where p is as in Equation 3.4. Given a continuum \widehat{K} given $\text{Vdiam}(\widehat{K}) \geq M_n$, we may take \widehat{z}_{TT} and \widehat{z}_{BB} with vertical distance equal to M_n . This means that

$$\text{Vdist}\left(\bigcup_{|t| \leq n} \widehat{I}^t(\widehat{\phi}_{\widehat{z}_{\text{TT}}}), \bigcup_{|t| \leq n} \widehat{I}^t(\widehat{\phi}_{\widehat{z}_{\text{BB}}})\right) \geq 2d_1 + p.$$

This means that there exists a strip $\mathbb{R} \times [y, y + 2d_1 + p] \subset \mathbb{R}^2$ between these two sets, which allows us to take a realizing point \widehat{z}^\rightarrow with vertical coordinate between $y + d_1$ and $y + d_1 + p$, whose associated canonically foliated strip \widehat{A}^\rightarrow will satisfy Equation 4.2, which concludes the proof. \square

REMARK 4.5. An identical proof works with \widehat{A}^\leftarrow for continua with large vertical diameter, and with $\widehat{A}^\uparrow, \widehat{A}^\downarrow$ for continua with large horizontal diameter.

DEFINITION 4.6 (Semianchor). Given $n \in \mathbb{Z}^+$, a CFS \tilde{A} , a Crossing continuum \tilde{K} and two leaves $\tilde{\phi}_B \subset \partial_B \tilde{A}, \phi_T \subset \partial_T \tilde{A}$, we will say that the quartet $(\tilde{A}, \tilde{K}, \tilde{\phi}_B, \phi_T)$ is an n -semianchor when there exist $\tilde{z}_{\text{TT}}, \tilde{z}_{\text{BB}} \in \tilde{K}$ such that

$$\tilde{f}^j(\tilde{\phi}_{\tilde{z}_{\text{TT}}}) \subset T_{\phi_T}(\tilde{A}), \quad \tilde{f}^j(\tilde{\phi}_{\tilde{z}_{\text{BB}}}) \subset B_{\phi_B}(\tilde{A}), \text{ for every } j \in \mathbb{Z}; |j| \leq n.$$

The following consequence of Lemma 4.4 is key to prove Theorem C.

LEMMA 4.7. Let $n \in \mathbb{Z}^+$. Then, there exists $M_n > 0$ such that for any \tilde{K} in $\tilde{\mathbb{D}}$ with $Vdiam(\tilde{K}) > M_n$, there exist $\tilde{A}, \tilde{\phi}_T \subset \partial_T \tilde{A}, \tilde{\phi}_B \subset \partial_B \tilde{A}$, such that

$$(\tilde{A}, \tilde{K}, \tilde{\phi}_B, \tilde{\phi}_T) \text{ is an } n\text{-semianchor.}$$

PROOF. Take $\hat{A}, \hat{z}_{TT}, \hat{z}_{BB}$ as in Lemma 4.4, and take natural lifts $\tilde{A}, \tilde{z}_{TT}, \tilde{z}_{BB}$. Note that $\tilde{z}_{TT} \in T_{\tilde{\phi}_T}(\tilde{A}), \tilde{z}_{BB} \in B_{\tilde{\phi}_B}(\tilde{A})$, for some unique leaves $\tilde{\phi}_T \subset \partial_T \tilde{A}, \tilde{\phi}_B \subset \partial_B \tilde{A}$. By the statement 4.2 in Lemma 4.4, we immediately obtain that

$$\tilde{I}^j(\tilde{\phi}_{z_{TT}}) \in T_{\tilde{\phi}_T}(\tilde{A}^\rightarrow) \text{ and } \tilde{I}^j(\tilde{\phi}_{z_{BB}}) \in B_{\tilde{\phi}_B}(\tilde{A}^\rightarrow), \text{ for every } j \text{ with } |j| \leq n,$$

which concludes the proof. \square

LEMMA 4.8. Let $n \in \mathbb{Z}^+$, $(\tilde{A}, \tilde{K}, \tilde{\phi}_B, \tilde{\phi}_T)$ an n -semianchor, and \tilde{z} a realizing point for \tilde{A} , with $\tilde{z} \in R_{\tilde{A}}(\tilde{K})$.

Then, for every $j \in \mathbb{Z}$ with $|j| \leq n$, we have that $\tilde{f}^j(\tilde{K})$ is a crossing continuum for \tilde{A} , and $\tilde{f}^j(\tilde{z}) \notin L_{\tilde{A}}(\tilde{f}^j(\tilde{K}))$.

PROOF. Note that $\Gamma_0 := \tilde{\phi}_{z_{TT}}^- \cup \tilde{K} \cup \tilde{\phi}_{z_{BB}}^+$ separates the ends of \tilde{A} , and leaves \tilde{z} on the same component as $r_{\tilde{A}}$. Name $R(\Gamma_0)$ to this component, and let $L(\Gamma_0)$ be the one containing the other end.

By Lemma 4.7, we know that if $|j| < n$, then $\tilde{f}^j(\tilde{\phi}_{z_{TT}}) \cap \tilde{A} = \emptyset$ and also $\tilde{f}^j(\tilde{\phi}_{z_{BB}}) \cap \tilde{A} = \emptyset$, which implies that $\tilde{f}^j(\tilde{K})$ is crossing for \tilde{A} through the leaves $\tilde{\phi}_T$ and $\tilde{\phi}_B$. Furthermore, $\tilde{f}^j(\tilde{z}) \in \tilde{f}^j(R(\Gamma_0))$, and $L_{\tilde{A}}(\tilde{f}^j(\tilde{K})) \subset \tilde{f}^j(L(\Gamma_0))$, which concludes the proof. \square

DEFINITION 4.9 (**Semianchored continuum**). Whenever $(\tilde{A}, \tilde{K}, \tilde{\phi}_B, \tilde{\phi}_T)$ is an n -semianchor, we will say that \tilde{K} is an n -semianchored continuum (or simply that it is n -semianchored).

We do the whole dynamical arguments in the universal covering $\tilde{\mathbb{D}}$, but we measure distances in \mathbb{R}^2 . Let us then define:

DEFINITION 4.10. We will say that $(\hat{A}, \hat{K}, \hat{\phi}_B, \hat{\phi}_T)$ is an n -semianchor if there exist lifts $(\tilde{A}, \tilde{K}, \tilde{\phi}_T, \tilde{\phi}_B)$ forming an n -semianchor. We will then say that \hat{K} is n -semianchored.

2. The Anchoring Lemma

LEMMA 4.11 (**Anchoring Lemma**). Let \tilde{A} be a canonically foliated strip, \tilde{K} a crossing continuum, and $\tilde{\phi}_T \subset T(\tilde{A}), \tilde{\phi}_B \subset B(\tilde{A})$ such that

- $\tilde{K} \cap \tilde{\phi}_T \neq \emptyset, \tilde{K} \cap \tilde{\phi}_B \neq \emptyset$
- $\tilde{A} \subset R(\tilde{\phi}_T) \cap R(\tilde{\phi}_B)$.

Then, we have that

- (1) For every $n > 0$, $\tilde{f}^n(\tilde{K})$ intersects both $\tilde{\phi}_T$ and $\tilde{\phi}_B$
- (2) For every leaf $\tilde{\phi} \subset \tilde{A}$ with $\tilde{\phi}_T, \tilde{\phi}_B \subset R(\tilde{\phi})$, there exists $n_0 > 0$ such that

$$\tilde{f}^n(\tilde{K}) \cap \tilde{\phi} \neq \emptyset, \text{ for every } n \geq n_0.$$

PROOF. Let $\tilde{z}_T \in \tilde{\phi}_T \cap \tilde{K}$, $\tilde{z}_B \in \tilde{\phi}_B \cap \tilde{K}$

(1) Fix an integer $n > 0$. Note that we have

$$\tilde{f}^n(\tilde{z}_T) \in L(\tilde{\phi}_T) \text{ and } \tilde{f}^n(\tilde{z}_B) \in L(\tilde{\phi}_B),$$

from where we obtain that

$$L(\tilde{\phi}_T) \cap \tilde{f}^n(\tilde{K}) \neq \emptyset \text{ and } L(\tilde{\phi}_B) \cap \tilde{f}^n(\tilde{K}) \neq \emptyset.$$

Given that $\tilde{A}^\rightarrow \subset R(\tilde{\phi}_T) \cap R(\tilde{\phi}_B)$, we conclude that $\tilde{f}^n(\tilde{K})$ is a crossing continuum for \tilde{A}^\rightarrow which intersects both $\tilde{\phi}_T$ and $\tilde{\phi}_B$.

(2) Let \tilde{z}_0 be a realizing point for \tilde{A} such that $\tilde{z}_0 \in R_{\tilde{A}}(\tilde{K})$, and let us name $\tilde{z}_n = \tilde{f}^n(\tilde{z}_0)$. Take two open arcs $\tilde{\gamma}_T \subset \tilde{\phi}_{\tilde{z}_T}^-$, $\tilde{\gamma}_B \subset \tilde{\phi}_{\tilde{z}_B}^+$, and note that

$$\Gamma_0 = \tilde{\gamma}_T \cup \tilde{K} \cup \tilde{\gamma}_B$$

separates the disk $\tilde{\mathbb{D}}$ into at least two connected components, one containing $L_{\tilde{A}}(\tilde{K})$, which we will call $L(\Gamma_0)$, and another one containing $R_{\tilde{A}}(\tilde{K})$, which we will call $R(\Gamma_0)$.

Fix a leaf $\tilde{\phi} \subset \tilde{A}$. Take a sufficiently large value of n_0 , such that $\tilde{z}_{n_0} \in L_{\tilde{A}}(\tilde{\phi})$. Note that for every $n > 0$, \tilde{z}_n still belongs to \tilde{A} and also

$$\tilde{z}_n \in R(\tilde{f}^n(\Gamma_0)) = \tilde{f}^n(R(\Gamma_0)),$$

which implies that

$$\tilde{f}^n(\Gamma_0) \cap \tilde{\phi} \neq \emptyset, \text{ for every } n \geq n_0.$$

Recalling that $\tilde{f}^n(\tilde{\phi}_T) \in L(\tilde{\phi}_T)$, $\tilde{f}^n(\tilde{\phi}_B) \in L(\tilde{\phi}_B)$, we obtain that

$$\tilde{f}^n(\tilde{\phi}_T) \cap \tilde{\phi} = \emptyset; \quad \tilde{f}^n(\tilde{\phi}_B) \cap \tilde{\phi} = \emptyset \text{ for every } n > 0$$

which concludes the proof. □

REMARK 4.12. If we exchange

$$\tilde{A} \subset R(\tilde{\phi}_T) \cap R(\tilde{\phi}_B) \text{ for } \tilde{A} \subset L(\tilde{\phi}_T) \cap L(\tilde{\phi}_B),$$

in the hypothesis of the Anchoring Lemma, we may proceed in the exact same fashion iterating \tilde{f} to the past, to obtain that

- (1) For every $n < 0$, $\tilde{f}^n(\tilde{K})$ intersects both $\tilde{\phi}_T$ and $\tilde{\phi}_B$
- (2) For every leaf $\tilde{\phi} \subset \tilde{A}$ with $\tilde{\phi}_T, \tilde{\phi}_B \subset L(\tilde{\phi})$, there exists $n_0 < 0$ such that

$$\tilde{f}^n(\tilde{K}) \cap \tilde{\phi} \neq \emptyset, \text{ for every } n \leq n_0.$$

DEFINITION 4.13 (**Stable and Unstable Anchors**).

- We will say the quartet $(\tilde{A}, \tilde{K}, \tilde{\phi}_B, \tilde{\phi}_T)$ is an unstable anchor (or simply u -anchor) when it satisfies the hypothesis of the Anchoring Lemma.
- Accordingly, we will say the quartet $(\tilde{A}, \tilde{K}, \tilde{\phi}_B, \tilde{\phi}_T)$ is a stable anchor (or s -anchor) when it satisfies the hypothesis of last remark.

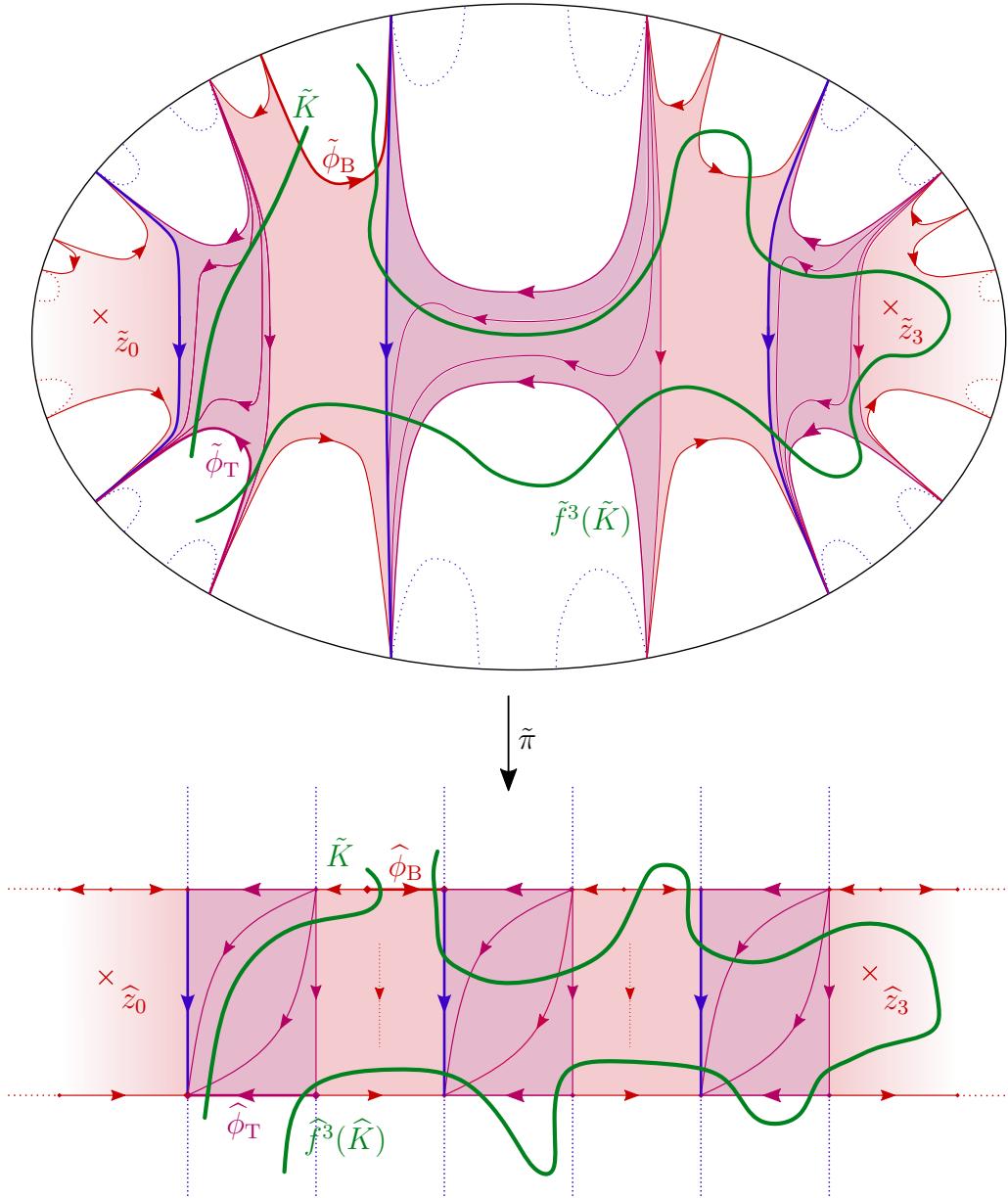


FIGURE 3. The sequence of iterates $\{\tilde{z}_n\}$ pushes \tilde{K} , the leaves $\tilde{\phi}_T, \tilde{\phi}_B$ serve as anchors fixing it. The blue leaves in \mathbb{R}^2 define fundamental domains in the canonically foliated strip. The intersections between \tilde{A}_k^\downarrow and \tilde{A}^\rightarrow are in purple, the intersections between \tilde{A}_k^\uparrow and \tilde{A}^\rightarrow are included in the red regions (and similarly for their projections to \mathbb{R}^2 .)

- For these two cases, we will say respectively that \tilde{K} is u -anchored (s -anchored) to \tilde{A} , by the leaves $\tilde{\phi}_B$ and $\tilde{\phi}_T$, or simply that \tilde{K} is u -anchored (s -anchored).

- We will say that \tilde{K} is an anchored continuum (or that it is anchored) if it is u -anchored or s -anchored.

DEFINITION 4.14. We will say that $(\hat{A}, \hat{K}, \hat{\phi}_B, \hat{\phi}_T)$ is a u -anchor (respectively s -anchor) whenever there exist respective lifts $(\tilde{A}, \tilde{K}, \tilde{\phi}_B, \tilde{\phi}_T)$ forming a u -anchor (respectively s -anchor).

The following is an almost direct consequence of the Anchoring Lemma.

PROPOSITION 4.15. *Let $(\hat{A}, \hat{K}, \hat{\phi}_B, \hat{\phi}_T)$ be a u -anchor. Then*

$$\lim_{n \rightarrow +\infty} \text{diam}(\hat{f}^n(\hat{K})) = \infty \text{ and in particular, } \mathcal{D}_{\hat{f}}(\hat{K}) = \infty$$

PROOF. Recall that by Lemma 3.4, we know that diameter of leaves $\hat{\phi} \in \hat{\mathcal{F}}$ is uniformly bounded. For every $j \in \mathbb{Z}^+$, let us take $\hat{\phi}_j \subset \hat{A}$ such that

$$\hat{\phi}_T, \hat{\phi}_B \subset \partial R_{\hat{A}}(\hat{\phi}_j); \text{ and } d(\hat{\phi}_j, \hat{\phi}_T) > j$$

By the Anchoring Lemma, we have there exists $n_j > 0$ such that for every $n > n_j$,

$$\hat{f}^n(\hat{K}) \cap \hat{\phi}_T \neq \emptyset, \quad \hat{f}^n(\hat{K}) \cap \hat{\phi}_j \neq \emptyset \implies \text{Hdiam}(\hat{f}^n(\hat{K})) > j$$

which concludes the proof. \square

Proceeding in the same fashion we obtain:

COROLLARY 4.16. *Let \hat{K} be an s -anchored continuum. Then*

$$\lim_{n \rightarrow -\infty} \text{diam}(\hat{f}^n(\hat{K})) = \infty, \text{ and in particular, } \mathcal{D}_{\hat{f}}(\hat{K}) = \infty$$

REMARK 4.17. If a continuum \hat{K} is crossing for \hat{A} through singularities, that is, when we take \tilde{K}, \tilde{A} we have that

$$\text{either } \tilde{K} \cap \partial_T \tilde{A} = \emptyset \text{ or } \tilde{K} \cap \partial_B \tilde{A} = \emptyset,$$

then the singularity serves as an anchor (remember that the boundary is fixed by \tilde{f}), and therefore we may proceed in the exact same fashion as in Lemma 4.11 to obtain $\mathcal{D}_{\tilde{f}}(\tilde{K}) = \infty$.

We can then assume without loss of generality, that our continua \hat{K} cross our canonically foliated strips \hat{A} through actual leaves of $\hat{\mathcal{F}}$.

3. Proofs of Theorems C and E

We have already proved that any anchored continuum has infinite dynamical diameter, and that every sufficiently large continuum is n -semianchored for a certain n . Let us finish the proof of Theorem C by gluing these two results together.

LEMMA 4.18. *For any 3-semianchored continuum \tilde{K} , there exists $0 \leq j \leq 3$ such that $\tilde{f}^j(\tilde{K})$ is anchored.*

PROOF. Let us assume without loss of generality that our 3-semianchor is given by $(\tilde{A}^\rightarrow, \tilde{K}, \tilde{\phi}_B^\rightarrow, \tilde{\phi}_T^\rightarrow)$. We will face one of the following four cases:

$$\tilde{A}^\rightarrow \subset R(\tilde{\phi}_T^\rightarrow) \cap R(\tilde{\phi}_B^\rightarrow), \quad \tilde{A}^\rightarrow \subset L(\tilde{\phi}_T^\rightarrow) \cap L(\tilde{\phi}_B^\rightarrow),$$

$$\tilde{A}^\rightarrow \subset R(\tilde{\phi}_T^\rightarrow) \cap L(\tilde{\phi}_B^\rightarrow), \quad \tilde{A}^\rightarrow \subset L(\tilde{\phi}_T^\rightarrow) \cap R(\tilde{\phi}_B^\rightarrow).$$

For the first two cases the lemma is proved, as the original quartet would already be respectively a u -anchor or an s -anchor. We will then assume that the fourth case holds, as the other one is analogous.

Recall that \tilde{A}^\rightarrow is crossed by infinitely many vertical CFS \tilde{A}_k^\downarrow with their corresponding realizing points \tilde{z}_k^\downarrow (see the construction of \tilde{A}^\downarrow in Chapter 3 for details). Moreover, we may take leaves $\tilde{\phi}_{T,k}^\downarrow \in \partial_T \tilde{A}_k^\downarrow \cap \tilde{A}^\rightarrow$ and recall the construction of fundamental domains \tilde{D}_k^\rightarrow of our strip \tilde{A}^\rightarrow , built in Chapter 3:

$$\tilde{D}_k^\rightarrow = \tilde{A}^\rightarrow \cap L(\tilde{\phi}_{T,k}^\downarrow) \cap R(\tilde{\phi}_{T,k+1}^\downarrow)$$

Write $\tilde{\phi}_{B,k}^\downarrow \in \partial_B \tilde{A}_k^\downarrow \cap \tilde{A}^\rightarrow$ for the other boundary component of the connected component of $\tilde{A}^\rightarrow \cap \tilde{A}^\downarrow$ which is bounded by $\tilde{\phi}_{T,k}^\downarrow$ (see Figure 4). Note that $\tilde{\phi}_B^\rightarrow$ belongs to the closure of a fundamental domain \tilde{D}_k^\rightarrow for a certain value of k , similarly $\tilde{\phi}_T^\rightarrow$ belongs to $\tilde{D}_{k'}^\rightarrow$ for a certain k' .

Case 1. $\tilde{K} \not\subset R(\tilde{\phi}_{B,k+1}^\downarrow)$. For this case we obtain that $\tilde{K} \cap \tilde{\phi}_{B,k+1}^\downarrow \neq \emptyset$ since \tilde{K} is connected. Then, we have that

$$(\tilde{A}_{k+1}^\downarrow, \tilde{K}, \tilde{\phi}_{B,k+1}^\downarrow, \tilde{\phi}_B^\rightarrow) \text{ is a } u\text{-anchor},$$

which finishes the proof for this case.

In identical fashion, we easily solve

Case 2. $\tilde{K} \not\subset L(\tilde{\phi}_{T,k'}^\downarrow)$. For this case we get that

$$(\tilde{A}_{k'}^\downarrow, \tilde{K}, \tilde{\phi}_T^\rightarrow, \tilde{\phi}_{T,k'}^\downarrow) \text{ is an } s\text{-anchor},$$

which also ends this case's analysis.

Case 3. $\tilde{K} \subset R(\tilde{\phi}_{B,k+1}^\downarrow)$ and $\tilde{K} \subset L(\tilde{\phi}_{T,k'}^\downarrow)$. For this case, we must have that $k' \leq k$: otherwise, with $k' > k$ we would have that \tilde{K} simultaneously holds cases 1 and 2, and it would then be s -anchored and u -anchored. See Figure 4 for details.

3.1. $k' = k$. Let us take \tilde{z}^\rightarrow a realizing point for \tilde{A}^\rightarrow with $\tilde{z}^\rightarrow \in \tilde{D}_{k-1}^\rightarrow$, and note that we must have $\tilde{z}^\rightarrow \in R_{\tilde{A}}(\tilde{K})$, because $\tilde{K} \subset L(\tilde{\phi}_{T,k'}^\downarrow)$. Now, given that

- $\tilde{f}^3(\tilde{z}^\rightarrow) \in \tilde{D}_{k+2}^\rightarrow \subset L(\tilde{\phi}_{B,k+1}^\downarrow)$
- $(\tilde{A}^\rightarrow, \tilde{K}, \tilde{\phi}_B^\rightarrow, \tilde{\phi}_T^\rightarrow)$ is a 3-semianchor,

we obtain that

- $\tilde{f}^3(\tilde{K}) \not\subset R(\tilde{\phi}_{B,k+1}^\downarrow)$
- $\tilde{f}^3(\tilde{K}) \cap \tilde{\phi}_{B,k+1}^\downarrow \neq \emptyset$, $\tilde{f}^3(\tilde{K}) \cap \tilde{\phi}_B^\rightarrow$

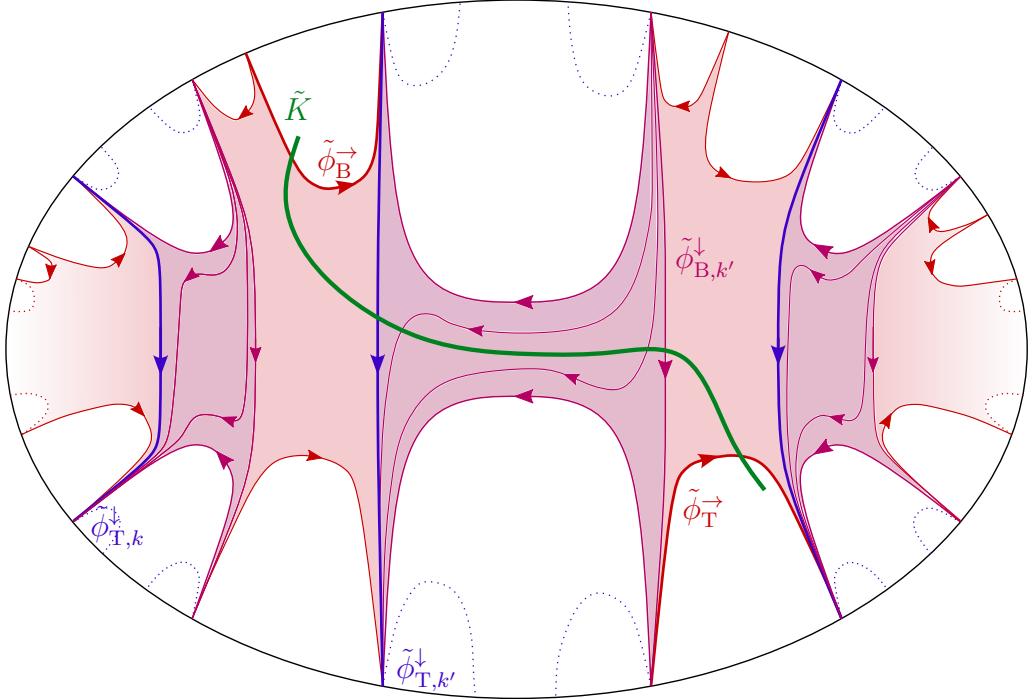


FIGURE 4. An example with $k' = k + 1$. Note that $(\tilde{A}_{k'}^\downarrow, \tilde{K}, \tilde{\phi}_T^\rightarrow, \tilde{\phi}_{T,k'}^\downarrow)$ is an s -anchor, and $(\tilde{A}_{k'}^\downarrow, \tilde{K}, \tilde{\phi}_{B,k'}, \tilde{\phi}_B^\rightarrow)$ is a u -anchor. This implies that $\lim_{n \rightarrow \pm\infty} \text{diam}(\widehat{f}^n(\tilde{K})) = \infty$.

from where we proceed as in Case 1 and conclude that

$$(\tilde{A}_{k+1}^\downarrow, \tilde{f}^3(\tilde{K}), \tilde{\phi}_{B,k+1}^\downarrow, \tilde{\phi}_B^\rightarrow) \text{ is a } u\text{-anchor,}$$

which concludes the analysis for this case. See Figure 5 for details.

3.2. $k' < k$. This implies that \tilde{A}_k^\downarrow separates $\tilde{\phi}_B^\rightarrow$ from $\tilde{\phi}_T^\rightarrow$. Let us define

$$\tilde{\phi}_{B,k}^\rightarrow = \partial_B \tilde{A}^\rightarrow \cap \tilde{A}_k^\downarrow, \quad \tilde{\phi}_{T,k}^\rightarrow = \partial_T \tilde{A}^\rightarrow \cap \tilde{A}_k^\downarrow.$$

If we have $\tilde{K} \cap \tilde{\phi}_{B,k}^\rightarrow \neq \emptyset$ (See Figure 6), we then get that

$$(\tilde{A}^\rightarrow, \tilde{K}, \tilde{\phi}_{B,k}^\rightarrow, \tilde{\phi}_T^\rightarrow) \text{ is an } s\text{-anchor.}$$

Finally, let assume that $\tilde{K} \cap \tilde{\phi}_{B,k}^\rightarrow = \emptyset$. Let us take $\tilde{z}^\downarrow \in R(\tilde{\phi}_{B,k}^\rightarrow) \cap \tilde{A}_k^\downarrow$, being the lift of z^\downarrow which holds that $\tilde{f}^2(\tilde{z}^\downarrow) \in L(\tilde{\phi}_{T,k}^\rightarrow)$. Given that

$$(\tilde{A}^\rightarrow, \tilde{K}, \tilde{\phi}_B^\rightarrow, \tilde{\phi}_T^\rightarrow) \text{ is a 2-semianchor}$$

we obtain that

$$(\tilde{A}_k^\downarrow, \tilde{K}, \tilde{\phi}_{B,k}^\downarrow, \tilde{\phi}_{T,k}^\downarrow) \text{ is also a 2-semianchor}$$

This fact implies that $\tilde{f}^2(\tilde{z}^\downarrow) \notin L_{\tilde{A}_k^\downarrow}(\tilde{K})$, which in particular shows that

$$\tilde{f}^2(\tilde{K}) \cap \tilde{\phi}_{T,k}^\rightarrow \neq \emptyset$$

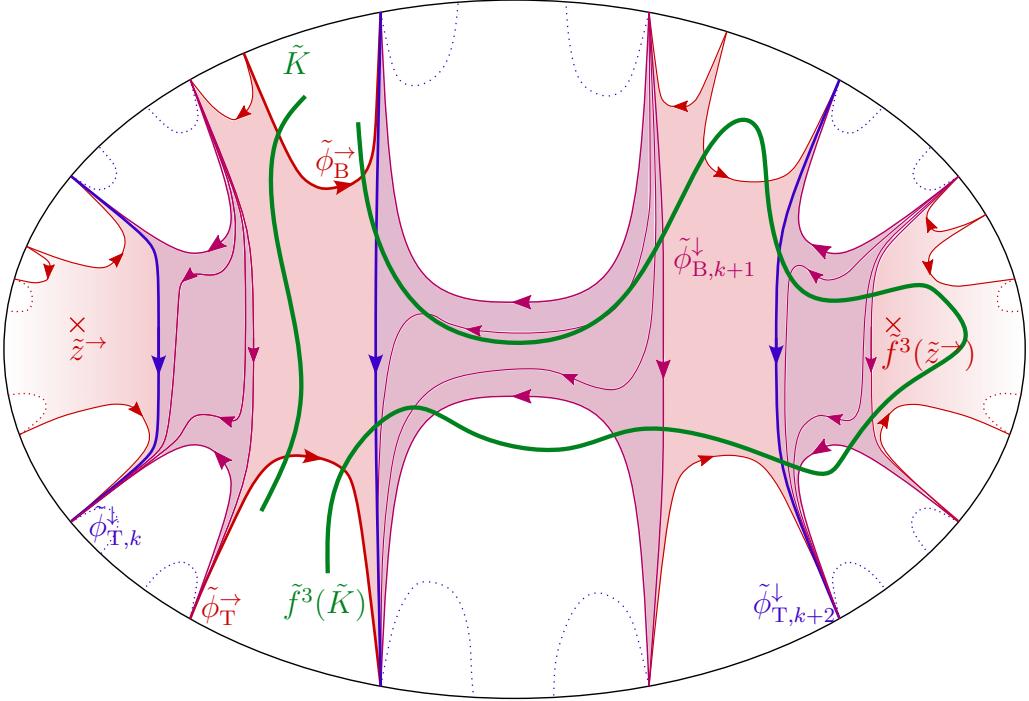


FIGURE 5. An example with $k' = k$. $\tilde{f}^3(\tilde{K})$ is u -anchored in many different ways, in particular $(\tilde{A}_{k+1}^\downarrow, \tilde{f}^3(\tilde{K}), \tilde{\phi}_{B,k+1}^\downarrow, \tilde{\phi}_B^\rightarrow)$ is a u -anchor.

from where we obtain that

$$(\tilde{A}^\rightarrow, \tilde{f}^2(\tilde{K}), \tilde{\phi}_B^\rightarrow, \tilde{\phi}_{T,k}^\rightarrow) \text{ is a } u\text{-anchor},$$

which concludes the proof. \square

In Case 3.2, $\tilde{f}^2(\tilde{K})$ is anchored to \tilde{A}^\rightarrow by construction. For Cases 1, 2 and 3.1, $f^j(\tilde{K})$ is anchored to some \tilde{A}_k^\downarrow (with $j = 0$ in Cases 1 and 2, $j = 3$ in Case 3.1). Repeating the argument from the last step of Case 3.2, it is easy to check that up to using two more iterations, we can anchor either $\tilde{f}^{j+2}(\tilde{K})$ or $\tilde{f}^{j-2}(\tilde{K})$ to \tilde{A}^\rightarrow .

This implies that

REMARK 4.19. If \hat{K} is 3-semianchored to \hat{A} , then at least one of the following occurs:

- For every $j \geq 5$, $\hat{f}^j(\hat{K})$ is u -anchored to \hat{A} , and so $\hat{f}^j(\hat{K}) \cap \text{cl}(\hat{A}) \neq \emptyset$,
- For every $j \leq -2$, $\hat{f}^j(\hat{K})$ is s -anchored to \hat{A} , and so $\hat{f}^j(\hat{K}) \cap \text{cl}(\hat{A}) \neq \emptyset$.

Also, note that combining Lemmas 4.7 and 4.18, we immediately obtain that

REMARK 4.20. Let \hat{f} be a lift to \mathbb{R}^2 of $f \in \text{Homeo}_0(\mathbb{T}^2)$, with $\vec{0} \in \text{int}(\rho(\hat{f}))$.

Then, there exists $M' > 0$ such that, for any continuum with $\text{diam}(\hat{K}) > M'$, there exists $0 \leq j \leq 3$ such that $\hat{f}^j(\hat{K})$ is anchored.

Let us now reiterate this argument and put everything together.

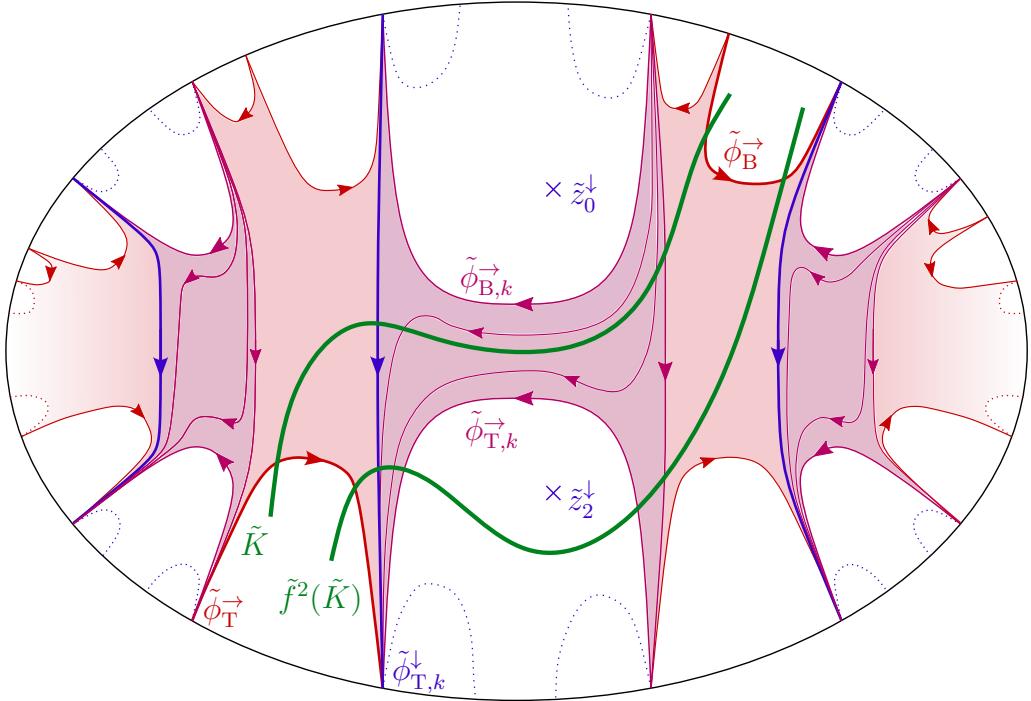


FIGURE 6. An example with $k' = k - 1$. Either \tilde{K} intersects $\tilde{\phi}_{B,k}^\rightarrow$, or $\tilde{f}^2(\tilde{K})$ intersects $\tilde{\phi}_{T,k}^\rightarrow$.

PROOF OF THEOREM C. By Remark 4.17, we may assume that \widehat{K} goes through no singularities. Take $M = 2M_3$ from Lemma 4.4 to see that any continuum \widehat{K} with $\text{diam}(\widehat{K}) > M$ is 3-semianchored, then use Lemma 4.18 to see that $\widehat{f}^j(\widehat{K})$ is anchored for some $0 \leq j \leq 3$, which by Proposition 4.15 implies that $\mathcal{D}_{\widehat{f}}(\widehat{K}) = \infty$, which concludes the proof. \square

Theorem E is now obtained as a consequence of Theorem C.

PROOF OF THEOREM E. By [KT13, Theorem B], we know that the lifted diameter of any periodic topological disk is finite (there is a shorter improved version of the proof of this result in [KLCT19, Theorem 6]). This implies that this disk is dynamically bounded, which then implies that its closure must be a dynamically bounded (inessential) continuum, whose diameter is uniformly bounded from Theorem C, which concludes the proof. \square

CHAPTER 5

The essential factor

The goal of this chapter is to build an explicit monotone semiconjugacy, for a torus homeomorphism f under the hypotheses of Theorem A, that is, a torus homeomorphism f which is isotopic to the identity, and such that $\rho(f)$ has nonempty interior. For the remainder of this section, let us assume these hypotheses hold and use the notation of said theorem. We will dedicate the following chapter to the basic understanding of the dynamics in the resulting quotient. In particular, let us recall that the dynamical diameter \mathcal{D} of a continuum $\hat{K} \in \mathbb{R}^2$ is defined as supremum of the diameters of the iterates of \hat{K} by a lift \hat{f} .

1. Dynamically bounded splitting

Let us define a relation $\sim_{\hat{f}}$ in \mathbb{R}^2 as follows:

DEFINITION 5.1. Let $\hat{z}, \hat{z}' \in \mathbb{R}^2$. We will say that $\hat{z} \sim_{\hat{f}} \hat{z}'$ if there exists a continuum $\hat{K} \in \mathbb{R}^2$ such that $\hat{z}, \hat{z}' \in \hat{K}$, and $\mathcal{D}_{\hat{f}}(\hat{K}) < \infty$.

It follows immediately that $\sim_{\hat{f}}$ is an equivalence relation, where we write $[\hat{z}]_{\hat{f}}$ for the equivalence class of \hat{z} under $\sim_{\hat{f}}$ (or simply $[z]$ and \sim when there is no ambiguity). Let us now understand the structure of these classes.

REMARK 5.2. For every $\hat{z} \subset \mathbb{R}^2$, $[\hat{z}] = \bigcup_{\hat{w} \in [\hat{z}]} \hat{K}_{\hat{w}}$, where $\hat{K}_{\hat{w}}$ is a dynamically bounded continuum containing both \hat{w} and \hat{z} . This implies $[\hat{z}]$ is connected.

LEMMA 5.3 (Uniformly bounded classes). *There exists $M > 0$, such that for every $\hat{z} \in \mathbb{R}^2$, we have $\text{diam}([\hat{z}]) < M$.*

PROOF. Take M as in Theorem C. Suppose by contradiction that we have $\text{diam}([\hat{z}]) > M$. Then, there exist $\hat{z}_1, \hat{z}_2 \in [\hat{z}]$ such that $d(\hat{z}_1, \hat{z}_2) > M$.

This implies that any continuum \hat{K} which contains both \hat{z}_1 and \hat{z}_2 has $\text{diam}(K) > M$. Using Theorem C, we instantly get that $\mathcal{D}(\hat{K}) = \infty$, which contradicts the fact that \hat{z}_1 and \hat{z}_2 belong to the same equivalence class. \square

COROLLARY 5.4. *For every $\hat{z} \in \mathbb{R}^2$, $[\hat{z}]$ is dynamically bounded. Furthermore, their closure is also dynamically bounded. This implies that $\text{cl}([\hat{z}]) \subset [\hat{z}]$, and in particular every class is closed.*

PROOF. The fact that every class is dynamically bounded is due to Lemma 5.3 and the fact that \hat{f} sends to classes to classes. The closure of class is also dynamically bounded because $\text{diam}(\text{cl}([\hat{z}])) = \text{diam}([\hat{z}])$. \square

REMARK 5.5. For every \widehat{z} , $[\widehat{z}]$ is filled, because $\text{diam}(\widehat{f}^j([\widehat{z}])) = \text{diam}(\text{Fill}(\widehat{f}^j([\widehat{z}])))$.

LEMMA 5.6. Let $\widehat{K} \subset \mathbb{R}^2$ be a dynamically bounded continuum for \widehat{f} . Then, for every deck transformation T , we have that $T(\widehat{K}) \cap \widehat{K} = \emptyset$.

PROOF. Suppose by contradiction that there exists a deck transformation $T(\widehat{z}) = \widehat{z} + v$ where $v \in \mathbb{Z}^2$ non-zero, such that $T(\widehat{K}) \cap \widehat{K} \neq \emptyset$. Start by building a necklace

$$\widehat{K}_T = \bigcup_{n \in \mathbb{Z}} T^n(\widehat{K}),$$

and note that $\text{diam}(\widehat{K}_T) = \infty$. Let us denote π_{v^\perp} to the projection onto the subspace generated by the vector v^\perp which is orthogonal to v . Given that $\mathcal{D}(\widehat{K}) < M$, we obtain that

$$\sup_{j \in \mathbb{Z}} \text{diam}(\pi_{v^\perp}(\widehat{f}^j(\widehat{K}_T))) = \sup_{j \in \mathbb{Z}} \text{diam}(\pi_{v^\perp}(\widehat{f}^j(\widehat{K}))) < M.$$

Note that the complement of \widehat{K}_T has two unbounded connected components U_+, U_- (think respectively of points with large positive and negative second coordinate in the positively-oriented base v, v^\perp). We shall use coord_{v^\perp} to denote this second coordinate. Again, up to taking a power and an adequate lift we will assume that $\vec{0} \in \text{int}(\rho(\widehat{f}))$. Then, we may take two periodic points $\widehat{z}_+ \in U_-$, $\widehat{z}_- \in U_+$ such that $\text{coord}_{v^\perp}(\rho(\widehat{z}_+)) > 0$ and $\text{coord}_{v^\perp}(\rho(\widehat{z}_-)) < 0$. This implies that for a sufficiently large value of j , we have that $\text{coord}_{v^\perp}(\widehat{f}^j(\widehat{z}_+)) - \text{coord}_{v^\perp}(\widehat{f}^j(\widehat{z}_-)) > 2M$, which on its turn implies that $\text{diam}(\pi_{v^\perp}(\widehat{f}^j(\widehat{K}_T))) > M$, which is a contradiction. \square

COROLLARY 5.7. For every \widehat{z} , we have that $\widehat{\pi}[\widehat{z}]$ is an inessential filled continuum, which is dynamically bounded.

PROOF. This is due to Lemmas 5.6 and 5.3 put together. \square

PROPOSITION 5.8. The equivalence relation $\sim_{\widehat{f}}$ is upper semicontinuous.

PROOF. First, note that it suffices to prove that the Hausdorff limit of a sequence of equivalence classes is entirely contained in an equivalence class.

Let $\{[\widehat{z}]_n\}_{n \in \mathbb{N}}$ be a sequence of equivalence classes such that $[\widehat{z}]_n \xrightarrow{n \rightarrow \infty} \widehat{K}$ for the Hausdorff topology. This implies that for every $\delta > 0$, there exists n_δ such that $\widehat{K} \subset B([\widehat{z}]_{n_\delta}, \delta)$.

Take $\varepsilon > 0$. Given that \widehat{f} is uniformly continuous, we get that for every $j \in \mathbb{N}$ there exists δ_j such that

$$\sup_{|i| \leq j} d_H(f^i([\widehat{z}]), f^i([\widehat{z}'])) < 1, \text{ whenever } d_H([\widehat{z}], [\widehat{z}']) < \delta_j.$$

For every $j \in \mathbb{Z}^+$, given that $\mathcal{D}([\widehat{z}]_{n_{\delta_j}}) \leq M$ and $d_H([\widehat{z}]_{n_{\delta_j}}, \widehat{K}) < \delta_j$, we obtain that

$$\sup_{|i| \leq j} \text{diam}(\widehat{f}^i(\widehat{K})) < M + 1,$$

and therefore $\mathcal{D}(\widehat{K}) < M+1$. We conclude that \widehat{K} is dynamically bounded, which concludes the proof. \square

Projecting the classes. We may now induce an equivalence class \sim_f for the torus, defined as

$$[z]_f = \widehat{\pi}([\widehat{z}]_{\widehat{f}}), \text{ where } \widehat{z} \text{ is any lift of } z,$$

which is equivalent to saying $z \sim_f z'$ if there exists a continuum K containing both z and z' , which is dynamically bounded.

From the results of last subsection, we recover the following

PROPOSITION 5.9 (Dynamically bounded Splitting). *Let \sim_f be an equivalence relation on the torus, whose classes are defined by $[z]_f = \widehat{\pi}([\widehat{z}]_{\widehat{f}})$. Then,*

- *Each equivalence class of \sim_f is an inessential filled continuum.*
- *Equivalence classes of \sim_f are uniformly dynamically bounded.*
- \sim_f is upper semicontinuous.

2. Monotone quotient

We have built a partition given by an equivalence relation which is upper semicontinuous, and whose classes are inessential filled continua. By a main result in [RS38], we know that the quotient topological space $T^2 = \mathbb{T}_{/\sim}^2$, equipped with the metric $d_{\sim}([z], [w]) = \min\{d(z', w') : z' \sim z, w' \sim w\}$ is homeomorphic to \mathbb{T}^2 (this is actually a generalization of a Moore's result for the sphere, which can be found in [Moo32]). Note that this induces a distance \widehat{d}_{\sim} on the universal covering \mathbb{R}^2 of T^2 (which is on its turn homeomorphic to \mathbb{R}^2), and allows us to measure the diameter of continua in T^2 . When there is no room for confusion, we will abuse notation and use d for all of these distances.

Given that f sends equivalence classes to equivalence classes, we immediately get natural dynamics g on the quotient space T^2 , defined as $g([z]) = [f(z)]$. Note that the quotient map $\pi_{\sim} : \mathbb{T}^2 \rightarrow T^2$ is monotone. We obtain the following diagram:

$$\begin{array}{ccc} \mathbb{T}^2 & \xrightarrow{f} & \mathbb{T}^2 \\ \pi_{\sim} \downarrow & & \downarrow \pi_{\sim} \\ T^2 & \xrightarrow{g} & T^2 \end{array}$$

DEFINITION 5.10. Given $f \in \text{Homeo}_0(\mathbb{T}^2)$ with $\text{int}(\rho(f)) \neq \emptyset$, and \sim the dynamically bounded equivalence relation, we will say that $g = f_{/\sim}$ is its essential factor.

REMARK 5.11. Given f under the General hypothesis and its essential factor g , we obtain lifts $\widehat{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\tilde{g} : \tilde{\mathbb{D}} \rightarrow \tilde{\mathbb{D}}$, proceeding in the same fashion as we did for f .

REMARK 5.12. A continuum K is essential in \mathbb{T}^2 if and only if $K_{\sim} = \pi_{\sim}(K)$ is essential in T^2 .

Keep in mind that the distance functions in \mathbb{T}^2 and T^2 are very much different. However, we may recover the following facts from the construction:

REMARK 5.13. Let $\gamma^\downarrow, \gamma^\rightarrow$ be the canonical π_1 generator of \mathbb{T}^2 , based at the origin. Then, their projections $\gamma_\sim^\downarrow, \gamma_\sim^\rightarrow$ to the quotient, are a π_1 generator of T^2 .

REMARK 5.14 (Canonical torus model). Our new torus T^2 is topologically conjugate to a canonical torus $S^1 \times S^1$. We may then think of T^2 as \mathbb{T}^2 in a canonical way: it is enough to choose a homeomorphism h such that $h(\gamma_\sim^\downarrow) = \gamma^\downarrow$, and $h(\gamma_\sim^\rightarrow) = \gamma^\rightarrow$

3. Quick dynamical consequences

We will start by stating some properties inherited by the essential factor g , due to the structure of f . The following is a purely topological fact.

CLAIM 5.15. *A continuum K has $\mathcal{D}_f(K) = \infty$ if and only if its projection K_\sim has $\mathcal{D}_g(K_\sim) = \infty$.*

PROOF. Having $\mathcal{D}_f(K) = \infty$ is equivalent to saying that there exists a sequence $\{n_j\}_{j \rightarrow \infty}$ such that $f^{n_j}(K)$ intersects a growing-to-infinity number of lifts $\hat{\gamma}$ of either γ^\downarrow or γ^\rightarrow . On its turn, this is equivalent to having that $g^{n_j}(K_\sim)$ intersects a growing-to-infinity number of lifts $\hat{\gamma}_\sim$ of either γ_\sim^\downarrow or γ_\sim^\rightarrow , which is equivalent to having $\mathcal{D}_g(K_\sim) = \infty$ \square

Note that not only our quotient has preserved the topological space, it has also preserved free homotopy classes of curves. We may also send the isotopy downstairs and note that

REMARK 5.16. The essential factor g is isotopic to the identity.

Rotation in the quotient. Recall that from the classical definition, given a standard plane torus \mathbb{T}^2 , we say that $v = (v_1, v_2) \in \rho(\hat{f})$ if and only if there exist $\hat{z}_k \in \mathbb{R}^2$, $n_k \rightarrow +\infty$ such that

$$\lim_{k \rightarrow \infty} \frac{\hat{f}^{n_k}(\hat{z}_k) - \hat{z}_k}{n_k} = v$$

Note that this is equivalent to *counting the intersections* with the canonical generators $\gamma^\downarrow, \gamma^\rightarrow$, that is, if we define $\hat{\gamma}_k$ to be a curve from \hat{z}_k to $\hat{f}^{n_k}(\hat{z}_k)$ in general position respect to the infinite lifts $\hat{\gamma}^\downarrow, \hat{\gamma}^\rightarrow$, and take $\gamma_k = \hat{\pi}(\hat{\gamma}_k)$ its projection to the torus, then

$$(5.1) \quad \lim_{k \rightarrow \infty} \frac{\hat{f}^{n_k}(\hat{z}_k) - \hat{z}_k}{n_k} = v \iff \lim_{k \rightarrow \infty} \frac{\gamma^\downarrow \wedge \gamma_k}{n_k} = v_1; \quad \lim_{k \rightarrow \infty} \frac{\gamma^\rightarrow \wedge \gamma_k}{n_k} = v_2$$

where \wedge denotes the classical intersection number with sign between two curves. This allows to understand rotation, which is more of a topological quantity rather than metric, for a homeomorphism of any torus, in particular for T^2 . Note that this notion is well defined, as the quotient yields the same result for the whole dynamically bounded class of \hat{z}_k .

PROPOSITION 5.17. *Let \hat{f} be a lift of $f \in \text{Homeo}_0(\mathbb{T}^2)$ with $\text{int}(\rho(f)) \neq \emptyset$, and let g be its essential factor. Then,*

- (1) *There exists a lift \hat{g} of g such that $\rho(\hat{g}) = \rho(\hat{f})$.*
- (2) *$\mathcal{D}_g(K_\sim) = \infty$ for every non-trivial continuum $K_\sim \subset \mathbb{T}^2$.*

PROOF. (1) As we have just explained above, $v = (v_1, v_2) \in \rho(\hat{g})$ if and only if

$$\lim_{k \rightarrow \infty} \frac{\gamma_\sim^\downarrow \wedge \gamma_{\sim,k}}{n_k} = v_1; \quad \lim_{k \rightarrow \infty} \frac{\gamma_\sim^\rightarrow \wedge \gamma_{\sim,k}}{n_k} = v_2$$

as in Equation 5.1 which lets us obtain the desired result given that the quotient π_\sim preserves the intersection number between curves, that is,

$$\gamma^\downarrow \wedge \gamma_k = \gamma_\sim^\downarrow \wedge \gamma_{\sim,k}; \quad \gamma^\rightarrow \wedge \gamma_k = \gamma_\sim^\rightarrow \wedge \gamma_{\sim,k}.$$

- (2) This is immediate from Claim 5.15 and the fact that the preimage $\pi_\sim^{-1}(K_\sim)$ of any non-trivial continuum $K_\sim \subset \mathbb{T}^2$ is dynamically unbounded.

□

From now on, and having taken the canonical model we already explained, we will abuse notation and write \mathbb{T}^2 for \mathbb{T}^2 , and similarly for the coverings \mathbb{R}^2 and $\tilde{\mathbb{D}}$. We will also eliminate the subindex \sim and write K instead of K_\sim for continua, z instead of z_\sim for points, and similarly for their lifts.

Let us now prove that g is tight, which follows easily from the fact that g is continuum-wise expansive.

PROPOSITION 5.18. *For each non-trivial continuum $K \subset \mathbb{T}^2$ we have that $h_{\text{top}}(g, K) > 0$.*

PROOF. Let us first prove that for every $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that: for every non-trivial continuum $K \subset \mathbb{T}^2$ with $\text{diam}(K) \geq \varepsilon$, there exists $-n_\varepsilon \leq n_K \leq n_\varepsilon$ such that

$$(5.2) \quad \text{diam}(g^{n_K}(K)) > 3\varepsilon$$

Suppose by contradiction that is not true. Then, there would exist a sequence of continua $\{K_n\}_{n \in \mathbb{N}}$, such that for every $n \in \mathbb{N}$ we have that

$$\text{diam}(K_n) \geq \varepsilon, \quad \text{diam}(g^j(K_n)) \leq 3\varepsilon, \quad \text{for every } -n \leq j \leq n$$

In this case, up to taking a subsequence we may assume that $K_n \rightarrow K$ in the Hausdorff topology. Then by the uniform continuity of \hat{g} we would have that

$$\mathcal{D}_g(K) \leq 3\varepsilon,$$

which contradicts the fact that g is infinitely continuum-wise expansive (see Proposition 5.17).

Let us now use the notation from Equation 1.1, and prove that for small values of ε , the number $s_n(\varepsilon, K)$ grows exponentially, that is,

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log(s_n(\varepsilon, K)) > 0.$$

We may assume that $\text{diam}(K) \geq \varepsilon$ up to taking an iterate by some power of g . It is now enough to prove that for every $j \in \mathbb{Z}^+$ we have that

$$(5.3) \quad s_{jn_\varepsilon}(\varepsilon, K) \geq 2^j$$

Take n_K from Equation 5.2. Given that $\text{diam}(g^{n_K}(K)) > 3\varepsilon$, we may take two subcontinua $K_0, K_1 \subset g^{n_K}(K)$ of diameter equal to ε , such that if we take one point in each of them, they are at distance greater than ε from each other. This means that we can take $z_0, z_1 \in K$ which are $(n_\varepsilon, \varepsilon)$ separated, that is $s_{n_\varepsilon}(\varepsilon, K) \geq 2$.

Reiterating this process for K_0 and K_1 we obtain two respective subcontinua $K_{00}, K_{01} \subset g^{n_{K_0}}(K_0)$, $K_{10}, K_{11} \subset g^{n_{K_1}}(K_1)$ of an iterate of each of them, such that points in K_{00} and points in K_{01} are at distance greater than ε from each other, and similarly for points in K_{10} with respect to points K_{11} . This allows to take four points $z_{00}, z_{01}, z_{10}, z_{11}$ in K which are $(2n_\varepsilon, \varepsilon)$ -separated. By repeating this process we obtain the equality from Equation 5.3, which concludes the proof. \square

Guided by the two properties held by the essential factor which were proved in Proposition 5.17, we shall introduce the following notion.

DEFINITION 5.19. We will say a (torus) homeomorphism $g \in \text{Homeo}_0(\mathbb{T}^2)$ is *frice* (i.e. fully rotational, infinitely continuum-wise expansive) if

- $\text{int}(\rho(g)) \neq \emptyset$,
- $\mathcal{D}_g(K) = \infty$ for every non-trivial continuum $K \subset \mathbb{T}^2$.

By Proposition 5.17 and Remark 5.16, we have that the essential factor of a torus homeomorphism in the General Hypothesis, is frice. Moreover, if we take a frice torus homeomorphism, then its essential factor will be the map itself, because the equivalence classes for \sim will be trivial. We then have that

REMARK 5.20. A torus homeomorphism g is frice, if and only if it is the essential factor of a torus homeomorphism f in the General Hypothesis.

For the sake of clarity, we will generally state future results for the essential factor by using the definition of frice (torus) homeomorphism instead.

Throughout the following sections, we will use some of the techniques and results we developed in Sections 3 and 4 for the context of frice homeomorphisms, as they satisfy the General Hypothesis. Keep in mind that in this context, any essential continuum is dynamically unbounded both to the past and to the future, therefore, all the anchoring techniques will be done for inessential continua.

CHAPTER 6

Stable and unstable sets

This brief chapter is devoted to introducing and studying the following three stability-related notions, which will be strongly used in Sections 7 and 8:

- Weakly stable and weakly unstable continua,
- Stable and unstable sets of a point,
- Stable and unstable continua.

1. Weak stability

DEFINITION 6.1. Let \hat{f} be a lift to \mathbb{R}^2 of a homeomorphism f in the General Hypothesis. We will say that a continuum \hat{K} is *weakly stable* for \hat{f} if there exists $n < 0$ such that $\hat{f}^n(\hat{K})$ is s -anchored, and analogously it is *weakly unstable* for \hat{f} if there exists $n > 0$ such that $\hat{f}^n(\hat{K})$ is u -anchored.

In this context, the projection $K = \hat{\pi}(\hat{K})$ to the torus will also be called a respectively a weakly stable (unstable) continuum for f .

We will generally use \hat{K}^{ws} and \hat{K}^{wu} for weakly stable and weakly unstable continua of the plane, respectively.

We saw in Proposition 5.17, that the dynamical diameter for a frice torus homeomorphism g , of every non-trivial continuum, is infinite. This fact, together with the structure of canonically foliated strips we built in Chapter 4 applied for g , gives us the following:

LEMMA 6.2. *Let \hat{g} be a lift to \mathbb{R}^2 of a frice torus homeomorphism g .*

Then, any non-trivial continuum $\hat{K} \subset \mathbb{R}^2$ is either weakly stable or weakly unstable for \hat{g} (it may very well be both).

PROOF. Up to taking a power and changing the lift, we assume that $\vec{0} \in \text{int}(\rho(\hat{g}))$. Let $\hat{K} \subset \mathbb{R}^2$ be a non-trivial continuum. By Proposition 5.17, we know that $\mathcal{D}_{\hat{g}}(\hat{K}) = \infty$. In particular, there exists j_0 such that $\text{diam}(\hat{g}^{j_0}(\hat{K})) > M'$, where M' is taken as in Remark 4.20. By this same remark, we obtain that there exists $j_0 \leq j \leq j_0 + 3$ such that $\hat{g}^j(\hat{K})$ is anchored, which concludes the proof. \square

2. Shape and size

DEFINITION 6.3. Let \hat{f} be a lift to \mathbb{R}^2 of a homeomorphism f in the General Hypothesis, and let $\hat{z} \in \mathbb{R}^2$. Then,

- (1) The ε -stable set of \hat{z} for \hat{f} , is defined as

$$W_{\varepsilon, f}^s(\hat{z}) = \{\hat{z}' \in \mathbb{R}^2 : d(\hat{f}^j(\hat{z}'), \hat{f}^j(\hat{z})) \leq \varepsilon, \text{ for every } j \geq 0\},$$

(2) Similarly, the ε -unstable set of \widehat{z} for \widehat{f} is defined as

$$W_{\varepsilon,f}^u(\widehat{z}) = \{\widehat{z}' \in \mathbb{R}^2 : d(\widehat{f}^j(\widehat{z}'), \widehat{f}^j(\widehat{z})) \leq \varepsilon, \text{ for every } j \leq 0\},$$

(3) For $z \in \mathbb{T}^2$, $z = \widehat{\pi}(\widehat{z})$, we define

$$W_{\varepsilon,f}^s(z) = \widehat{\pi}(W_{\varepsilon,f}^s(\widehat{z})), \quad W_{\varepsilon,f}^u(z) = \widehat{\pi}(W_{\varepsilon,f}^u(\widehat{z})).$$

(4) $K_{\varepsilon,f}^s(\widehat{z})$ is defined as the connected component of $W_{\varepsilon,f}^s(\widehat{z})$ containing \widehat{z} ,

(5) $K_{\varepsilon,f}^u(\widehat{z})$ is defined as the connected component of $W_{\varepsilon,f}^u(\widehat{z})$ containing \widehat{z} ,

(6) For $z \in \mathbb{T}^2$, $z = \widehat{\pi}(\widehat{z})$, we define

$$K_{\varepsilon,f}^s(z) = \widehat{\pi}(K_{\varepsilon,f}^s(\widehat{z})), \quad K_{\varepsilon,f}^u(z) = \widehat{\pi}(K_{\varepsilon,f}^u(\widehat{z})).$$

We will omit the homeomorphism as a subindex when there is no room for ambiguity.

DEFINITION 6.4 (Stable set of a point). Let \widehat{f} be a lift to \mathbb{R}^2 of a torus homeomorphism f in the General Hypothesis, and let $\widehat{z} \in \mathbb{R}^2$. We define the *stable set of \widehat{z}* and the *unstable set of \widehat{z}* for \widehat{f} , respectively as

$$K^s(\widehat{z}) = \bigcup_{\varepsilon > 0} K_\varepsilon^s(\widehat{z}), \quad K^u(\widehat{z}) = \bigcup_{\varepsilon > 0} K_\varepsilon^u(\widehat{z}).$$

Note that the orbit of every $\widehat{z}' \in K^s(\widehat{z})$ remains at a bounded distance from the orbit of \widehat{z} . Again, for $z \in \mathbb{T}^2$ with $z = \widehat{\pi}(\widehat{z})$, we define

$$K^s(z) = \widehat{\pi}(K^s(\widehat{z})).$$

DEFINITION 6.5 (Stable Continuum). Let \widehat{f} be a lift to \mathbb{R}^2 of a torus homeomorphism f in the General Hypothesis. We will say that $\widehat{K} \subset \mathbb{R}^2$ is a *stable continuum* if $\widehat{K} \subset K^s(\widehat{z})$ for some $z \in \mathbb{R}^2$, and it will be an *unstable continuum* if $\widehat{K} \subset K^u(\widehat{z})$ for some $z \in \mathbb{R}^2$.

While the definitions in this section can be applied in a broader context, the work in what follows will be done for Fréchet torus homeomorphisms, which we shall denote by g .

Recall that by Proposition 5.17, we have that any such g is continuum-wise expansive (i.e. there exists a positive uniform lower bound for the dynamical diameter of non-trivial continua), and so is any lift \widehat{g} of g to \mathbb{R}^2 . We may then apply a result from Rodríguez Hertz, which can be found in [Her04, Theorem 0.2], and obtain the following:

LEMMA 6.6. *Let g be a Fréchet torus homeomorphism. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\inf_{z \in \mathbb{T}^2} \text{diam}(K_\varepsilon^s(z)) \geq \delta, \quad \inf_{z \in \mathbb{T}^2} \text{diam}(K_\varepsilon^u(z)) \geq \delta$$

In particular, for every $z \in \mathbb{T}^2$ we have that $K^s(z)$ is nonempty.

Note that this result for the diameter of stable sets has its natural analogous in \mathbb{R}^2 when applied to a lift \widehat{g} of g .

LEMMA 6.7. Let \widehat{g} be a lift to \mathbb{R}^2 of a frice torus homeomorphism g . Then, any non-trivial stable continuum is also weakly stable, and any non-trivial unstable continuum is also weakly unstable. In particular, they project onto inessential continua.

PROOF. It suffices to prove the result for stable continua, as the other case is analogous. Let $\widehat{K} \subset K^s(\widehat{z})$ be a non-trivial stable continuum for \widehat{g} . Note that there exists $\varepsilon > 0$ such that $\widehat{K} \subset K_\varepsilon^s(\widehat{z})$, which implies that

$$\sup_{j \in \mathbb{Z}^+} \text{diam}(\widehat{g}^j(\widehat{K})) \leq 2\varepsilon.$$

Given the diameter of \widehat{K} does not grow to infinity for future iterates, we obtain by Proposition 4.15 that \widehat{K} is not weakly unstable. Then, Remark 6.2 yields that \widehat{K} is weakly stable, which concludes the proof. \square

LEMMA 6.8. Let \widehat{g} be a lift to \mathbb{R}^2 of a frice torus homeomorphism. Then, for every open set $\widehat{V} \subset \mathbb{R}^2$, there exist two non-trivial continua $\widehat{K}^{ws}, \widehat{K}^{wu} \subset \widehat{V}$, which are respectively weakly stable and weakly unstable continua for \widehat{g} .

PROOF. Let $\widehat{V} \subset \mathbb{R}^2$ be an open set. Take $\widehat{z} \in \widehat{V}$, let $\varepsilon < d(\widehat{z}, \partial\widehat{V})$, and note that $K_\varepsilon^s(\widehat{z})$ is a stable continuum, which is non-trivial by Lemma 6.6, and is also included in \widehat{V} . By Lemma 6.7, we obtain that $\widehat{K}^{ws} = K_\varepsilon^s(\widehat{z})$ is a weakly stable continuum. We proceed for weakly unstable continua in identical fashion and thus conclude the proof. \square

As a consequence we obtain the following two results:

REMARK 6.9. Let \widehat{g} be a lift to \mathbb{R}^2 of a frice torus homeomorphism g . Then, any closed ball in \mathbb{R}^2 is simultaneously weakly stable and weakly unstable for \widehat{g} .

REMARK 6.10. Let \widehat{g} be a lift to \mathbb{R}^2 of a frice torus homeomorphism g . Then, for every $\varepsilon > 0$, both $K_\varepsilon^s(\widehat{z})$ and $K_\varepsilon^u(\widehat{z})$ are nonempty, filled and have empty interior. Moreover, any stable or unstable continuum is filled and has empty interior.

3. Dynamical properties

We will study some properties of the stable sets, which have their natural analogous versions for unstable sets. We obtain results using the anchoring techniques developed in Chapter 4, the most important one of the section being Proposition 6.14.

LEMMA 6.11. For every $\widehat{z}' \in K^s(\widehat{z})$, we have that

$$\lim_{j \rightarrow +\infty} d(\widehat{g}^j(\widehat{z}'), \widehat{g}^j(\widehat{z})) = 0$$

PROOF. Suppose by contradiction that there exists $\varepsilon > 0$, and $\widehat{z}' \in K_\varepsilon^s(\widehat{z})$ such that $\lim_{j \rightarrow +\infty} d(\widehat{g}^j(\widehat{z}'), \widehat{g}^j(\widehat{z})) \neq 0$. This implies that there exists a sequence $\{j_n\}_{n \in \mathbb{N}}$ of future iterates, with $j_n \rightarrow +\infty$ and some $\delta > 0$ such that

$$\text{diam}(\widehat{g}^{j_n}(K_\varepsilon^s(\widehat{z}))) \geq \delta.$$

We shall use \widehat{K}_n to denote the connected component of $\widehat{g}^{j_n}(K_\varepsilon^s(\widehat{z})) \cap B(\widehat{g}^{j_n}(\widehat{z}), \delta)$ containing $\widehat{g}^{j_n}(\widehat{z})$, which is a continuum of diameter between δ and 2δ , and we will write $K_n = \widehat{\pi}(\widehat{K}_n) \subset \mathbb{T}^2$. Now, given that the space of continua of a compact space, equipped with the Hausdorff topology, is compact, we have that up to taking a subsequence,

$$K_n \xrightarrow[\text{Hff}]{n \rightarrow +\infty} K.$$

where $\delta \leq \text{diam}(K) \leq 2\delta$.

Fix $m > 0$, and let us check that $\text{diam}(g^m)(K) \leq 4\varepsilon$. For this, take $\delta_{m,\varepsilon} > 0$ from the uniform continuity of g , such that any two points $\delta_{m,\varepsilon}$ -near, must be ε -near for m iterates of g . Now, simply take n large enough such that

$$\text{d}_H(K_n, K) < \delta_{m,\varepsilon}.$$

Take any pair of points $w', w'' \in K$ and note that there exist $z', z'' \in K_n$ with

$$\text{d}(z', w') < \delta_{m,\varepsilon}, \text{ and } \text{d}(z'', w'') < \delta_{m,\varepsilon}.$$

Moreover,

$$\text{d}(g^m(z'), g^m(z'')) \leq \text{d}(g^{m+j_n}(z), g^m(z'')) + \text{d}(g^m(z'), g^{m+j_n}(z)) \leq 2\varepsilon,$$

from where we obtain that

$$\begin{aligned} & \text{d}(g^m(w'), g^m(w'')) \\ & \leq \text{d}(g^m(w'), g^m(z')) + \text{d}(g^m(z'), g^m(z'')) + \text{d}(g^m(z''), g^m(w'')) \leq \varepsilon + 2\varepsilon + \varepsilon \leq 4\varepsilon. \end{aligned}$$

Let us prove the same assertion for a negative value $-m < 0$. Note that any two points $z', z'' \in K_n$ are 2ε -near for all of its future, but we also have that for any $m \leq j_n$,

$$\text{d}(g^{-m}(z'), g^{-m}(z'')) \leq \text{d}(g^{-m+j_n}(z), g^{-m}(z'')) + \text{d}(g^{-m}(z'), g^{-m+j_n}(z)) \leq 2\varepsilon,$$

simply because $z', z'' \in g^{j_n}(K_\varepsilon^s(z))$.

The proof for negative values $-m < 0$ is then analogous: it is enough to note that given any pair of points $w', w'' \in K$, there exist $z', z'' \in K_n$ with

$$\text{d}(z', w') < \delta_{m,\varepsilon}, \quad \text{d}(z'', w'') < \delta_{m,\varepsilon}, \quad \text{and } j_n \geq m,$$

and repeat the previous computations for positive values of m .

We have just constructed a non-trivial dynamically bounded continuum K , which is a direct contradiction with the second item in Proposition 5.17. \square

Recall that by Lemma 6.7, we know that the projections $K_\varepsilon^s(z) = \widehat{\pi}(K_\varepsilon^s(\widehat{z}))$, $K^u(z) = \widehat{\pi}(K^u(\widehat{z}))$ to the torus, are inessential continua.

The following result strongly uses the techniques developed in Chapter 4.

LEMMA 6.12. *Fix $d > 0$. Then, there exists $L_d > 0$, such that for every stable continuum \widehat{K} with $\text{diam}(\widehat{K}) > L_d$, we have that*

$$\text{diam}(\widehat{g}^j(\widehat{K})) > d, \text{ for every } j < 0.$$

PROOF. As in Chapter 4, let us work with vertical diameter Vdiam without loss of generality. Take \widehat{A}^\rightarrow a canonically foliated strip. Take M_3 from Lemma 4.7, and taking

$$L_d = 2M_3 + 2d + 1,$$

we obtain that any continuum \widehat{K} with $\text{Vdiam}(\widehat{K}) > L_d$ is 3-semianchored to two lifts $\widehat{A}_k^\rightarrow, \widehat{A}_{k'}^\rightarrow$ of the same canonically foliated strip A^\rightarrow , and such that the vertical distance d_V between the closure of these strips holds

$$d_V(\widehat{A}_k^\rightarrow, \widehat{A}_{k'}^\rightarrow) \geq 2d.$$

Given that no iterate of \widehat{K} can be u -anchored since \widehat{K} is contained in a stable set, we obtain from Corollary 4.16 and Remark 4.19 that the past iterates of \widehat{K} intersect the closure of both strips, from where we obtain

$$\text{diam}(\widehat{g}^j(\widehat{K})) \geq 2d, \text{ for every } j < 0,$$

which concludes the proof. \square

The following result is the contrapositive of what we have just proved.

COROLLARY 6.13. *Fix $d > 0$. Then, there exists $L_d > 0$, such that for any stable continuum K with $\text{diam}(\widehat{K}) \leq d$, we have that*

$$\text{diam}(\widehat{g}^j)(\widehat{K}) \leq L_d, \text{ for every } j > 0.$$

We finish the section by proving the following key result, which will be used in Chapter 8.

PROPOSITION 6.14 (**Hausdorff limit of small stable continua**). *Let $\{\widehat{K}_n\}_{n \in \mathbb{N}}$ be a sequence of stable continua $\widehat{K}_n \subset K^s(\widehat{z}_n)$, such that $\text{diam}(\widehat{K}_n) \leq d$ for every $n > 0$ and some $d > 0$. Suppose that*

$$\widehat{K}_n \xrightarrow[n \rightarrow +\infty]{\text{Hff}} \widehat{K}.$$

Then, \widehat{K} is also a stable continuum.

The proof is essentially the same as in Lemma 6.11.

PROOF. Fix $\widehat{w} \in \widehat{K}$. We will prove that for any other $\widehat{w}' \in \widehat{K}$ and every $j > 0$, we have that $d(\widehat{g}^j(\widehat{w}), \widehat{g}^j(\widehat{w}')) \leq L_d$, which is enough to finish the lemma as it implies that $\widehat{K} \subset K_{L_d}^s(w)$.

Fix $j_0 > 0$ and $\varepsilon > 0$, and take $\delta_{j_0, \varepsilon}$ from the uniform continuity of \widehat{g} , as in Lemma 6.11. Take a large enough value of n such that $d_H(\widehat{K}_n, \widehat{K}) < \delta_{j_0, \varepsilon}$. We may then finish with a similar argument to the one in Lemma 6.11: first, take $z, z' \in \widehat{K}_n$ such that

$$d(\widehat{z}, \widehat{w}) < \delta_{j_0, \varepsilon}, \quad d(\widehat{z}', \widehat{w}') < \delta_{j_0, \varepsilon},$$

which means that

$$d(\widehat{g}^j(\widehat{z}), \widehat{g}^j(\widehat{w})) < \varepsilon, \quad d(\widehat{g}^j(\widehat{z}'), \widehat{g}^j(\widehat{w}')) < \varepsilon, \text{ for every } 0 \leq j \leq j_0.$$

By Corollary 6.13, we know that

$$d(\widehat{g}^j(\widehat{z}), \widehat{g}^j(\widehat{z}')) \leq L_d, \text{ for every } j \geq 0.$$

Then, by the triangle inequality, we obtain that

$$d(\widehat{g}^j(\widehat{w}), \widehat{g}^j(\widehat{w}')) < L_d + 2\varepsilon, \text{ for every } 0 \leq j \leq j_0.$$

Taking j_0 growing to ∞ and ε going to 0, we obtain the desired result. \square

We finish the chapter with a sufficient condition for continua to be weakly (un)stable.

LEMMA 6.15. *Let \widehat{f} be a lift of a torus homeomorphism $f \in \text{Homeo}_0(\mathbb{T}^2)$ with $\vec{0} \in \text{int}(\rho(\widehat{f}))$. Let $\widehat{z}_1, \widehat{z}_2$ be two periodic points with different associated rotation directions, and let $d > 0$.*

Then, every continuum \widehat{K} such that

$$\widehat{K} \cap K_m^s(\widehat{z}_1) \neq \emptyset, \quad \widehat{K} \cap K_m^s(\widehat{z}_2) \neq \emptyset,$$

is weakly unstable.

PROOF. The proof uses the anchoring techniques from Chapter 4. Let the respective rotation vectors for the periodic points be given by

$$\rho(\widehat{z}_1) = (a_1, b_1), \quad \rho(\widehat{z}_2) = (a_2, b_2), \quad \text{with } \frac{b_1}{a_1} \neq \frac{b_2}{a_2}.$$

We will assume without loss of generality that the respective rotation vectors hold $a_1, a_2, b_1, b_2 < 0$. The proof for the other cases is analogous, up to taking a different CFS.

Let us take a continuum $\widehat{K} \subset \widehat{V}$ such that

$$\widehat{K} \cap K_m^s(\widehat{z}_1) \neq \emptyset, \quad \widehat{K} \cap K_m^s(\widehat{z}_2) \neq \emptyset$$

and note that

$$\widehat{C} := K_m^s(\widehat{z}_1) \cup \widehat{K} \cup K_m^s(\widehat{z}_2) \text{ is connected.}$$

Note that

$$\sup_{j \in \mathbb{Z}^+} \text{diam}(\widehat{f}^j(K_m^s(\widehat{z}_1))) \leq 2m, \quad \sup_{j \in \mathbb{Z}^+} \text{diam}(\widehat{f}^j(K_m^s(\widehat{z}_2))) \leq 2m.$$

Up to taking sufficiently large negative values of k , we have that $\widehat{C} \subset B(\widehat{A}_k^\rightarrow)$.

Note that, given the diameter of leaves $\widehat{\phi} \in \widehat{\mathcal{F}}$ is uniformly bounded, and that the slopes of the vectors $\rho(\widehat{z}_1)$ and $\rho(\widehat{z}_2)$ are different, we know that we may take $k < 0$ with $\widehat{C} \subset B(\widehat{A}_k^\rightarrow)$, and such that the sets of leaves $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$ are nonempty and disjoint, where

$$\widehat{\Phi}_1 = \{\widehat{\phi} \subset \partial_B \widehat{A}_k^\rightarrow : \widehat{\phi} \cap \widehat{I}_{\widehat{\mathcal{F}}}^+(\widehat{z}_1) \neq \emptyset\},$$

$$\widehat{\Phi}_2 = \{\widehat{\phi} \subset \partial_B \widehat{A}_k^\rightarrow : \widehat{\phi} \cap \widehat{I}_{\widehat{\mathcal{F}}}^+(\widehat{z}_2) \neq \emptyset\}.$$

Given that $b_1, b_2 < 0$, we can take $j_0 > 0$ such that

$$\widehat{f}^{j_0}(\widehat{z}_1) \in T(\widehat{A}_k^\rightarrow), \quad \widehat{f}^{j_0}(\widehat{z}_2) \in T(\widehat{A}_k^\rightarrow),$$

$$d(\widehat{f}^{j_0}(\widehat{z}_1), \widehat{A}_k^\rightarrow) > m, \quad d(\widehat{f}^{j_0}(\widehat{z}_2), \widehat{A}_k^\rightarrow) > m,$$

which means that

$$(6.1) \quad \widehat{f}^{j_0}(K_m^s(\widehat{z}_2)) \subset T(\widehat{A}_k^\rightarrow), \quad \widehat{f}^{j_0}(K_m^s(\widehat{z}_1)) \subset T(\widehat{A}_k^\rightarrow)$$

Let us check that $\widehat{f}^{j_0}(\widehat{K})$ is u -anchored. Take two leaves $\widehat{\phi}_1 \in \widehat{\Phi}_1$, $\widehat{\phi}_2 \in \widehat{\Phi}_2$. The key observation here is that given $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$ are disjoint, we have that for every pair of lifts $\tilde{\phi}_1, \tilde{\phi}_2$ of $\widehat{\phi}_1, \widehat{\phi}_2 \subset \widehat{A}_k^\rightarrow$ which hold that $\tilde{I}_{\tilde{\mathcal{F}}}^+(\tilde{z}_1) \cap \tilde{\phi}_1 \neq \emptyset$, $\tilde{I}_{\tilde{\mathcal{F}}}^+(\tilde{z}_2) \cap \tilde{\phi}_2 \neq \emptyset$, we have that $\tilde{\phi}_2 \subset R(\tilde{\phi}_1)$, $\tilde{\phi}_1 \subset R(\tilde{\phi}_2)$, which means that if we take two lifts $\tilde{A}_{k,1}^\rightarrow, \tilde{A}_{k,2}^\rightarrow$ of $\widehat{A}_k^\rightarrow$ such that

$$\tilde{I}_{\tilde{\mathcal{F}}}^+(\tilde{z}_1) \cap \tilde{A}_{k,1}^\rightarrow \neq \emptyset, \quad \tilde{I}_{\tilde{\mathcal{F}}}^+(\tilde{z}_2) \cap \tilde{A}_{k,2}^\rightarrow \neq \emptyset,$$

we will have that

$$\tilde{A}_{k,2}^\rightarrow \subset B(\tilde{A}_{k,1}^\rightarrow), \quad \tilde{A}_{k,1}^\rightarrow \subset B(\tilde{A}_{k,2}^\rightarrow).$$

See Figure 7 for details.

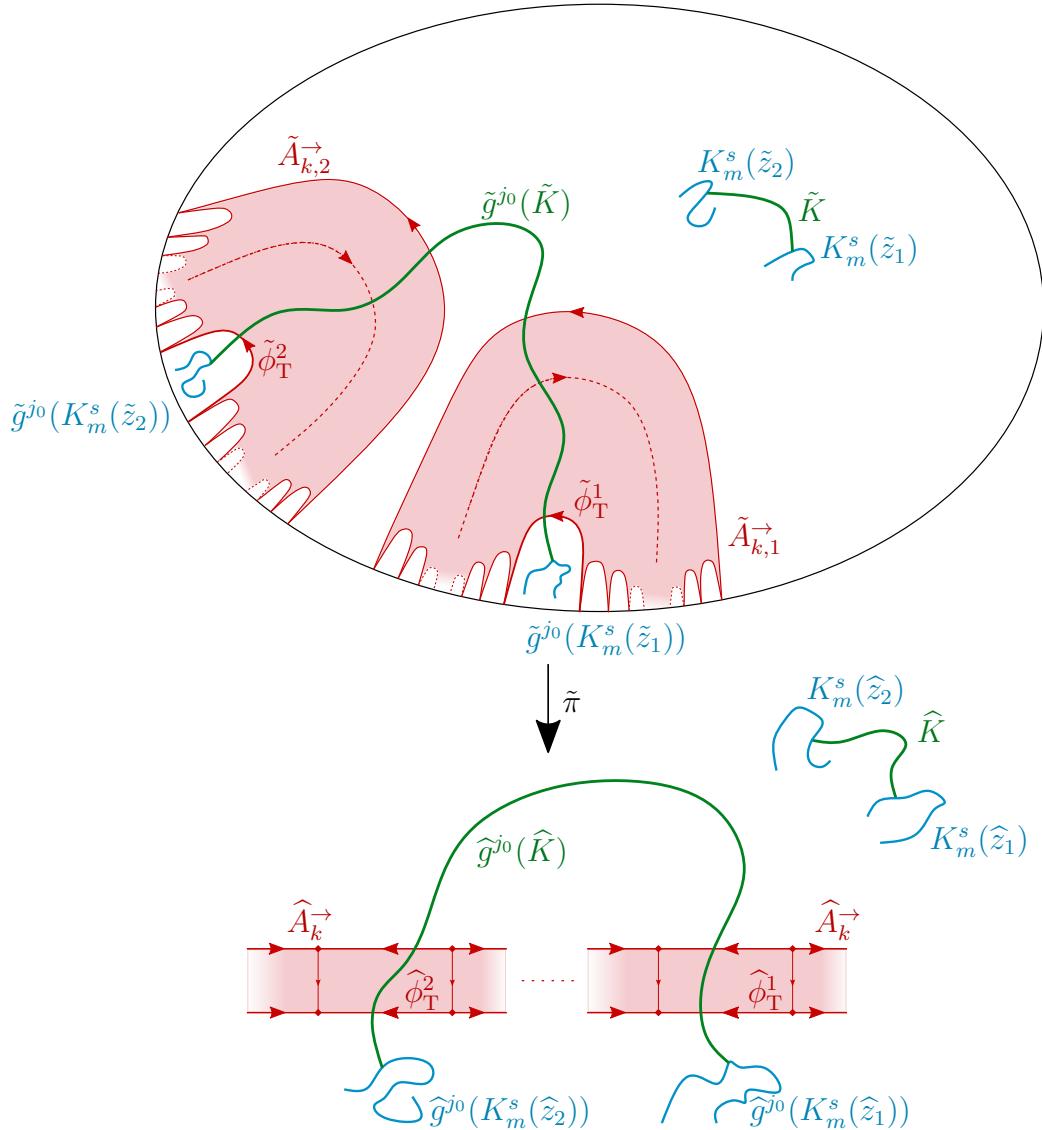


FIGURE 7. Stable sets appear in light blue, the continua connecting each respective pair of stable sets appear in green. Note that both $(\tilde{A}_{k,1}^\rightarrow, \tilde{f}^{j_0}(\tilde{K}), \tilde{\phi}_T^2, \tilde{\phi}_T^1)$ and $(\tilde{A}_{k,2}^\rightarrow, \tilde{f}^{j_0}(\tilde{K}), \tilde{\phi}_T^1, \tilde{\phi}_T^2)$ are u -anchors.

This fact, together with equation 6.1, shows that there exist two lifts $\tilde{A}_{k,1}^\rightarrow, \tilde{A}_{k,2}^\rightarrow$ of $\widehat{A}_k^\rightarrow$, two leaves $\tilde{\phi}_T^1, \tilde{\phi}_T^2 \subset T(\widehat{A}_k^\rightarrow)$ with their respective lifts $\tilde{\phi}_T^1 \subset T(\tilde{A}_{k,1}^\rightarrow)$ and $\tilde{\phi}_T^2 \subset T(\tilde{A}_{k,2}^\rightarrow) \subset B(\tilde{A}_{k,1}^\rightarrow)$, such that by taking natural lifts we obtain

$$(6.2) \quad \tilde{f}^{j_0}(K_m^s(\tilde{z}_1)) \subset T_{\tilde{\phi}_T^1}(\tilde{A}_{k,1}^\rightarrow), \quad \tilde{f}^{j_0}(K_m^s(\tilde{z}_2)) \subset T_{\tilde{\phi}_T^2}(\tilde{A}_{k,2}^\rightarrow) \subset B(\tilde{A}_k^\rightarrow),$$

which implies that

$$\tilde{f}^{j_0}(\tilde{K}) \cap \tilde{\phi}_T^1 \neq \emptyset, \quad \tilde{f}^{j_0}(\tilde{K}) \cap \tilde{\phi}_T^2 \neq \emptyset.$$

We also obtain that

$$\tilde{A}_{k,1}^\rightarrow \subset R(\tilde{\phi}_T^1), \quad \tilde{A}_{k,1}^\rightarrow \subset R(\tilde{\phi}_T^2),$$

which means that

$$(\tilde{A}_{k,1}^\rightarrow, \tilde{f}^{j_0}(\tilde{K}), \tilde{\phi}_T^2, \tilde{\phi}_T^1) \text{ is a } u\text{-anchor},$$

which shows that \widehat{K} is a weakly unstable continuum, and thus finishes the proof. \square

CHAPTER 7

The essential factor is rotationally mixing

So far, we have obtained the following three of the six properties described in Theorem A for the essential factor, proved in Proposition 5.17:

- There exists a lift \widehat{g} of g such that $\rho(\widehat{g}) = \rho(\widehat{f})$,
- g is infinitely continuum-wise expansive.
- g is tight: for each non-trivial continuum $K \subset \mathbb{T}^2$, we have $h_{\text{top}}(g, K) > 0$,

One of the goals of the chapter is to prove the following property:

- g is topologically mixing, and if $\vec{0} \in \text{int}(\rho(\widehat{g}))$, \widehat{g} is topologically mixing,

the other one being to prove Theorem D, which we will do at the end of the Section.

Let us show that for the first purpose, it is enough to prove Proposition 7.1, as we immediately discuss.

PROPOSITION 7.1. *Let $g \in \text{Homeo}_0(\mathbb{T}^2)$ be a frice torus homeomorphism. Then, if \widehat{g} is a lift of g with $\vec{0} \in \text{int}(\rho(\widehat{g}))$, then \widehat{g} is topologically mixing.*

COROLLARY 7.2. *Any frice torus homeomorphism g is rotationally mixing. In particular, this holds for the essential factor of a torus homeomorphism in the General Hypothesis.*

PROOF. Recalling that $\rho(\widehat{g}^j) = j\rho(\widehat{g})$, we know that there exists a power g^j and an adequate lift \widehat{g}^j which has $\vec{0} \in \rho(\widehat{g}^j)$, and is therefore mixing, which means that g is rotationally mixing. \square

REMARK 7.3. Assuming Proposition 7.1 holds, we have obtained that g^j is mixing, which implies that the original g is also mixing (see Lemma 2.6).

COROLLARY 7.4. *The essential factor g is strictly toral, and $\text{Ess}(g) = \mathbb{T}^2$.*

PROOF. Note that g is non-wandering because it is mixing. Given that g already satisfies the General Hypothesis, we conclude that it is strictly toral. Moreover, g is also transitive, again because it is mixing. By Remark 2.16, we conclude that $\text{Ess}(g) = \mathbb{T}^2$, as desired. \square

The idea for the proof of Proposition 7.1 is to use the existence of stable and unstable sets for every point, together with a refinement of the anchoring techniques, mainly developed in Proposition 7.5. For this matter, we will use the results and notation developed in Sections 3 and 4.

1. Total anchoring

The following proposition roughly states that for homeomorphisms in the General Hypothesis ($f \in \text{Homeo}_0(\mathbb{T}^2)$, $\text{int}(\rho(f)) \neq \emptyset$), if we take a lift \hat{f} with $\vec{0} \in \text{int}(\rho(\hat{f}))$, the orbit of any weakly unstable continuum is u -anchored to every canonically foliated strip in \mathcal{A} , and even more, the pair of leaves which serve as anchors can be chosen to be almost anywhere in the boundary of the chosen strip.

PROPOSITION 7.5 (Total anchoring). *Let \hat{f} be a lift to \mathbb{R}^2 of $f \in \text{Homeo}_0(\mathbb{T}^2)$, such that $\vec{0} \in \text{int}(\rho(\hat{f}))$. Fix a weakly unstable continuum $\hat{K} \subset \mathbb{R}^2$, a canonically foliated strip $\hat{A} \subset \mathbb{R}^2$ with $\hat{A} \in \hat{\mathcal{A}}$, and a leaf $\hat{\phi} \subset \hat{A}$. Then,*

- (1) *there exists $j > 0$ such that $\hat{f}^j(\hat{K})$ is u -anchored to \hat{A} .*
- (2) *there exist two lifts $\tilde{A}, \tilde{\phi}$ of \hat{A} and $\hat{\phi}$, two leaves $\tilde{\phi}_B, \tilde{\phi}_T \subset \partial R_{\tilde{A}}(\tilde{\phi})$, and some $j' > 0$, such that*

$$(\tilde{A}, \tilde{f}^{j'}(\tilde{K}), \tilde{\phi}_B, \tilde{\phi}_T) \text{ is a } u\text{-anchor, and } \tilde{f}^{j'}(\tilde{K}) \subset R_{\tilde{A}}(\tilde{\phi}).$$

The key improvement of this statement compared to the results obtained in Chapter 4, is that the orbit of an anchored continuum will have iterates which are anchored to any canonically foliated strip, and the leaves serving as anchors are *as much to the right as we want*.

REMARK 7.6. From the second item we immediately conclude that

$$(\hat{A}, \hat{f}^{j'}(\hat{K}), \hat{\phi}_B, \hat{\phi}_T) \text{ is a } u\text{-anchor.}$$

We also have that $\hat{\phi}_B, \hat{\phi}_T \subset \partial R_{\hat{A}}(\hat{\phi})$.

REMARK 7.7. While the statement of Proposition 7.5 is written for the original f , it works identically for any frice torus homeomorphism, and in particular for the essential factor g .

PROOF OF PROPOSITION 7.5.

(1). Let us assume without loss of generality that \hat{K} is u -anchored to \hat{A}_0^\rightarrow , and let us take two leaves $\hat{\phi}_{B,0}^\rightarrow, \hat{\phi}_{T,0}^\rightarrow$ such that

$$(\hat{A}_0^\rightarrow, \hat{K}, \hat{\phi}_{B,0}^\rightarrow, \hat{\phi}_{T,0}^\rightarrow) \text{ is a } u\text{-anchor.}$$

Fix a lift \tilde{A}_0^\rightarrow , and the natural lifts $\tilde{K}, \tilde{\phi}_{B,0}^\rightarrow, \tilde{\phi}_{T,0}^\rightarrow$ such that

$$(\tilde{A}_0^\rightarrow, \tilde{K}, \tilde{\phi}_{B,0}^\rightarrow, \tilde{\phi}_{T,0}^\rightarrow) \text{ is a } u\text{-anchor.}$$

We will start by proving that there exists some vertical CFS $\hat{A}^\downarrow \in \hat{\mathcal{A}}^\downarrow$ (recall Equations 3.5 and 3.6), such that a future iterate of \hat{K} is u -anchored to \hat{A}^\downarrow . Recall that fixing a realizing point \tilde{z}^\rightarrow , we obtain a partition of \hat{A}^\rightarrow in fundamental domains, and also that the family $\hat{\mathcal{A}}^\downarrow$ is naturally indexed by \mathbb{Z} . For each of these CFS $\hat{A}_k^\downarrow \in \hat{\mathcal{A}}^\downarrow$, we may take a lift $\tilde{A}_k^\downarrow \subset \tilde{\mathbb{D}}$, which intersects \tilde{A}_0^\rightarrow . Let us recover the structure of fundamental domains we built in Chapter 3:

$$(7.1) \quad \tilde{D}_k^\rightarrow = \tilde{A}_0^\rightarrow \cap L(\tilde{\phi}_{T,k}^\downarrow) \cap R(\tilde{\phi}_{T,k+1}^\downarrow).$$

Note that there exists k such that $\tilde{\phi}_{B,0}^\rightarrow \subset R(\tilde{\phi}_{T,k}^\downarrow)$, $\tilde{\phi}_{T,0}^\rightarrow \subset R(\tilde{\phi}_{T,k}^\downarrow)$.

By the same argument of the Anchoring Lemma (see for example Lemma 4.8), we have that there exists $j_0 > 0$ such that

$$\tilde{f}^{j_0}(\tilde{K}) \cap \tilde{\phi}_{B,k}^\downarrow \neq \emptyset.$$

Then, we can conclude that

$$(7.2) \quad (\tilde{A}_k^\downarrow, \tilde{f}^{j_0}(\tilde{K}), \tilde{\phi}_{B,k}^\downarrow, \tilde{\phi}_{T,0}^\rightarrow) \text{ is a } u\text{-anchor.}$$

Now, note that similarly \tilde{A}_k^\downarrow is on its turn partitioned in fundamental domains by taking lifts \tilde{A}_m^\rightarrow of each of the respective \hat{A}_m^\rightarrow which intersect \tilde{A}_k^\downarrow , given by:

$$\tilde{D}_m^\downarrow = \tilde{A}_k^\downarrow \cap L(\tilde{\phi}_{B,m}^\rightarrow) \cap R(\tilde{\phi}_{B,m-1}^\rightarrow)$$

and observe that $\tilde{\phi}_{B,k}^\downarrow \subset \tilde{D}_0^\downarrow$, and therefore for $m \neq 0$, we may take leaves $\tilde{\phi}_{B,m}^\rightarrow \in \partial_B \tilde{A}_m^\rightarrow \cap \tilde{A}_k^\downarrow$ and obtain that,

$$\begin{aligned} \tilde{\phi}_{T,0}^\rightarrow &\subset R(\tilde{\phi}_{B,-1}^\rightarrow); \quad \tilde{\phi}_{B,k}^\downarrow \subset R(\tilde{\phi}_{B,-1}^\rightarrow), \\ \tilde{\phi}_{B,-1}^\rightarrow &\in R(\tilde{\phi}_{T,0}^\rightarrow) \cap R(\tilde{\phi}_{B,k}^\downarrow). \end{aligned}$$

Again by the Anchoring Lemma, for every $m < 0$ there exists $j_m > j_0$ such that

$$\tilde{f}^{j_m}(\tilde{K}) \cap \tilde{\phi}_{T,m}^\rightarrow \neq \emptyset, \quad \text{where } \tilde{\phi}_{T,m}^\rightarrow \subset \tilde{A}_k^\downarrow \cap \partial_T \tilde{A}_m^\rightarrow,$$

and we may then conclude that

$$(\tilde{A}_m^\rightarrow, \tilde{f}^{j_m}(\tilde{K}), \tilde{\phi}_{T,0}^\rightarrow, \tilde{\phi}_{T,m}^\rightarrow) \text{ is a } u\text{-anchor.}$$

We have just proved that $\tilde{f}^{j_0}(\hat{K})$ is u -anchored to some \hat{A}_k^\downarrow , and that $\tilde{f}^{j_m}(\hat{K})$ is u -anchored for \hat{A}_m^\rightarrow , for every $m < 0$. In the same fashion, we obtain that $\tilde{f}^{j'_0}(\hat{K})$ is u -anchored to some $\hat{A}_{k'}^\uparrow$ and that $\tilde{f}^{j'_m}(\hat{K})$ is u -anchored for \hat{A}_m^\rightarrow , for every $m > 0$. Therefore, the thesis is true for every \hat{A}_m^\rightarrow .

Proceeding inductively, for example from the fact that $\tilde{f}^{j_0}(\hat{K})$ is u -anchored to \hat{A}_k^\downarrow , we conclude that there is a future iterate of \hat{K} which is on its turn u -anchored to some $\hat{A}_{k''}^\leftarrow$. Repeating the same argument which let us conclude that there are future iterates of \hat{K} u -anchored to each of the \hat{A}_m^\rightarrow , we conclude that there are also future iterates u -anchored for every other $\hat{A}_m^{\uparrow, \leftarrow, \downarrow}$, which concludes the proof.

(2). Let us fix a canonically foliated strip \hat{A}_0^\rightarrow and a fundamental domain $\hat{D}_r \subset \hat{A}_0^\rightarrow$ such that $\phi \subset \hat{D}_r$. As in the first part, and let us assume that

$$(\tilde{A}_0^\rightarrow, \tilde{K}, \tilde{\phi}_{B,0}^\rightarrow, \tilde{\phi}_{T,0}^\rightarrow) \text{ is a } u\text{-anchor.}$$

Take a fundamental domain $\hat{D}'_r \subset \hat{A}_0^\rightarrow$ to the very right of $\hat{\phi}$, more precisely, use Lemma 3.16 with $n = 3$, to obtain r' such that for every leaf $\hat{\phi}'$ in $\hat{D}_{r'+1}$, and any natural set of lifts $\tilde{\phi}', \tilde{\phi}$ to \mathbb{D} (belonging to the same lift \tilde{A}_0^\rightarrow of \hat{A}_0^\rightarrow), we have that

$$(7.3) \quad \tilde{f}^j(\tilde{\phi}') \subset R(\tilde{\phi}), \quad \text{for every } 0 \leq j \leq 3.$$

We start by iterating thrice the process we built in the last item -which lead us to Equation 7.2-, *turning to the right* successively for \widehat{A}_k^\downarrow , \widehat{A}_m^\leftarrow for some m such that $\widehat{A}_m^\leftarrow \cap \widehat{A}_0^\rightarrow = \emptyset$, and lastly for \widehat{A}_n^\uparrow such that

$$\widehat{A}_n^\uparrow \cap \widehat{A}_0^\rightarrow \subset \bigcup_{i \leq r'} \widehat{D}_i^\rightarrow.$$

More precisely, we can obtain $l_3 > l_2 > l_1 > 0$ such that

$(\widehat{A}_k^\downarrow, \widehat{f}^{l_1}(\widehat{K}), \widehat{\phi}_{B,k}^\downarrow, \widehat{\phi}_{T,0}^\rightarrow)$ is a u -anchor (direct from Equation 7.2),

$(\widehat{A}_m^\leftarrow, \widehat{f}^{l_2}(\widehat{K}), \widehat{\phi}_{B,m}^\leftarrow, \widehat{\phi}_{T,0}^\rightarrow)$ is a u -anchor (reiterate the technique),

$(\widehat{A}_n^\uparrow, \widehat{f}^{l_3}(\widehat{K}), \widehat{\phi}_{B,n}^\uparrow, \widehat{\phi}_{T,0}^\rightarrow)$ is a u -anchor (repeat it once again.)

Note that the process we built allows $\widehat{\phi}_{T,0}^\rightarrow$ to always be the top leaf of the respective anchors, the only one changing being the bottom one. See Figure 8 for details.

Now, simply observe that the future iterates of \widehat{K} eventually end up intersecting every leaf in \widehat{A}_n^\uparrow , in particular there exists j_B such that

$$\widehat{f}^{j_B}(\widehat{K}) \cap \widehat{\phi}_B^\rightarrow \neq \emptyset,$$

where we take $\widehat{\phi}_B^\rightarrow \subset \partial_B \widehat{A}_0^\rightarrow \cap \widehat{A}_n^\uparrow$ as the *last intersection* between $\widehat{A}_0^\rightarrow$ and \widehat{A}_n^\uparrow , that is, take a realizing point \widehat{z}^\uparrow for \widehat{A}_n^\uparrow , take

$$\underline{t} = \sup\{t \in \mathbb{R} : \widehat{I}_{\widehat{f}}^t(\widehat{z}^\uparrow) \cap \widehat{A}_0^\rightarrow \neq \emptyset\},$$

and then define $\widehat{\phi}_B^\rightarrow$ as the leaf through $\widehat{I}_{\widehat{f}}^{\underline{t}}(\widehat{z}^\uparrow)$. Note that the orientation of $\widehat{\phi}_B^\rightarrow$ implies that there exists a new lift $\tilde{A}_0'^\rightarrow$ of $\widehat{A}_0^\rightarrow$, and some lift $\tilde{\phi}_B^\rightarrow$ of $\widehat{\phi}_B^\rightarrow$ such that

$$\tilde{A}_0'^\rightarrow \subset R(\tilde{\phi}_B^\rightarrow); \quad \tilde{\phi}_B^\rightarrow \subset \partial_B \tilde{A}_0'^\rightarrow; \quad \text{and} \quad \tilde{f}^{j_B}(\tilde{K}) \cap \tilde{\phi}_B^\rightarrow \neq \emptyset.$$

Then, notice that by construction,

$$\tilde{\phi}_{T,0}^\rightarrow \subset T(\tilde{A}_0'^\rightarrow); \quad \text{and} \quad \tilde{A}_0'^\rightarrow \subset R(\tilde{\phi}_{T,0}^\rightarrow),$$

which implies that

$$(\tilde{A}_0'^\rightarrow, \tilde{f}^{j_B}(\tilde{K}), \tilde{\phi}_B^\rightarrow, \tilde{\phi}_{T,0}^\rightarrow) \text{ is a } u\text{-anchor.}$$

Fix $\tilde{\phi}$ the lift of $\widehat{\phi}$ which belongs to $\tilde{A}_0'^\rightarrow$, \tilde{D}'_r the corresponding lift of \widehat{D}_r , and call $T(\tilde{D}'_r)$ to the subset of $T(\tilde{A}_0'^\rightarrow)$ bounded by the boundary leaves of \tilde{D}'_r . Notice that by construction, $\tilde{\phi}_{T,0}^\rightarrow \subset T(\tilde{D}'_r) \subset R(\tilde{\phi})$, because $\tilde{\phi} \subset \tilde{D}'_r$.

We can finally apply the Anchoring Lemma once more, and obtain $\tilde{\phi}_T \subset \partial_T \tilde{D}'_{r+1} \subset R(\tilde{\phi}')$, and $j' > j_B$ such that

$$(\tilde{A}_0'^\rightarrow, \tilde{f}^{j'}(\tilde{K}), \tilde{\phi}_B, \tilde{\phi}_T) \text{ is a } u\text{-anchor.}$$

Now, if $\tilde{f}^{j'}(\tilde{K}) \subset R(\tilde{\phi})$, the proof is finished. Otherwise we have that $\tilde{f}^{j'}(\tilde{K}) \cap R(\tilde{\phi}) \neq \emptyset$, which implies that $\tilde{f}^{j'}(\tilde{K}) \cap \tilde{\phi}_{T,r'+2}^\downarrow \neq \emptyset$, where $\tilde{\phi}_{T,r'+2}^\downarrow \subset \partial \tilde{D}'_{r+1}$.

In this case, let us take $j'' < j' + 3$, such that

$$\tilde{f}^{j''}(\tilde{K}) \cap \tilde{\phi}_{T,r'+2}^\downarrow = \emptyset, \quad \text{but} \quad \tilde{f}^{j''+1}(\tilde{K}) \cap \tilde{\phi}_{T,r'+2}^\downarrow \neq \emptyset.$$

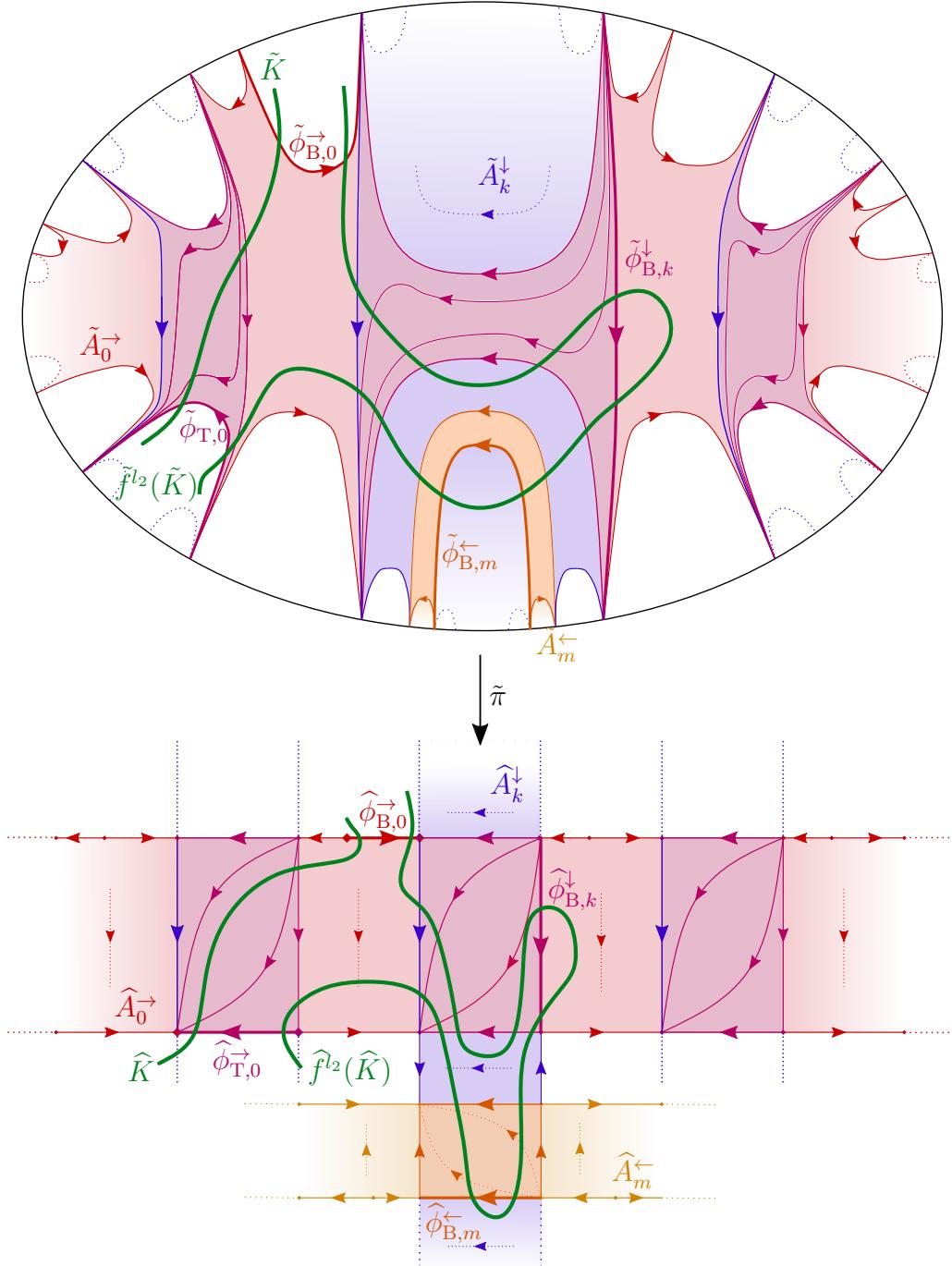


FIGURE 8. A simple example with $m = -1$. Note that both $(\tilde{A}_k^\downarrow, \tilde{f}^{l_2}(\tilde{K}), \tilde{\phi}_{T,k}^\downarrow, \tilde{\phi}_{B,0}^\rightarrow)$ and $(\tilde{A}_m^\leftarrow, \tilde{f}^{l_2}(\tilde{K}), \tilde{\phi}_{T,m}^\leftarrow, \tilde{\phi}_{B,0}^\rightarrow)$ are u -anchors.

One last time, by the Anchoring Lemma for the two CFS $\tilde{\phi}_{r'+1}^\uparrow, \tilde{\phi}_{r'+1}^\downarrow$, we have that

$$\tilde{f}^{j''+3}(\tilde{K}) \cap \tilde{\phi}_{T'}^\rightarrow \neq \emptyset, \quad \tilde{f}^{j''+3}(\tilde{K}) \cap \tilde{\phi}_{B'}^\rightarrow \neq \emptyset,$$

where $\tilde{\phi}_T' \subset \partial_T \tilde{D}'_{r'+1} \cap \tilde{\phi}_{r'+1}^\downarrow$, and $\tilde{\phi}_B' \subset \partial_B \tilde{D}'_{r'+1} \tilde{\phi}_{r'+1}^\uparrow$, which implies by Lemma 3.16 that $\tilde{f}^{j''+3}(\tilde{K}) \subset R(\tilde{\phi})$, and that

$$(\tilde{A}'_0, \tilde{f}^{j''+3}(\tilde{K}), \tilde{\phi}_B', \tilde{\phi}_T') \text{ is a } u\text{-anchor},$$

which concludes the proof. \square

Note that all techniques we have used, immediately yield us an analogous result for weakly stable continua.

2. Mixing pairs of continua

Let us prove one last result before moving to the proof of Proposition 7.1. We emphasize that this result already has the mixing-like structure, and that it strongly uses Proposition 7.5.

LEMMA 7.8. *Let \hat{f} be a lift to \mathbb{R}^2 of a torus homeomorphism $f \in \text{Homeo}_0(\mathbb{T}^2)$, such that $\vec{0} \in \text{int}(\rho(\hat{f}))$, and let $\hat{K}^{ws}, \hat{K}^{wu}$ be a pair of respectively weakly stable and weakly unstable continua for \hat{f} . Then, there exists $l_0 \geq 0$ such that*

$$\hat{f}^l(\hat{K}^{wu}) \cap \hat{K}^{ws} \neq \emptyset, \forall l \geq l_0.$$

PROOF. Fix a canonically foliated strip $\hat{A} \in \hat{\mathcal{A}}$. By Proposition 7.5, there exists $l_- < 0$, and two leaves $\hat{\phi}_B^{ws}, \hat{\phi}_T^{ws}$ such that

$$(\hat{A}, \hat{f}^{l_-}(\hat{K}^{ws}), \hat{\phi}_B^{ws}, \hat{\phi}_T^{ws}) \text{ is a } u\text{-anchor.}$$

Take $\phi \subset \hat{A}$ such that

$$\hat{\phi}_B^{ws}, \hat{\phi}_T^{ws} \subset \partial L_{\hat{A}}(\hat{\phi}),$$

and again by Proposition 7.5, there exists $l_+ > 0$ and two leaves $\hat{\phi}_B^{wu}, \hat{\phi}_T^{wu}$ such that $\hat{\phi}_B^{wu} \subset \partial R_{\hat{A}}(\hat{\phi}), \hat{\phi}_T^{wu} \subset \partial R_{\hat{A}}(\hat{\phi})$ and such that

$$(\hat{A}, \hat{f}^{l_+}(\hat{K}^{wu}), \hat{\phi}_B^{wu}, \hat{\phi}_T^{wu}) \text{ is a } u\text{-anchor.}$$

This implies that we may take natural lifts such that

$$(\tilde{A}, \tilde{f}^{l_+}(\tilde{K}^{wu}), \tilde{\phi}_B^{wu}, \tilde{\phi}_T^{wu}) \text{ is a } u\text{-anchor,}$$

$$(\tilde{A}, \tilde{f}^{l_-}(\tilde{K}^{ws}), \tilde{\phi}_B^{ws}, \tilde{\phi}_T^{ws}) \text{ is an } s\text{-anchor,}$$

and that there exists $\tilde{\phi} \subset \tilde{A}$ lift of $\hat{\phi}$ such that

$$\tilde{\phi}_B^{wu} \subset R(\tilde{\phi}); \tilde{\phi}_T^{wu} \subset R(\tilde{\phi}); \tilde{\phi}_B^{ws} \subset L(\tilde{\phi}); \tilde{\phi}_T^{ws} \subset L(\tilde{\phi}).$$

$$\tilde{f}^{l_+}(\tilde{K}^{wu}) \subset R(\tilde{\phi}); \tilde{f}^{l_-}(\tilde{K}^{ws}) \subset L(\tilde{\phi}).$$

See Figure 9 for details.

Take a leaf $\tilde{\phi}' \subset \tilde{A}$ such that $\tilde{\phi}_B^{ws} \subset R(\tilde{\phi}')$; $\tilde{\phi}_T^{ws} \subset R(\tilde{\phi}')$. By the Anchoring Lemma, we know there exists $l'_+ > l_+$ such that

$$\tilde{f}^l(\tilde{K}^{wu}) \cap \tilde{\phi}' \neq \emptyset \text{ for every } l \geq l'_+.$$

This implies that

$$\tilde{f}^l(\tilde{K}) \cap (\tilde{\phi}_B^{ws} \cup \tilde{f}^{l_-}(\tilde{K}^{ws}) \cup \tilde{\phi}_T^{ws}) \neq \emptyset \text{ for every } l \geq l'_+,$$

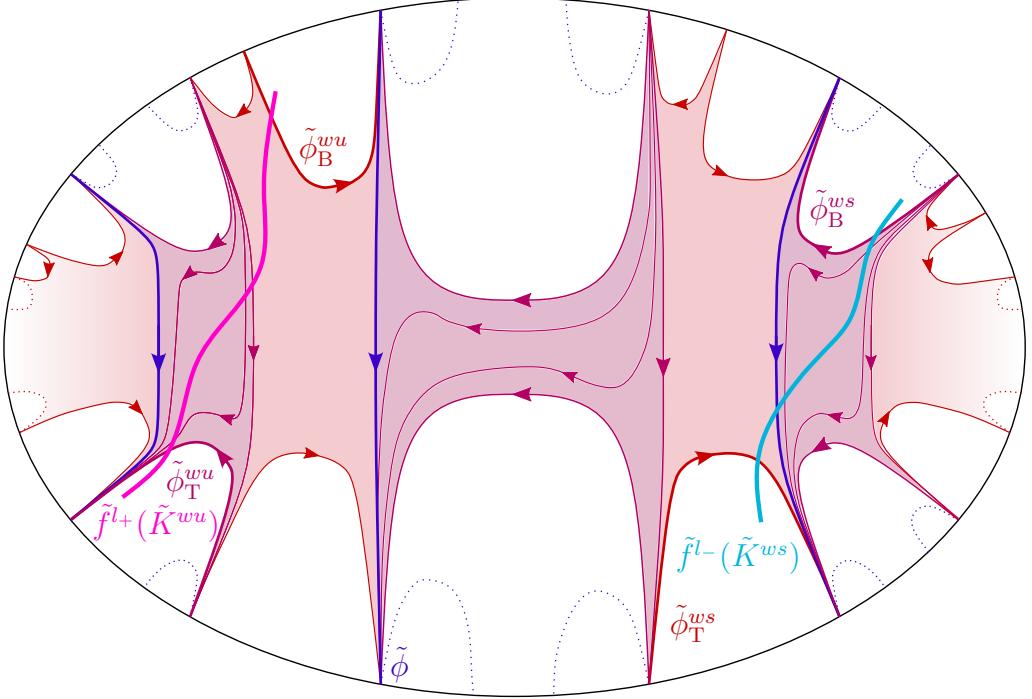


FIGURE 9. Configuration after having anchored both \widehat{K}^{ws} , \widehat{K}^{wu} to the same canonically foliated strip. For this particular example, we have that $\tilde{f}^l(\tilde{K}^{wu}) \cap \tilde{K}^{ws} \neq \emptyset$, for every $l \geq l_+ - l_- + 4$.

as the latter union separates the disk $\tilde{\mathbb{D}}$. Now, the key is to observe that

$$\tilde{f}^{l+}(\tilde{K}^{wu}) \subset R(\tilde{\phi}) \subset L(\tilde{\phi}_B^{ws}) \cap L(\tilde{\phi}_T^{ws}),$$

which implies that for large enough values of l (say $l \geq l''_+ \geq l'_+$), we have that

$$\tilde{f}^l(\tilde{K}^{wu}) \cap (\tilde{\phi}_B^{ws} \cup \tilde{\phi}_T^{ws}) = \emptyset,$$

which on its turn implies that

$$\tilde{f}^l(\tilde{K}^{wu}) \cap \tilde{f}^{l-}(\tilde{K}^{ws}) \neq \emptyset, \text{ for every } l \geq l''_+,$$

from where we conclude that

$$\tilde{f}^l(\widehat{K}^{wu}) \cap \widehat{K}^{ws} \neq \emptyset \text{ for every } l \geq l_0 = l''_+ - l_- > 0,$$

which concludes the proof. \square

Note that this result, together with Lemma 6.8, show that

REMARK 7.9. Let \widehat{z} be the lift of a periodic point of a frice torus homeomorphism g . Then $K^s(\widehat{z})$ and $K^u(\widehat{z})$ are dense in \mathbb{R}^2 .

We now finish with the proof of Proposition 7.1.

PROOF OF PROPOSITION 7.1. Take two open sets $\widehat{V}_1, \widehat{V}_2 \subset \mathbb{R}^2$. By Lemma 6.8, we may take $\widehat{K}^{wu} \subset \widehat{V}_1, \widehat{K}^{ws} \subset \widehat{V}_2$, respectively weakly unstable and weakly stable continua. By Lemma 7.8, there exists $l_0 > 0$ such that

$$\widehat{g}^l(\widehat{K}^{wu}) \cap \widehat{K}^{ws} \neq \emptyset \text{ for every } l \geq l_0,$$

which implies that

$$\widehat{g}^l(\widehat{V}_1) \cap \widehat{V}_2 \neq \emptyset \text{ for every } l \geq l_0,$$

which completes the proof. \square

3. Proof of Theorem D

The following result is central for the proof of Theorem D, and it is nothing more than an adaptation of Lemma 6.8, which worked for finite torus homeomorphisms, to the context of transitive ones.

LEMMA 7.10. *Let \widehat{f} be a lift of a transitive homeomorphism $f \in \text{Homeo}_0(\mathbb{T}^2)$, with $\vec{0} \in \text{int}(\rho(\widehat{f}))$. Then, for every open set $\widehat{V} \subset \mathbb{R}^2$, there exist two non-trivial continua $\widehat{K}^{ws}, \widehat{K}^{wu} \subset \widehat{V}$, which are respectively weakly stable and weakly unstable continua for f .*

PROOF. Let $V = \widehat{\pi}(\widehat{V})$, and assume up to taking a connected component, that \widehat{V} is connected. By the realization result in [Fra89], let us take the lifts $\widehat{z}_1, \widehat{z}_2$ of two f -periodic points z_1, z_2 with respective rotation vectors $(a_1, b_1), (a_2, b_2)$ with $a_1, b_1, b_2 < 0, a_2 > 0$.

Take the respective projections $\widehat{z}_{1\sim} = \pi_\sim(\widehat{z}_1), \widehat{z}_{2\sim} = \pi_\sim(\widehat{z}_2)$ to \mathbb{R}_\sim^2 for the essential factor. By Remark 7.9, we know that $K^s(z_{1\sim}), K^s(z_{2\sim})$ are both dense in \mathbb{R}_\sim^2 . In particular, by taking their preimage by the semiconjugacy π_\sim , we will obtain that

$$K'^s(\widehat{z}_1) = \pi_\sim^{-1}(K^s(\widehat{z}_{1\sim})) \text{ goes through every fundamental domain,}$$

and similarly for $K'^s(\widehat{z}_2) = \pi_\sim^{-1}(K^s(\widehat{z}_{2\sim}))$

Given that f is transitive, by [KT13, Corollary E] we know that $\text{Ess}(f) = \mathbb{T}^2$, so there exists $k > 0$ such that

$$\underline{V} := \bigcup_{j=0}^{j=k} f^j(V) \text{ is fully essential.}$$

which means there exist two respective points in the finite orbits of z_1 and z_2 (which we will give the same name for the sake of simplicity and without loss of generality), such that

$$K'^s(z_1) \cap V \neq \emptyset, K'^s(z_2) \cap V \neq \emptyset.$$

Up to retaking the lifts $\widehat{z}_1, \widehat{z}_2$, we may assume without loss of generality that $K'^s(\widehat{z}_1) \cap \widehat{V} \neq \emptyset, K'^s(\widehat{z}_2) \cap \widehat{V} \neq \emptyset$. In particular, there exists $m > 0$ such that

$$K_m'^s(\widehat{z}_1) \cap \widehat{V} \neq \emptyset, K_m'^s(\widehat{z}_2) \cap \widehat{V} \neq \emptyset,$$

where we define

$$K_m'^s(\widehat{z}_1) = \pi_\sim^{-1}(K_m^s(\widehat{z}_{1\sim})), \quad K_m'^s(\widehat{z}_2) = \pi_\sim^{-1}(K_m^s(\widehat{z}_{2\sim}))$$

Let us take a curve $\widehat{\gamma} \subset \widehat{V}$ which goes from a point in $K_m'^s(\widehat{z}_1)$ to a point in $K_m'^s(\widehat{z}_2)$, we then obtain that

$$\widehat{C} := K_m'^s(\widehat{z}_1) \cup \widehat{\gamma} \cup K_m'^s(\widehat{z}_2) \text{ is connected.}$$

Note that there exists some $d > 0$ such that

$$\sup_{j \in \mathbb{Z}^+} \text{diam}(\widehat{f}^j(K_m'^s(\widehat{z}_1))) \leq d, \quad \sup_{j \in \mathbb{Z}^+} \text{diam}(\widehat{f}^j(K_m'^s(\widehat{z}_2))) \leq d,$$

because each iterate intersects a uniformly bounded number of fundamental domains. This shows that

$$K_m'^s(\widehat{z}_1) \subset K_d^s(\widehat{z}_1), \quad K_m'^s(\widehat{z}_2) \subset K_d^s(\widehat{z}_2),$$

which allows us to use Lemma 6.15 and conclude that $\widehat{\gamma}$ is a weakly unstable continuum. In identical fashion we can build $\widehat{\gamma}' \subset \widehat{V}$ a weakly stable continuum, which concludes the proof. \square

We now finish the chapter with the proof of Theorem D. Let us recall the desired statement:

Theorem D. Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ be a transitive homeomorphism.

- (1) If $\rho(f)$ has nonempty interior, then f is mixing.
- (2) Furthermore, if f has a lift \widehat{f} with $\vec{0} \in \text{int}(\rho(\widehat{f}))$, then \widehat{f} is also mixing.

PROOF. Note that the second item implies the first. Suppose we have proven the second: we know that a power f^j of f has a lift \widehat{f}^j with $\vec{0}$ in the interior of its rotation set, which by the second item will be mixing, which implies that f^j is mixing, which by Lemma 2.6 then implies that f is also mixing.

Let us then take \widehat{f} with $\vec{0} \in \text{int}(\rho(\widehat{f}))$, and two open sets $\widehat{V}_1, \widehat{V}_2 \in \mathbb{R}^2$. By Lemma 7.10, we can take $\widehat{K}^{wu} \in \widehat{V}_1, \widehat{K}^{ws} \in \widehat{V}_2$, and by Lemma 7.8, there exists $l_0 > 0$ such that

$$\widehat{f}^l(\widehat{K}^{wu}) \cap \widehat{K}^{ws} \neq \emptyset \text{ for every } l \geq l_0,$$

which implies that

$$\widehat{f}^l(\widehat{V}_1) \cap \widehat{V}_2 \neq \emptyset \text{ for every } l \geq l_0,$$

which completes the proof. \square

CHAPTER 8

Heteroclinic pseudo-rectangles

We have already proven four out of the six properties in Theorem A for the essential factor g . In this chapter we construct the main tool for the two properties which are still left to prove:

- For each open set $U \subset \mathbb{T}^2$, g has a Markovian horseshoe $X \subset U$,
- g is area preserving.

Assuming we have proven the density of Markovian horseshoes, the proof of the second item appears as a consequence almost automatically; the extensive proof is given in Proposition 10.1.

The proof of the first item is much more technical, and will require fine use of the anchoring techniques together with the construction of a new tool, to which this chapter is devoted.

The idea is to build a family of filled continua, which we will call fitted heteroclinic pseudo-rectangles (FHPN). Proposition 8.9, which shows the existence of an FHPN inside any open set of the plane for any piecewise torus homeomorphism, is the key result of the section. This will be crucial to prove the existence of a dense family of Markovian horseshoes for the essential factor g : Proposition 9.1 will ensure that inside every FHPN there exists the Markovian horseshoe we are looking for.

Heuristically, the idea resides on adapting to this context, the classical notion of heteroclinic intersection for stable-unstable manifolds of hyperbolic periodic points.

1. Topological structure

DEFINITION 8.1. Let S be a surface. We will say $R \subset S$ is a *rectangle* if it is homeomorphic to $[0, 1]^2$, as the image a homeomorphism $h : [0, 1]^2 \rightarrow h([0, 1]^2) = R \subset S$. We will call *sides* of R to the image of the sides of $[0, 1]^2$ by h . We will call *horizontal* and *vertical sides* to the sets R_H and R_V , respectively the images of the horizontal and vertical sides of $[0, 1]^2$ by h .

DEFINITION 8.2 (Heteroclinic pseudo-rectangle). An inessential continuum $P \subset \mathbb{T}^2$ will be a *heteroclinic pseudo-rectangle* (HPR) for a torus homeomorphism f in the General Hypothesis with associated MDTD (f, I, \mathcal{F}) if:

- (1) P contains no singularities of \mathcal{F} ,
- (2) P is the closure of an open disk D ,
- (3) For any $z \in \partial P$, any neighbourhood U of z intersects the fully essential component P_∞ of the complement of P ,

- (4) (Sides). $\partial P = K_1^s \cup K_1^u \cup K_2^s \cup K_2^u$, respectively two stable and two unstable continua, contained in the (un)stable sets of four periodic points $z_1^s, z_2^s, z_1^u, z_2^u$, with different non-trivial associated rotation vectors.
- (5) $K_i^s \cap K_j^u \neq \emptyset$, for every $i, j = 1, 2$.

REMARK 8.3. From the fourth item of last definition, and the definition of stable set, we automatically obtain that $K_1^\sigma \cap K_2^\sigma = \emptyset$, where $\sigma = s, u$.

Note that any lift \widehat{P} of a heteroclinic pseudo-rectangle is homeomorphic to P , we will then say that $\widehat{P} \subset \mathbb{R}^2$ is a heteroclinic pseudo-rectangle if it is a connected component of $\widehat{\pi}^{-1}(P)$, where P is a heteroclinic pseudo-rectangle. For $\widehat{P} \subset \mathbb{R}^2$, the third condition is equivalent to saying that for any neighbourhood \widehat{U} of a point $\widehat{z} \in \partial \widehat{P}$, the set \widehat{U} intersects the only unbounded component \widehat{P}_∞ of the complement of \widehat{P} . In the same fashion, \tilde{P} is a heteroclinic pseudo-rectangle when it is a lift of a heteroclinic pseudo-rectangle \widehat{P} .

Prime end model. Given a heteroclinic pseudo-rectangle $P = \text{cl}(D)$, we will denote $\check{P} \simeq \text{cl}(\mathbb{D})$ to the Caratheodory compactification (see Definition 2.37) of the disk D , and we will call h be the Riemann map from \mathbb{D} to D .

Let us define $X_1^s \subset \partial \mathbb{D}$ as the set of accessible prime ends \check{z} in $\partial \check{P}$ for which there exists a ray r landing at \check{z} , with $h(r)$ landing at $z \in K_1^s$. Then we define $\check{K}_1^s := \text{cl}(X_1^s)$. We define $\check{K}_2^s, \check{K}_1^u$, and \check{K}_2^u in the same fashion.

The following two lemmas state that P looks as a rectangle when seen in the prime end topology.

LEMMA 8.4. \check{K}_i^σ is a closed interval for every $\sigma \in \{s, u\}$, $i \in \{1, 2\}$, and their one-to-one intersections are either empty or single points.

PROOF. Recall for the proof that K_i^σ, K_j^σ are at a positive distance. Note that by construction, each \check{K}_i^σ is a closed subset of the circle.

- (1) $\check{K}_i^\sigma \neq \mathbb{S}^1$.

Take $z \in K_j^\sigma$ in the *opposite side* of P , and take a neighbourhood U of z which does not intersect K_i^σ (note that $d(K_i^\sigma, K_j^\sigma) > 0$). Given the set of accessible points in ∂P is dense (see Remark 2.39), take an accessible point $z' \in U$, and note that $\check{z}' \notin \check{K}_i^\sigma$.

- (2) $\check{K}_i^\sigma \neq \emptyset$.

By symmetry, it is enough to prove this for \check{K}_1^s . Note that there exists $z \in K_1^s$ with a neighbourhood U of z such that it intersects neither K_1^u nor K_2^u , as we would otherwise could write K_1^s as the union of the two disjoint closed sets $K_1^s \cap K_1^u, K_1^s \cap K_2^u$, which would be a contradiction, because K_1^s is connected. Then, by Remark 2.39, U must contain an accessible point $z' \in K_1^s$, and we then have $\check{z}' \in \check{K}_1^s$.

- (3) $\check{K}_1^s \cup \check{K}_2^s \cup \check{K}_1^u \cup \check{K}_2^u = \mathbb{S}^1$.

This is because the set of accessible prime ends is dense in $\partial \check{P} = \mathbb{S}^1$ (See Theorem 2.40).

- (4) \check{K}_i^σ is connected.

Suppose by contradiction that it has two different connected components. This means we can take four cyclically-ordered accessible prime ends $\check{z}, \check{w}, \check{z}', \check{w}'$, such that $\check{z}, \check{z}' \in \check{K}_i^\sigma$, and $\check{w}, \check{w}' \notin \check{K}_i^\sigma$. We may then build respective rays $r_{\check{z}}, r_{\check{w}}, r_{\check{z}'}, r_{\check{w}'}$ landing respectively at $\check{z}, \check{w}, \check{z}', \check{w}'$, such that

- The rays $h(r_{\check{z}}), h(r_{\check{w}}), h(r_{\check{z}'}), h(r_{\check{w}'})$ respectively land at z, w, z', w' with $z, z' \in K_i^\sigma, w, w' \notin K_i^\sigma$.
- $r_{\check{z}}^{-1} \cdot r_{\check{z}'} \wedge r_{\check{w}}^{-1} \cdot r_{\check{w}'} = 1 \implies h(r_{\check{z}}^{-1}) \cdot h(r_{\check{z}'}) \wedge h(r_{\check{w}}^{-1}) \cdot h(r_{\check{w}'}) = 1$

Take sufficiently small neighbourhoods $U_w, U_{w'}$ respectively of w, w' such that they intersect neither the crosscut $h(r_{\check{z}}^{-1}) \cdot h(r_{\check{z}'})$ nor K_i^{sigma} . Note that from the heteroclinic pseudo-rectangle definition, both U_w and $U_{w'}$ intersect P_∞ . This means we can build a curve γ from $\partial U_{w'}$ to ∂U_w which is contained in P_∞ except for its endpoints, and is therefore disjoint from $h(r_{\check{z}}^{-1}) \cdot h(r_{\check{z}'})$. We may then complete it with two simple arcs $\eta_w, \eta_{w'}$ respectively contained in $U_w, U_{w'}$ except for its endpoints, to obtain a simple closed curve γ' which holds

$$h(r_{\check{z}}^{-1}) \cdot h(r_{\check{z}'}) \wedge \gamma' = h(r_{\check{z}}^{-1}) \cdot h(r_{\check{z}'}) \wedge h(r_{\check{w}}^{-1}) \cdot h(r_{\check{w}'}) = 1,$$

which implies that γ' separates z from z' , which is a contradiction as K_i^σ is connected.

So far we have proved that each \check{K}_i^σ is a nonempty closed interval, and that their union is the full circle. The only property left to check is that each intersection $\check{K}_i^\sigma, \check{K}_j^\tau$ has at most one point.

- (5) No intersection $\check{K}_i^\sigma, \check{K}_j^\tau$ contains an interval.

Again by symmetry, let us suppose by contradiction that there exists an interval $I \subset \check{K}_1^s \cap \check{K}_1^u$. Take two accessible prime ends $\check{z}, \check{z}' \in \check{K}_1^s \cap I$, and respective rays $r_{\check{z}}, r_{\check{z}'} : [0, 1] \rightarrow \check{P}$ as in last item. Let θ_n be an arc of circumference centered at the origin, connecting $r_{\check{z}}$ to $r_{\check{z}'}$ and at a distance at most $1/n$ of I . Take their images $h(\theta_n)$ by h . Up to taking a subsequence, the Hausdorff limit Θ of $h(\theta_n)$ is a non-trivial continuum. Provided this continuum is contained in K_1^s , we obtain our contradiction: we could repeat the construction with \check{K}_1^u and we could take subarcs $\theta'_n \subset \theta_n$ converging to a subcontinuum $\Theta' \subset K_1^u \cap K_1^s$, which would mean we would have a dynamically bounded non-trivial continuum for g .

Let us then prove that the Hausdorff limit of θ_n is contained in K_1^s . This is again a connection argument. Suppose by contradiction that there exists $w \notin K_1^s$ and $w_n \in \theta_n$ such that $w_n \rightarrow w$. We would have a neighbourhood U_w of w which is disjoint from K_1^s . We then take n sufficiently large, such that $w_n \in U_w$, and build a curve from $h(r_{\check{z}})(\vec{0})$ to w , which is disjoint from K_1^s . As in last item, we may complete this curve to a simple closed curve γ' which would separate K_1^s . This finishes the proof.

□

Let us define

$$\text{Imp}(\check{K}_i^\sigma) := \bigcup_{\check{z} \in K_i^\sigma} \text{Imp}(\check{z}).$$

The following result exists to give a more complete understanding of the set of HPR.

LEMMA 8.5. $K_i^\sigma \subset \text{Imp}(\check{K}_i^\sigma)$.

PROOF. First, let us check that $\text{Imp}(\check{K}_i^\sigma)$ is a closed set. From Lemma 8.4, we know that \check{K}_i^σ is an interval $[a, b]$. For sufficiently large values of n , let us take $\check{\gamma}_n$ a crosscut connecting two accessible prime ends $\check{z}_n, \check{z}'_n \notin \check{K}_i^\sigma$, such that $d_H(\check{\gamma}_n, \check{K}_i^\sigma) < \frac{1}{n}$. This defines a chain $\{\check{\gamma}_n\}_{n \geq n_0}$ of crosscuts, which on its turn defines a nested sequence $\{\text{cl}(\check{d}_n)\}_{n \geq n_0}$ of closed regions, which define a nested sequence of closed sets $\text{cl}(d_n) = \text{cl}(h(d_n))$, which hold $\bigcap_{n \geq n_0} \text{cl}(d_n) = K \subset \partial P$.

Let us prove that $\text{Imp}(\check{K}_i^\sigma) \subset K$. Take $z \in \text{Imp}(\check{K}_i^\sigma)$, and take a chain of crosscuts η_n defining the prime end $\check{z} \in \check{K}_i^\sigma$ containing z . Given that $\text{diam}(\eta_n) \rightarrow 0$ as n goes to infinity, we have that up to taking a subsequence, the corresponding sequence $\{\check{\eta}_n\}$ of crosscuts, holds $\check{\eta}_n \subset \check{d}_n$, which means that $\eta_n \subset d_n$, which proves that $\text{Imp}(\check{z}) \subset K$.

For the converse inclusion, take $z \subset K$, and note that for every n we may take a nested sequence $\{\eta'_n\}$ of crosscuts with diameter going to 0, defining regions d'_n and such that $d'_n \subset d_n$ for every n . This implies that $\check{d}'_n \subset \check{d}_n$, which proves that $z \subset \text{Imp}(\check{K}_i^\sigma)$.

Now, suppose by contradiction that there exists $z \in K_i^\sigma$ which is not on the impression of \check{K}_i^σ . Then we would have that

$$d(z, \text{Imp}(\check{K}_i^\sigma)) = \delta > 0,$$

and we may then take a neighbourhood U_z of z such $U_z \cap \text{Imp}(\check{K}_i^\sigma) = \emptyset$. As in the second item of Lemma 8.4, we have that there exists $z' \subset U_z$ which is separated from the other three sides, that is, with a neighbourhood $U_{z'} \subset U_z$ with an accessible point $z'' \in K_i^\sigma$, which means that $z'' \in \text{Imp}(\check{K}_i^\sigma)$, which would be a contradiction. \square

We will strongly use the following two lemmas for the two main results of Chapter 9: Proposition 9.1 and Proposition 9.3.

LEMMA 8.6. *Let P be a heteroclinic pseudo-rectangle. For every $\varepsilon > 0$ there exists a rectangle $R \subset P$ with sides $R_1^s, R_2^s, R_1^u, R_2^u$ such that*

$$d(z_i^\sigma, K_i^\sigma) < \varepsilon, \text{ for every } z_i^\sigma \subset R_i^\sigma; \sigma = s, u; i = 1, 2.$$

PROOF. Go to the prime-end compactification $\check{P} = \text{cl}(\mathbb{D})$, and let us assume without loss of generality that $\check{K}_1^s, \check{K}_1^u, \check{K}_2^s, \check{K}_2^u$ are positively and cyclically ordered. For every pair of values $(i, j) \in \{1, 2\}^2$, let \check{z}_{ij} be the only prime end in $\check{K}_i^s \cap \check{K}_j^u$. Fix a pair (i, j) . Given that $\text{Imp}(\check{z}_{ij})$ is defined by a chain of crosscuts with diameter going to 0, we may take γ_{ij} a crosscut with $\text{diam}(\gamma_{ij}) < \frac{\varepsilon}{2}$, and note

that the endpoints belong respectively to K_i^s and K_j^u provided ε is sufficiently small, because the endpoints $\check{z}_{ij}^s, \check{z}_{ij}^u$ of $\check{\gamma}_{ij}$, belong respectively to \check{K}_1^s and \check{K}_j^u . We shall use $\check{I}_{ij} \subset \partial\mathbb{D}$ to denote the interval determined by the endpoints of $\check{\gamma}_{ij}$, which contains \check{z}_{ij} .

Given that the set of accessible prime ends is dense, we may take accessible prime ends $\check{z}'_{ij}^s \in \check{K}_i^s \cap \check{I}_{ij}$, $\check{z}'_{ij}^u \in \check{K}_j^u \cap \check{I}_{ij}$ such that

$$\check{z}_{ij}^s, \check{z}'_{ij}^s, \check{z}_{ij}^u, \check{z}'_{ij}^u \text{ are cyclically ordered,}$$

and take disjoint respective rays $\check{r}_{ij}^s, \check{r}_{ij}^u$ from the origin to $\check{z}'_{ij}^s, \check{z}'_{ij}^u$, such that their images by h land at the respective points $z'_{ij}^s \in K_i^s, z'_{ij}^u \in K_j^u$, and such that they each intersect $\check{\gamma}_{ij}$ exactly once.

Take \check{U}_1^s the angular region determined by \check{r}_{12}^s and \check{r}_{11}^s whose boundary is contained in the interior of \check{K}_1^s . We now proceed as in Item (5) from Lemma 8.4: take a sequence of curves $\{\check{\theta}_{1,n}^s\}_{n \in \mathbb{Z}^+}$ from \check{r}_{12}^s to \check{r}_{11}^s and uniformly approaching the boundary, and note that up to taking a subsequence, the Hausdorff limit of $h(\check{\theta}_{1,n}^s)$ is included in K_1^s . This means we may take n_1^s such that

- $h(\check{\theta}_{1,n_1^s}^s) \subset B(K_1^s, \frac{\varepsilon}{2})$,
- $\check{\theta}_{1,n}^s \cap \check{\gamma}_{12} \neq \emptyset, \check{\theta}_{1,n}^s \cap \check{\gamma}_{11} \neq \emptyset$.

Now, define $\check{\theta}_1^s$ as a minimal-for-inclusion subarc of $\check{\theta}_{1,n_1^s}^s$ going from $\check{\gamma}_{11}$ to $\check{\gamma}_{12}$, and define $\check{\theta}_1^u, \check{\theta}_2^s$, and $\check{\theta}_2^u$ in the same fashion. Define $\check{\gamma}'_{ij}$ as the subarc of $\check{\gamma}_{ij}$ which goes from $\check{\theta}_i^s$ to $\check{\theta}_j^u$. See Figure 10 for details.

Finally, note that taking the correct orientation for each of these eight arcs, we have that

$$\check{\theta}_1^s \cdot \check{\gamma}'_{11} \cdot \check{\theta}_1^u \cdot \check{\gamma}'_{21} \cdot \check{\theta}_2^s \cdot \check{\gamma}'_{22} \cdot \check{\theta}_2^u \cdot \check{\gamma}'_{12}$$

is a simple closed curve, and moreover we may define

$$R_1^s = h(\check{\theta}_1^s \cdot \check{\gamma}'_{11}), \quad R_1^u = h(\check{\theta}_1^u \cdot \check{\gamma}'_{21}), \quad R_2^s = h(\check{\theta}_2^s \cdot \check{\gamma}'_{22}), \quad R_2^u = h(\check{\theta}_2^u \cdot \check{\gamma}'_{12}),$$

and note that it is a rectangle satisfying the required properties. \square

LEMMA 8.7. *Let $P \subset \mathbb{T}^2$ be an HPR. Let K be a continuum, and let $Q = K \cap P$ be a compact set such that*

- $Q \cap K_1^u = \emptyset, Q \cap K_2^u = \emptyset$
- Q separates the unstable sides: for any continuum $Q' \subset P$ from K_1^u to K_2^u , we have that $Q' \cap Q \neq \emptyset$.

Then, there exists a subcontinuum $Q^u \subset Q$ connecting the stable sides, that is,

$$Q^u \cap K_1^s \neq \emptyset, \quad Q^u \cap K_2^s \neq \emptyset$$

PROOF. Let $\delta_1 > 0$ be the distance between Q and the union of the unstable sides. Define Q_1 as the closure of the union of every connected component of Q which intersects K_1^s , and define Q_2 in the same fashion. Suppose by contradiction that these two compact sets are disjoint, and obtain that $d(Q_1, Q_2) = \delta_2 > 0$. Take $\delta = \frac{\min\{\delta_1, \delta_2\}}{4}$, and take respective finite coverings $\mathcal{U}_1, \mathcal{U}_2$ of Q_1 and Q_2 , by

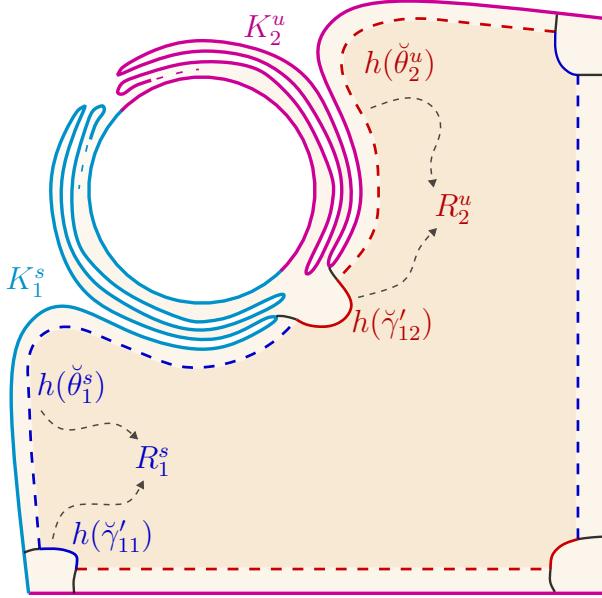


FIGURE 10. The stable and unstable sides of the HPR appear respectively in light blue and pink. The sides of the new rectangle appear in blue and red. Dotted blue and red lines come from taking curves near the sides in the prime-end model. Solid blue and red lines come from subarcs of crosscuts whose respective cross-sections contain each of the vertices of the rectangle in the prime-end model.

balls of radius δ . Let U_1 be the union of the balls from \mathcal{U}_1 , define U_2 in the same fashion, and note that $\text{cl}(U_1) \cap \text{cl}(U_2) = \emptyset$ by construction.

Going to the prime-end compactification \check{P} of P , and because of Lemma 8.4, we can take two crosscuts $\check{\gamma}_1$ with endpoints in \check{K}_1^s , $\check{\gamma}_2$ with endpoints in \check{K}_2^u , such that they are disjoint and each of them separates U_1 from U_2 . This means that the remaining connected component of the complement of these two crosscuts contains both \check{K}_1^u and \check{K}_2^u , which means there exists two accessible prime ends and then a crosscut $\check{\gamma}_2$ from \check{K}_1^u to \check{K}_2^u , such that its image by h is an arc disjoint from Q and connecting K_1^u to K_2^u , which contradicts the hypothesis. \square

2. Dense family of fitted heteroclinic pseudo-rectangles

We will dedicate this section to prove that inside any open set in \mathbb{R}^2 there is a *fitted* heteroclinic pseudo-rectangle, we only need the fitted condition to make the arguments in Chapter 9 simpler. Although our stable and unstable sets are dynamically defined, the proof of this density is mostly topological.

DEFINITION 8.8. We will say a heteroclinic pseudo-rectangle is *fitted* if

$$\rho(z_1^s) = (a_1^s, b_1^s), \quad \rho(z_2^s) = (a_2^s, b_2^s), \quad \rho(z_1^u) = (0, b_1^u), \quad \rho(z_2^u) = (0, b_2^u),$$

where

$$a_1^s < 0, \quad a_2^s < 0, \quad b_1^s < 0, \quad b_2^s < 0, \quad b_1^u < 0, \quad b_2^u > 0, \quad \text{and} \quad \frac{b_1^s}{a_1^s} \neq \frac{b_2^s}{a_2^s}.$$

We will sometimes call them FHPR for the sake of simplicity.

Note from Definition 8.2 that an HPR for a power g^j of g , is also an HPR for g . Therefore we may assume up to taking some power, that g has a lift \widehat{g} such that $\vec{0} \in \rho(\widehat{g})$. We will then prove that:

PROPOSITION 8.9. *Let \widehat{g} be a lift to \mathbb{R}^2 of a frice torus homeomorphism g . If $\vec{0} \in \text{int}(\rho(\widehat{g}))$, then for every open set $\widehat{U} \subset \mathbb{R}^2$, there exists a fitted heteroclinic pseudo-rectangle $\widehat{P} \subset \widehat{U}$ for \widehat{g} .*

PROOF. This is quite an intricate construction, so we will divide it into steps.

Step 1. Accumulating small stable continua.

Let $(\widehat{g}, \widehat{I}, \widehat{\mathcal{F}})$ be an MDTD for \widehat{g} . Recall that the set $\text{Sing}(\widehat{\mathcal{F}})$ of singularities of the foliation, has no interior and does not disconnect the plane, because every non-trivial continuum is dynamically unbounded for g (see Proposition 5.17). Let us then assume up to taking a smaller set that \widehat{U} is an euclidean ball $\widehat{U} = B(\widehat{z}, \varepsilon)$ with no singularities inside, and take $\widehat{U}' = B(\widehat{z}, \frac{\varepsilon}{2})$. Given that $\vec{0} \in \rho(\widehat{g})$, and using the realization of rational vectors in $\text{int}(\rho(\widehat{g}))$ in [Fra89], we shall take $\{\widehat{z}_n\}_{n \in \mathbb{Z}^+}$ a sequence of lifts of periodic points for g , such that

$$\rho_g(\widehat{z}_n) = (a_n, b_n),$$

such that $a_n < 0$ and $b_n < 0$ for every n , and such that the sequence of slopes $\{\frac{b_n}{a_n}\}_{n \in \mathbb{Z}^+}$ is injective. Notice that

- Each $\widehat{K}^s(\widehat{z}_n)$ is dense in \mathbb{R}^2 because of the results of Lemma 7.8 and Lemma 6.8.
- $\widehat{K}^s(\widehat{z}_n) \cap \widehat{K}^s(\widehat{z}_m) = \emptyset$ whenever $m \neq n$, because of Lemma 6.11 and the fact that our \widehat{z}_n have different rotation vectors, and therefore their orbits separate linearly in \mathbb{R}^2 .

In particular, $\widehat{K}^s(\widehat{z}_n) \cap \widehat{U}' \neq \emptyset$ for every $n > 0$, which means that we may take a sequence of stable continua $\{\widehat{K}_n^s\}_{n \in \mathbb{Z}^+}$, with $\widehat{K}_n^s \subset \widehat{K}^s(\widehat{z}_n)$ and such that \widehat{K}_n^s is a stable continuum with

$$\widehat{K}_n^s \cap \partial \widehat{U} \neq \emptyset, \quad \widehat{K}_n^s \cap \partial \widehat{U}' \neq \emptyset,$$

minimal for the inclusion. Note that

$$\frac{\varepsilon}{2} \leq \text{diam}(\widehat{K}_n^s) \leq 2\varepsilon \text{ for every } n,$$

which implies that, up to taking a subsequence, we have that there exists \widehat{K}^s such that

$$\widehat{K}_n^s \xrightarrow[\text{Hff}]{} \widehat{K}^s.$$

We remark that \widehat{K}^s cannot intersect $\widehat{K}_{n_1}^s$ and $\widehat{K}_{n_2}^s$ for $n_1 \neq n_2$: if it did intersect

both, it would be a weakly unstable continuum by Lemma 6.15, but we know \widehat{K}^s is a stable continuum by Proposition 6.14, so it would be a contradiction. We may therefore assume that \widehat{K}^s is disjoint from all \widehat{K}_n^s .

Step 2. Understanding the local picture.

Notice that \widehat{K}^s intersects both $\partial\widehat{U}$ and $\partial\widehat{U}'$. Moreover, \widehat{K}^s is a stable continuum by Proposition 6.14, and is therefore filled and has empty interior by Remark 6.10, which implies that it is inessential when seen as a subset of the annulus $\widehat{S} = \text{cl}(\widehat{U}) \setminus \widehat{U}'$. This means that up to a change of coordinates, \widehat{K}^s is disjoint from an open *radial region* $\widehat{V}_0 \subset \widehat{S}$ (i.e. an open region contained between two disjoint simple curves from $\partial\widehat{U}$ to $\partial\widehat{U}'$).

\widehat{K}^s could *a priori* separate the annulus, but we know that $\widehat{S} \setminus \widehat{K}^s$ has only one connected component which contains our sequence of continua \widehat{K}_n^s , as we could otherwise disconnect \widehat{K}^s using \widehat{K}_n^s and \widehat{K}_m^s included in different connected components of its complement. Then, up to taking a subsequence, every continuum \widehat{K}_n^s belongs to the same connected component \widehat{V} of $\widehat{S} \setminus (\widehat{V}_0 \cup \widehat{K}^s)$.

Note that \widehat{V} is a disk, and $\partial\widehat{V} = \partial_R\widehat{V} \cup \partial_T\widehat{V} \cup \partial_L\widehat{V} \cup \partial_B\widehat{V}$ where

- $\partial_R\widehat{V}$ is a closed arc of $\partial\widehat{U}$,
- $\partial_T\widehat{V}$ is a closed arc from \widehat{U} to \widehat{U}' , whose interior is in \widehat{S} ,
- $\partial_L\widehat{V}$ is a closed arc of $\partial\widehat{U}'$,
- $\partial_B\widehat{V} = \widehat{K}^s$.

We can take a closed arc $\widehat{\eta}$ contained in $\widehat{S} \setminus \text{cl}(\widehat{V})$, except for its endpoints which belong respectively to $\partial_L\widehat{V} \cap \widehat{K}^s$, $\partial_R\widehat{V} \cap \widehat{K}^s$, and denote by \widehat{V}' the rectangle whose sides are $\partial_R\widehat{V}$, $\partial_T\widehat{V}$, $\partial_L\widehat{V}$ and $\widehat{\eta}$. By the Jordan-Schoenflies Theorem, we can take a new change of coordinates sending \widehat{V}' to the square $[0, 1] \times [0, 1]$; we can therefore assume that \widehat{V} is *almost* the square $[0, 1] \times [0, 1]$, more precisely

$$\partial_R\widehat{V} = \{1\} \times [0, 1], \quad \partial_T\widehat{V} = [0, 1] \times \{1\}, \quad \partial_L\widehat{V} = \{0\} \times [0, 1], \quad \partial_B\widehat{V} \subset [0, 1] \times [0, 1].$$

See Figure 11 for details.

Let use the following notation:

$$\partial_L\widehat{K}_n^s = \widehat{K}_n^s \cap \partial_L\widehat{V}, \quad \partial_R\widehat{K}_n^s = \widehat{K}_n^s \cap \partial_R\widehat{V}.$$

Each of these two sets is naturally contained in a respective closed segment I_n^L, I_n^R , which are minimal for the inclusion. Again, up to taking a subsequence, we may assume that

$\{I_n^L\}_{n \in \mathbb{Z}^+}, \{I_n^R\}_{n \in \mathbb{Z}^+}$ are decreasing-to-zero sequences of disjoint closed segments.

Note that

$$d(\widehat{K}_n^s, \widehat{K}_{n+1}^s) = \delta_n > 0$$

because they are disjoint compact sets. Then, there exists an open connected component \widehat{V}_n between \widehat{K}_n^s and \widehat{K}_{n+1}^s , that is, there exists a curve γ_n from $\partial_L\widehat{V}$ to $\partial_R\widehat{V}$ which separates \widehat{K}_n^s from \widehat{K}_{n+1}^s . Note that there exists only one of these components for each value of n , as they would otherwise separate \widehat{K}_n^s or \widehat{K}_{n+1}^s .

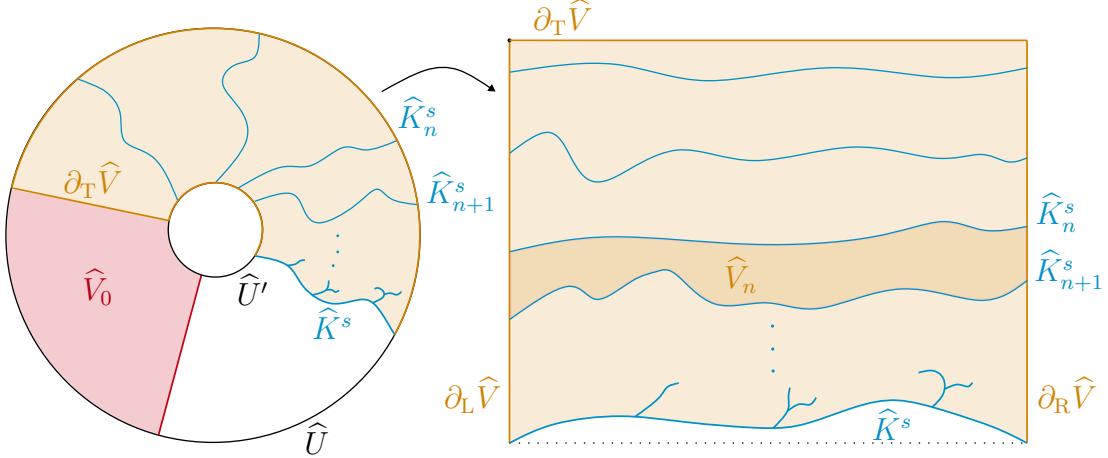


FIGURE 11. The complement of every \widehat{K}_n^s has well-defined upper and lower components, each of them accumulating at $\partial_L \widehat{V}$ and $\partial_R \widehat{V}$.

We will call \widehat{V}_n^+ to the connected component of $\widehat{V} \setminus \widehat{K}_n^s$ which contains \widehat{V}_n , and we will call \widehat{V}_n^- to the connected component of $\widehat{V} \setminus \widehat{K}_n^s$ which contains \widehat{V}_{n-1} .

Fix $n > 0$, and note that if $\delta < \delta_n$, then the filling of the δ -neighbourhood of \widehat{K}_n^s does not intersect \widehat{K}_{n+1}^s . This means there exists a curve γ_n included in this filled neighbourhood except for its endpoints (in particular $\gamma_n \subset \widehat{V}_n$ except for its endpoints), which respectively belong to $\partial_L \widehat{V}_n$, and $\partial_R \widehat{V}_n$.

This proves that we may take a sequence of curves $\{\gamma_n^i\}_{i \in \mathbb{Z}^+}$, each of them contained in \widehat{V}_n except for its endpoints, and intersecting both $\partial_L \widehat{V}$ and $\partial_R \widehat{V}$, such that the Hausdorff limit is a continuum $\widehat{K}_n^+ \subset \widehat{K}_n^s$ which also intersects $\partial_L \widehat{V}$ and $\partial_R \widehat{V}$. In a identical fashion, by approximating \widehat{K}_n^- with curves contained in \widehat{V}_{n+1} except for its endpoints, we obtain $\widehat{K}_n^- \subset \widehat{K}_n^s$ which also intersects $\partial_L \widehat{V}$ and $\partial_R \widehat{V}$. Given that \widehat{K}_n^s is minimal for the inclusion, we obtain that $\widehat{K}_n^+ = \widehat{K}_n^s$, and similarly $\widehat{K}_n^- = \widehat{K}_n^s$. This proves that

$$(8.1) \quad \widehat{K}_n^s \subset \partial \widehat{V}_{n-1}, \quad \widehat{K}_n^s \subset \partial \widehat{V}_n$$

Step 3. Intercalating unstable continua.

For each of these \widehat{V}_n , in our current coordinates, take a small euclidean ball \widehat{W}_n contained in this component, centered at w_n with x -coordinate equal to $\frac{1}{2}$, and radius much smaller than $\frac{1}{4}$.

We now take a sequence $\{\widehat{z}'_n\}_{n \in \mathbb{Z}^+}$ of periodic points such that

$$\rho_g(\widehat{z}'_n) = \left(0, \frac{(-1)^n}{n}\right),$$

As in Step 1, we obtain that

- Each $\widehat{K}^u(\widehat{z}'_n)$ is dense in \mathbb{R}^2 ,
- $\widehat{K}^u(\widehat{z}'_n) \cap \widehat{K}^u(\widehat{z}'_m) = \emptyset$ whenever $m \neq n$

In particular $\widehat{K}^u(\widehat{z}'_n)$ must intersect \widehat{W}_n . Let us then take a sequence of unstable continua $\{\widehat{K}_n^u\}_{n \in \mathbb{Z}^+}$ with $\widehat{K}_n^u \subset \widehat{K}^u(\widehat{z}'_n)$, such that

$$\widehat{K}_n^u \cap \partial \widehat{V} \neq \emptyset, \quad \widehat{K}_n^u \cap \partial \widehat{W}_n \neq \emptyset,$$

minimal for the inclusion, with that property.

Step 4. Finding many transversal stable-unstable intersections.

Fix an integer $j > 0$. Let us define \widehat{V}_n^j as the connected component of $\widehat{V} \setminus (\widehat{K}_{n-j}^s \cup \widehat{K}_{n+j}^s)$ which is *between* \widehat{K}_{n-j}^s and \widehat{K}_{n+j}^s , that is, the only component of $\widehat{V} \setminus (\widehat{K}_{n-j}^s \cup \widehat{K}_{n+j}^s)$ where there is a curve joining $\partial_L \widehat{V}$ to $\partial_R \widehat{V}$, which separates \widehat{K}_{n-j}^s from \widehat{K}_{n+j}^s .

More over, let use the following notation:

$$\partial_L \widehat{V}_n^j = \partial \widehat{V}_n^j \cap \partial_L \widehat{V}, \quad \partial_R \widehat{V}_n^j = \partial \widehat{V}_n^j \cap \partial_R \widehat{V},$$

which are both segments respectively included in $\partial_L \widehat{V}$ and $\partial_R \widehat{V}$.

We will prove the following:

LEMMA 8.10. *For up to finitely many values of n ,*

$$(8.2) \quad \text{either } \widehat{K}_n^u \cap \widehat{K}_{n-j}^s \neq \emptyset, \text{ or } \widehat{K}_n^u \cap \widehat{K}_{n+j}^s \neq \emptyset,$$

$$(8.3) \quad \widehat{K}_n^u \cap \partial_L \widehat{V}_n^j = \emptyset, \quad \widehat{K}_n^u \cap \partial_R \widehat{V}_n^j = \emptyset$$

PROOF. Suppose by contradiction that the result from Equation 8.2 does not hold. Then, up to taking a subsequence, we would have that \widehat{K}_n^u is included in \widehat{V}_n^j . Note that this implies that

$$(8.4) \quad d_n = \sup\{d(z, \widehat{K}^s) : z \in \widehat{K}_n^u\} \xrightarrow{n \rightarrow \infty} 0.$$

We may assume up to taking a new subsequence, that every K_n^u would intersect $\partial_L \widehat{V}_n^l$. Given that in the current coordinates, we have that for every $n > 0$,

$$\frac{1}{4} \leq \text{diam}(\widehat{K}_n^u) \leq \sqrt{2},$$

we would have by Proposition 6.14, that

$$\widehat{K}_n^u \xrightarrow[n \rightarrow +\infty]{\text{Hff}} \widehat{K}^u.$$

By Equation 8.4, we obtain that \widehat{K}^u would be an unstable continuum contained in \widehat{K}^s , which would be also be non-trivial.

Then, our set K^u would be simultaneously stable and unstable, therefore being dynamically bounded for g , which would be a contradiction. This proves that the results from 8.2 hold.

Let us now suppose by contradiction that Equation 8.3 does not hold. The argument is completely analogous: in that case, we could find an infinite subsequence $\{\widehat{K}_n^u\}$ which would converge in the Hausdorff topology, to a non trivial unstable continuum $\widehat{K}^u \subset \widehat{K}^s$, which yields the already-obtained contradiction and thus concludes the proof. \square

Step 5. Locating the sides.

Start by applying Lemma 8.10 to the value $j = 5$. Take a sufficiently large value of n such that we avoid the finitely many pathological values from Step 4, that is, such that Equations 8.2 and 8.3 are held for every $n' \geq n$. We suppose without loss of generality that n is even, and we will work with the strip between \widehat{K}_n^s and \widehat{K}_{n+9}^s .

By Equation 8.2, we have that of the following is true:

$$\widehat{K}_{n+4}^u \cap \widehat{K}_n^s \neq \emptyset, \text{ or } \widehat{K}_{n+4}^u \cap \widehat{K}_{n+9}^s \neq \emptyset.$$

We assume that the first one holds, as the proof for the other case is analogous. This implies that

$$\widehat{K}_{n+4}^u \cap \widehat{K}_{n+1}^s \neq \emptyset, \quad \widehat{K}_{n+4}^u \cap \widehat{K}_{n+3}^s \neq \emptyset, \quad \widehat{K}_{n+4}^u \cap \widehat{K}_{n+4}^s \neq \emptyset.$$

Again by Equation 8.2, we have that

$$\widehat{K}_{n+1}^u \cap \widehat{K}_n^s \neq \emptyset, \text{ or } \widehat{K}_{n+1}^u \cap \widehat{K}_{n+4}^s \neq \emptyset.$$

Once again we assume that the first one holds, as the analysis for the other case is analogous. This implies that

$$\widehat{K}_{n+1}^u \cap \widehat{K}_n^s \neq \emptyset.$$

Putting everything together, so far we have obtained that

$$(8.5) \quad \widehat{K}_{n+4}^u \cap \widehat{K}_n^s \neq \emptyset, \quad \widehat{K}_{n+4}^u \cap \widehat{K}_{n+1}^s \neq \emptyset, \quad \widehat{K}_{n+1}^u \cap \widehat{K}_n^s \neq \emptyset, \quad \widehat{K}_{n+1}^u \cap \widehat{K}_{n+4}^s \neq \emptyset.$$

Let $m = n+4$, and let us take an unstable subcontinuum $\widehat{K}'_m^u \subset \widehat{K}_m^u$ intersecting both \widehat{K}_n^s and \widehat{K}_{n+1}^s , minimal for the inclusion. Each of these continua are contained in the closure of \widehat{V}_n . For $m' = n+1$, define $\widehat{K}'_{m'}^u \subset \widehat{K}_{m'}^u$ in identical fashion.

Proceeding as in Step 4, we will define

$$\partial_L \widehat{V}_n = \partial \widehat{V}_n \cap \partial_L V, \quad \partial_R \widehat{V}_n = \partial \widehat{V}_n \cap \partial_R V$$

By Equation 8.3 we have that

$$\widehat{K}'_m^u \cap \partial_L \widehat{V}_n = \emptyset, \quad \widehat{K}'_m^u \cap \partial_R \widehat{V}_n = \emptyset, \quad \widehat{K}'_{m'}^u \cap \partial_L \widehat{V}_n = \emptyset, \quad \widehat{K}'_{m'}^u \cap \partial_R \widehat{V}_n = \emptyset.$$

This implies that, when seen as a subset of the closure of \widehat{V}_n , the unstable continuum \widehat{K}'_m^u (and similarly $\widehat{K}'_{m'}^u$) separates $\partial_L \widehat{V}_n$ from $\partial_R \widehat{V}_n$: otherwise there would exist a curve from $\partial_L \widehat{V}_n$ to $\partial_R \widehat{V}_n$ included in \widehat{V}_n which would separate \widehat{K}'_m^u , which would be a contradiction.

Thus, one of the following symmetrical statements holds:

$$(8.6) \quad \widehat{K}'_m^u \text{ separates } \partial_L \widehat{V}_n \text{ from } \widehat{K}'_{m'}^u, \text{ and } \widehat{K}'_{m'}^u \text{ separates } \partial_R \widehat{V}_n \text{ from } \widehat{K}'_m^u,$$

$$(8.7) \quad \widehat{K}'_{m'}^u \text{ separates } \partial_L \widehat{V}_n \text{ from } \widehat{K}'_m^u, \text{ and } \widehat{K}'_m^u \text{ separates } \partial_R \widehat{V}_n \text{ from } \widehat{K}'_{m'}^u.$$

Once again we can assume that the first one holds without loss of generality. We shall then say \widehat{K}'_m^u is *on the left* of $\widehat{K}'_{m'}^u$.

Step 6. Building the heteroclinic pseudo-rectangle.

We will now concentrate exclusively on the connected component \widehat{V}_n and the four continua

$$\widehat{K}_n^s, \widehat{K}_{n+1}^s, \widehat{K}_m'^u \text{ and } \widehat{K}_{m'}'^u.$$

Recall that $K_m'^u$ is on the left of $K_{m'}'^u$, and that \widehat{V}_n is a disk.

Note that $\widehat{K}_m'^u$ separates the disk \widehat{V}_n into (at least) two connected components $L(\widehat{K}_m'^u)$, $R(\widehat{K}_m'^u)$ being, respectively, the components containing $\partial_L \widehat{V}_n$ and $\partial_R \widehat{V}_n$. Each of these connected components is a disk, as they are simply connected because their boundary is connected.

Note that, again, $\widehat{K}_{m'}'^u$ separates $R(\widehat{K}_m'^u)$ into at least two connected components $L(\widehat{K}_{m'}'^u) \cap R(\widehat{K}_m'^u)$ and $R(\widehat{K}_{m'}'^u)$.

Given that both $\widehat{K}_m'^u$ and $\widehat{K}_{m'}'^u$ are minimal for the inclusion, we may replicate the argument from Equation 8.1: as in the very end of Step 2, take a sequence of curves $\{\gamma_m^i\}_{i \in \mathbb{Z}^+}$ such that each of them belongs to $L(\widehat{K}_m'^u)$, except for its endpoints, which are respectively in \widehat{K}_n^s and \widehat{K}_{n+1}^s , and such that there is a subsequence of $\{\gamma_m^i\}$ which converges to a subcontinuum of $\widehat{K}_m'^u$ in the Hausdorff topology, and recall that $\widehat{K}_m'^u$ is minimal for inclusion by construction. Repeat this process for the other components to obtain

$$(8.8) \quad K_m'^u \subset \partial L(K_m'^u), \quad K_m'^u \subset \partial R(K_m'^u), \quad K_{m'}'^u \subset \partial L(K_{m'}'^u), \quad K_{m'}'^u \subset \partial R(K_{m'}'^u)$$

Let us define $\widehat{D} = L(\widehat{K}_m'^u) \cap R(\widehat{K}_{m'}'^u)$, $\widehat{P} = \text{cl}(\widehat{D})$, and let us prove that it is a fitted heteroclinic pseudo-rectangle.

Take $\widehat{K}_n'^s = \partial \widehat{D} \cap \widehat{K}_n^s$ and $\widehat{K}_{n+1}'^s = \partial \widehat{D} \cap \widehat{K}_{n+1}^s$. Each of these sets is connected: if $\widehat{K}_n'^s$ is not connected, then it would be contained in the closure of two disjoint closed angular regions R_1, R_2 of the disk \widehat{D} , which on their turn define two disjoint open regions R'_1, R'_2 , which are the connected components in \widehat{D} of the complement of the regions R_1, R_2 . Then, because of Equations 8.1 and 8.8, we can repeat the argument in Item (4) of Lemma 8.4 to obtain two simple curves $\gamma : \gamma' : [0, 1] \rightarrow \text{cl}(\widehat{V})$ which are included in \widehat{V} except for their endpoints, such that

- $\gamma \cap \gamma'$ is a single transversal intersection contained in D (and therefore $\gamma \wedge \gamma' = \pm 1$),
- γ has both endpoints in $\partial_T \widehat{V}$, and γ' has both endpoints in $\partial_L \widehat{V} \cup \partial_R \widehat{V}$,

which would be a contradiction. Proceeding in identical fashion we get that $\widehat{K}_{n+1}'^s$ is also connected.

We finally have that have that

- \widehat{P} contains no singularities as \widehat{U} contains no singularities,
- \widehat{P} is the closure of an open disk \widehat{D} ,
- $\partial \widehat{P} = \widehat{K}_m'^u \cup \widehat{K}_{m'}'^u \cup \widehat{K}_n'^s \cup \widehat{K}_{n+1}'^s$,
- Each neighbourhood of a boundary point intersects the unbounded component of the complement of P , because of Equations 8.1 and 8.8.

- $\widehat{K}'^s_n \cap \widehat{K}'^u_m \neq \emptyset$ by construction, and the same happens for the other three desired intersections.

The five items from Definition 8.2 are then held, as well as the condition from Definition 8.8 because m is even and m' is odd, and therefore \widehat{P} is a fitted heteroclinic pseudo-rectangle, which concludes the proof. \square

CHAPTER 9

The essential factor has a dense family of horseshoes

The goal of this chapter is to prove the following:

PROPOSITION 9.1. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ with $\text{int}(\rho(f)) \neq \emptyset$, and let \widehat{g} a lift of the essential factor, with $\vec{0} \in \text{int}(\rho(\widehat{g}))$. Then, for every open set $\widehat{U} \subset \mathbb{R}^2$, \widehat{g} has a Markovian horseshoe $\widehat{X} \subset \widehat{U}$.*

The idea resides on using the density of the family of fitted heteroclinic pseudo-rectangles, and proving that inside each one of these, there exists a Markovian horseshoe. Unlike the proof of Proposition 8.9, which is almost purely topological, this one heavily relies on dynamical notions, and we will use the techniques and results developed in Sections 4 and 7.

The path for this result is a little bit technical, so we divide it into two big steps, dedicating one of the following subsections to each of them:

- (1) Proving that inside every FHPR there exists a topological horseshoe, in the sense of [KY01] (this is Proposition 9.2). These horseshoes capture some of the rotational behaviour of g .
- (2) To ensure the density of periodic points, we prove in a similar way that inside each FHPR there is a Markovian horseshoe, as in Definition 2.28. This is done in Proposition 9.3.

1. Topological horseshoes

The central result to be proved in this section is the following:

PROPOSITION 9.2. *Let P be a fitted heteroclinic pseudo-rectangle. Then, for every $m > 0$, there exists $j > 0$ such that P contains a topological horseshoe in m symbols for g , with period j .*

As we have seen in Chapter 2, we will use the notion of topological horseshoe defined by Kennedy and Yorke in [KY01], for the restriction to a rectangle $R \subset P$ (taken from Lemma 8.6), of a positive iterate g^j of g . We need to check the 5 Horseshoe Hypothesis, that is

- (1) \mathbb{T}^2 is a separable metric space.
- (2) $R \subset P \subset \mathbb{T}^2$ is locally connected and compact.
- (3) The map $g : R \rightarrow \mathbb{T}^2$ is continuous.
- (4) The sets $R_1^s \subset P$ and $R_2^s \subset P$ are disjoint, nonempty and compact.
- (5) (*Multiple crossing.*) $g^j|_R$ has crossing number greater than or equal to $m > 1$.

that is, every connection contains at least m mutually disjoint preconnections.

We recall that, from Lemma 8.6, each of the sides of the rectangle R are ε -near the respective stable and unstable continuum $K_1^s, K_2^s, K_1^u, K_2^u$. Given ε can be supposed to be as small as we want, we will proceed to prove the *multiple crossing* for the actual HPR P : the actual needed value for ε will be automatic from the construction, and specified at the end of it.

Thus, in our context, a connection will be a continuum $Q^u \subset P$ intersecting both K_1^s and K_2^s . Note that Q^u is not necessarily an unstable continuum, it is simply parallel to the unstable sides in a weak sense.

PROOF OF PROPOSITION 9.2. Fix $m > 0$. Let us divide this proof into steps.

Step 1. Reduction to transverse crossing.

Reasoning as in Lemma 8.7, it is enough to prove the

Transverse crossing property. For every connection \widehat{Q}^u , there exists an integer $j > 0$ and m different lifts $\widehat{P}_1, \dots, \widehat{P}_m$ of P to \mathbb{R}^2 , such that for every integer l with $1 \leq l \leq m$, and every continuum $\widehat{Q}_l^s \subset \widehat{P}_l$ connecting the unstable sides $\widehat{K}_{1,l}^u, \widehat{K}_{2,l}^u \subset \widehat{P}_l$, we have that

$$\widehat{g}^j(\widehat{Q}^u) \cap \widehat{Q}_l^s \neq \emptyset.$$

Step 2. Past anchoring of the stable sides.

Let us define an *extended stable side* $\widehat{K}_1^{*s} \subset K^s(\widehat{z}_1)$ as a stable continuum which is minimal for the inclusion, such that

$$\widehat{K}_1^s \subset \widehat{K}_1^{*s}, \quad \widehat{z}_1 \in \widehat{K}_1^{*s},$$

and let

$$d_1^s = \text{diam}(\widehat{K}_1^{*s}).$$

Define in a similar fashion $\widehat{K}_2^{*s}, \widehat{K}_1^{*u}, \widehat{K}_2^{*u}, d_2^s, d_1^u, d_2^u$, respectively the other extended stable side and the two extended unstable sides. Let $\bar{d} = \max\{d_1^s, d_2^s, d_1^u, d_2^u\}$. Let us write

$$\widehat{P}^* = \widehat{P} \cup \widehat{K}_1^{*s} \cup \widehat{K}_2^{*s} \cup \widehat{K}_1^{*u} \cup \widehat{K}_2^{*u},$$

and call it the *extended heteroclinic pseudo-rectangle*.

Now, using the result and notation from Proposition 3.3 we can take two CFS $\widehat{A}_k^\rightarrow, \widehat{A}_{k'}^\rightarrow \in \widehat{\mathcal{A}}^\rightarrow$, such that

$$(9.1) \quad \widehat{P}^* \subset B(\widehat{A}_k^\rightarrow), \quad \widehat{P}^* \subset T(\widehat{A}_{k'}^\rightarrow).$$

Now, take a sufficiently large negative integer $j_- < 0$, such that

$$(9.2) \quad \widehat{g}^{j_-}(\widehat{z}_1^u) \subset B(\widehat{A}_{k'}^\rightarrow), \quad \widehat{g}^{j_-}(\widehat{z}_2^u) \subset T(\widehat{A}_k^\rightarrow),$$

$$(9.3) \quad d(\widehat{g}^{j_-}(\widehat{z}_1^u), \widehat{A}_{k'}^\rightarrow) > L_{\bar{d}}, \quad d(\widehat{g}^{j_-}(\widehat{z}_2^u), \widehat{A}_k^\rightarrow) > L_{\bar{d}},$$

where $L_{\bar{d}}$ is given by Corollary 6.13. This implies that

$$(9.4) \quad \widehat{g}^{j_-}(\widehat{K}_1^u) \subset \widehat{g}^{j_-}(\widehat{K}_1^{*u}) \subset B(\widehat{A}_{k'}^\rightarrow), \quad \widehat{g}^{j_-}(\widehat{K}_2^u) \subset \widehat{g}^{j_-}(\widehat{K}_2^{*u}) \subset T(\widehat{A}_k^\rightarrow).$$

Take a natural set of lifts \tilde{P}^* , \tilde{K}_2^{*s} , \tilde{K}_2^{*s} , \tilde{K}_1^{*u} , \tilde{K}_2^{*u} , \tilde{A}_k^\rightarrow , $\tilde{A}_{k'}^\rightarrow$ to $\tilde{\mathbb{D}}$, respectively of \hat{P}^* , \tilde{K}_2^{*s} , \tilde{K}_2^{*s} , \hat{K}_1^{*u} , \hat{K}_2^{*u} , \hat{A}_k^\rightarrow and $\hat{A}_{k'}^\rightarrow$.

Given that the *past* transverse path by the isotopy of any point, only crosses leaves from left to right, and by Equation 9.4, we obtain that there exist two leaves

$$\tilde{\phi}_T^s \subset \partial_T \tilde{A}_k^\rightarrow, \quad \tilde{\phi}_B^s \subset \partial_B \tilde{A}_{k'}^\rightarrow,$$

such that

$$(9.5) \quad \tilde{g}^{j_-}(\tilde{K}_1^u) \subset B_{\tilde{\phi}_B^s}(\tilde{A}_{k'}^\rightarrow) \subset B(\tilde{A}_k^\rightarrow), \quad \tilde{g}^{j_-}(\tilde{K}_2^u) \subset T_{\tilde{\phi}_T^s}(\tilde{A}_k^\rightarrow) \subset T(\tilde{A}_{k'}^\rightarrow),$$

$$(9.6) \quad \tilde{A}_k^\rightarrow \cup \tilde{A}_{k'}^\rightarrow \subset L(\tilde{\phi}_B^s), \quad \tilde{A}_k^\rightarrow \cup \tilde{A}_{k'}^\rightarrow \subset L(\tilde{\phi}_T^s).$$

This conveniently implies that any continuum intersecting both $\tilde{\phi}_B^s$ and $\tilde{\phi}_T^s$, will be automatically s -anchored to \tilde{A}_k^\rightarrow and $\tilde{A}_{k'}^\rightarrow$ by those same two leaves. See Figure 12 for details.

Now, note that from Equation 9.5, and the heteroclinic pseudo-rectangle definition, we obtain the following key partial result:

$$(9.7) \quad \text{Any } \tilde{Q}^s \subset \tilde{g}^{j_-}(\tilde{P}) \text{ from } \tilde{g}^{j_-}(\tilde{K}_1^u) \text{ to } \tilde{g}^{j_-}(\tilde{K}_2^u), \text{ must intersect } \tilde{\phi}_B^s \text{ and } \tilde{\phi}_T^s.$$

Note that the value $j_- = j_-(k, k') < 0$ exists for every pair (k, k') satisfying Equation 9.1. Be aware that we will use this fact in the next step, where we potentially retake the value of k .

Step 3. Anchoring of the unstable sides to the same canonically foliated strip.

The rest of the proof is similar to the ending of Lemma 7.8.

Nevertheless, we need one more key observation: there exists $M > 0$ such that if $d(\hat{P}^*, \hat{A}_k^\rightarrow) > M$, then

$$(9.8) \quad \text{the sets of leaves } \hat{\Phi}_1^s, \hat{\Phi}_2^s \text{ and } \hat{\Phi}_2^u \text{ are disjoint,}$$

where

$$\begin{aligned} \hat{\Phi}_1^s &= \{\hat{\phi} \subset \partial_B \hat{A}_k^\rightarrow : \hat{\phi} \cap \hat{I}_{\hat{\mathcal{F}}}^+(\hat{z}_1^s) \neq \emptyset\}, \\ \hat{\Phi}_2^s &= \{\hat{\phi} \subset \partial_B \hat{A}_k^\rightarrow : \hat{\phi} \cap \hat{I}_{\hat{\mathcal{F}}}^+(\hat{z}_2^s) \neq \emptyset\}, \\ \hat{\Phi}_2^u &= \{\hat{\phi} \subset \partial_B \hat{A}_k^\rightarrow : \hat{\phi} \cap \hat{I}_{\hat{\mathcal{F}}}^-(\hat{z}_2^u) \neq \emptyset\}. \end{aligned}$$

This is due to the following facts put together:

- The diameter of the leaves is uniformly bounded,
- Fundamental domains from canonically foliated strips are also bounded
- The slopes of the rotation vectors of the periodic points in the extended stable sides are different, so the $x - \text{coordinate}$ of the intersection of the two transverse paths with horizontal lines separate linearly.

Take a value of k satisfying Equation 9.8, and take $j_+ = j_+(k) > 0$ such that

$$(9.9) \quad \hat{g}^{j_+}(\hat{z}_1^s) \subset T(\hat{A}_k^\rightarrow), \quad \hat{g}^{j_+}(\hat{z}_2^s) \subset T(\hat{A}_k^\rightarrow)$$

$$(9.10) \quad d(\hat{g}^{j_+}(\hat{z}_1^s), \hat{A}_k^\rightarrow) > L_{\bar{d}}, \quad d(\hat{g}^{j_+}(\hat{z}_2^s), \hat{A}_k^\rightarrow) > L_{\bar{d}},$$

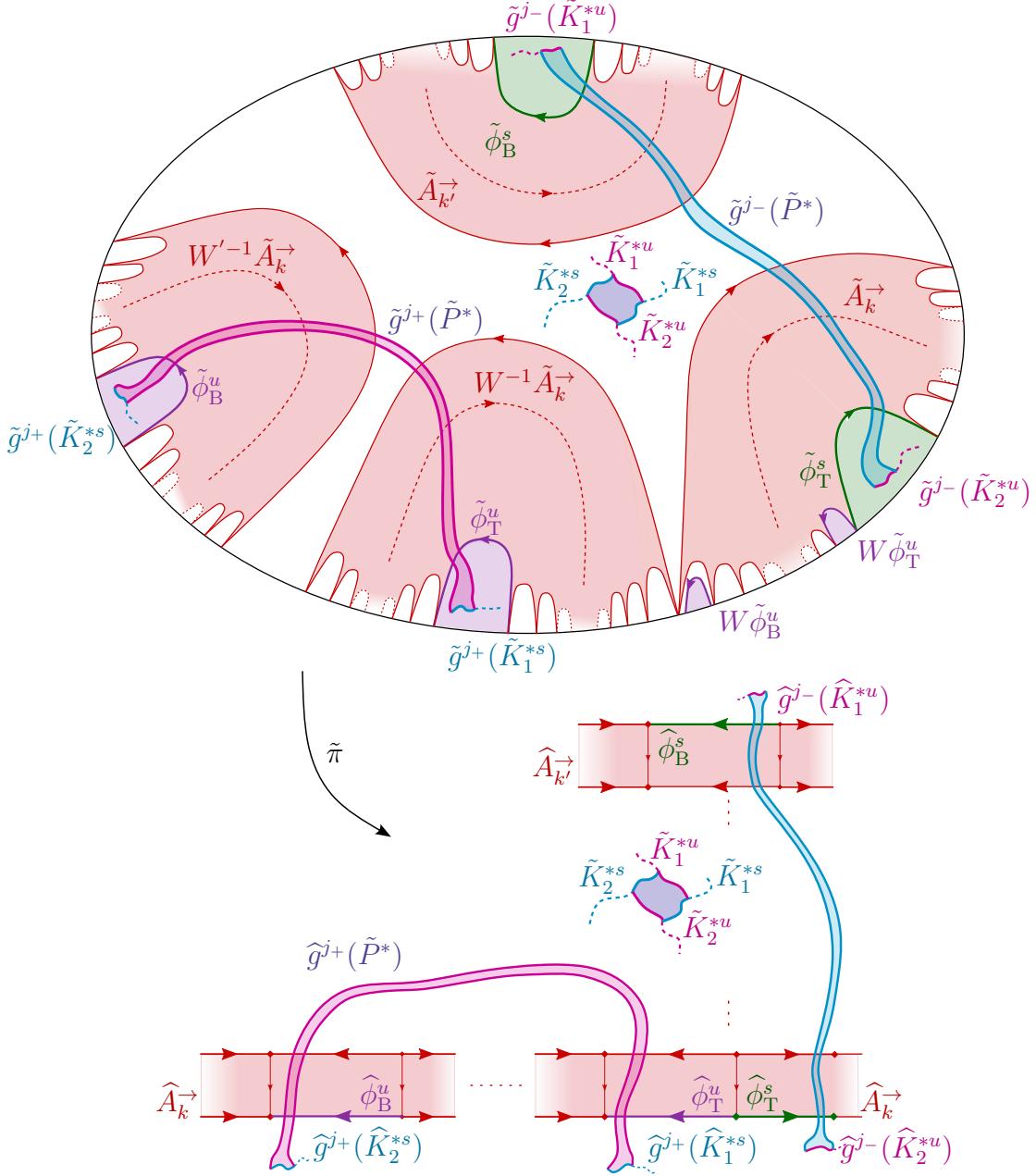


FIGURE 12. Stable sides are solid light blue, unstable sides are solid pink. Dotted lines complete the extended stable and unstable sides. Note that any continuum $\tilde{Q}^s \subset \tilde{g}^{j-}(\tilde{P})$ from $\tilde{g}^{j-}(\tilde{K}_1^u)$ to $\tilde{g}^{j-}(\tilde{K}_2^u)$ holds that $(\tilde{A}_k^-, \tilde{Q}^s, \tilde{\phi}_B^s, \tilde{\phi}_T^s)$ is an s -anchor.

where $L_{\bar{d}}$ is once again taken from Corollary 6.13. As in Step 2, this implies that

$$(9.11) \quad \hat{g}^{j+}(\hat{K}_1^s) \subset \hat{g}^{j+}(\hat{K}_1^{*s}) \subset T(\hat{A}_k^-), \quad \hat{g}^{j+}(\hat{K}_2^s) \subset \hat{g}^{j+}(\hat{K}_2^{*s}) \subset T(\hat{A}_k^-).$$

Recall that, from Corollary 6.13, the future orbit of the extended stable sides will be at a uniformly bounded distance from their respective periodic points. This, together with the result in Equation 9.11, implies that the future orbits of \tilde{K}_1^s and \tilde{K}_2^s will only go through different lifts of \tilde{A}_k^\rightarrow in $\tilde{\mathbb{D}}$, let us call them respectively $W'^{-1}\tilde{A}_k^\rightarrow$, and $W^{-1}\tilde{A}_k^\rightarrow$.

Note that the extended stable sides are stable continua of the lifts of two periodic points with different rotational directions. Reasoning as in Lemma 6.15 (see in particular Equation 6.2 and the lines leading to it), this fact shows that there exist two leaves

$$(9.12) \quad \tilde{\phi}_B^u \subset \partial_T W'^{-1}\tilde{A}_k^\rightarrow \subset B(W'^{-1}\tilde{A}_k^\rightarrow), \quad \tilde{\phi}_T^u \subset \partial_T W^{-1}\tilde{A}_k^\rightarrow,$$

such that

$$(9.13) \quad \tilde{g}^{j+}(\tilde{K}_2^s) \subset L(\tilde{\phi}_B^u) \subset B(W'^{-1}\tilde{A}_k^\rightarrow), \quad \tilde{g}^{j+}(\tilde{K}_1^s) \subset L(\tilde{\phi}_T^u) = T_{\tilde{\phi}_T^u}(W^{-1}\tilde{A}_k^\rightarrow).$$

Once again, see Figure 12 for details. It can be helpful to compare it with Figure 7, from Lemma 6.15.

Now, by retaking the lifts $\tilde{P}' = W\tilde{P}$, and therefore taking the corresponding lifts $\tilde{g}^{j+}(\tilde{P}') = \tilde{g}^{j+}(W\tilde{P})$, $\tilde{\phi}'_B^u = W\tilde{\phi}_B^u$, $\tilde{\phi}'_T^u = W\tilde{\phi}_T^u$, we obtain analogous results to the ones in Equation 9.13, more precisely

$$(9.14) \quad \tilde{g}^{j+}(\tilde{K}'^s) \subset L(\tilde{\phi}_B^u) \subset B(\tilde{A}_k^\rightarrow), \quad \tilde{g}^{j+}(\tilde{K}'^s) \subset L(\tilde{\phi}_T^u) = T_{\tilde{\phi}_T^u}(\tilde{A}_k^\rightarrow).$$

Moreover, we also have that

$$(9.15) \quad (\tilde{\phi}'_B^u \cup \tilde{g}^{j+}(\tilde{P}') \cup \tilde{\phi}'_T^u) \subset L(\tilde{\phi}_B^s), \quad (\tilde{\phi}'_B^u \cup \tilde{g}^{j+}(\tilde{P}') \cup \tilde{\phi}'_T^u) \subset L(\tilde{\phi}_T^s)$$

, which on its turn implies that

$$(9.16) \quad \tilde{g}^{j+}(\tilde{P}') \subset L(\tilde{\phi}_B^s), \quad \tilde{g}^{j+}(\tilde{P}') \subset L(\tilde{\phi}_T^s), \text{ for every } j \geq j_+$$

provided k is a sufficiently large negative integer. It may help the heuristics of the proof to see that the configuration we have just obtained is virtually the same as the one obtained in Figure 9, from Lemma 7.8.

Note that we immediately obtain an analogous of Equation 9.7, that is

$$(9.17) \quad \text{Any connection } \tilde{Q}^u \subset \tilde{g}^{j+}(\tilde{P}'), \text{ must intersect } \tilde{\phi}'_B^u \text{ and } \tilde{\phi}'_T^u.$$

See Figure 13 for details.

Step 4. Finding the transverse crossing.

The remainder of the proof is very similar to the one in Lemma 7.8.

Take any connection $\tilde{Q}^u \subset \tilde{g}^{j+}(\tilde{P}')$ (i.e. a continuum connecting the stable sides), and any continuum $\tilde{Q}^s \subset \tilde{g}^{j-}(\tilde{P})$ connecting the respective unstable sides. Take a minimal-for-inclusion subcontinuum \tilde{Q}'^s intersecting both $\tilde{\phi}_B^s$ and $\tilde{\phi}_T^s$. Now, as in the Anchoring Lemma, take

$$\tilde{z}_B^s \in \tilde{Q}'^s \cap \tilde{\phi}_B^s, \quad \tilde{z}_T^s \in \tilde{Q}'^s \cap \tilde{\phi}_T^s,$$

and define

$$\Gamma_s = \tilde{\phi}_{\tilde{z}_B^s}^+ \cup \tilde{Q}'^s \cup \tilde{\phi}_{\tilde{z}_T^s}^-.$$

We may then take a new leaf $\tilde{\phi}_1 \subset \tilde{A}_k^\rightarrow$ such that

$$\Gamma_s \subset R(\tilde{\phi}_1),$$

which in particular means that Γ_s separates $\tilde{\phi}_B^{u'}$ and $\tilde{\phi}_T^{u'}$, from $\tilde{\phi}_1$. Note that

$$(9.18) \quad (\tilde{A}_k^\rightarrow, \tilde{Q}^u, \tilde{\phi}_B^{u'}, \tilde{\phi}_T^{u'}) \text{ is a } u\text{-anchor},$$

from where we obtain that there exists $j_1 > 0$ such that

$$(9.19) \quad \tilde{g}^{j_1}(\tilde{Q}^u) \cap \tilde{\phi}_1 \neq \emptyset, \text{ for every } j > j_1.$$

This, together with Equation 9.16 and the fact that $\tilde{Q}^s \subset R(\tilde{\phi}_1)$, lets us conclude that

$$(9.20) \quad \tilde{g}^{j_1}(\tilde{Q}^u) \cap \tilde{Q}^s \neq \emptyset, \text{ for every } j \geq j_1.$$

Let us write $\hat{P}_1 = \hat{P}$. To finish proving the Transverse Crossing Property, simply take $m - 1$ horizontal translates $\hat{P}_2, \dots, \hat{P}_m$ of \hat{P} , where $\hat{P}_l = \hat{P} + (l - 1, 0)$, and note that taking the natural lifts $\tilde{P}_2, \dots, \tilde{P}_m$ such that $\tilde{g}^{j_-}(\tilde{P}_2), \dots, \tilde{g}^{j_-}(\tilde{P}_m)$ are s -anchored to \tilde{A}_k^\rightarrow , we may take a new leaf $\tilde{\phi}_m \subset \tilde{A}_k^\rightarrow$, which is *much further to the visual right*, such that

$$\tilde{g}^{j_-}(\tilde{P}_l) \subset R(\tilde{\phi}_m), \text{ for every } l \text{ with } 1 \leq l \leq m,$$

and conclude that there exists $j_m > 0$ such that

$$\tilde{g}^{j_m}(\tilde{Q}^u) \cap \tilde{Q}_l^s \neq \emptyset, \text{ for every } j \geq j_m, 0 \leq l \leq m$$

where $\tilde{Q}_l^s \subset \tilde{g}^{j_-}(\tilde{P}_l)$, (See Figure 13 for details). This proves the Transverse Crossing Property for every $j \geq j_+ - j_- + j_m$.

Note that we can take ε in Lemma 8.6, such that the new sides $\tilde{R}_1^s, \tilde{R}_2^s, \tilde{R}_1^u, \tilde{R}_2^u$ of the rectangle \tilde{R} still get anchored in the same way as the sides $\tilde{K}_1^s, \tilde{K}_2^s, \tilde{K}_1^u, \tilde{K}_2^u$ of the HPR \hat{P} (we use the uniform continuity of \hat{g} and the fact that we have to control distances for a finite number of iterates of \hat{g}). This concludes the proof. \square

Recall that, from the Kennedy-Yorke topological horseshoe definition, we obtain positive topological entropy for $g|_P$, but we do not automatically recover the existence of periodic points inside P .

2. Markovian horseshoes

The key result from this section comes next. It is heavily based on the techniques displayed at Proposition 9.2

PROPOSITION 9.3. *Let $\hat{P} \subset \mathbb{R}^2$ be a fitted heteroclinic pseudo-rectangle and let $m > 0$ be an integer. Then, there exists a rectangle $\tilde{R} \subset \hat{P}$ which contains a Markovian horseshoe in m symbols for \hat{g} .*

Before starting, recall Definition 2.27 of Markovian intersection.

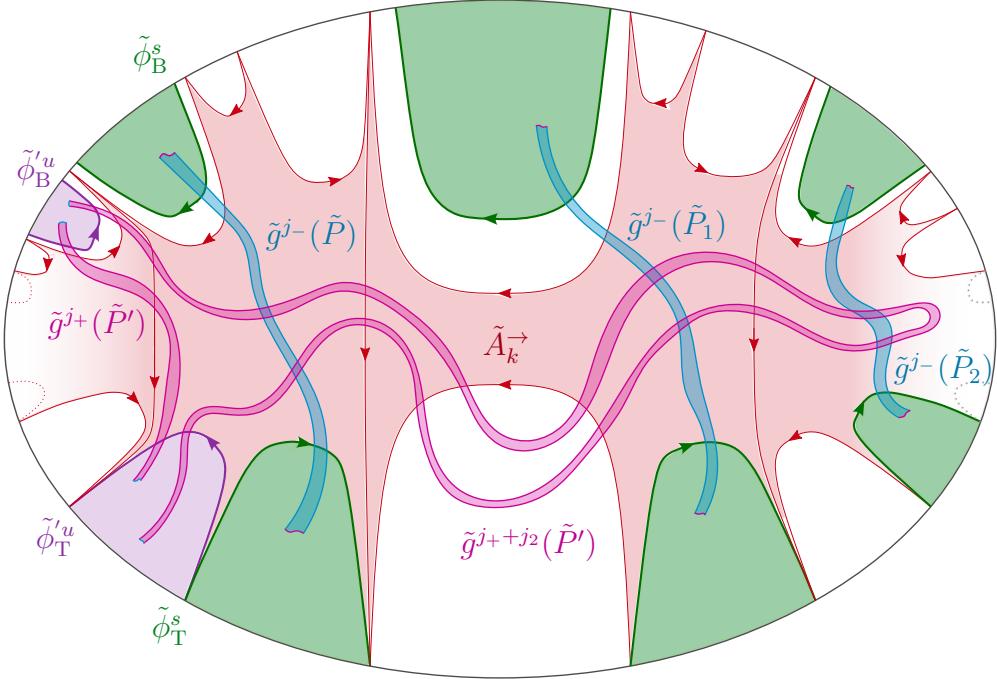


FIGURE 13. Once again, stable sides are light blue, unstable sides are pink. The unstable sides of $\tilde{g}^{j-}(\tilde{P})$ and its translates, are included in the green regions, the stable sides of $\tilde{g}^{j+}(\tilde{P}')$ are included in the purple regions. The future orbit of $\tilde{g}^{j+}(\tilde{P}')$ can not intersect any of the green leaves.

DEFINITION 1 (Markovian Intersection). *Let $R_1, R_2 \subset S$ be two rectangles. We will say that $R_1 \cap R_2$ is a Markovian intersection if there exists a homeomorphism h from a neighbourhood of $R_1 \cup R_2$ to an open subset of \mathbb{R}^2 , such that*

- $h(R_2) = [0, 1]^2$;
- Either $h(R_1^+) \subset \{(x, y) \in \mathbb{R}^2 : y > 1\}$ and $h(R_1^-) \subset \{(x, y) \in \mathbb{R}^2 : y < 0\}$; or $h(R_1^+) \subset \{(x, y) \in \mathbb{R}^2 : y < 0\}$ and $h(R_1^-) \subset \{(x, y) \in \mathbb{R}^2 : y > 1\}$;
- $h(R_1) \subset \{(x, y) \in \mathbb{R}^2 : y > 1\} \cup [0, 1]^2 \cup \{(x, y) \in \mathbb{R}^2 : y < 0\}$.

Once again we will make good use of the anchoring techniques. For the sake of clarity, we will divide the proof into steps.

PROOF OF PROPOSITION 9.3.

Step 1. Constructing an Initial Configuration.

Fix a lift \tilde{P} of \hat{P} , and take the lifts $\tilde{K}_1^s, \tilde{K}_2^s, \tilde{K}_1^u, \tilde{K}_2^u$ of its sides. Start by recovering a partial configuration we obtained in Proposition 9.2, that is, take

$$\tilde{A}_k^\rightarrow, \tilde{A}_{k'}^\rightarrow, \tilde{\phi}_B^s, \tilde{\phi}_T^s, \tilde{\phi}_B'^u, \tilde{\phi}_T'^u, j_- < 0, j_+ > 0, W,$$

such that they hold every result from Equation 9.5 to Equation 9.13. See Figure 14 for details.

The key idea is to use Proposition 7.5. From its statement, let us take $\tilde{g}^{j+}(\widehat{P}) = \tilde{\pi}(\tilde{g}^{j+}(\tilde{P}))$ as an unstable continuum, $\widehat{A}_k^\rightarrow = \tilde{\pi}(\tilde{A}_k^\rightarrow)$ as our CFS, and a leaf $\widehat{\phi} \subset \widehat{A}_k^\rightarrow$ such that $\tilde{g}^{j-}(\widehat{P}) \cap \widehat{A}_k^\rightarrow \subset L_{\widehat{A}_k^\rightarrow}(\widehat{\phi})$. Then, the second item of Proposition 7.5 allows us, by looking at the future iterates of $\tilde{g}^{j+}(\tilde{P})$ by \tilde{g} , to find a new lift $W_1\tilde{P}$ by a deck transformation W_1 -think of W_1 as the deck transformation associated to a large loop going through four CFS, one in each of the four families in $\widehat{\mathcal{A}}$, as in [Proposition 7.5, Step (2)]-, some $j_1 > 0$ and three leaves

$$\tilde{\phi} \subset W_1\tilde{A}_k^\rightarrow, \quad \tilde{\phi}'_B \subset \partial_B W_1\tilde{A}_k^\rightarrow, \quad \tilde{\phi}'_T \subset \partial_T W_1\tilde{A}_k^\rightarrow,$$

(where $\tilde{\phi}$ is the lift of $\widehat{\phi}$ which belongs to $W_1\tilde{A}_k^\rightarrow$, and $\tilde{\phi}'_B, \tilde{\phi}'_T$ are respectively the *anchoring* leaves which appear in the statement of Proposition 7.5 as $\tilde{\phi}_B$ and $\tilde{\phi}_T$), such that the following five equations hold:

$$(9.21) \quad W_1\tilde{A}_k^\rightarrow \subset R(\tilde{\phi}'_B) \cap R(\tilde{\phi}'_T), \quad W_1\tilde{A}_k^\rightarrow \subset L(W_1\tilde{\phi}_B^s) \cap L(W_1\tilde{\phi}_T^s),$$

$$(9.22) \quad \tilde{g}^{j-}(W_1\tilde{P}) \subset L(\tilde{\phi}), \quad \tilde{g}^{j++j_1}(\tilde{P}) \subset R(\tilde{\phi}),$$

To check Equations 9.21 and 9.22, think of W_1 as the covering transformation associated to a loop in \mathbb{R}^2 as in Proposition 7.5: once we project to \mathbb{R}^2 and get that $\tilde{g}^j(\widehat{P})$ is u -anchored to $\widehat{A}_k^\rightarrow$, we use four CFS, one in each of our four directions and make the anchored continuum *turn to the right* each time it enters a new CFS, until it enters again $\widehat{A}_k^\rightarrow$ coming from $T(\widehat{A}_k^\rightarrow)$. This is thoroughly described in [Proposition 7.5, Step (2)].

$$(9.23) \quad \tilde{g}^{j-}(W_1\tilde{K}_1^u) \subset R(W_1\tilde{\phi}_B^s), \quad \tilde{g}^{j-}(W_1\tilde{K}_2^u) \subset R(W_1\tilde{\phi}_T^s),$$

$$(9.24) \quad \tilde{g}^{j++j_1}(\tilde{K}_1^s) \subset L(\tilde{\phi}_B^u) \subset T(W_1\tilde{A}_k^\rightarrow), \quad \tilde{g}^{j++j_1}(\tilde{K}_2^s) \subset L(\tilde{\phi}_T^u) \subset T(W_1\tilde{A}_k^\rightarrow),$$

These two equations come respectively as a consequence from Equations 9.7 and 9.13, both appearing in Proposition 9.2.

$$(9.25) \quad \text{For any continuum } \tilde{Q}^u \subset \tilde{P} \text{ from } \tilde{K}_1^s \text{ to } \tilde{K}_2^s, \text{ we have that}$$

$$(W_1\tilde{A}_k^\rightarrow, \tilde{g}^{j++j_1}(\tilde{Q}^u), \tilde{\phi}'_B, \tilde{\phi}'_T) \text{ is a } u\text{-anchor.}$$

For this last equations, we are applying Proposition 7.5 to the continuum \tilde{Q}^u , and using the fact that $\tilde{g}^{j+}(\tilde{Q}^u)$ is u -anchored to $W^{-1}\tilde{A}_k^\rightarrow$. See Figures 14 and 15 for details.

For the sake of simplicity, let us rename $\tilde{O} := \tilde{g}^{j-}(W_1\tilde{P})$, and $\tilde{O}' := \tilde{g}^{j++j_1}(\tilde{P})$ for the remainder of the proof. Then, provided

- (1) the statements from Equation 9.21 to Equation 9.25 are held,
- (2) $\tilde{O}' = \tilde{g}^l(W_1^{-1}\tilde{O})$, where $l > 0$, and W_1 is a deck transformation,

we will say that (\tilde{O}, \tilde{O}') is a pair in the *Initial Configuration* relative to W_1 , and with \tilde{O} *on the left* of \tilde{O}' . Note that any set $\tilde{Q} \subset \tilde{O}$ has its natural copy $\tilde{Q}' := \tilde{g}^l(W_1^{-1}\tilde{Q}) \subset \tilde{O}'$, and vice versa.

Let us also rename $\tilde{\phi}_B := W_1\tilde{\phi}_B^s$, $\tilde{\phi}_T := W_1\tilde{\phi}_T^s$, and $\tilde{A} := W_1\tilde{A}_k^\rightarrow$. See Figure 15.

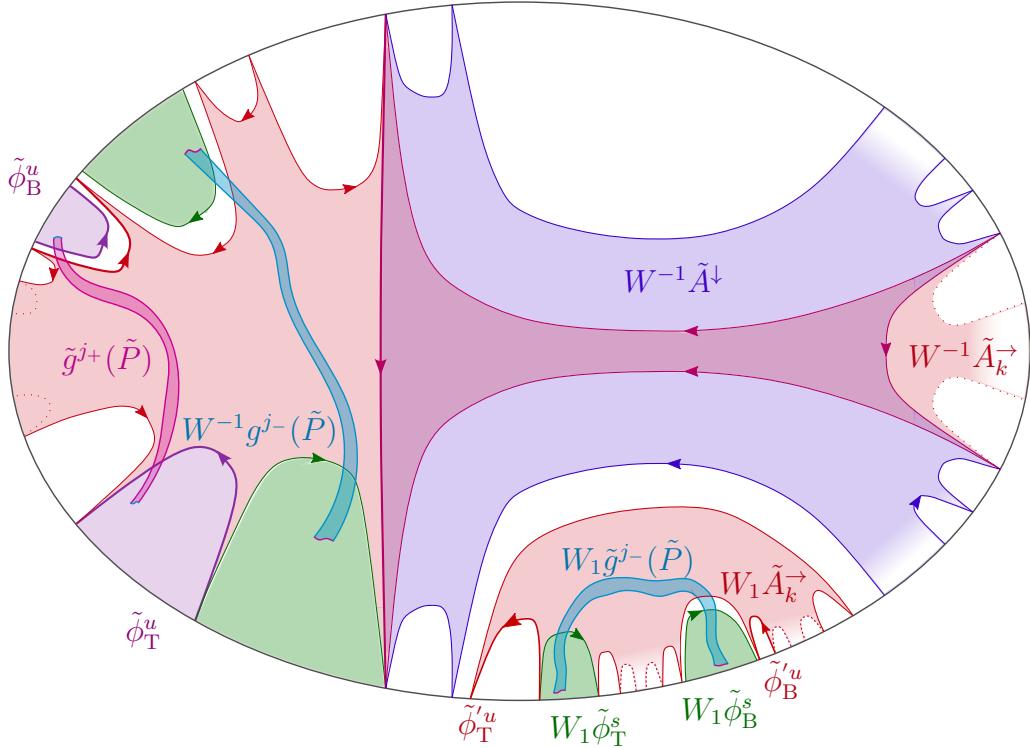


FIGURE 14. We recover the configuration and notation from Proposition 9.2, use the Total Anchoring from Proposition 7.5 to anchor a future iterate of $\tilde{g}^{j+}(\tilde{P})$ to the CFS $W_1\tilde{A}_k^-$. the deck transformation W is taken as in Proposition 9.2. Compare this with Figure 12

Step 2. Finding Markovian intersections.

We will prove that given a pair (\tilde{O}, \tilde{O}') in the Initial Configuration, there exists $j > 0$ and a rectangle $\tilde{R} \subset \tilde{O}$ (with its natural copy $\tilde{R}' \subset \tilde{O}'$) such that $\tilde{g}^j(\tilde{R}') \cap \tilde{R}$ is Markovian.

First, note that by Lemma 8.6, we may take a rectangle $\tilde{R} \subset \tilde{O}$, with horizontal sides $\tilde{I}_1^s, \tilde{I}_2^s$ sufficiently close to the stable sides $\tilde{K}_1^s, \tilde{K}_2^s$ of the pseudo-rectangle \tilde{O} , and similarly vertical sides $\tilde{I}_1^u, \tilde{I}_2^u$ sufficiently close to the unstable sides $\tilde{K}_1^u, \tilde{K}_2^u$, and their corresponding copies $\tilde{R}' \subset \tilde{O}'$ with sides $\tilde{I}_1'^s, \tilde{I}_2'^s, \tilde{I}_1'^u, \tilde{I}_2'^u$, such that, (recalling Equations 9.23 and 9.24), we obtain

$$(9.26) \quad \tilde{I}_1'^s \subset L(\tilde{\phi}_B^u) \subset T(\tilde{A}), \quad \tilde{I}_2'^s \subset L(\tilde{\phi}_T^u) \subset T(\tilde{A}),$$

$$(9.27) \quad \tilde{I}_1'^u \subset R(\tilde{\phi}_B), \quad \tilde{I}_2'^u \subset R(\tilde{\phi}_T),$$

and therefore we have that

$$(9.28) \quad \text{For any continuum } \tilde{Q}'^u \subset \tilde{R}' \text{ from } \tilde{I}_1'^s \text{ to } \tilde{I}_2'^s,$$

$(\tilde{A}, \tilde{Q}'^u, \tilde{\phi}_B^u, \tilde{\phi}_T^u)$ is a u -anchor.

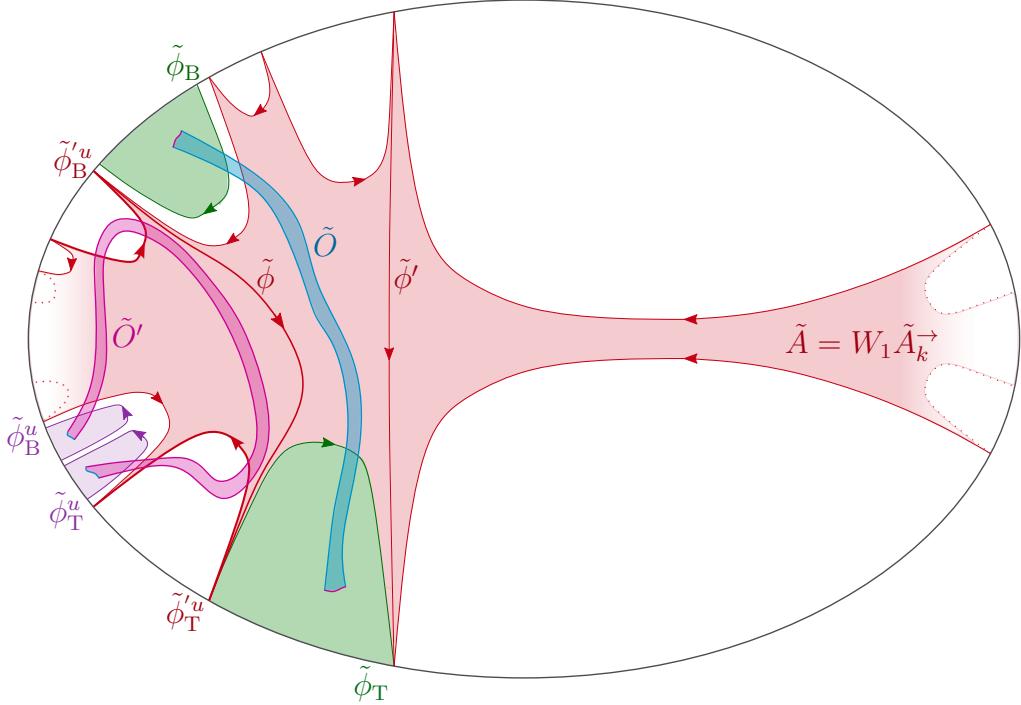


FIGURE 15. An Initial Configuration, with $\tilde{O}' = \tilde{g}^l(W_1^{-1}\tilde{O})$. The leaf $\tilde{\phi}$ separates \tilde{O} from \tilde{O}' , and the leaf $\tilde{\phi}'$ helps us find the desired Markovian intersection.

Take a leaf $\tilde{\phi}' \subset \tilde{A}$, such that $\tilde{O} \subset R(\tilde{\phi}')$. By the Anchoring Lemma (4.11), there exists $j > 0$ such that $\tilde{g}^j(\tilde{Q}'^u) \cap \tilde{\phi}' \neq \emptyset$, for any such a continuum \tilde{Q}'^u as in Equation 9.28. Given that $\tilde{R}' \subset R(\tilde{\phi}) \subset L(\tilde{\phi}_B) \cap L(\tilde{\phi}_T)$, we obtain that $\tilde{g}^j(\tilde{R}') \subset L(\tilde{\phi}_B) \cap L(\tilde{\phi}_T)$, which implies that $\tilde{g}^j(\tilde{R}')$ intersects neither $\tilde{\phi}_B$ nor $\tilde{\phi}_T$. Let us show that using this fact and Lemma 8.7, there must be a curve $\gamma'^s \subset \tilde{g}^j(\tilde{R}')$ parallel to the stable sides (i.e. going from one unstable side to the other, without intersecting the stable sides), such that $\gamma'^s \subset L_{\tilde{A}}(\tilde{R})$.

The application of Lemma 8.7 is as follows: take $\tilde{Q} := \tilde{\phi}' \cap \tilde{g}^j(\tilde{R}')$. Now, note that \tilde{Q} is in the hypothesis of Lemma 8.7, as it does not intersect the stable sides and it also *separates the stable sides* (i.e. it intersects every continuum going from one stable side to the other). Then by this same lemma we obtain that \tilde{Q} must have a subcontinuum $\tilde{Q}' \subset \tilde{g}^j(\tilde{R}')$ connecting the stable sides. As $\tilde{\phi}'$ is a line (i.e. homeomorphic to \mathbb{R} by a proper application) in $\tilde{\mathbb{D}}$, we recover that \tilde{Q}' can be taken as a curve, which we will be our desired γ'^s .

Observe that γ'^s divides $\tilde{g}^j(\tilde{R}')$ into two complementary rectangles. Define $\tilde{g}^j(\underline{\tilde{R}'})$ as any of the two, say for example the one with horizontal sides $\tilde{g}^j(\tilde{I}'^s_1)$, γ'^s and for vertical sides the corresponding two subarcs γ'_1^u , γ'_2^u of $\tilde{g}^j(\tilde{I}'^u_1)$, $\tilde{g}^j(\tilde{I}'^u_2)$, which go from $\tilde{g}^j(\tilde{I}'^s_1)$ to γ'^s .

Finally, define \tilde{R} as the natural copy of $\tilde{g}^j(\underline{\tilde{R}'})$ which is included in \tilde{R} , and note that as $\tilde{\phi}_B$ and $\tilde{\phi}_T$ respectively separate each of the vertical sides of $\underline{\tilde{R}}$ from $\tilde{g}^j(\underline{\tilde{R}'})$,

then

$$\tilde{g}^j(\tilde{R}') \cap \tilde{R} \text{ is Markovian.}$$

Step 3. Iterating the process.

The rest of the proof is inductive. We will give all the details to go from $m = 1$ to $m = 2$, and we will show this is enough. Recall that $\text{Dom}(\widehat{I})$ is an infinitely-punctured plane, and therefore the group of deck transformations of the universal covering is free. Given that every subgroup of a free group is also free, it is enough to find $l > 0$, two deck transformations W_1, W_2 with no relations whatsoever (take for example the ones induced by *looping* around increasing sets of singularities), and using again Proposition 7.5 obtain two pairs (\tilde{O}, \tilde{O}') , $(W_2\tilde{O}, \tilde{g}^l(\tilde{O}'))$ which are in the Initial Configuration respectively relative to W_1 and W_2W_1 in the respective canonically foliated strips \tilde{A} , $\tilde{W}_2\tilde{A}$, and a leaf $\tilde{\phi} \subset \tilde{A}$ such that

$$(9.29) \quad \tilde{O} \subset R(\tilde{\phi}), \quad W_2\tilde{O} \subset L(\tilde{\phi}).$$

To finish the proof, observe that given $(W_2\tilde{O}, \tilde{g}^l(\tilde{O}'))$ is in the Initial Configuration, there exists $j > 0$ and a rectangle $\tilde{R}' \subset \tilde{O}'$ such that

$$(9.30) \quad \tilde{g}^{l+j}(\tilde{R}') \cap W_2\tilde{R} \text{ is Markovian.}$$

Note that the corresponding new stable (horizontal) side $\gamma'^s \subset \tilde{g}^{l+j}(\tilde{R}')$ holds

$$\gamma'^s \subset L_{W_2\tilde{A}}(W_2\tilde{R}) \subset L(\tilde{\phi}),$$

which implies that $R(\tilde{\phi}_B) \cup \tilde{R} \cup R(\tilde{\phi}_T)$ separates the stable sides of $\tilde{g}^{l+j}(\tilde{R}')$, which shows that

$$(9.31) \quad \tilde{g}^{l+j}(\tilde{R}') \cap \tilde{R} \text{ is also Markovian,}$$

and finishes the proof for $m = 2$. Figure 16 illustrates this phenomenon.

For $m > 2$, the argument is identical: take successive deck transformations W_1, \dots, W_m with no relations, and positive integers l_2, \dots, l_m such that

- (1) (\tilde{O}, \tilde{O}') is in the Initial Configuration relative to W_1 , in the canonically foliated strip \tilde{A} ,
- (2) There exists a leaf $\tilde{\phi} \subset \tilde{A}$ such that $\tilde{O} \subset R(\tilde{\phi})$, $W_2\tilde{O} \subset L(\tilde{\phi})$,
- (3) For every $2 \leq i \leq m$, $(\tilde{g}^{l_i}(\tilde{O}'), W_i \dots W_2\tilde{O})$ is in the Initial Configuration relative to $W_i \dots W_1$, in the canonically foliated strip $W_i \dots W_2\tilde{A}$,
- (4) For every $2 \leq i \leq m-1$ there exists a leaf $\tilde{\phi}_i \subset W_i \dots W_2\tilde{A}$ such that $W_i \dots W_2\tilde{O} \subset R(\tilde{\phi}_i)$, $W_{i+1}W_i \dots W_2\tilde{O} \subset L(\tilde{\phi}_i)$.

To finish the proof, take $j > 0$ and a rectangle \tilde{R}' such that

$$\tilde{g}^{l_m+j}(\tilde{R}') \cap W_m \dots W_2(\tilde{R}) \text{ is Markovian,}$$

which on its turn implies that

$$\tilde{g}^{l_m+j}(\tilde{R}') \cap W_i \dots W_2(\tilde{R}) \text{ is Markovian for every } 2 \leq i \leq m,$$

and that

$$\tilde{g}^{l_m+j}(\tilde{R}') \cap \tilde{R} \text{ is also Markovian.}$$

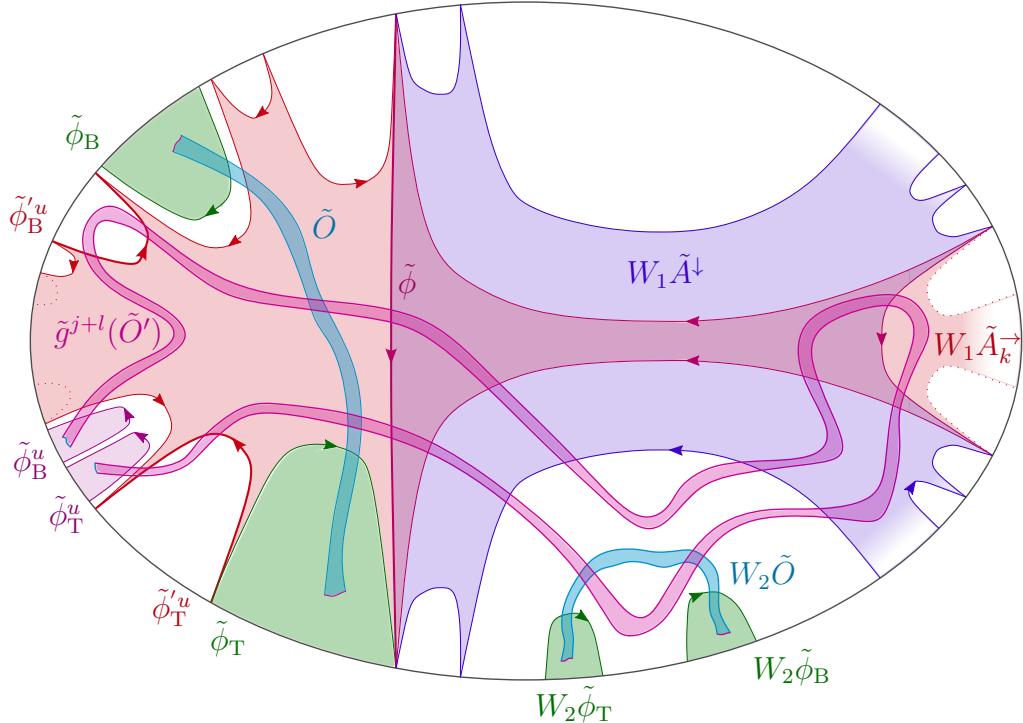


FIGURE 16. We use a similar strategy to the one in Proposition 7.5: *push* the pseudo-rectangle until it enters a new copy of the same canonically foliated strip, and then anchor it there.

This finishes the proof. □

PROOF OF PROPOSITION 9.1. By Proposition 8.9, we know that fitted heteroclinic pseudo-rectangles are dense, and by Proposition 9.3 we know that inside each of these there is a Markovian horseshoe, which concludes the proof. □

As each Markovian horseshoe $\widehat{R} \subset \mathbb{R}^2$ contains periodic points (at least one per periodic sequence by the two-sided shift σ_m in m -symbols), we conclude that

COROLLARY 9.4. *If $\vec{0} \in \text{int}(\rho(\widehat{g}))$, then $\text{Per}(\widehat{g})$ is dense in \mathbb{R}^2*

which in particular implies that

COROLLARY 9.5. *If $\vec{0} \in \text{int}(\rho(\widehat{g}))$, then $\text{Per}_0(g)$ is dense in \mathbb{T}^2*

where $\text{Per}_0(g)$ is the set of periodic points of g with trivial associated rotation vector. Note that these periodic points *do* turn around the singularities of the isotopy.

REMARK 9.6. Apart from the General Hypothesis holding for g (namely $g \in \text{Homeo}_0(\mathbb{T}^2)$, $\text{int}(\rho(g)) \neq \emptyset$), and that $\vec{0} \in \text{int}(\rho(\widehat{g}))$, the properties of g on which the construction of this chapter depend on are the ones in Chapter 5, in particular

- $\mathcal{D}_g(K) = \infty$ for every non-trivial continuum K (which gives us the existence of stable and unstable continua for every point, which have uniformly large diameter),

and the ones derived from it, as the existence of stable sets and the total anchoring of Proposition 7.5.

This allows us to recover the following result, which was a suggestion by Pierre-Antoine Guihéneuf.

COROLLARY 9.7. *Let v be a rational vector in $\text{int}(\rho(\widehat{g}))$. Then,*

$$\{\widehat{z} \in \mathbb{R}^2 : \exists q \in \mathbb{Z}, w = qv \in \mathbb{Z}^2 \text{ s.t. } \widehat{g}^q(\widehat{z}) = \widehat{z} + w\} \text{ is dense in } \mathbb{R}^2.$$

PROOF. Let q be an integer such that $w = qv \in \mathbb{Z}^2$, and note that $w \in \text{int}(\rho(\widehat{g}^q))$. Now, take a new lift $\widehat{g}_w^q = \widehat{g}^q - w$, which is naturally isotopic to the identity as \widehat{g}^q also is. Last, note that \widehat{g}_w^q holds every property from Remark 9.6, which means that

$$X_v(\widehat{g}) = \text{Per}(\widehat{g}_w^q) \text{ is dense in } \mathbb{R}^2,$$

which concludes the proof. \square

Let us define $\text{Per}_v(g) = \widehat{\pi}(X_v(\widehat{g}))$, where \widehat{g} is a lift of g , i.e. the set of periodic points of g with rotation vector equal to v . We have then obtained that

COROLLARY 9.8. *For every rational vector $v \in \text{int}(\rho(\widehat{g}))$, we have that $\text{Per}_v(g)$ is dense.*

CHAPTER 10

Proof of Theorem A

1. The essential factor is conservative

We will prove that, up to a continuous change of coordinates, the essential factor of a torus homeomorphism f in the General Hypothesis preserves the Lebesgue measure. More precisely,

PROPOSITION 10.1. *Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be an isotopic-to-the-identity homeomorphism, such that $\rho(f)$ has nonempty interior, and let $g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be its essential factor (see Definition 5.10 and Remark 5.14). Then, there exists a homeomorphism $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that hgh^{-1} preserves the Lebesgue measure on \mathbb{T}^2 .*

PROOF. The whole idea of the proof is to find an atomless measure μ with total support, which is invariant by g . Once we have done that, we know by Oxtoby-Ulam's Theorem (see [Rou12, Section 2]), that the desired change of coordinates exists. We will strongly use the density of Markovian horseshoes for g .

Start by taking a countable base $\mathcal{U} = \{U_n\}_{n \in \mathbb{Z}^+}$ for the usual topology in \mathbb{T}^2 (i.e. balls of rational radius centered at rational points), and fix $n > 0$. By Proposition 8.9, we know that there exists a heteroclinic pseudo-rectangle P included in U_n , and hence by Proposition 9.2 we know that there exists a compact subset $\Lambda \subset P$ which is invariant by some power g^k of g , and such that it is an extension of the Bernoulli shift σ in 2 symbols by the surjection π , that is

$$\begin{array}{ccc} \Lambda & \xrightarrow{g^k} & \Lambda \\ \pi \downarrow & & \downarrow \pi \\ \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \end{array}$$

where $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$. Now, let us find a minimal subshift σ_α using a well-known technique. Choose an irrational number $0 < \alpha < 1$, and build a Sturmian sequence $\omega \in \Sigma_2$ with density of 1's equal to α : the j -th coordinate of ω will be 1 if and only if

$$\lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor = 1,$$

and it will be 0 otherwise.

We know that the closure X of the orbit of ω , is invariant by σ and is also minimal for σ , from where we can obtain a minimal subshift $\sigma|_X$ (in particular it has no periodic orbits).

Take its preimage $\pi^{-1}(X) \subset U_n$ and look at the restriction $g^k|_{\pi^{-1}(X)}$. We know that it has no periodic orbits because its projection does not have either. By the Krylov–Bogolyubov theorem, we know there exists a g^k -invariant probability measure μ_n supported in $\pi^{-1}(X)$, which must have no atoms, as their existence would on its turn imply the existence of periodic orbits for $g^k|_{\pi^{-1}(X)}$. Let us then define

$$\mu'_n = \frac{1}{n} \sum_{j=1}^{j=n} g_*(\mu_n).$$

Note that $\text{supp}(\mu'_n) \cap U_n \neq \emptyset$ since $\mu'_n(U_n) \geq 1/n$, and that μ'_n is a g -invariant measure with no atoms. Let us now define the probability measure μ as

$$\mu = \sum_{n \in \mathbb{Z}^+} \frac{\mu'_n}{2^n}.$$

Now, note that the probability μ has the following properties:

- It is invariant by g because it is the countable sum of g -invariant measures,
- It has total support: its support intersects every open ball because $\{U_n\}_{n \in \mathbb{Z}^+}$ is a base for the usual topology,
- It has no atoms because each of the countably many μ_n has no atoms.

By Oxtoby-Ulam's Theorem, we know there exists a change of coordinates $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that $h_*(\mu) = \text{Leb}$, the usual Lebesgue measure for \mathbb{T}^2 , which concludes the proof. \square

2. The proof

We have finally gathered all the pieces to prove Theorem A.

PROOF OF THEOREM A. Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be an isotopic-to-the-identity homeomorphism whose rotation set has nonempty interior, and let $g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be its essential factor. Let us check that we have already obtained the required properties:

- $\rho(\widehat{g}) = \rho(\widehat{f})$ for some well-chosen lifts. This comes from Proposition 5.17.
 - For every non-trivial continuum K , $\mathcal{D}_g(K) = \infty$. This is also proved in Proposition 5.17.
 - g is tight. This is Proposition 5.18.
 - g is rotationally mixing. This is proved in Proposition 7.1.
 - Density of Markovian horseshoes for g . This is proved in Proposition 9.1.
- We also obtain that $\overline{\text{Per}(g)} = \mathbb{T}^2$.

By Proposition 10.1, we obtain that for some change of coordinates h , the conjugate hgh^{-1} is area preserving (we can assume that h is isotopic to the identity without loss of generality). Note that these first five properties are invariant under this topological conjugacy h . For the sake of simplicity, rename hgh^{-1} as g , which holds the six desired properties and thus finishes the proof. \square

Bibliography

- [ABP20] J. Alonso, J. Brum, and A. Pasquelli. Generic rotation sets in hyperbolic surfaces, 2020.
- [AZ14] Salvador Addas-Zanata. Area-preserving diffeomorphisms of the torus whose rotation sets have non-empty interior, 2014.
- [AZ15] Salvador Addas-Zanata. Uniform bounds for diffeomorphisms of the torus and a conjecture of boyland. *Journal of the London Mathematical Society*, 91(2):537–553, February 2015.
- [B06] Francois Béguin. Ensemble de rotation des homéomorphismes du tore. *Course Notes*, 2006.
- [BCLR20] François Béguin, Sylvain Crovisier, and Frédéric Le Roux. Fixed point sets of isotopies on surfaces. *Journal of the European Mathematical Society (EMS Publishing)*, 22(6), 2020.
- [BdCH15] Philip Boyland, André de Carvalho, and Toby Hall. New rotation sets in a family of torus homeomorphisms. *Inventiones mathematicae*, 204(3):895–937, October 2015.
- [BdCH23] Philip Boyland, André de Carvalho, and Toby Hall. The dynamics of measurable pseudo-anosov maps, 2023.
- [BHM⁺22] Jonathan Bowden, Sebastian Hensel, Kathryn Mann, Emmanuel Militon, and Richard Webb. Rotation sets and actions on curves. *Advances in Mathematics*, 408:108579, 2022.
- [BHW22] Jonathan Bowden, Sebastian Hensel, and Richard Webb. Quasi-morphisms on surface diffeomorphism groups. *Journal of the American Mathematical Society*, 35(1):211–231, 2022.
- [Bow71] Rufus Bowen. Entropy for group endomorphisms and homogeneous spaces. *Transactions of the American Mathematical Society*, 153:401–414, 1971.
- [Chi79] Boris V Chirikov. A universal instability of many-dimensional oscillator systems. *Physics Reports*, 52(5):263–379, 1979.
- [CL66] Edward Foyle Collingwood and Arthur J Lohwater. *The theory of cluster sets*, volume 56. Cambridge University Press, 1966.
- [dCP02] Andre de Carvalho and Miguel Paternain. Monotone quotients of surface diffeomorphisms. *Mathematical Research Letters*, 10, 12 2002.
- [FM12] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton University Press, 2012.
- [Fra69] John Franks. Anosov diffeomorphisms on tori. *Transactions of the American Mathematical Society*, 145:117–124, 1969.
- [Fra89] John M. Franks. Realizing rotation vectors for torus homeomorphisms. *Transactions of the American Mathematical Society*, 311:107–115, 1989.
- [Fri82] David Fried. The geometry of cross sections to flows. *Topology*, 21(4):353–371, 1982.
- [GK17] Pierre-Antoine Guihéneuf and Andres Koropecki. Stability of the rotation set of area-preserving toral homeomorphisms. *Nonlinearity*, 30(3):1089–1096, February 2017.
- [GM22] Pierre-Antoine Guihéneuf and Emmanuel Militon. Homotopic rotation sets for higher genus surfaces. *to appear in Memoirs of the AMS*, 2022.
- [GM23] Pierre-Antoine Guiheneuf and Emmanuel Militon. Parabolic isometries of the fine curve graph of the torus. *arXiv preprint arXiv:2302.08184*, 2023.

- [GSGL23] Alejo García-Sassi, Pierre-Antoine Guihéneuf, and Pablo Lessa. Geodesic tracking and the shape of ergodic rotation sets. *arXiv preprint, 2312.06249*, 2023.
- [Han85] Michael Handel. Global shadowing of pseudo-anosov homeomorphisms. *Ergodic Theory and Dynamical Systems*, 5(3):373–377, 1985.
- [Her04] Jana Rodriguez Hertz. Continuum-wise expansive homeomorphisms on peano continua. 2004.
- [Jäg08] T. Jäger. Linearization of conservative toral homeomorphisms. *Inventiones mathematicae*, 176(3):601–616, December 2008.
- [Kat80] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 51:137–173, 1980.
- [KH95] Anatole Katok and Boris Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1995.
- [KLCT19] Andres Koropecki, Patrice Le Calvez, and Fabio Tal. A triple boundary lemma for surface homeomorphisms. *Proceedings of the American Mathematical Society*, 147(2):681–686, 2019.
- [KT13] Andres Koropecki and Fabio Armando Tal. Strictly toral dynamics. *Inventiones mathematicae*, 196(2):339–381, May 2013.
- [Kwa92] Jaroslaw Kwapisz. Every convex polygon with rational vertices is a rotation set. *Ergodic Theory and Dynamical Systems*, 12(2):333–339, 1992.
- [Kwa95] J Kwapisz. A toral diffeomorphism with a nonpolygonal rotation set. *Nonlinearity*, 8(4):461–476, July 1995.
- [KY01] Judy Kennedy and James Yorke. Topological horseshoes. *Transactions of the American Mathematical Society*, 353(6):2513–2530, 2001.
- [LC05a] Patrice Le Calvez. An equivariant foliated version of Brouwer’s translation theorem. *Publ. Math., Inst. Hautes Étud. Sci.*, 102:1–98, 2005.
- [LC05b] Patrice Le Calvez. Une version feuilletée équivariante du théorème de translation de brouwer. *Publications Mathématiques de l’IHÉS*, 1(102):1–98, 2005.
- [LCT18] Patrice Le Calvez and Fabio Tal. Forcing theory for transverse trajectories of surface homeomorphisms. *Invent. Math.*, 212(2):619–729, 2018.
- [LCT22] Patrice Le Calvez and Fabio Tal. Topological horseshoes for surface homeomorphisms. *Duke Math. J.*, 171(12):2519–2626, 2022.
- [Lel19] Gabriel Lellouch. *Sur les ensembles de rotation des homéomorphismes de surface en genre ≥ 2* . Theses, Sorbonne Université, April 2019.
- [Les11] Pablo Lessa. Rotation vectors for homeomorphisms of non-positively curved manifolds. *Nonlinearity*, 24(11):3237–3266, 2011.
- [LKFA90] P. Leboeuf, J. Kurchan, M Feingold, and D.P. Arovas. Phase-space localization - topological aspects of quantum chaos. *Physical Review Letters*, 65(25):3076–3079, December 1990.
- [LM91] J. Llibre and R. S. Mackay. Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity. *Ergodic Theory Dyn. Syst.*, 1991.
- [LMP⁺21] Adele Long, Dan Margalit, Anna Pham, Yvon Verberne, and Claudia Yao. Automorphisms of the fine curve graph. *arXiv preprint arXiv:2108.04872*, 2021.
- [Mat97] S. Matsumoto. Rotation sets of surface homeomorphisms. *Bol. Soc. Bras. Mat.*, 28:89–101, 1997.
- [Mil06] John Milnor. *Dynamics in One Complex Variable. (AM-160): Third Edition. (AM-160)*. Princeton University Press, 2006.
- [Moo32] Robert Lee Moore. *Foundations of point set theory*, volume 13. American Mathematical Soc., 1932.
- [MZ89] Michał Misiurewicz and Krystyna Zieman. Rotation Sets for Maps of Tori. *Journal of the London Mathematical Society*, s2-40(3):490–506, 12 1989.

- [MZ91] Michał Misiurewicz and Krystyna Zieman. Rotation sets and ergodic measures for torus homeomorphisms. *Fundamenta Mathematicae*, 137(1):45–52, 1991.
- [Pas13] Alejandro Passeggi. Rational polygons as rotation sets of generic homeomorphisms of the two torus. *Journal of the London Mathematical Society*, 89(1):235–254, October 2013.
- [Pol92] Mark Pollicott. Rotation sets for homeomorphisms and homology. *Transactions of the American Mathematical Society*, 331:881–894, 1992.
- [Pol93] Mark Pollicott. *Lectures on Ergodic Theory and Pesin Theory on Compact Manifolds*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1993.
- [PPS18] Alejandro Passeggi, Rafael Potrie, and Martín Sambarino. Rotation intervals and entropy on attracting annular continua. *Geom. Topol.*, 22(4):2145–2186, 2018.
- [Rou12] Frédéric Le Roux. On closed subgroups of the group of homeomorphisms of a manifold, 2012.
- [RS38] J. H. Roberts and N. E. Steenrod. Monotone transformations of two-dimensional manifolds. *Annals of Mathematics*, 39(4):851–862, 1938.
- [RW22] Frédéric Le Roux and Maxime Wolff. Automorphisms of some variants of fine graphs. *arXiv preprint arXiv:2210.05460*, 2022.
- [Sch57] Sol Schwartzman. Asymptotic cycles. *Annals of Mathematics*, 66(2):270–284, 1957.
- [ST21] Guilherme Salomão and Fabio Tal. Non-existence of sublinear diffusion for a class of torus homeomorphisms. *Ergodic Theory and Dynamical Systems*, 42:1–31, 01 2021.
- [Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bulletin of the American Mathematical Society*, 19:417–431, 1988.