Second Order Diff Equations.

Second order diff equations depend on the second derivative,

Ex: $\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$ ⇐ Spring-mass system,

To treat 2^{nd} order diff equations, we convert them into a system of 1^{st} order diff equations,

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \rightarrow \left[\begin{array}{l} \frac{dx}{dt} = v = g(v, x, t) \\ \\ \frac{dv}{dt} = -\frac{kx}{m} = f(v, x, t) \end{array}\right.$$

In general any 2^{nd} order diff. eq. can be written as,

$$\frac{d^2 y}{dt^2} = f(\dot{y}, y, t) = f(v, y, t) \quad , \quad \text{where } \dot{y} = v$$

Two first order diff equations,

$$\boxed{\frac{dy}{dt} = v \quad , \quad \frac{dv}{dt} = f(v, y, t)}$$

Then the Euler method applied to this system takes the form

$$\begin{bmatrix} v_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} v_n + f(v_n, y_n, t_n)\, h \\ y_n + v_n h \end{bmatrix} = \begin{bmatrix} v_n + f(v_n, y_n, t_n)\, h \\ y_n + g(v_n, y_n, t_n)\, h \end{bmatrix}$$

Given the initial conditions v_0, y_0, t_0, and $h = t_{n+1} - t_n$.

And the 4^{th} order Runge-Kutta takes the form,

$$\begin{bmatrix} v_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} v_n + \frac{h}{6}\left[K_{1v} + 2K_{2v} + 2K_{3v} + K_{4v}\right] \\ y_n + \frac{h}{6}\left[K_{1y} + 2K_{2y} + 2K_{3y} + K_{4y}\right] \end{bmatrix}$$

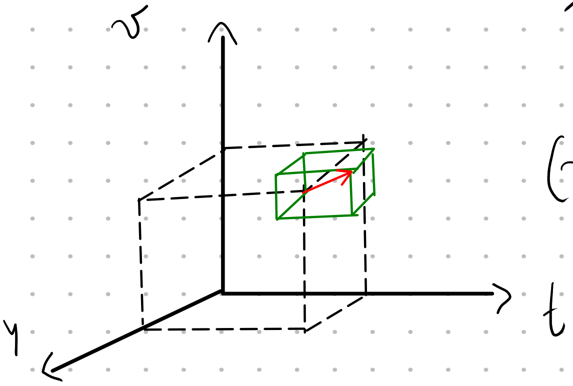
with, $K_{1y} = g(v_n, y_n, t_n)$, $K_{1v} = f(v_n, y_n, t_n)$

$$K_{2y} = g\left(v_n + K_{1v}\frac{h}{2}, y_n + K_{1y}\frac{h}{2}, t_n + \frac{h}{2}\right), K_{2v} = f\left(v_n + K_{1v}\frac{h}{2}, y_n + K_{1y}\frac{h}{2}, t_n + \frac{h}{2}\right)$$

$$K_{3y} = g\left(v_n + K_{2v}\frac{h}{2}, y_n + K_{2y}\frac{h}{2}, t_n + \frac{h}{2}\right), K_{2v} = f\left(v_n + K_{2v}\frac{h}{2}, y_n + K_{2y}\frac{h}{2}, t_n + \frac{h}{2}\right)$$

$$K_{4y} = g\left(v_n + K_{3v}h, y_n + K_{3y}h, t_n + h\right), K_{4v} = \left(v_n + K_{3v}h, y_n + K_{3y}h, t_n + h\right)$$

We can draw the method as,



moves from

$(v_0, y_0, t_0) \to (v_1, y_1, t_1) \to \dots \to (v_n, y_n, t_n) \to (v_{n+1}, y_{n+1}, t_{n+1})$

According to Runge-Kutta Rule,