First Order Differential Equations and Euler Method

First order diff equations,

Rate of change of populations.

$$\frac{dp}{dt} = kp$$

Movement with damping

$$\frac{dv}{dt} + \frac{b}{m}v = 0 \Rightarrow \frac{dv}{dt} = -\frac{b}{m}v$$

Second order diff. equations,

Spring mass system

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

We can write any first order diff equation in the form,

$$\frac{dy}{dt} = f(y,t)$$

Example:

$$\frac{dy}{dt} = t\sqrt{y}$$

Analytic solution:

$$\frac{dy}{\sqrt{y}} = t\,dt \rightarrow \int y^{-1/2}\,dy = \int t\,dt \rightarrow \frac{y^{1/2}}{1/2} = \frac{t^2}{2} + C$$

$$y_{(t)} = \left(\frac{t^2}{4} + C\right)^2 \rightarrow y(t) = \left(\frac{t^2}{4} + \sqrt{y_0}\right)^2$$

We can solve the diff. eq. numerically using the numerical derivative.

$$\frac{dy}{dt} = f(y,t)$$

$$\frac{y(t_{n+1}) - y(t_n)}{t_{n+1} - t_n} = f(y,t)$$

$\boxed{y(t_{n+1}) = y(t_n) + f(y(t_n), t_n)\, h}$ with $h = t_{n+1} - t_n$

this is the Euler Method,

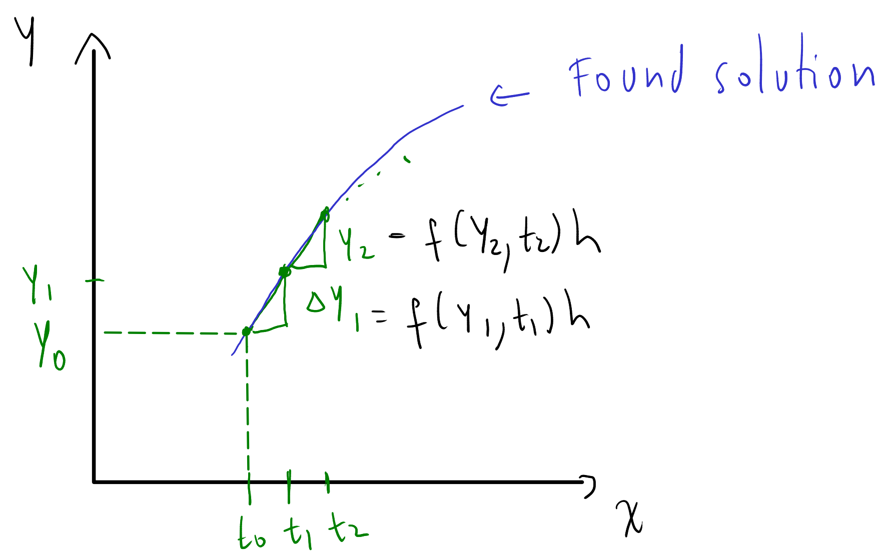
Start at some point t_0, evolve the time $t_0 \to t_1 \to \dots t_n \to t_{n+1}$ to construct the solution $y_0 \to y_1 \to \dots \to y_n \to y_{n+1}$

We may interpret the term $f(y,t)$ as Δy

Since $\frac{\Delta y}{h} = \frac{dy}{dt} \to \Delta y = \frac{dy}{dt} h = f(y,t)\, h$

Hence, the method takes the form

$$y(t_{n+1}) = y(t_n) + \Delta y$$



Improved Euler Method

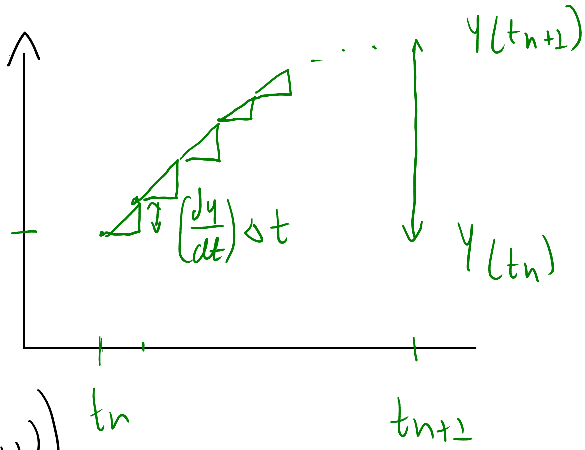
We have,

$$\frac{dy}{dt} = f(y,t)$$

$$\int_{t_n}^{t_{n+1}} y'(t)\,dt = \int_{t_n}^{t_{n+1}} f(y,t)\,dt$$

Using the fundamental theorem of calculus,

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(y,t)\,dt$$



We may approximate this integral with the trapezoid integral rule,

$$y_{n+1} - y_n = \frac{h}{2}\Big(f(y_n,t_n) + \underbrace{f(y_{n+1},t_{n+1})}\Big)$$

↳ to approximate this part we use the conventional Euler method,

$$y_{n+1} - y_n = \frac{h}{2}\Big(f(y_n,t_n) + f\big(y_n + f(y_n,t_n)h,\ t_n + h\big)\Big)$$

Hence,

$$\boxed{\begin{array}{l} y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \\ k_1 = h\,f(y_n, t_n) \quad ; \quad k_2 = h\,f\big(y_n + f(y_n,t_n)h,\ t_n + h\big) \end{array}}$$

This is the improved Euler method.

We can also interpret Euler method, as a Taylor approximation,

$$y(t) = y(t_0) + y'(t_0)(t-t_0) + \frac{y''(t_0)}{2!}(t-t_0)^2 + \cdots +.$$

$$y_{n+1} = y_n + f(y_n, t_n)h + O(h^2)$$