Wave Equation

We want to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

We set, u_i^n ← time, ← space

From the second derivative numerical, we have

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

We wish to solve u for later times, hence, we solve for u_i^{n+1}

$$u_i^{n+1} = a u_i^n - u_i^{n-1} + r(u_{i+1}^n + u_{i-1}^n) \qquad (1)$$

Where,

$$r = \frac{(\Delta t)^2 c^2}{(\Delta x)^2}, \quad a = 2(1 - r)$$

Since, u_i^{n+1} depend on time n, $n-1$, we use the initial velocity to find u_i^1 from u_i^0, and then find u_i^2 from u_i^1 & u_i^0

We have the following boundary conditions,

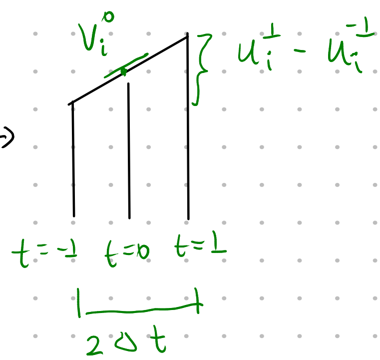
$$u(t,0) = l_0, \quad u(t,L) = l_L, \quad u(0,x) = f(x), \quad \frac{d u(0,x)}{dt} = V_0(x)$$

We look for u_i^1,

We have that,

$$\frac{\partial u(0,x)}{\partial t} = V_0(x)$$

Then, $$\frac{u_i^1 - u_i^{-1}}{2\Delta t} = V_i^0 \longrightarrow$$



Hence,

$$u_i^{-1} = u_i^1 - 2V_i^0 \Delta t$$

Setting, $n=0$ in eq. (1)

$$u_i^1 = a u_i^0 - u_i^{-1} + r\left(u_{i+1}^0 + u_{i-1}^0\right)$$

Replacing u_i^{-1},

$$u_i^1 = a u_i^0 - \left(u_i^1 - 2 v_i^0 \Delta t\right) + r\left[u_{i+1}^0 + u_{i-1}^0\right]$$

Solving for u_i^1, we find the general sol of the wave eq.,

$$\boxed{\begin{aligned} u_i^1 &= \frac{a}{2} u_i^0 + \frac{r}{2}\left(u_{i+1}^0 + u_{i-1}^0\right) + v_i^0 \Delta t \\ u_i^{n+1} &= a u_i^n - u_i^{n-1} + r\left(u_{i+1}^n + u_{i-1}^n\right) \end{aligned}}$$

With $a = 2(1-r)$, $r = \frac{(\Delta t)^2 c^2}{(\Delta x)^2}$

the method converges, when,

$a > 0$ $\quad 2\left(1 - \frac{c^2 \Delta t^2}{\Delta x^2}\right) > 0$

Meaning, $\quad \Delta t < \frac{\Delta x}{c}$

Numerically, we build a matrix $U[\mathrm{len}(t), \mathrm{len}(x)]$