Finite Differences Method

Heat Equation.

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

Solution:

$$u = f(x)\, g(t)$$

$$\frac{1}{\beta} \frac{\partial g(t)}{\partial t} f(x) = g(t) \frac{\partial^2 f(x)}{\partial x^2}$$

$$\frac{1}{\beta} \frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} = -k$$

$$\Rightarrow \quad f''(x) + \omega^2 f(x) = 0 \qquad \text{Harmonic motion}$$

$$f(x) = A\cos(\sqrt{k}\,x) + B\sin(\sqrt{k}\,x)$$

And,

$$g'(t) = -k\, g(t)$$

$$\frac{dg}{g} = -k\, dt$$

$$\ln\left(g(t)/g(0)\right) = -kt$$

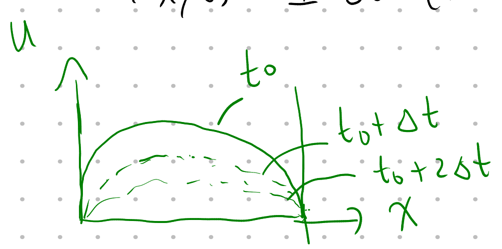
$$g(t) = g(0)\, e^{-kt}$$

Hence,

$$u(x,t) = \left[A\sin(\sqrt{k}\,x) + B\cos(\sqrt{k}\,x)\right] e^{-kt}$$

Imposing boundary conditions

$$u(x,0) = 1\,\text{sen}(\pi x) \;\rightarrow\; u(x,t) = \sin(\pi x)\, e^{-\pi t}$$



Finite Differences Method

We may write the first derivative numerically in two forms,

$$f'(x) = \frac{f(x+h) - f(x)}{h} \quad ; \quad f(x) = \frac{f(x+h) - f(x-h)}{2h} \leftarrow \text{Central derivative,}$$

The second derivative might be written in the form,

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Proof: From taylor we have,

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2} + O(h^3)$$

$$+ \quad f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 + O(h^3)$$

$$f(x+h) + f(x-h) = 2f(x) + f''(x)h^2$$

$$\Rightarrow \quad f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

We may write the heat eq. in the form

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} \quad ; \quad u_i^n \quad \begin{array}{l} n \leftarrow \text{time} \\ i \leftarrow \text{position} \end{array}$$

then,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \beta \left(\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{(\Delta x)^2} \right)$$

So,

$$u_i^{n+1} = u_i^n + \frac{\beta \Delta t}{\Delta x^2}(u_{i+1}^n + u_{i-1}^n - 2u_i^n)$$

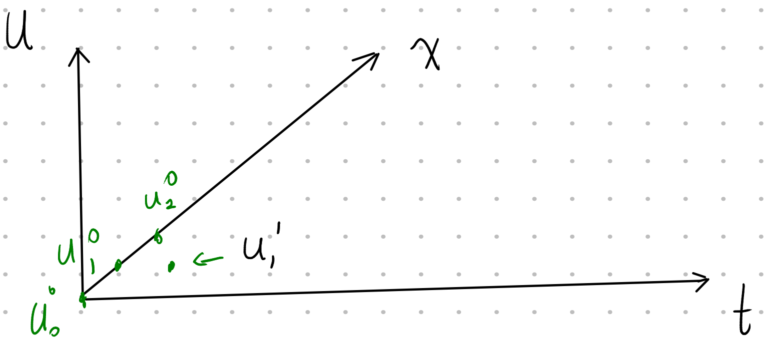
$$u_i^{n+1} = \left(1 - \frac{2\beta \Delta t}{\Delta x^2}\right) u_i^n + \frac{\beta \Delta t}{\Delta x^2}(u_{i+1}^n + u_{i-1}^n)$$

We set, $a = \left(1 - \frac{2\beta\Delta t}{\Delta x^2}\right)$, $r = \frac{\beta \Delta t}{\Delta x^2}$

$$u_i^{n+1} = a u_i^n + r\left(u_{i+1}^n + u_{i-1}^n\right)$$

For convergence of the method we require,

$$\left(1 - \frac{2\beta\Delta t}{\Delta x^2}\right) > 0 \quad \rightarrow \quad \Delta t < \frac{(\Delta x)^2}{2\beta}$$



$$u_1^1 = a u_1^0 + r\left(u_2^0 + u_0^0\right)$$

We require two initial conditions, $u(x,0)$, $u(0,t)$