Runge-Kutta Methods.

Improved Euler method.

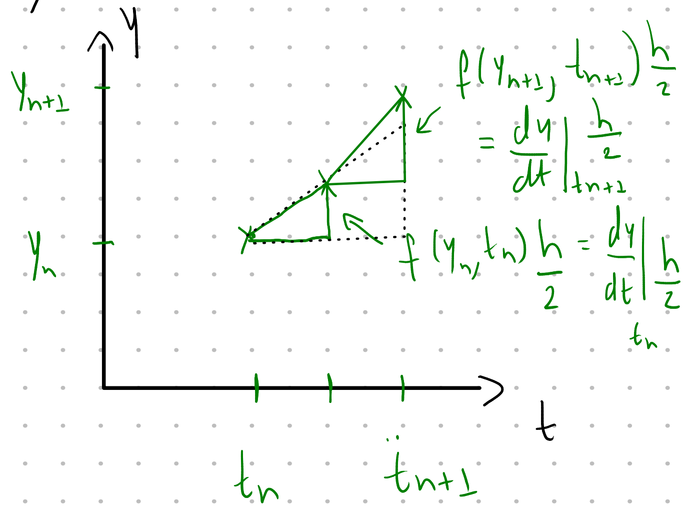
$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

$$k_1 = f(y_n, t_n) \quad k_2 = f(y_n + f(y_n, t_n)h, t_n + h)$$

We have that

$$f(y_{n+1}, t_{n+1}) = \left.\frac{dy}{dt}\right|_{t_{n+1}}$$

$$\Rightarrow f(y_n + f(y_n, t_n)h, t_n + h) = \left.\frac{dy}{dt}\right|_{t_{n+1}}$$



We aproximate y_{n+1}, by moving half step along the derivative in t_n, y_n and half step along the derivative in t_{n+1}, y_{n+1}, to approximate the derivative along t_{n+1}, y_{n+1}, we compute y_{n+1}, using a complete step ; $y_{n+1} = y_n + f(y_n, t_n)h$; as in the conventional Euler method.

Runge-Kutta Method

We have,

$$\frac{dy}{dt} = f(y, t)$$

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y, t)\, dt$$

We approximate this integral, using Simpons' Method,

$$\int_a^b f(x)\, dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

$$\int_{t_n}^{t_{n+1}} f(y,t)\,dt = \frac{h}{6}\left[f(y_n,t_n) + 4f\left(\frac{y_{n+1}+y_n}{2}, \frac{t_{n+1}+t_n}{2}\right) + f(y_{n+1}, t_{n+1})\right]$$

Now,

$$4f\left(\frac{y_{n+1}+y_n}{2}, \frac{t_{n+1}+t_n}{2}\right) = 2f\left(\frac{y_{n+1}+y_n}{2}, \frac{t_{n+1}+t_n}{2}\right) + 2f\left(\frac{y_{n+1}+y_n}{2}, \frac{t_{n+1}+t_n}{2}\right)$$

$$2f\left(\frac{y_{n+1}+y_n}{2}, \frac{t_{n+1}+t_n}{2}\right) = 2f\left(\frac{y_n + y_n + f(y_n,t_n)h}{2}, \frac{2t_n + h}{2}\right)$$

$$= 2f\left(y_n + f(y_n,t_n)\frac{h}{2}, t_n + \frac{h}{2}\right) \approx$$

The second half, takes advantage of the last approximation

$$2f\left(\frac{y_{n+1}+y_n}{2}, \frac{t_{n+1}+t_n}{2}\right) = 2f\left(\frac{y_n + y_n + \frac{h}{2} f\left(y_n + f(y_n,t_n)\frac{h}{2}, t_n + \frac{h}{2}\right)}{2}, t_n + \frac{h}{2}\right)$$

$$= 2f\left(y_n + \frac{k_2}{2}, t_n + \frac{h}{2}\right) = 2K_3$$

Finally, to compute

$f(y_{n+1}, t_{n+1}) = f(y_n + K_3, t_n + h)$

Where, we used the last approximation,

Hence, the final Runge-Kutta method takes the form

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(t_n, y_n)$$

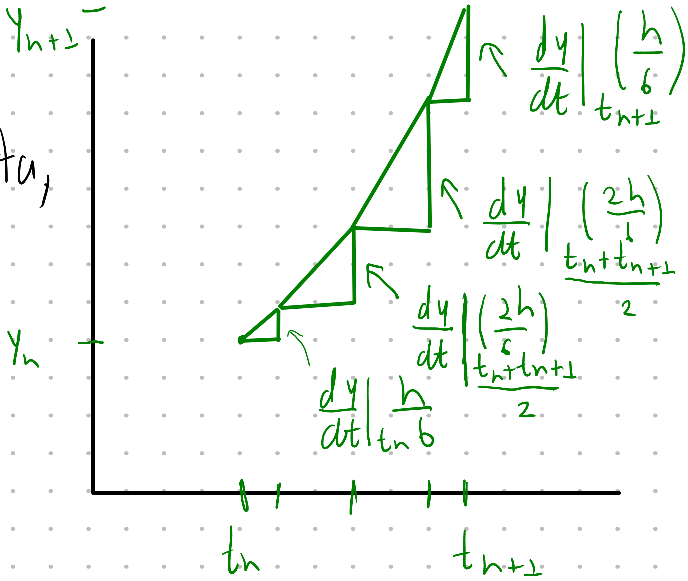
$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(t_n + h, y_n + k_3)$$

Pictorially,

4th Order Runge-Kutta,



; taking advantage of the previous approximations