SPAZIO VETTORIALE

Def: Sia $(\mathbb{K}, +_{\mathbb{K}}, \cdot_{\mathbb{K}})$ un campo, sia $(V, +_V)$ gruppo abeliano. Dico che V è **SPAZIO VETTORIALE** su campo $\mathbb{K}$ ($\mathbb{K}$-spazio vettoriale) se:

$$\exists \ \cdot : \mathbb{K} \times V \longrightarrow V$$
$$(k, \underline{v}) \longmapsto k \cdot \underline{v}$$

t.c.

1. $\forall h, k \in \mathbb{K} \quad \forall v \in V \quad (\underbrace{h +_{\mathbb{K}} k}_{\in \mathbb{K}}) \cdot \underline{v}$ $\quad \times V$

   $\underbrace{h \cdot \underline{v}}_{\in V} +_V \underbrace{k \underline{v}}_{\in V}$

2. $\forall k \in \mathbb{K} \quad \forall u, v \in V$

   $(\underline{u} +_V \underline{v}) \cdot k \qquad \underbrace{k \underline{u}}_{\in V} +_V \underbrace{k \underline{v}}_{\in V}$

3. $\forall k, h \in \mathbb{R} \quad \forall \underline{v} \in V$

   $(\underbrace{k \cdot_{\mathbb{K}} h}_{\in \mathbb{K}}) \cdot \underline{v} = (\underbrace{h \underline{v}}_{\in V}) \cdot k$

4. $\forall \underline{v} \in V \qquad \underbrace{1_{\mathbb{K}}}_{\text{el. neutro del prodotto in } \mathbb{K}} \cdot \underline{v} = \underline{v}$

Dato un campo $(\mathbb{K}, +_{\mathbb{K}}, \cdot_{\mathbb{K}})$ dico che un insieme V è spazio vettoriale su $\mathbb{K}$ se:

$$\exists +_V : V \times V \to V, \quad \exists \cdot : \mathbb{K} \times V \to V \quad \underline{t.c}$$

① $+_V$ è associativa (cioè $\forall \underline{v}, \underline{u}, \underline{w} \in V$ $(\underline{v} +_V \underline{u}) +_V w =$
$= \underline{v} +_V (\underline{u} +_V w)$)

② $+_V$ ha elemento neutro

③ ogni elemento in V ha inverso rispetto a $+_V$
($\forall \underline{v} \in V \ \exists -\underline{v} \ \ \underline{v} + (-\underline{v}) = \underline{0}$)

④ $+_V$ è commutativa ($\forall \underline{u}, \underline{v} \in V \ \ \underline{u} + \underline{v} = \underline{v} + \underline{u}$)

⑤ $\forall h, k \in \mathbb{K}, \ \forall \underline{u} \in V \ \ (h +_{\mathbb{K}} k) \cdot \underline{u} = h\underline{u} +_V k\underline{u}$

⑥ $\forall h \in \mathbb{K} \ \forall \underline{u}, \underline{v} \in V \ \ h(\underline{v} +_V \underline{u}) = h\underline{v} +_V h\underline{u}$

⑦ $\forall h, k \in \mathbb{K} \ \forall \underline{v} \in V \ \ (k \cdot h)\underline{v} = k(h\underline{v})$

⑧ $\forall \underline{v} \in V \ \ 1_{\mathbb{K}} \cdot \underline{v} = \underline{v}$

Chiamo gli elementi di V vettori $(\underline{v} \in V)$ e gli elementi di $\mathbb{K}$ scalari

$\underline{es}$ $\mathbb{K} = \mathbb{R}$ $\quad V = \mathbb{R}^2$ $\quad (\mathbb{R}^2, +_{\mathbb{R}^2})$ è gruppo abeliano

$$+_{\mathbb{R}^2} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \mapsto \begin{pmatrix} a +_{\mathbb{R}} c \\ b +_{\mathbb{R}} d \end{pmatrix}$$

$$\underline{0}_{\mathbb{R}^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\forall \underline{x} \in \mathbb{R}^2 \qquad -\underline{x} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\cdot : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ ⇝ mostro ⑤, ⑥, ⑦, ⑧

$(k, \underline{x}) \to k\underline{x}$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad k\underline{x} = \begin{pmatrix} kx_1 \\ kx_2 \end{pmatrix}$

⑤ se $\underline{v} = \underline{x} \in \mathbb{R}^2 \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$(h+k)\underline{x} = (h+k)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (h +_{\mathbb{R}} k) \cdot_{\mathbb{R}} x_1 \\ (h +_{\mathbb{R}} k) \cdot_{\mathbb{R}} x_2 \end{pmatrix} =$$

$$= \begin{pmatrix} hx_1 + kx_1 \\ hx_2 + kx_2 \end{pmatrix} = \begin{pmatrix} hx_1 \\ hx_2 \end{pmatrix} + \begin{pmatrix} kx_1 \\ kx_2 \end{pmatrix} =$$

prop. distributiva in $\mathbb{R}$

$$= h\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + k\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h\underline{x} + k\underline{x}$$

⑥ $\forall \underline{x}, \underline{y} \in \mathbb{R}^2 \quad h(\underline{x} + \underline{y}) = h\underline{x} + h\underline{y}$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$h\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} +_{\mathbb{R}^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = h\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

$$= \begin{pmatrix} h(x_1 + y_1) \\ h(x_2 + y_2) \end{pmatrix} = \begin{pmatrix} hx_1 + hy_1 \\ hx_2 + hy_2 \end{pmatrix}$$

prop. distr. in $\mathbb{R}$

$$= \begin{pmatrix} hx_1 \\ hx_1 \end{pmatrix} + \begin{pmatrix} hy_1 \\ hy_2 \end{pmatrix} = h\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + h\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= h\underline{x} + h\underline{y}$$

$$⑦ \quad (h \cdot k) \cdot \underline{x} = (k \cdot h)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} (h \cdot k) x_1 \\ (h \cdot k) x_2 \end{pmatrix} = \begin{pmatrix} h \cdot (k x_1) \\ h \cdot (k x_2) \end{pmatrix}$$

prop. ass in $\mathbb{R}$

$$= h\begin{pmatrix} k x_1 \\ k x_2 \end{pmatrix} = h\left(k\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = h \cdot (k \cdot \underline{x})$$

$$⑧ \quad \forall\, \underline{v} \cdot V \quad 1_{\mathbb{K}} \cdot \underline{v} = \underline{v}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad 1 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \cdot x_1 \\ 1 \cdot x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow (\mathbb{R}^2, +_{\mathbb{R}^2})$ è spazio vettoriale sul campo $\mathbb{R}$

<u>Def</u>: Considero $+_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$\cdot : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$k, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} k x_1 \\ \vdots \\ k x_n \end{pmatrix}$$

<u>Prop</u>: $(\mathbb{R}^n, +_{\mathbb{R}^n})$ con l'operazione $\cdot$ è spazio vettoriale sul campo $\mathbb{R}$

<u>Def</u>: $M_{a \times b}(\mathbb{R}) = \{$ matrici con a righe, b colonne a coefficienti in $\mathbb{R} \}$

$$+ : M_{a \times b}(\mathbb{R}) \times M_{a \times b}(\mathbb{R}) \longrightarrow M_{a \times b}(\mathbb{R})$$

$$\underset{[a_{ij}]}{A}, \quad \underset{[b_{ij}]}{B} \longrightarrow \underset{[a_{ij} + b_{ij}]}{A + B}$$

$$\bullet \quad \mathbb{R} \times M_{a\times b}(\mathbb{R}) \longrightarrow M_{a\times b}(\mathbb{R})$$

$$(k, A) \longrightarrow kA$$

$$k, [a_{ij}] \longrightarrow [k a_{ij}]$$

Prop: $(M_{a\times b}(\mathbb{R}), +_{M_{a\times b}(\mathbb{R})})$ con $\cdot$ è spazio vettoriale su $\mathbb{R}$

Es $\quad M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$$2M + 3N = 2\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 9 & 0 \\ 3 & 3 & 6 \end{bmatrix}$$

Def: $\mathbb{R}[x] = \{$ polinomi a coefficienti reali $\}$

$$\overset{\cup}{p(x)} = a_0 + a x + a_2 x^2 + \dots + a_d x^d \quad a_i \in \mathbb{R}$$

$$+_{\mathbb{R}[x]} : \mathbb{R}[x] \times \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$$

$$\underset{\displaystyle \sum_{i=0}^{d} a_i x^i}{\overset{p(x)}{\|}} \qquad \underset{\displaystyle \sum_{j=0}^{r} a_j x^j}{\overset{q(x)}{\|}} \longrightarrow \overset{p(x) + q(x)}{\sum_{h=0}^{\max(d,r)} (a_h + b_h) x^h}$$

dove pongo a_i o b_j uguali a 0 se non compaiono in $p(x)$, $q(x)$

Es $\quad p(x) = x + 1 \quad q(x) = x^3 + x^2 + 2$

$$p(x) + q(x) = \sum_{h=0}^{3} (a_h + b_h) x^h$$

$$= \underset{h=0}{(1+2)} + \underset{h=1}{(1+0)x} + \underset{h=2}{(0+1)x^2} + \underset{h=3}{(0+1)x^3} = 3 + x + x^2 + x^3$$

$\cdot : \mathbb{R} \times \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$

$$k \, , \, p(x) \longmapsto k\,p(x)$$

$$\sum_{i=0}^{d} a_i x^i \longrightarrow \sum_{i=0}^{d} (k a_i) x^i$$

<u>Prop</u>: $(\mathbb{R}[x], +_{\mathbb{R}[x]})$ è gruppo abeliano e con

$\cdot : \mathbb{R} \times \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ è spazio vettoriale su $\mathbb{R}$

<u>Def</u>: sia S un insieme, sia $(V, +_V)$, uno spazio vettoriale su campo $\mathbb{K}$

⊙ $F = \{ f : S \to V \text{ funzion} \}$

⊙ $+_F : F \times F \to F$ $\qquad f : S \to V \quad g : S \to V$

$$(f, g) \longmapsto f+g \qquad f+g : S \to V$$

per definire $f+g$ devo dire chi è l'immagine di ogni:

$$s \in S \quad (f+g)(s) = \underset{V}{\underset{\cap}{f(s)}} +_V \underset{V}{\underset{\cap}{g(s)}} \in V$$

⊙ $\cdot : \mathbb{R} \times F \to F$

$$(k, f) \longmapsto kf \qquad kf : S \to V$$

$$(kf)(s) = \underset{\mathbb{K}}{\underset{\cap}{k}} \underbrace{f(s)}_{V} \in V$$

Prop: $(F, +_F)$ è gruppo abeliano e con

$\cdot : \mathbb{R} \times F \to F$ è uno spazio vettoriale su $\mathbb{R}$

Dim: si deve far vedere che valgono ① ... ⑧

① + è associativa

$(f+g)+h = f(g+h)$ = fra funzioni

$$((f +_F g) +_F h)(s) = (f+g)(s) +_V h(s)$$

$$= (f(s) +_V g(s)) +_V h(s) = f(s) +_V (g(s) +_V h(s))$$

vale ass. di $+_V$

$$= f(s) +_V (g +_F h)(s) = (f +_F (g +_F h))(s)$$

$$\underline{0}_F : S \to V$$

$$\underline{0}_F(s) = \underline{0}_V \quad \forall s \in S$$

Def: $\mathbb{K}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ t.c. } x_i \in \mathbb{K} \right\}$

$$+_{\mathbb{K}^n} : \mathbb{K}^n \times \mathbb{K}^n \longrightarrow \mathbb{K}^n$$

$$(\underline{x}, \underline{y}) \to (\underline{x} + \underline{y}) \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 +_{\mathbb{K}} y_1 \\ \vdots \\ x_n +_{\mathbb{K}} y_n \end{pmatrix}$$

$$\cdot_{\mathbb{K}^n} : \mathbb{K} \times \mathbb{K}^n \longrightarrow \mathbb{K}^n$$

$$(k, \underline{x}) \to k\underline{x} \qquad k \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} kx_1 \\ \vdots \\ kx_n \end{pmatrix}$$

similmente definisco $M_{a,b}(\mathbb{K})$, $\mathbb{K}[x]$

insieme delle matrici a coefficienti in $\mathbb{K}$ — polinomi con coefficienti in $\mathbb{K}$

Oss: $\mathbb{R}^n$ ha un numero infinito di elementi

$$\mathbb{Z}_2^3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ t.c. } x_i \in \mathbb{Z}_2 \right\} \quad \mathbb{Z}_2 = \{ [0]_2, [1]_2 \}$$

gli elementi sono

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\mathbb{Z}_2$ é campo $\Rightarrow (\mathbb{Z}_2)^3$ è spazio vettoriale su campo $\mathbb{Z}_2$ con 8 elementi

Prop: $\mathbb{K}^n$, $M_{a \times b}(\mathbb{K})$, $\mathbb{K}[x]$ con le operazioni viste prima sono spazi vettoriali sul campo $\mathbb{K}$

Proprietà: Sia V un $\mathbb{K}$ - spazio vettoriale

$$\forall \underline{v} \in V \quad \underset{\mathbb{K}}{0_{\mathbb{K}}} \cdot \underset{V}{\underline{v}} = \underline{0}_V$$

dim: $0_{\mathbb{K}} = 0_{\mathbb{K}} + 0_{\mathbb{K}}$

$$0_{\mathbb{K}} \cdot \underline{v} = (0_{\mathbb{K}} + 0_{\mathbb{K}}) \cdot \underline{v}$$

$$\overset{(5)}{=} 0_{\mathbb{K}} \cdot \underline{v} + 0_{\mathbb{K}} \cdot \underline{v}$$

sommo ad entrambi membri $\underbrace{-0_{\mathbb{K}} \underline{v}}_{\in V}$

$$0_{\mathbb{K}} \underline{v} +_V (-0_{\mathbb{K}} \underline{v}) = 0_{\mathbb{K}} \underline{v} +_V 0_{\mathbb{K}} \underline{v} +_V (-0_{\mathbb{K}} \underline{v})$$

$$= 0_{\mathbb{K}} \underline{v} +_V \underline{0}_V = 0_{\mathbb{K}} \underline{v}$$

$$= 0_V = 0_{\mathbb{K}} \underline{v}$$

② $\forall k \in \mathbb{K} \quad k \cdot \underline{0}_V = \underline{0}_V$

dim: $\underline{0}_V = \underline{0}_V + \underline{0}_V$

$$k\underline{0}_V = k(\underline{0}_V + \underline{0}_V) = k\underline{0}_V + k\underline{0}_V$$

sommo $-k\underline{0}_V$

$$k\underline{0}_V +_V (-k\underline{0}_V) = k\underline{0}_V +_V k\underline{0}_V +_V$$