MANDATORY 3

STA 510

Aleksander B. Jakobsen
Student number 236227
University of Stavanger

Fall 2020

1b) $p(x) = 2x\, e^{-x^2}$

Inverse transform method requires CDF:

$$\int_0^x 2xe^{-x^2}\, dx$$

In order to solve this integral, I look at the expression $2xe^{-x^2}$, and think "What must i derivate to get this?". I realize that the expression $(e^{-x^2})'$ yields $-2xe^{-x^2}$. Comparing this to the integral, there is only a difference in the minus sign. Therefore:

$$F(x) = \int_0^x 2xe^{-x^2} = -e^{-x^2}\Big|_0^x = 1 - e^{-x^2} = U$$

$$1 - e^{-x^2} = U$$

$$-e^{-x^2} = U - 1$$

$$e^{-x^2} = 1 - U$$

$$\log e^{-x^2} = \log(1-U)$$

$$-x^2 = \log(1-U)$$

$$x^2 = -\log(1-U)$$

$$\underline{\underline{x = \sqrt{-\log(1-U)}}}$$

See R code for further solution.

1c) $\int_0^\infty e^{\sqrt{x}} e^{-20(x-4)^2} dx$

This can be written as

$$\int_0^\infty e^{\sqrt{x}} e^{-20(x-4)^2} \cdot \frac{\sqrt{2\pi}\sqrt{\frac{1}{40}}}{\sqrt{2\pi}\sqrt{\frac{1}{40}}} dx$$

$$= \int_0^\infty \sqrt{2\pi} \cdot \sqrt{\frac{1}{40}} \cdot e^{\sqrt{x}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{40}}} \cdot e^{-20(x-4)^2}}_{\text{Normal dist with mean}=4 \text{ and sigma}=\sqrt{\frac{1}{40}}} dx$$

Note:

Sigma is $\sqrt{\frac{1}{40}}$ fits because

$$e^{-\frac{1}{2}\left(\frac{x-4}{\sqrt{\frac{1}{40}}}\right)^2} = e^{-\frac{1}{2}\frac{(x-4)^2}{\frac{1}{40}}}$$

$$= e^{-\frac{40}{2}(x-4)^2} = e^{-20(x-4)^2} .$$

4a) Find the distribution of conditional posteriors $\alpha|\beta,\tau,\vec{y}$ and $\beta|\alpha,\tau,\vec{y}$.

Conditional posterior τ, i.e $\tau|\alpha,\beta,\vec{y}$ is a Gamma distribution with shape parameter $\frac{n}{2}+1$ and scale $\frac{1}{\frac{1}{2}\sum_{i=1}^{n}(y_i-\alpha-\beta x_i)^2+1}$

This can be written as

$$\pi(\tau|\alpha,\beta,\vec{y}) \sim \text{Gamma}\left(\frac{n}{2}+1, \frac{1}{\frac{1}{2}\sum(y_i-\alpha-\beta x_i)^2+1}\right)$$

Further, we know from the problem description that the joint posterior of $\vec{\Theta}=(\alpha,\beta,\tau)$ has a log-density kernel

$$\log g(\vec{\Theta}) = \frac{n}{2}\log(\tau) - \frac{\tau}{2}\sum_{i=1}^{n}(y_i-\alpha-\beta x_i)^2 - \frac{\alpha^2+\beta^2}{200} - \tau$$

As in lecture note "Markov chain Monte Carlo, part 3", lets consider the shape of the above expression as a function of α only. We get:

$$\log\pi(\alpha|\beta,\tau,\vec{y}) = \text{constant} - \frac{\tau}{2}\sum_{i=1}^{n}(y_i-\alpha-\beta x_i)^2 - \frac{\alpha^2}{200}.$$

Lets consider the expession $(y_i-\alpha-\beta x_i)^2$:

$$(y_i-\alpha-\beta x_i)^2 = (y_i-\alpha-\beta x_i)(y_i-\alpha-\beta x_i)$$
$$= (y_i^2 - y_i\alpha - y_i\beta x_i - y_i\alpha + \alpha^2 + \alpha\beta x_i - y_i\beta x_i + \alpha\beta x_i + \beta^2 x_i^2)$$
$$= (y_i^2 - 2y_i\alpha - 2y_i\beta x_i + \alpha^2 + 2\alpha\beta x_i + \beta^2 x_i^2)$$

This yields

$$\log\pi(\alpha|\beta,\tau,\vec{y}) = \text{constant} - \frac{\tau}{2}\left(n\alpha^2 + \alpha\sum_{i=1}^{n}(2\beta x_i - 2y_i)\right) - \frac{\alpha^2}{200},$$

where terms not including α is added to the constant since we are considering the expression as a function of α only.

We may further write the expression as

$$= \text{constant} - \frac{\tau n \alpha^2}{2} + \frac{\tau\alpha}{2}\sum_{i=1}^{n}(2\beta x_i - 2y_i) - \frac{\alpha^2}{200}$$

$$= \text{constant} - \tau\alpha\sum_{i=1}^{n}(\beta x_i - y_i) - \left(\frac{\tau n}{2} + \frac{1}{200}\right)\alpha^2$$

From lecture note "Markov chain Monte Carlo, part 3", we know that $\log p(x) = a + bx + cx^2 \Rightarrow x \sim N\left(-\frac{b}{2c}, -\frac{1}{2c}\right)$.

Thus, we get that

$$b = -\tau\sum_{i=1}^{n}(\beta x_i - y_i) \quad \text{and} \quad c = -\left(\frac{\tau n}{2} + \frac{1}{200}\right)$$

And the posterior distribution $\alpha | \beta, \tau, \vec{y}$ thus becomes

$$\pi(\alpha | \beta, \tau, \vec{y}) \sim N\left(-\frac{b}{2c}, -\frac{1}{2c}\right),$$ where b and c

is as defined above. Or completly written out:

$$\underline{\underline{N\left(\frac{-\tau\sum_{i=1}^{n}(\beta x_i - y_i)}{(\tau n + \frac{1}{100})}, \frac{1}{(\tau n + \frac{1}{100})}\right)}}$$

Now lets find the conditional posterior $\beta | \alpha, \tau, \vec{y}$:

A similar process as finding $\alpha | \beta, \tau, \vec{y}$ is conducted. We start by considering the shape of the log density kernel from the problem description ~~for~~ as a function of β only. This yields:

$$\log \pi(\beta | \alpha, \tau, \vec{y}) = \text{constant} - \frac{\tau}{2} \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2 - \frac{\beta^2}{200}$$

Recall that $(y_i - \alpha - \beta x_i)^2$ gives

$(y_i^2 - 2y_i\alpha - 2y_i\beta x_i + \alpha^2 + 2\alpha\beta x_i + \beta^2 x_i^2)$.

This yields

$$\log \pi(\beta | \alpha, \tau, \vec{y}) = \text{constant} - \frac{\tau}{2} \sum_{i=1}^{n} (2\alpha\beta x_i - 2y_i\beta x_i + \beta^2 x_i^2) - \frac{\beta^2}{200},$$

where terms not including β is added to the constant since we are considering the expression as a function of β only. (My pen died so I had to switch colors)

We may rewrite the expression

~~log π(β|α,τ~~

$$\log \pi(\beta | \alpha, \tau, \vec{y}) = \text{constant} - \frac{\tau}{2} \sum_{i=1}^{n} (2\alpha\beta x_i - 2y_i\beta x_i) - \frac{\tau}{2} \sum_{i=1}^{n} \beta^2 x_i^2 - \frac{\beta^2}{200}$$

$$= \text{constant} - \tau\beta \sum_{i=1}^{n} x_i(\alpha - y_i) - \frac{\tau}{2} \sum_{i=1}^{n} \beta^2 x_i^2 - \frac{\beta^2}{200}$$

$$= \text{constant} - \tau\beta \sum_{i=1}^{n} x_i(\alpha - y_i) - \left(\frac{\tau}{2} \sum_{i=1}^{n} x_i^2 + \frac{1}{200}\right) \beta^2$$

Following the same principles as above, we get

$$b = -\tau \sum_{i=1}^{n} x_i(\alpha - y_i) \quad \text{and} \quad c = -\left(\frac{\tau}{2} \sum_{i=1}^{n} x_i^2 + \frac{1}{200}\right)$$

The posterior distribution $\beta | \alpha, \tau, \vec{y}$ thus becomes

$\pi(\beta | \alpha, \tau, \vec{y}) \sim N\left(-\frac{b}{2c}, -\frac{1}{2c}\right)$, wher b and c is as defined above. Or completly written out:

$$N\left(\frac{-\tau \sum_{i=1}^{n} x_i(\alpha - y_i)}{\tau \sum_{i=1}^{n} x_i + \frac{1}{100}}, \frac{1}{\tau \sum_{i=1}^{n} x_i + \frac{1}{100}} \right)$$

Source as used in the .R file

1. rdrr.io; effectiveSize: Effective sample size for estimating the mean [online]. Accessed 04/11/20. Available at: https://rdrr.io/cran/coda/man/effectiveSize.html
2. Stack Overflow; Find position of first value greater than X in a vector [online]. Accessed 05/11/20. Available at: https://stackoverflow.com/questions/29388334/find-position-of-first-value-greater-than-x-in-a-vector
3. Stack Overflow; boxplots from two dataframes in R [online]. Accessed 07/11/20. Available at: https://stackoverflow.com/questions/45150841/boxplots-from-two-dataframes-in-r
4. Wikipedia Commons; Boxplot vs PDF. [online]. Accessed 12/11/20. Available at: https://commons.wikimedia.org/wiki/File:Boxplot_vs_PDF.svg