STA 510

MANDATORY 2

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① $\int_{-1}^{1}\int_{-1}^{1} I_D(x,y)\,dx\,dy = \pi$

where $I_D(x,y) = \begin{cases} 1 & , x^2+y^2 \le 1 \\ 0 & , \text{otherwise} \end{cases}$

Crude first attempt : $\Theta_{CMC} = \frac{4}{n}\sum_{i=1}^{n} I_D(X_i, Y_i)$

X_i , Y_i independent

a) Rewrite $\int_{-1}^{1}\int_{-1}^{1} I_D(x,y)\,dx\,dy$ as

$$\int_{b_1}^{b_2}\int_{a_1}^{a_2} I_D(x,y)\,dx\,dy = (a_2-a_1)(b_2-b_1)\int_{b_1}^{b_2}\int_{a_1}^{a_2} I_D(x,y)\cdot\frac{1}{(b_2-b_1)}\cdot\frac{1}{(a_2-a_1)}$$

$$= (a_2-a_1)(b_2-b_1)\,E\big(I_D(X,Y)\big)$$

and a crude estimate is

$$\frac{(a_2-a_1)(b_2-b_1)}{n}\sum_{i=1}^{n} I_D(X_i,Y_i) = \frac{4}{n}\sum_{i=1}^{n} I_D(X_i,Y_i)$$

Inserting values for a and b, we get

$$\frac{(1-(-1))(1-(-1))}{n}\sum_{i=1}^{n} I_D(X_i,Y_i) = \frac{4}{n}\sum_{i=1}^{n} I_D(X_i,Y_i)$$

and thus, $\frac{4}{n}\sum_{i=1}^{n} I_D(X_i,Y_i)$ is a crude monte carlo estimator

Show that $1_D(X_i, Y_i)$ has a Bernoulli distribution with success probability $\frac{\pi}{4}$.

$$1_D(X_i, Y_i) = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{wit prob } q = 1-p. \end{cases}$$

$1_D(X_i, Y_i)$ is either 0 or 1

The area of the cicle is π.

If "success", then $1_D(X_i, Y_i) = 1$

Assuming success, then the integral becomes:

$$\int_{-1}^{1}\int_{-1}^{1} dx\,dy = \int_{-1}^{1} x\Big|_{-1}^{1} dy = \int_{-1}^{1} 2\,dy = 4$$

Then the success probability is the prob of the cicle of area π ~~to~~ ~~fall~~ divided by square area ~~4~~, hence $p = \frac{\pi}{4}$

Since this is a "either / or" or "success / failure", and $p = \frac{\pi}{4}$, yielding $q = 1 - p = 1 - \frac{\pi}{4}$, then this is a Bernoulli distribution

b) $\sum_{i=1}^{n} 1_D(X_i, Y_i)$ has a binomial distribution, since it is the repetition of Bernolli trials, all with "success" $= \frac{\pi}{4}$ and "failure" $= 1 - \frac{\pi}{4}$.

Thus $\sum_{i=1}^{n} 1_D =$ Binomial

Expectation for Binomial: np

Variance for Binomial: $np(1-p)$

$$E\left(\frac{4}{n}\sum_{i=1}^{n} 1_D(X_i, Y_i)\right) = \frac{4}{n} E\left(\sum_{i=1}^{n} 1_D(X_i, Y_i)\right) = \frac{4}{\cancel{n}} \cdot \cancel{n} p$$

$$= 4 \cdot \frac{\pi}{4} = \pi$$

$$Var\left(\frac{4}{n}\sum_{i=1}^{n} 1_D(X_i, Y_i)\right) = \left(\frac{4}{n}\right)^2 Var\left(\sum_{i=1}^{n} 1_D(X_i, Y_i)\right)$$

$$= \frac{16}{n^2} \cdot np(1-p) = \frac{16}{n} \cdot \frac{\pi}{4}\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{n}(4 - \pi)$$

1e) ~~I expect the antithetic variables will reduce the Monte Carlo variance, because the covariances, $Cov(X_i, V_i)$ and $Cov(Y_i, W_i)$ should be negative, and thus reduce variance.~~

Function is for a circle, and is thereby not monotonic. Thus, variance will not be reduced.

2b)

Find smallest possible λ_{max}:

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} \frac{297(1+\cos(2\pi(t+1/10)))(1-\exp(-t/10)/2)}{10} + \frac{3}{5}$$

Evaluate the lim where there are t's.

$$\lim_{t\to\infty} \cos(2\pi(t+1/10)) = 1$$

$$\lim_{t\to\infty} \exp(-t/10) = \lim_{t\to\infty} \frac{1}{\exp(t/10)} = 0$$

Thus

$$\lim_{t\to\infty} \lambda(t) = \frac{297(1+1)(1-0)}{10} + \frac{3}{5} = \underline{\underline{60 = \lambda_{max}}}$$

4a)

$$f_R(r) = \begin{cases} \dfrac{e^{-\frac{(r-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma(1-F(l))} & \text{if } r \geq l \\ 0 & \text{if } r < l \end{cases}$$

where F is the cdf of a $N(\mu, \sigma^2)$ r.v.

$$G_R(r) = \int_{-\infty}^{r} f_R(r)\,dr = \int_{l}^{r} \frac{e^{-\frac{(r-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma(1-F(l))}\,dr$$

$$= \frac{1}{1-F(L)} \underbrace{\int_{L}^{r} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2} dr}_{\text{This is pdf of a normal r.v.}}$$

$$= \frac{1}{1-F(l)} \cdot \underbrace{\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{r-\mu}{\sigma}\right)\right]}_{\text{cdf of a } N(\mu,\sigma^2) \text{ r.v.}}\Bigg|_{r=L}^{r=r}$$

$$= \frac{1}{1-F(l)} \cdot F(r) - F(L)$$

$$\underline{\underline{G_R(r) = \frac{F(r) - F(l)}{1 - F(l)}}}$$

$G_R^{-1}(u) = ?$

$$\frac{F(r) - F(l)}{1 - F(l)} = u$$

$$F(r) = u(1 - F(l)) + F(l)$$

$$F^{-1}(F(r)) = \underline{\underline{F^{-1}(u(1 - F(l)) + F(l)) = G_R^{-1}(u)}}$$

4c) To simulat S:

$$\int_{\frac{0.5}{R}}^{S} \frac{10R}{3} dS = \frac{10RS}{3} \Big|_{\frac{0.5}{R}}^{S} = U[0,1]$$

$$= \frac{10RS}{3} - \frac{10R}{3} \cdot \frac{0.5}{R} = U[0,1]$$

$$\frac{10RS}{3} = U[0,1] + \frac{5}{3}$$

$$S = \frac{3 \cdot U[0,1]}{10} + \frac{1}{2R}$$

This is used in code to simulate S|R.

Source as used in the .R file

1. B. Efron and R. Tibishirani. *An introduction to the Bootstrap*. Chapman & Hall/CRC, Boca Raton, FL, 1993.
2. Rizzo, M.L. *Statistical computing with R, 2nd ed.* Chapman & Hall/CRC, Boca Raton, FL, 2019.