Question 1:

In[60]:= `ClearAll;`

First we need to define the limits of the solid (f(x,y) limits).

In[61]:= $f[x\_, y\_] := \frac{61}{5} - \frac{12}{5} x + x^2 - \frac{12}{5} y + \frac{6}{5} (x * y) + \frac{2}{5} y^2$

In[62]:= $g[x\_, y\_] := 15 - x^2 - \frac{6}{5} (x * y) - \frac{2}{5} y^2$

Then we need to solve this system of equations for the limits of y.

In[63]:= `Solve[f[x, y] == g[x, y], y]`

Out[63]= $\left\{\left\{y \to \frac{1}{2}\left(3 - 3x - \sqrt{23 - 6x - x^2}\right)\right\}, \left\{y \to \frac{1}{2}\left(3 - 3x + \sqrt{23 - 6x - x^2}\right)\right\}\right\}$

In[64]:= $u[x\_] := \frac{1}{2}\left(3 - 3x - \sqrt{23 - 6x - x^2}\right)$

In[65]:= $v[x\_] := \frac{1}{2}\left(3 - 3x + \sqrt{23 - 6x - x^2}\right)$

Finally we solve that system for the limits of x.

In[66]:= $\text{Solve}[23 - 6x - x^2 == 0]$

Out[66]= $\left\{\left\{x \to -3 - 4\sqrt{2}\right\}, \left\{x \to -3 + 4\sqrt{2}\right\}\right\}$

In[67]:= $a = -3 - 4\sqrt{2}$

Out[67]= $-3 - 4\sqrt{2}$

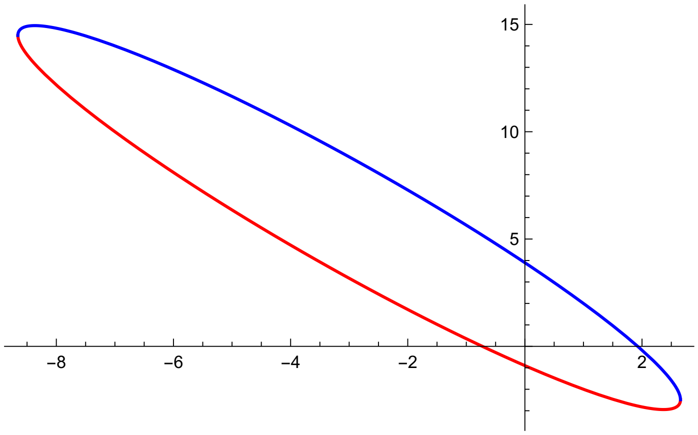
In[68]:= $b = -3 + 4\sqrt{2}$

Out[68]= $-3 + 4\sqrt{2}$

We can project the solid onto the 2d plane to get a better idea of the limits of x and y.

In[69]:=
```
Plot[{u[x], v[x]}, {x, a, b}, PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 0, 1]}]
```

Out[69]=



Next to find the mass of the solid, we solve the triple integral for the equation of the area we are looking for.

In[71]:=
$$\int_a^b \int_{u[x]}^{v[x]} \int_{f[x,y]}^{g[x,y]} (x + 2\,(y \wedge 2) - 3\,z)\, dz\, dy\, dx$$

Out[71]=
$$\frac{50\,432\,\pi}{15}$$

Question 2:

```
ClearAll;
```

In[342]:=
```
region = Show[Graphics[{Thickness[.005],
    Line[{{-1, -1/2}, {1, -1/2}, {1, 1/2}, {-1, 1/2}, {-1, -1/2}}]}]]
```

In[376]:=

In[377]:=

In[378]:=
```
bndry = {{-0.2257, 0.3676}, {-0.2474, 0.3676}, {-0.2734, 0.3676},
   {-0.3125, 0.3589}, {-0.3472, 0.3502}, {-0.3776, 0.3459}, {-0.4167, 0.3372},
   {-0.4557, 0.3328}, {-0.5078, 0.3241}, {-0.5425, 0.3155}, {-0.5816, 0.3068},
   {-0.6076, 0.2807}, {-0.6163, 0.259}, {-0.6467, 0.2243}, {-0.6597, 0.1939},
   {-0.7205, 0.1245}, {-0.7595, 0.1115}, {-0.7812, 0.08543}, {-0.7986, 0.06807},
   {-0.8247, 0.03769}, {-0.8464, 0.007309}, {-0.8507, -0.04043}, {-0.8724, -0.1186},
   {-0.8767, -0.1489}, {-0.8681, -0.175}, {-0.8637, -0.201}, {-0.842, -0.2531},
   {-0.8116, -0.2705}, {-0.7595, -0.2574}, {-0.7292, -0.2444}, {-0.6901, -0.2271},
   {-0.6641, -0.2271}, {-0.612, -0.2488}, {-0.5903, -0.2791}, {-0.5642, -0.3182},
   {-0.5165, -0.3529}, {-0.4818, -0.3703}, {-0.4427, -0.3703}, {-0.3863, -0.3703},
   {-0.3342, -0.3746}, {-0.3082, -0.3746}, {-0.2604, -0.366}, {-0.2214, -0.3703},
   {-0.1953, -0.3703}, {-0.1519, -0.3312}, {-0.0434, -0.3095}, {-0.01736, -0.3052},
   {-0.02604, -0.2965}, {0.01736, -0.2878}, {0.1085, -0.2922}, {0.191, -0.3009},
   {0.23, -0.2878}, {0.3212, -0.2748}, {0.3602, -0.2748}, {0.4601, -0.2314},
   {0.4905, -0.2227}, {0.5078, -0.2097}, {0.3906, -0.1923}, {0.3038, -0.2097},
   {0.2648, -0.2097}, {0.2387, -0.2054}, {0.1693, -0.1967}, {0.1519, -0.1837},
   {0.1649, -0.1663}, {0.178, -0.1446}, {0.2257, -0.1186}, {0.2778, -0.07082},
   {0.3125, -0.01439}, {0.3472, 0.05505}, {0.3602, 0.09411}, {0.3559, 0.1419},
   {0.3082, 0.1809}, {0.2778, 0.1896}, {0.2561, 0.1809}, {0.1953, 0.1505},
   {0.1432, 0.1505}, {0.1562, 0.1853}, {0.191, 0.2026}, {0.2387, 0.2547},
   {0.2908, 0.2938}, {0.3255, 0.3285}, {0.3429, 0.3589}, {0.3168, 0.3719},
   {0.1953, 0.3459}, {0.1519, 0.3068}, {0.07378, 0.2634}, {-0.07378, 0.2156},
   {-0.09549, 0.207}, {-0.1432, 0.1939}, {-0.191, 0.1939}, {-0.2734, 0.1853},
   {-0.3299, 0.1766}, {-0.2865, 0.233}, {-0.217, 0.2938}, {-0.1997, 0.3415}}
```

Out[378]=

```
{{-0.2257, 0.3676}, {-0.2474, 0.3676}, {-0.2734, 0.3676}, {-0.3125, 0.3589},
 {-0.3472, 0.3502}, {-0.3776, 0.3459}, {-0.4167, 0.3372}, {-0.4557, 0.3328},
 {-0.5078, 0.3241}, {-0.5425, 0.3155}, {-0.5816, 0.3068}, {-0.6076, 0.2807},
 {-0.6163, 0.259}, {-0.6467, 0.2243}, {-0.6597, 0.1939}, {-0.7205, 0.1245},
 {-0.7595, 0.1115}, {-0.7812, 0.08543}, {-0.7986, 0.06807}, {-0.8247, 0.03769},
 {-0.8464, 0.007309}, {-0.8507, -0.04043}, {-0.8724, -0.1186},
 {-0.8767, -0.1489}, {-0.8681, -0.175}, {-0.8637, -0.201}, {-0.842, -0.2531},
 {-0.8116, -0.2705}, {-0.7595, -0.2574}, {-0.7292, -0.2444}, {-0.6901, -0.2271},
 {-0.6641, -0.2271}, {-0.612, -0.2488}, {-0.5903, -0.2791}, {-0.5642, -0.3182},
 {-0.5165, -0.3529}, {-0.4818, -0.3703}, {-0.4427, -0.3703}, {-0.3863, -0.3703},
 {-0.3342, -0.3746}, {-0.3082, -0.3746}, {-0.2604, -0.366}, {-0.2214, -0.3703},
 {-0.1953, -0.3703}, {-0.1519, -0.3312}, {-0.0434, -0.3095}, {-0.01736, -0.3052},
 {-0.02604, -0.2965}, {0.01736, -0.2878}, {0.1085, -0.2922}, {0.191, -0.3009},
 {0.23, -0.2878}, {0.3212, -0.2748}, {0.3602, -0.2748}, {0.4601, -0.2314},
 {0.4905, -0.2227}, {0.5078, -0.2097}, {0.3906, -0.1923}, {0.3038, -0.2097},
 {0.2648, -0.2097}, {0.2387, -0.2054}, {0.1693, -0.1967}, {0.1519, -0.1837},
 {0.1649, -0.1663}, {0.178, -0.1446}, {0.2257, -0.1186}, {0.2778, -0.07082},
 {0.3125, -0.01439}, {0.3472, 0.05505}, {0.3602, 0.09411}, {0.3559, 0.1419},
 {0.3082, 0.1809}, {0.2778, 0.1896}, {0.2561, 0.1809}, {0.1953, 0.1505},
 {0.1432, 0.1505}, {0.1562, 0.1853}, {0.191, 0.2026}, {0.2387, 0.2547},
 {0.2908, 0.2938}, {0.3255, 0.3285}, {0.3429, 0.3589}, {0.3168, 0.3719},
 {0.1953, 0.3459}, {0.1519, 0.3068}, {0.07378, 0.2634}, {-0.07378, 0.2156},
 {-0.09549, 0.207}, {-0.1432, 0.1939}, {-0.191, 0.1939}, {-0.2734, 0.1853},
 {-0.3299, 0.1766}, {-0.2865, 0.233}, {-0.217, 0.2938}, {-0.1997, 0.3415}}
```

In[379]:=

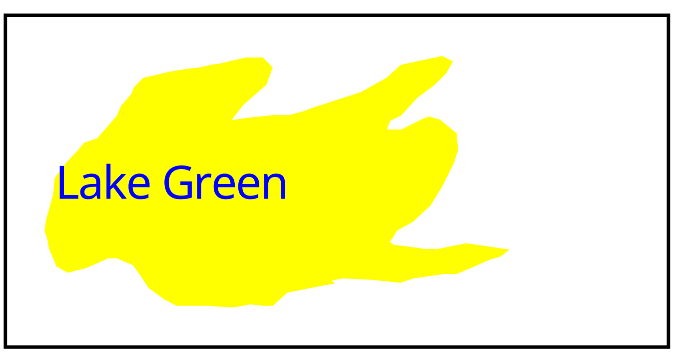
```
lake = Graphics[{Yellow, Polygon[bndry]}];
```

In[380]:=

```
Show[region, lake,
 Graphics[{Blue,
   Text[

    StyleForm["Lake Green", FontFamily → "Script", FontSize → 24], {-1/2, 0}]
   }], PlotRange → All]
```

Out[380]=



This is the lake generated from the code above. Below is the length of the vertex list.

In[381]:=
```
n = Length[bndry] - 1
```
Out[381]=
```
94
```

We define area as the simplified integral of x wrt to y.

In[382]:= $\texttt{areaInt[P\_, Q\_]} := \frac{(P[[1]] + Q[[1]])}{2} (Q[[2]] - P[[2]])$

In[383]:= $\texttt{area} = \sum_{k=1}^{n} \texttt{areaInt[bndry[[k]], bndry[[k + 1]]]}$

Out[383]=
```
0.658327
```

We repeat the process used to find the area for the two center of mass coordinates.

In[384]:= $\texttt{Simplify}\left[\int_0^1 ((((1 - t)\, P[1] + t * Q[1]) ^ 2) (Q[2] - P[2]))\, dt\right]$

Out[384]=
$$\frac{1}{3} \left(P[1]^2 + P[1] \times Q[1] + Q[1]^2\right) (-P[2] + Q[2])$$

In[385]:= $\texttt{xc[P\_, Q\_]} := \left(\left(P[[1]]^2 + P[[1]] \times Q[[1]] + Q[[1]]^2\right) / 3\right) (-P[[2]] + Q[[2]])$

In[386]:= $\texttt{x} = \frac{1}{2\, \texttt{area}} \sum_{k=1}^{n} \texttt{xc[bndry[[k]], bndry[[k + 1]]]}$

Out[386]=
```
-0.24423
```

In[387]:= $\texttt{Simplify}\left[\int_0^1 ((((1 - t)\, P[2] + t * Q[2]) ^ 2) (Q[1] - P[1]))\, dt\right]$

Out[387]=
$$\frac{1}{3} (-P[1] + Q[1]) \left(P[2]^2 + P[2] \times Q[2] + Q[2]^2\right)$$

In[388]:= $\texttt{yc[P\_, Q\_]} := \left((-P[[1]] + Q[[1]]) \left(P[[2]]^2 + P[[2]] \times Q[[2]] + Q[[2]]^2\right)\right) / 3$

In[389]:= $\texttt{y} = \frac{-1}{2\, \texttt{area}} \sum_{k=1}^{n} \texttt{yc[bndry[[k]], bndry[[k + 1]]]}$

Out[389]=
```
-0.0257504
```

In[390]:=
```
centroid = {x, y}
```
Out[390]=
```
{-0.24423, -0.0257504}
```

In[391]:=

We can plot these coordinates as the center of the lake.

In[392]:=
```
Show[region, lake,
 Graphics[{Blue,
   Text[

    StyleForm["Lake Green", FontFamily → "Script", FontSize → 24], {-1/2, 0}],
      {PointSize[.025], Red, Point[centroid]}}], PlotRange → All]
```

Out[392]=