

Noisy matrix decomposition via convex relaxation: Optimal rates in high dimensions

Alekh Agarwal[†] Sahand Negahban[†] Martin J. Wainwright^{†,*}
alekh@eecs.berkeley.edu sahand@eecs.berkeley.edu wainwrig@stat.berkeley.edu

Department of EECS[†] Department of Statistics^{*}
UC Berkeley UC Berkeley

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Abstract

We analyze a class of estimators based on convex relaxation for solving high-dimensional matrix decomposition problems. The observations are the noisy realizations of the sum of an (approximately) low rank matrix Θ^* with a second matrix Γ^* endowed with a complementary form of low-dimensional structure; this set-up includes many statistical models of interest, including forms of factor analysis, multi-task regression with shared structure, and robust covariance estimation. We derive a general theorem that gives upper bounds on the Frobenius norm error for an estimate of the pair (Θ^*, Γ^*) obtained by solving a convex optimization problem that combines the nuclear norm with a general decomposable regularizer. Our results are based on imposing a “spikiness” condition that is related to but milder than singular vector incoherence. We specialize our general result to two cases that have been studied in past work: low rank plus an entrywise sparse matrix, and low rank plus a columnwise sparse matrix. For both models, our theory yields non-asymptotic Frobenius error bounds for both for deterministic and stochastic noise matrices, and applies to matrices Θ^* that can be exactly or approximately low rank, and matrices Γ^* that can be exactly or approximately sparse. Moreover, for the case of stochastic noise matrices, we establish matching lower bounds on the minimax error, showing that our results cannot be improved beyond constant factors. The sharpness of our theoretical predictions is confirmed by numerical simulations.

1 Introduction

The focus of this paper is a class of high-dimensional matrix decomposition problems of the following variety. Suppose that we observe a matrix $Y \in \mathbb{R}^{d_1 \times d_2}$ that is (approximately) equal to the sum of two unknown matrices: how to recover good estimates of the pair? Of course, this problem is ill-posed in general, so that it is necessary to impose some kind of low-dimensional structure on the matrix components, one example being rank constraints. The framework of this paper supposes that one matrix component (denoted Θ^*) is low-rank, either exactly or in an approximate sense, and allows for general forms of low-dimensional structure for the second component Γ^* . Two particular cases of structure for Γ^* that have been considered in past work are elementwise sparsity [9, 8, 7] and column-wise sparsity [18, 28].

Problems of matrix decomposition are motivated by a variety of applications. Many classical methods for dimensionality reduction, among them factor analysis and principal components analysis (PCA), are based on estimating a low-rank matrix from data. Different

forms of robust PCA can be formulated in terms of matrix decomposition using the matrix Γ^* to model the gross errors [9, 7, 28]. Similarly, certain problems of robust covariance estimation can be described using matrix decompositions with a column/row-sparse structure, as we describe in this paper. The problem of low rank plus sparse matrix decomposition also arises in Gaussian covariance selection with hidden variables [8], in which case the inverse covariance of the observed vector can be decomposed as the sum of a sparse matrix with a low rank matrix. Matrix decompositions also arise in multi-task regression [31, 21, 26], involve solving a collection of regression problems, referred to as tasks, over a common set of features. For some features, one expects their weighting to be preserved across features, which can be modeled by a low-rank constraint, whereas other features are expected to vary across tasks, which can be modeled by a sparse component [6, 2]. See Section 2.1 for further discussion of these motivating applications.

In this paper, we study a noisy linear observation that can be used to describe a number of applications in a unified way. Let \mathfrak{X} be an operator that maps matrices in $\mathbb{R}^{d_1 \times d_2}$ to matrices in $\mathbb{R}^{n_1 \times n_2}$. In the simplest of cases, this observation operator is simply the identity mapping, so that we necessarily have $n_1 = d_1$ and $n_2 = d_2$. However, as we discuss in the sequel, it is useful for certain applications, such as multi-task regression, to consider more general linear operators of this form. In this paper, we study matrix decomposition for the general linear observation model

$$Y = \mathfrak{X}(\Theta^* + \Gamma^*) + W, \quad (1)$$

where Θ^* and Γ^* are unknown $d_1 \times d_2$ matrices, and $W \in \mathbb{R}^{n_1 \times n_2}$ is some type of observation noise; it is potentially dense, and can either be deterministic or stochastic. The matrix Θ^* is assumed to be either exactly low-rank, or well-approximated by a low-rank matrix, whereas the matrix Γ^* is assumed to have a complementary type of low-dimensional structure, such as sparsity. As we discuss in Section 2.1, a variety of interesting statistical models can be formulated as instances of the observation model (1). Such models include versions of factor analysis involving non-identity noise matrices, robust forms of covariance estimation, and multi-task regression with some features shared across tasks, and a sparse subset differing across tasks. Given this observation model, our goal is to recover accurate estimates of the decomposition (Θ^*, Γ^*) based on the noisy observations Y . In this paper, we analyze simple estimators based on convex relaxations involving the nuclear norm, and a second general norm \mathcal{R} .

Most past work on the model (1) has focused on the noiseless setting ($W = 0$), and for the identity observation operator (so that $\mathfrak{X}(\Theta^* + \Gamma^*) = \Theta^* + \Gamma^*$). Chandrasekaran et al. [9] studied the case when Γ^* is assumed to be sparse, with a relatively small number $s \ll d_1 d_2$ of non-zero entries. In the noiseless setting, they gave sufficient conditions for exact recovery for an adversarial sparsity model, meaning the non-zero positions of Γ^* can be arbitrary. Subsequent work by Candes et al. [7] analyzed the same model but under an assumption of random sparsity, meaning that the non-zero positions are chosen uniformly at random. In very recent work, Xu et al. [28] have analyzed a different model, in which the matrix Γ^* is assumed to be columnwise sparse, with a relatively small number $s \ll d_2$ of non-zero columns. Their analysis guaranteed approximate recovery for the low-rank matrix, in particular for the uncorrupted columns. After initial posting of this work, we became aware of recent work by Hsu et al. [14], who derived Frobenius norm error bounds for the case of exact elementwise sparsity. As we discuss in more detail in Section 3.3, in this special case, our bounds are based on milder conditions, and yield sharper rates for problems where the rank and sparsity scale with the dimension.

Our main contribution is to provide a general result (Theorem 1) on approximate recovery of the unknown decomposition from noisy observations, valid for structural constraints on Γ^* imposed via a decomposable regularizer. The class of decomposable regularizers, introduced in past work by Negahban et al. [19], includes the elementwise ℓ_1 -norm and columnwise $(2, 1)$ -norm as special cases, as well as various other regularizers used in practice. Our main result is stated in Theorem 1: it provides finite-sample guarantees for estimates obtained by solving a class of convex programs formed using a composite regularizer. The resulting Frobenius norm error bounds consist of multiple terms, each of which has a natural interpretation in terms of the estimation and approximation errors associated with the sub-problems of recovering Θ^* and Γ^* . We then specialize Theorem 1 to the case of elementwise or columnwise sparsity models for Γ^* , thereby obtaining recovery guarantees for matrices Θ^* that may be either exactly or approximately low-rank, as well as matrices Γ^* that may be either exactly or approximately sparse. We provide non-asymptotic error bounds for general noise matrices W both for elementwise and columnwise sparse models (see Corollaries 1 through Corollary 5). To the best of our knowledge, these are the first results that apply to this broad class of models, allowing both for noisiness ($W \neq 0$), and matrix components that are only approximately low-rank and/or sparse.

In addition, the error bounds obtained by our analysis are sharp, and cannot be improved in general. More precisely, for the case of stochastic noise matrices, we prove that the squared Frobenius errors achieved by our estimators are minimax-optimal (see Theorem 2). An interesting feature of our analysis is that, in contrast to previous work [9, 28, 7], we do *not* impose incoherence conditions on the singular vectors of Θ^* ; rather, we control the interaction with a milder condition involving the dual norm of the regularizer. In the special case of elementwise sparsity, this dual norm enforces an upper bound on the “spikiness” of the low-rank component, and has proven useful in the related setting of noisy matrix completion [20]. This constraint is not strong enough to guarantee identifiability of the models (and hence exact recovery in the noiseless setting), but it does provide a bound on the degree of non-identifiability. We show that this same term arises in both the upper and lower bounds on the problem of approximate recovery that is of interest in the noisy setting.

The remainder of the paper is organized as follows. In Section 2, we set up the problem in a precise way, and describe the estimators. Section 3 is devoted to the statement of our main result on achievability, as well as its various corollaries for special cases of the matrix decomposition problem. We also state a matching lower bound on the minimax error for matrix decomposition with stochastic noise. In Section 4, we provide numerical simulations that illustrate the sharpness of our theoretical predictions. Section 5 is devoted to the proofs of our results, with certain more technical aspects of the argument deferred to the appendices, and we conclude with a discussion in Section 6.

Notation: For the reader’s convenience, we summarize here some of the standard notation used throughout this paper. For any matrix $M \in \mathbb{R}^{d_1 \times d_2}$, we define the Frobenius norm $\|M\|_F := \sqrt{\sum_{j=1}^{d_1} \sum_{k=1}^{d_2} M_{jk}^2}$, corresponding to the ordinary Euclidean norm of its entries. We denote its singular values by $\sigma_1(M) \geq \sigma_2(M) \geq \dots \geq \sigma_d(M) \geq 0$, where $d = \min\{d_1, d_2\}$. Its nuclear norm is given by $\|M\|_N = \sum_{j=1}^d \sigma_j(M)$.

2 Convex relaxations and matrix decomposition

In this paper, we consider a family of regularizers formed by a combination of the *nuclear norm* $\|\Theta\|_N := \sum_{j=1}^{\min\{d_1, d_2\}} \sigma_j(\Theta)$, which acts as a convex surrogate to a rank constraint for Θ^* (e.g., see Recht et al. [25] and references therein), with a *norm-based regularizer* $\mathcal{R} : \mathbb{R}^{d_1 \times d_2} \rightarrow \mathbb{R}_+$ used to constrain the structure of Γ^* . We provide a general theorem applicable to a class of regularizers \mathcal{R} that satisfy a certain decomposability property [19], and then consider in detail a few particular choices of \mathcal{R} that have been studied in past work, including the elementwise ℓ_1 -norm, and the columnwise $(2, 1)$ -norm (see Examples 4 and 5 below).

2.1 Some motivating applications

We begin with some motivating applications for the general linear observation model with noise (1).

Example 1 (Factor analysis with sparse noise). In a factor analysis model, random vectors $Z_i \in \mathbb{R}^d$ are assumed to be generated in an i.i.d. manner from the model

$$Z_i = LU_i + \varepsilon_i, \quad \text{for } i = 1, 2, \dots, n, \quad (2)$$

where $L \in \mathbb{R}^{d_1 \times r}$ is a loading matrix, and the vectors $U_i \sim N(0, I_{r \times r})$ and $\varepsilon_i \sim N(0, \Gamma^*)$ are independent. Given n i.i.d. samples from the model (2), the goal is to estimate the loading matrix L , or the matrix LL^T that projects onto column span of L . A simple calculation shows that the covariance matrix of Z_i has the form $\Sigma = LL^T + \Gamma^*$. Consequently, in the special case when $\Gamma^* = \sigma^2 I_{d \times d}$, then the range of L is spanned by the top r eigenvectors of Σ , and so we can recover it via standard principal components analysis.

In other applications, we might no longer be guaranteed that Γ^* is the identity, in which case the top r eigenvectors of Σ need not be close to column span of L . Nonetheless, when Γ^* is a sparse matrix, the problem of estimating LL^T can be understood as an instance of our general observation model (1) with $d_1 = d_2 = d$, and the identity observation operator \mathfrak{X} (so that $n_1 = n_2 = d$). In particular, if we let the observation matrix $Y \in \mathbb{R}^{d \times d}$ be the sample covariance matrix $\frac{1}{n} \sum_{i=1}^n Z_i Z_i^T$, then some algebra shows that $Y = \Theta^* + \Gamma^* + W$, where $\Theta^* = LL^T$ is of rank r , and the random matrix W is a re-centered form of Wishart noise [1]—in particular, the zero-mean matrix

$$W := \frac{1}{n} \sum_{i=1}^n Z_i Z_i^T - \{LL^T + \Gamma^*\}. \quad (3)$$

When Γ^* is assumed to be elementwise sparse (i.e., with relatively few non-zero entries), then this constraint can be enforced via the elementwise ℓ_1 -norm (see Example 4 to follow). ♣

Example 2 (Multi-task regression). Suppose that we are given a collection of d_2 regression problems in \mathbb{R}^{d_1} , each of the form $y_j = X\beta_j^* + w_j$ for $j = 1, 2, \dots, d_2$. Here each $\beta_j^* \in \mathbb{R}^{d_1}$ is an unknown regression vector, $w_j \in \mathbb{R}^n$ is observation noise, and $X \in \mathbb{R}^{n \times d_1}$ is the design matrix. This family of models can be written in a convenient matrix form as $Y = XB^* + W$, where $Y = [y_1 \ \dots \ y_{d_2}]$ and $W = [w_1 \ \dots \ w_{d_2}]$ are both matrices in $\mathbb{R}^{n \times d_2}$ and $B^* := [\beta_1^* \ \dots \ \beta_{d_2}^*] \in \mathbb{R}^{d_1 \times d_2}$ is a matrix of regression vectors. Following standard terminology in multi-task learning, we refer to each column of B^* as a *task*, and each row of B^* as a *feature*.

In many applications, it is natural to assume that the feature weightings—i.e., that is, the vectors $\beta_j^* \in \mathbb{R}^{d_2}$ —exhibit some degree of shared structure across tasks [2, 31, 21, 26]. This type of shared structure can be modeled by imposing a low-rank structure; for instance, in the extreme case of rank one, it would enforce that each β_j^* is a multiple of some common underlying vector. However, many multi-task learning problems exhibit more complicated structure, in which some subset of features are shared across tasks, and some other subset of features vary substantially across tasks [2, 5]. For instance, in the Amazon recommendation system, tasks correspond to different classes of products, such as books, electronics and so on, and features include ratings by users. Some ratings (such as numerical scores) should have a meaning that is preserved across tasks, whereas other features (e.g., the label “boring”) are very meaningful in applications to some categories (e.g., books) but less so in others (e.g., electronics).

This kind of structure can be captured by assuming that the unknown regression matrix B^* has a low-rank plus row-sparse decomposition—namely, $B^* = \Theta^* + \Gamma^*$ where Θ^* is low-rank and Γ^* is row-sparse, with a relatively small number of non-zero rows (corresponding to the features that differ significantly across tasks). With this model structure, we then define the observation operator $\mathfrak{X} : \mathbb{R}^{d_1 \times d_2} \rightarrow \mathbb{R}^{n \times d_2}$ (so that $n_1 = n$ and $n_2 = d_2$ in our general notation via $A \mapsto XA$). In this way, we obtain another instance of the linear observation model (1). In Example 5 to follow, we discuss an appropriate regularizer for enforcing column (or row) sparsity. ♣

Example 3 (Robust covariance estimation). For $i = 1, 2, \dots, n$, let $U_i \in \mathbb{R}^d$ be samples from a zero-mean distribution with unknown covariance matrix Θ^* . When the vectors U_i are observed without any form of corruption, then it is straightforward to estimate Θ^* by performing PCA on the sample covariance matrix. Imagining that $j \in \{1, 2, \dots, d\}$ indexes different individuals in the population, now suppose that the data associated with some subset S of individuals is arbitrarily corrupted. This adversarial corruption can be modeled by assuming that we observe the vectors $Z_i = U_i + v_i$ for $i = 1, \dots, n$, where each $v_i \in \mathbb{R}^d$ is a vector supported on the subset S . Letting $Y = \frac{1}{n} \sum_{i=1}^n Z_i Z_i^T$ be the sample covariance matrix of the corrupted samples, some algebra shows that it can be decomposed as $Y = \Theta^* + \Gamma^* + W$, where $W := \frac{1}{n} \sum_{i=1}^n U_i U_i^T - \Theta^*$ is again a type of re-centered Wishart noise, and the remaining term can be written as

$$\Delta = \frac{1}{n} \sum_{i=1}^n v_i v_i^T + \frac{1}{n} \sum_{i=1}^n (U_i v_i^T + v_i U_i^T). \quad (4)$$

Note that Δ itself is not a column-sparse or row-sparse matrix; however, since each vector $v_i \in \mathbb{R}^d$ is supported only on some subset $S \subset \{1, 2, \dots, d\}$, we can write $\Delta = \Gamma^* + (\Gamma^*)^T$, where Γ^* is a column-sparse matrix with entries only in columns indexed by S . This structure can be enforced by the use of the column-sparse regularizer (12), as described in Example 5 to follow. ♣

2.2 Convex relaxation for noisy matrix decomposition

Given the observation model $Y = \mathfrak{X}(\Theta^* + \Gamma^*) + W$, it is natural to consider an estimator based on solving the regularized least-squares program

$$\min_{(\Theta, \Gamma)} \left\{ \frac{1}{2} \|Y - \mathfrak{X}(\Theta + \Gamma)\|_F^2 + \lambda_d \|\Theta\|_N + \mu_d \mathcal{R}(\Gamma) \right\}.$$

Here (λ_d, μ_d) are non-negative regularizer parameters, to be chosen by the user. Our theory also provides choices of these parameters that guarantee good properties of the associated estimator. Although this estimator is reasonable, it turns out that an additional constraint yields an equally simple estimator that has attractive properties, both in theory and in practice.

In order to understand the need for an additional constraint, it should be noted that without further constraints, the model (1) is unidentifiable, even in the noiseless setting ($W = 0$). Indeed, as has been discussed in past work [9, 7, 28], no method can recover the components (Θ^*, Γ^*) unless the low-rank component is “incoherent” with the matrix Γ^* . For instance, supposing for the moment that Γ^* is a sparse matrix, consider a rank one matrix with $\Theta_{11}^* \neq 0$, and zeros in all other positions. In this case, it is clearly impossible to disentangle Θ^* from a sparse matrix. Past work on both matrix completion and decomposition [9, 7, 28] has ruled out these types of troublesome cases via conditions on the singular vectors of the low-rank component Θ^* , and used them to derive sufficient conditions for exact recovery in the noiseless setting (see the discussion following Example 4 for more details).

In this paper, we impose a related but milder condition, previously introduced in our past work on matrix completion [20], with the goal of performing approximate recovery. To be clear, this condition does not guarantee identifiability, but rather provides a bound on the *radius of non-identifiability*. It should be noted that non-identifiability is a feature common to many high-dimensional statistical models.¹ Moreover, in the more realistic setting of noisy observations and/or matrices that are not exactly low-rank, such approximate recovery is the best that can be expected. Indeed, one of our main contributions is to establish minimax-optimality of our rates, meaning that no algorithm can be substantially better over the matrix classes that we consider.

For a given regularizer \mathcal{R} , we define the quantity $\kappa_d(\mathcal{R}) := \sup_{V \neq 0} \mathcal{R}(V) / \|V\|_F$, which measures the relation between the regularizer and the Frobenius norm. Moreover, we define the associated dual norm

$$\mathcal{R}^*(U) := \sup_{\mathcal{R}(V) \leq 1} \langle V, U \rangle, \quad (5)$$

where $\langle V, U \rangle := \text{trace}(V^T U)$ is the trace inner product on the space $\mathbb{R}^{d_1 \times d_2}$. Our estimators are based on constraining the interaction between the low-rank component and Γ^* via the quantity

$$\varphi_{\mathcal{R}}(\Theta) := \kappa_d(\mathcal{R}^*) \mathcal{R}^*(\Theta). \quad (6)$$

More specifically, we analyze the family of estimators

$$\min_{(\Theta, \Gamma)} \left\{ \frac{1}{2} \|Y - \mathfrak{X}(\Theta + \Gamma)\|_F^2 + \lambda_d \|\Theta\|_N + \mu_d \mathcal{R}(\Gamma) \right\}, \quad (7)$$

subject to $\varphi_{\mathcal{R}}(\Theta) \leq \alpha$ for some fixed parameter α .

2.3 Some examples

Let us consider some examples to provide intuition for specific forms of the estimator (7), and the role of the additional constraint.

¹For instance, see the paper [24] for discussion of non-identifiability in high-dimensional sparse regression.

Example 4 (Sparsity and elementwise ℓ_1 -norm). Suppose that Γ^* is assumed to be sparse, with $s \ll d_1 d_2$ non-zero entries. In this case, the sum $\Theta^* + \Gamma^*$ corresponds to the sum of a low rank matrix with a sparse matrix. Motivating applications include the problem of factor analysis with a non-identity but sparse noise covariance, as discussed in Example 1, as well as certain formulations of robust PCA [7], and model selection in Gauss-Markov random fields with hidden variables [8]. Given the sparsity of Γ^* , an appropriate choice of regularizer is the elementwise ℓ_1 -norm

$$\mathcal{R}(\Gamma) = \|\Gamma\|_1 := \sum_{j=1}^{d_1} \sum_{k=1}^{d_2} |\Gamma_{jk}|. \quad (8)$$

With this choice, it is straightforward to verify that

$$\mathcal{R}^*(Z) = \|Z\|_\infty := \max_{j=1, \dots, d_1} \max_{k=1, \dots, d_2} |Z_{jk}|, \quad (9)$$

and moreover, that $\kappa_d(\mathcal{R}^*) = \sqrt{d_1 d_2}$. Consequently, in this specific case, the general convex program (7) takes the form

$$\min_{(\Theta, \Gamma)} \left\{ \frac{1}{2} \|Y - \mathfrak{X}(\Theta + \Gamma)\|_F^2 + \lambda_d \|\Theta\|_N + \mu_d \|\Gamma\|_1 \right\} \quad \text{such that } \|\Theta\|_\infty \leq \frac{\alpha}{\sqrt{d_1 d_2}}. \quad (10)$$

The constraint involving $\|\Theta\|_\infty$ serves to control the “spikiness” of the low rank component, with larger settings of α allowing for more spiky matrices. Indeed, this type of spikiness control has proven useful in analysis of nuclear norm relaxations for noisy matrix completion [20]. To gain intuition for the parameter α , if we consider matrices with $\|\Theta\|_F \approx 1$, as is appropriate to keep a constant signal-to-noise ratio in the noisy model (1), then setting $\alpha \approx 1$ allows only for matrices for which $|\Theta_{jk}| \approx 1/\sqrt{d_1 d_2}$ in all entries. If we want to permit the maximally spiky matrix with all its mass in a single position, then the parameter α must be of the order $\sqrt{d_1 d_2}$. In practice, we are interested in settings of α lying between these two extremes. ♣

Past work on ℓ_1 -forms of matrix decomposition has imposed singular vector incoherence conditions that are related to but different from our spikiness condition. More concretely, if we write the SVD of the low-rank component as $\Theta^* = UDV^T$ where D is diagonal, and $U \in \mathbb{R}^{d_1 \times r}$ and $V \in \mathbb{R}^{d_2 \times r}$ are matrices of the left and right singular vectors. Singular vector incoherence bounds quantities such as

$$\|UU^T - \frac{r}{d_1} I_{d_1 \times d_1}\|_\infty, \quad \|VV^T - \frac{r}{d_2} I_{d_2 \times d_2}\|_\infty, \quad \text{and} \quad \|UV^T\|_\infty. \quad (11)$$

all of which measure the degree of “coherence” between the singular vectors and the canonical basis. A remarkable feature of such conditions is that they have no dependence on the *singular values* of Θ^* . This lack of dependence makes sense in the noiseless setting, where exact recovery is the goal. For noisy models, in contrast, one should only be concerned with recovering components with “large” singular values. In this context, our bound on the maximum element $\|\Theta^*\|_\infty$, or equivalently on the quantity $\|UDV^T\|_\infty$, is natural. Note that it imposes no constraint on the matrices UU^T or VV^T , and moreover it uses the diagonal matrix of singular values as a weight in the ℓ_∞ bound. Moreover, we note that there are many matrices for which $\|\Theta^*\|_\infty$ satisfies a reasonable bound, whereas the incoherence measures are poorly behaved (e.g., see Section 3.4.2 in the paper [20] for one example).

Example 5 (Column-sparsity and block columnwise regularization). Other applications involve models in which Γ^* has a relatively small number $s \ll d_2$ of non-zero columns (or a relatively small number $s \ll d_1$ of non-zero rows). Such applications include the multi-task regression problem from Example 2, the robust covariance problem from Example 3, as well as a form of robust PCA considered by Xu et al. [28]. In this case, it is natural to constrain Γ via the $(2, 1)$ -norm regularizer

$$\mathcal{R}(\Gamma) = \|\Gamma\|_{2,1} := \sum_{k=1}^{d_2} \|\Gamma_k\|_2, \quad (12)$$

where Γ_k is the k^{th} column of Γ (or the $(1, 2)$ -norm regularizer that enforces the analogous constraint on the rows of Γ). For this choice, it can be verified that

$$\mathcal{R}^*(U) = \|U\|_{2,\infty} := \max_{k=1,2,\dots,d_2} \|U_k\|_2, \quad (13)$$

where U_k denotes the k^{th} column of U , and that $\kappa_d(\mathcal{R}^*) = \sqrt{d_2}$. Consequently, in this specific case, the general convex program (7) takes the form

$$\min_{(\Theta, \Gamma)} \left\{ \frac{1}{2} \|Y - \mathfrak{X}(\Theta + \Gamma)\|_F^2 + \lambda_d \|\Theta\|_N + \mu_d \|\Gamma\|_{2,1} \right\} \quad \text{such that } \|\Theta\|_{2,\infty} \leq \frac{\alpha}{\sqrt{d_2}}. \quad (14)$$

As before, the constraint $\|\Theta\|_{2,\infty}$ serves to limit the “spikiness” of the low rank component, where in this case, spikiness is measured in a columnwise manner. Again, it is natural to consider matrices such that $\|\Theta^*\|_F \approx 1$, so that the signal-to-noise ratio in the observation model (1) stays fixed. Thus, if $\alpha \approx 1$, then we are restricted to matrices for which $\|\Theta_k^*\|_2 \approx \frac{1}{\sqrt{d_2}}$ for all columns $k = 1, 2, \dots, d_2$. At the other extreme, in order to permit a maximally “column-spiky” matrix (i.e., with a single non-zero column of ℓ_2 -norm roughly 1), we need to set $\alpha \approx \sqrt{d_2}$. As before, of practical interest are settings of α lying between these two extremes. ♣

3 Main results and their consequences

In this section, we state our main results, and discuss some of their consequences. Our first result applies to the family of convex programs (7) whenever \mathcal{R} belongs to the class of decomposable regularizers [19]. Accordingly, we begin in Section 3.1 by defining the notion of decomposability, and then illustrating how the elementwise- ℓ_1 and columnwise- $(2, 1)$ -norms, as discussed in Examples 4 and 5 respectively, are both instances of decomposable regularizers. Section 3.2 contains the statement of our main achievable result, whereas Sections 3.3 and 3.4 are devoted to its consequences for the cases of elementwise sparsity and columnwise sparsity, respectively. In Section 3.5, we state matching lower bounds on the minimax errors in the case of stochastic noise.

3.1 Decomposable regularizers

The notion of decomposability is defined in terms of a pair of subspaces, which (in general) need not be orthogonal complements. Here we consider a special case of decomposability that is sufficient to cover the examples of interest in this paper:

Definition 1. Given a subspace $\mathcal{M} \subseteq \mathbb{R}^{d_1 \times d_2}$ and its orthogonal complement \mathcal{M}^\perp , a norm-based regularizer \mathcal{R} is *decomposable with respect* $(\mathcal{M}, \mathcal{M}^\perp)$ if

$$\mathcal{R}(U + V) = \mathcal{R}(U) + \mathcal{R}(V) \quad \text{for all } U \in \mathcal{M}, \text{ and } V \in \mathcal{M}^\perp. \quad (15)$$

To provide some intuition, the subspace \mathcal{M} should be thought of as the nominal *model subspace*; in our results, it will be chosen such that the matrix Γ^\star lies within or close to \mathcal{M} . The orthogonal complement \mathcal{M}^\perp represents deviations away from the model subspace, and the equality (15) guarantees that such deviations are penalized as much as possible.

As discussed at more length in Negahban et al. [19], a large class of norms are decomposable with respect to interesting² subspace pairs. Of particular relevance to us is the decomposability of the elementwise ℓ_1 -norm $\|\Gamma\|_1$ and the columnwise $(2, 1)$ -norm $\|\Gamma\|_{2,1}$ as well as the row-sparse $(1, 2)$ norm $\|\Gamma\|_{1,2}$, as previously discussed in Examples 4 and 5 respectively.

Decomposability of $\mathcal{R}(\cdot) = \|\cdot\|_1$: Beginning with the elementwise ℓ_1 -norm, given an arbitrary subset $S \subseteq \{1, 2, \dots, d_1\} \times \{1, 2, \dots, d_2\}$ of matrix indices, consider the subspace pair

$$\mathcal{M}(S) := \{U \in \mathbb{R}^{d_1 \times d_2} \mid U_{jk} = 0 \text{ for all } (j, k) \notin S\}, \quad \text{and} \quad \mathcal{M}^\perp(S) := (\mathcal{M}(S))^\perp. \quad (16)$$

It is then easy to see that for any pair $U \in \mathcal{M}(S), U' \in \mathcal{M}^\perp(S)$, we have $\|U + U'\|_1 = \|U\|_1 + \|U'\|_1$, showing that the elementwise ℓ_1 -norm is decomposable with respect to the pair $(\mathcal{M}(S), \mathcal{M}^\perp(S))$.

Decomposability of $\mathcal{R}(\cdot) = \|\cdot\|_{2,1}$: Similarly, the columnwise $(2, 1)$ -norm is also decomposable with respect to appropriately defined subspaces, indexed by subsets $C \subseteq \{1, 2, \dots, d_2\}$ of column indices. Indeed, using V_k to denote the k^{th} column of the matrix V , define

$$\mathcal{M}(C) := \{V \in \mathbb{R}^{d_1 \times d_2} \mid V_k = 0 \text{ for all } k \notin C\}, \quad (17)$$

and $\mathcal{M}^\perp(C) := (\mathcal{M}(C))^\perp$. Again, it is easy to verify that for any pair $V \in \mathcal{M}(C), V' \in \mathcal{M}^\perp(C)$, we have $\|V + V'\|_{2,1} = \|V\|_{2,1} + \|V'\|_{2,1}$, thus verifying the decomposability property.

For any decomposable regularizer and subspace $\mathcal{M} \neq \{0\}$, we define the compatibility constant

$$\Psi(\mathcal{M}, \mathcal{R}) := \sup_{U \in \mathcal{M}, U \neq 0} \frac{\mathcal{R}(U)}{\|U\|_F}. \quad (18)$$

This quantity measures the compatibility between the Frobenius norm and the regularizer over the subspace \mathcal{M} . For example, for the ℓ_1 -norm and the set $\mathcal{M}(S)$ previously defined (16), an elementary calculation yields $\Psi(\mathcal{M}(S); \|\cdot\|_1) = \sqrt{s}$.

3.2 Results for general regularizers and noise

We begin by stating a result for a general decomposable regularizer \mathcal{R} and a general noise matrix W for the case of an identity observation operator. Following this statement, we describe how this result is easily extended to general observation operators. In later subsections,

²Note that any norm is (trivially) decomposable with respect to the pair $(\mathcal{M}, \mathcal{M}^\perp) = (\mathbb{R}^{d_1 \times d_2}, \{0\})$.

we specialize this result to particular choices of regularizers, and then to stochastic noise matrices. In all our results, we measure error using the *squared Frobenius norm* summed across both matrices

$$e^2(\hat{\Theta}, \hat{\Gamma}) := \|\hat{\Theta} - \Theta^*\|_F^2 + \|\hat{\Gamma} - \Gamma^*\|_F^2. \quad (19)$$

With this notation, we have the following result:

Theorem 1. *Given observations Y from the model $Y = \Theta^* + \Gamma^* + W$, suppose that we solve the convex program (7) with regularization parameters (λ_d, μ_d) such that*

$$\lambda_d \geq 4\|W\|_{\text{op}}, \quad \text{and} \quad \mu_d \geq 4\mathcal{R}^*(W) + \frac{4\alpha}{\kappa_d}. \quad (20)$$

Then there is a universal constant c_1 such that for any matrix pair (Θ^, Γ^*) with $\varphi_{\mathcal{R}}(\Theta^*) \leq \alpha$, for all integers $r = 1, 2, \dots, \min\{d_1, d_2\}$, and for any \mathcal{R} -decomposable pair $(\mathcal{M}, \mathcal{M}^\perp)$, any optimal solution $(\hat{\Theta}, \hat{\Gamma})$ satisfies*

$$e^2(\hat{\Theta}, \hat{\Gamma}) \leq \underbrace{c_1 \lambda_d^2 \left\{ r + \frac{1}{\lambda_d} \sum_{j=r+1}^d \sigma_j(\Theta^*) \right\}}_{\mathcal{K}_{\Theta}^*} + \underbrace{c_1 \mu_d^2 \left\{ \Psi^2(\mathcal{M}; \mathcal{R}) + \frac{1}{\mu_d} \mathcal{R}(\Pi_{\mathcal{M}^\perp}(\Gamma^*)) \right\}}_{\mathcal{K}_{\Gamma}^*}. \quad (21)$$

It is worthwhile making some remarks about this theorem.

Deterministic guarantee: To be clear, Theorem 1 is a deterministic statement that applies to any optimum of the convex program (7). Note that the upper bound (21) involves two components, \mathcal{K}_{Θ}^* and \mathcal{K}_{Γ}^* , which correspond (respectively) to the complexities associated with the sub-problems of recovering Θ^* and Γ^* . Each component is further divided into a sum of two terms, which have an interpretation as the estimation error and the approximation error. Considering for instance the term \mathcal{K}_{Θ}^* , the quantity $\lambda_d^2 r$ corresponds to the *estimation error* associated with a rank r matrix, whereas the term $\lambda_d \sum_{j=r+1}^d \sigma_j(\Theta^*)$ corresponds to the *approximation error* associated with representing Θ^* (which might be full rank) by a matrix of rank r . A similar interpretation applies to the two components associated with Γ^* .

A family of upper bounds: Since the inequality (21) corresponds to a family of upper bounds indexed by r and the subspace \mathcal{M} , these quantities can be chosen adaptively, depending on the structure of the matrices (Θ^*, Γ^*) , so as to obtain the tightest possible upper bound. In the simplest case, the matrix Θ^* is exactly low rank (say rank r), and Γ^* lies within a \mathcal{R} -decomposable subspace \mathcal{M} . In this case, the approximation errors vanish, and Theorem 1 guarantees that the squared Frobenius error is at most

$$e^2(\hat{\Theta}; \hat{\Gamma}) \leq c_1 \lambda_d^2 r + c_1 \mu_d^2 \Psi^2(\mathcal{M}; \mathcal{R}). \quad (22)$$

We illustrate some consequences of this special case of Theorem 1 in the sections to follow.

Extensions to general observation operators \mathfrak{X} : Theorem 1 can be extended to observation operators beyond the identity mapping, such as the mapping $A \mapsto XA$ that underlies the multi-task regression problem (see Example 2). Of course, some conditions are required on the observation operator; for instance, it is sufficient to for have a lower curvature condition, meaning there is a constant $\rho > 0$ such that $\|\mathfrak{X}(A)\|_F^2 \geq \rho \|A\|_F^2$ for all $A \in \mathbb{R}^{d_1 \times d_2}$. Clearly, this condition holds trivially for the identity operator with $\rho = 1$. Moreover, it can be shown (e.g., [21], Lemma 2) to hold with high probability for the multi-task regression problem of Example 2.

Finally, the adjoint \mathfrak{X}^* , which is a linear map from $\mathbb{R}^{n_1 \times n_2}$ to $\mathbb{R}^{d_1 \times d_2}$, plays an important role in the specification of the regularization parameters. In particular, the appropriate generalization of the regularization condition (20) is

$$\lambda_d \geq 4\|\mathfrak{X}^*(W)\|_{\text{op}}, \quad \text{and} \quad \mu_d \geq 4\mathcal{R}^*(\mathfrak{X}^*(W)) + \frac{4\alpha}{\kappa_d}.$$

Under this condition, a straightforward modification of our proof technique shows that the Frobenius error is bounded as in Theorem 1, multiplied by an additional factor $(1/\rho)$ corresponding to the inverse curvature.

3.3 Results for ℓ_1 -norm regularization

Theorem 1 holds for any regularizer that is decomposable with respect to some subspace pair. As previously noted, an important example of a decomposable regularizer is the elementwise ℓ_1 -norm, which is decomposable with respect to subspaces of the form (16). We remind the reader that the corollaries apply to the case of $Y = \Theta^* + \Gamma^* + W$, that is, where \mathfrak{X} is identity.

Corollary 1. *Suppose that we solve the convex program (10) with regularization parameters (λ_d, μ_d) such that*

$$\lambda_d \geq 4\|W\|_{\text{op}}, \quad \text{and} \quad \mu_d \geq 4\|W\|_{\infty} + \frac{4\alpha}{\sqrt{d_1 d_2}}. \quad (23)$$

Then there is a universal constant c_1 such that for any matrix pair (Θ^, Γ^*) with $\|\Theta^*\|_{\infty} \leq \frac{\alpha}{\sqrt{d_1 d_2}}$ and for all integers $r = 1, 2, \dots, \min\{d_1, d_2\}$, and $s = 1, 2, \dots, (d_1 d_2)$, we have*

$$e^2(\hat{\Theta}, \hat{\Gamma}) \leq c_1 \lambda_d^2 \left\{ r + \frac{1}{\lambda_d} \sum_{j=r+1}^d \sigma_j(\Theta^*) \right\} + c_1 \mu_d^2 \left\{ s + \frac{1}{\mu_d} \sum_{(j,k) \notin S} |\Gamma_{jk}^*| \right\}, \quad (24)$$

where S is an arbitrary subset of matrix indices of cardinality at most s .

Remarks: This result follows directly by specializing Theorem 1 to the elementwise ℓ_1 -norm. As noted in Example 4, for this norm, we have $\kappa_d = \sqrt{d_1 d_2}$, so that the choice (23) satisfies the conditions of Theorem 1. The dual norm is given by the elementwise ℓ_{∞} -norm $\mathcal{R}^*(\cdot) = \|\cdot\|_{\infty}$. As observed in Section 3.1, the ℓ_1 -norm is decomposable with respect to subspace pairs of the form $(\mathcal{M}(S), \mathcal{M}^{\perp}(S))$, for an arbitrary subset S of matrix indices. Moreover, for any subset S of cardinality s , we have $\Psi^2(\mathcal{M}(S)) = s$. It is easy to verify that with this choice, we have $\Pi_{\mathcal{M}^{\perp}}(\Gamma^*) = \sum_{(j,k) \notin S} |\Gamma_{jk}^*|$, from which the claim follows.

It is worth noting the inequality (21) corresponds to a family of upper bounds indexed by r and the subset S . For any fixed integer $s \in \{1, 2, \dots, (d_1 d_2)\}$, it is natural to let S index the largest s values (in absolute value) of Γ^* . Moreover, the choice of the pair (r, s) can be further

adapted to the structure of the matrix. For instance, when Θ^* is exactly low rank, and Γ^* is exactly sparse, then one natural choice is $r = \text{rank}(\Theta^*)$, and $s = |\text{supp}(\Gamma^*)|$. With this choice, both the approximation terms vanish, and Corollary 1 guarantees that any solution $(\hat{\Theta}, \hat{\Gamma})$ of the convex program (10) satisfies

$$\|\hat{\Theta} - \Theta^*\|_F^2 + \|\hat{\Gamma} - \Gamma^*\|_F^2 \lesssim \lambda_d^2 r + \mu_d^2 s. \quad (25)$$

Further specializing to the case of noiseless observations ($W = 0$), yields a form of approximate recovery—namely

$$\|\hat{\Theta} - \Theta^*\|_F^2 + \|\hat{\Gamma} - \Gamma^*\|_F^2 \lesssim \alpha^2 \frac{s}{d_1 d_2}. \quad (26)$$

This guarantee is weaker than the exact recovery results obtained in past work on the noiseless observation model [9, 7]; however, these papers imposed incoherence requirements on the singular vectors of the low-rank component Θ^* that are more restrictive than the conditions of Theorem 1.

Our elementwise ℓ_∞ bound is a weaker condition than incoherence, since it allows for singular vectors to be coherent as long as the associated singular value is not too large. Moreover, the bound (26) is unimprovable up to constant factors, due to the non-identifiability of the observation model (1), as shown by the following example.

Example 6. [Unimprovability for elementwise sparse model] Consider a given sparsity index $s \in \{1, 2, \dots, (d_1 d_2)\}$, where we may assume without loss of generality that $s \leq d_2$. We then form the matrix

$$\Theta^* := \frac{\alpha}{\sqrt{d_1 d_2}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \underbrace{[1 \ 1 \ 1 \ \dots \ 0 \ \dots \ 0]}_{f^T}, \quad (27)$$

where the vector $f \in \mathbb{R}^{d_2}$ has exactly s ones. Note that $\|\Theta^*\|_\infty = \frac{\alpha}{\sqrt{d_1 d_2}}$ by construction, and moreover Θ^* is rank one, and has s non-zero entries. Since up to s entries of the noise matrix Γ^* can be chosen arbitrarily, “nature” can always set $\Gamma^* = -\Theta^*$, meaning that we would observe $Y = \Theta^* + \Gamma^* = 0$. Consequently, based on observing only Y , the pair (Θ^*, Γ^*) is indistinguishable from the all-zero matrices $(0_{d_1 \times d_2}, 0_{d_1 \times d_2})$. This fact can be used to show that no method can have squared Frobenius error lower than $\approx \frac{\alpha^2 s}{d_1 d_2}$; see Section 3.5 for a precise statement. Therefore, the bound (26) cannot be improved unless one is willing to impose further restrictions on the pair (Θ^*, Γ^*) . We note that the singular vector incoherence conditions, as imposed in past work [9, 7, 14] and used to guarantee exact recovery, would exclude the matrix (27), since its left singular vector is the unit vector $e_1 \in \mathbb{R}^{d_1}$. ♣

Results for stochastic noise matrices: Our discussion thus far has applied to general noise matrices W . More concrete results can be obtained by assuming that W is stochastic. Our first stochastic result applies to a noise matrix W generated with i.i.d. $N(0, \nu^2/(d_1 d_2))$ entries.³

³To be clear, we state our results in terms of the noise scaling $\nu^2/(d_1 d_2)$ since it corresponds to a model with constant signal-to-noise ratio when the Frobenius norms of Θ^* and Γ^* remain bounded, independently of the dimension. The same results would hold if the noise were not rescaled, modulo the appropriate rescalings of the various terms.

Corollary 2. Suppose Θ^* has rank at most r , and Γ^* has at most s non-zero entries. If the noise matrix W has i.i.d. $N(0, \nu^2/(d_1 d_2))$ entries, and we solve the convex program (10) with regularization parameters

$$\lambda_d = \frac{8\nu}{\sqrt{d_1}} + \frac{8\nu}{\sqrt{d_2}}, \quad \text{and} \quad \mu_d = 16\nu \sqrt{\frac{\log(d_1 d_2)}{d_1 d_2}} + \frac{4\alpha}{\sqrt{d_1 d_2}}, \quad (28)$$

then with probability greater than $1 - \exp(-2 \log(d_1 d_2))$, any optimal solution $(\hat{\Theta}, \hat{\Gamma})$ satisfies

$$e^2(\hat{\Theta}, \hat{\Gamma}) \leq \underbrace{c_1 \nu^2 \left(\frac{r(d_1 + d_2)}{d_1 d_2} \right)}_{\mathcal{K}_{\Theta}^*} + \underbrace{c_1 \nu^2 \left(\frac{s \log(d_1 d_2)}{d_1 d_2} \right)}_{\mathcal{K}_{\Gamma}^*} + c_1 \frac{\alpha^2 s}{d_1 d_2} \quad (29)$$

Remarks: In the statement of this corollary, the settings of λ_d and μ_d are based on upper bounding $\|W\|_{\infty}$ and $\|W\|_{\text{op}}$, using large deviation bounds and some non-asymptotic random matrix theory. With a slightly modified argument, the bound (29) can be sharpened slightly by reducing the logarithmic term to $\log(\frac{d_1 d_2}{s})$. As shown in Theorem 2 to follow in Section 3.5, this sharpened bound is minimax-optimal, meaning that no estimator (regardless of its computational complexity) can achieve much better estimates for the matrix classes and noise model given here.

It is also worth observing that both terms in the bound (29) have intuitive interpretations. Considering first the term \mathcal{K}_{Θ}^* , we note that the numerator term $r(d_1 + d_2)$ is of the order of the number of free parameters in a rank r matrix of dimensions $d_1 \times d_2$. The multiplicative factor $\frac{\nu^2}{d_1 d_2}$ corresponds to the noise variance in the problem. On the other hand, the term \mathcal{K}_{Γ}^* measures the complexity of estimating s non-zero entries in a $d_1 \times d_2$ matrix. Note that there are $\binom{d_1 d_2}{s}$ possible subsets of size s , and consequently, the numerator includes a term that scales as $\log \binom{d_1 d_2}{s} \approx s \log(d_1 d_2)$. As before, the multiplicative pre-factor $\frac{\nu^2}{d_1 d_2}$ corresponds to the noise variance. Finally, the second term within \mathcal{K}_{Γ}^* —namely the quantity $\frac{\alpha^2 s}{d_1 d_2}$ —arises from the non-identifiability of the model, and as discussed in Example 6, it cannot be avoided without imposing further restrictions on the pair (Γ^*, Θ^*) .

We now turn to analysis of the sparse factor analysis problem: as previously introduced in Example 1, this involves estimation of a covariance matrix that has a low-rank plus elementwise sparse decomposition. In this case, given n i.i.d. samples from the unknown covariance matrix $\Sigma = \Theta^* + \Gamma^*$, the noise matrix $W \in \mathbb{R}^{d \times d}$ is a recentered Wishart noise (see equation (3)). We can use tail bounds for its entries and its operator norm in order to specify appropriate choices of the regularization parameters λ_d and μ_d . We summarize our conclusions in the following corollary:

Corollary 3. Consider the factor analysis model with $n \geq d$ samples, and regularization parameters

$$\lambda_d = 16 \|\sqrt{\Sigma}\|_2 \sqrt{\frac{d}{n}}, \quad \text{and} \quad \mu_d = 32 \rho(\Sigma) \sqrt{\frac{\log d}{n}} + \frac{4\alpha}{d}, \quad \text{where } \rho(\Sigma) = \max_j \Sigma_{jj}. \quad (30)$$

Then with probability greater than $1 - c_2 \exp(-c_3 \log(d))$, any optimal solution $(\hat{\Theta}, \hat{\Gamma})$ satisfies

$$e^2(\hat{\Theta}, \hat{\Gamma}) \leq c_1 \left\{ \|\Sigma\|_2 \frac{rd}{n} + \rho(\Sigma) \frac{s \log d}{n} \right\} + c_1 \frac{\alpha^2 s}{d^2}.$$

We note that the condition $n \geq d$ is necessary to obtain consistent estimates in factor analysis models, even in the case with $\Gamma^* = I_{d \times d}$ where PCA is possible (e.g., see Johnstone [15]). Again, the terms in the bound have a natural interpretation: since a matrix of rank r in d dimensions has roughly rd degrees of freedom, we expect to see a term of the order $\frac{rd}{n}$. Similarly, since there are $\log \binom{d^2}{s} \approx s \log d$ subsets of size s in a $d \times d$ matrix, we also expect to see a term of the order $\frac{s \log d}{n}$. Moreover, although we have stated our choices of regularization parameter in terms of $\|\Sigma\|_2$ and $\rho(\Sigma)$, these can be replaced by the analogous versions using the sample covariance matrix $\hat{\Sigma}$. (By the concentration results that we establish, the population and empirical versions do not differ significantly when $n \geq d$.)

Comparison to Hsu et al. [14] This recent work focuses on the problem of matrix decomposition with the $\|\cdot\|_1$ -norm, and provides results both for the noiseless and noisy setting. All of their work focuses on the case of exactly low rank and exactly sparse matrices, in contrast to our Theorem 1 that provides an upper bound for general matrix pairs. Most relevant is comparison of our ℓ_1 -results with exact rank-sparsity constraints to their Theorem 3, which provides various error bounds (in nuclear and Frobenius norm) for such models with additive noise. These bounds are obtained using an estimator similar to our program (10), and in parts of their analysis, they enforce bounds on the ℓ_∞ -norm of the solution. However, this is not done directly with a constraint on Θ as in our estimator, but rather by penalizing the difference $\|Y - \Gamma\|_\infty$, or by thresholding the solution.

Apart from these minor differences, there are two major differences between our results, and those of Hsu et al. First of all, their analysis involves three quantities (α, β, γ) that measure singular vector incoherence, and must satisfy a number of inequalities. In contrast, our analysis is based only on a single condition: the “spikiness” condition on the low-rank component Θ^* . As we have seen, this constraint is weaker than singular vector incoherence, and consequently, unlike the result of Hsu et al., we do not provide exact recovery guarantees for the noiseless setting. However, it is interesting to see (as shown by our analysis) that a very simple spikiness condition suffices for the approximate recovery guarantees that are of interest for noisy observation models. Given these differing assumptions, the underlying proof techniques are quite distinct, with our methods leveraging the notion of restricted strong convexity introduced by Negahban et al. [19].

The second (and perhaps most significant) difference is in the sharpness of the results for the noisy setting, and the permissible scalings of the rank-sparsity pair (r, s) . As will be clarified in Section 3.5, the rates that we establish for low-rank plus elementwise sparsity for the noisy Gaussian model (Corollary 2) are minimax-optimal up to constant factors. In contrast, the upper bounds in Theorem 3 of Hsu et al. involve the *product* rs , and hence are sub-optimal as the rank and sparsity scale. These terms appear only additively both our upper and minimax lower bounds, showing that an upper bound involving the product rs is sub-optimal. Moreover, the bounds of Hsu et al. (see Section IV.D) are limited to matrix decompositions for which the rank-sparsity pair (r, s) are bounded as

$$rs \lesssim \frac{d_1 d_1}{\log(d_1) \log(d_2)} \quad (31)$$

This bound precludes many scalings that are of interest. For instance, if the sparse component Γ^* has a nearly constant fraction of non-zeros (say $s \asymp \frac{d_1 d_2}{\log(d_1) \log(d_2)}$ for concreteness), then the bound (31) restricts to Θ^* to have constant rank. In contrast, our analysis allows for high-dimensional scaling of both the rank r and sparsity s simultaneously; as can be

seen by inspection of Corollary 2, our Frobenius norm error goes to zero under the scalings $s \asymp \frac{d_1 d_2}{\log(d_1) \log(d_2)}$ and $r \asymp \frac{d_2}{\log(d_2)}$.

3.4 Results for $\|\cdot\|_{2,1}$ regularization

As another illustration of the consequences of Theorem 1, we now turn to the columnwise $(2,1)$ -norm previously defined in Example 5, and methods based on solving the convex program (14). As before, specializing Theorem 1 to this decomposable regularizer yields the following guarantee:

Corollary 4. *Suppose that we solve the convex program (14) with regularization parameters (λ_d, μ_d) such that*

$$\lambda_d \geq 4\|W\|_{\text{op}}, \quad \text{and} \quad \mu_d \geq 4\|W\|_{2,\infty} + \frac{4\alpha}{\sqrt{d_2}}. \quad (32)$$

Then there is a universal constant c_1 such that for any matrix pair (Θ^, Γ^*) with $\|\Theta^*\|_{2,\infty} \leq \frac{\alpha}{\sqrt{d_2}}$ and for all integers $r = 1, 2, \dots, d$ and $s = 1, 2, \dots, d_2$, we have*

$$\|\hat{\Theta} - \Theta^*\|_{\text{F}}^2 + \|\hat{\Gamma} - \Gamma^*\|_{\text{F}}^2 \leq c_1 \lambda_d^2 \left\{ r + \frac{1}{\lambda_d} \sum_{j=r+1}^d \sigma_j(\Theta^*) \right\} + c_1 \mu_d^2 \left\{ s + \frac{1}{\mu_d} \sum_{k \notin C} \|\Gamma_k^*\|_2 \right\}, \quad (33)$$

where $C \subseteq \{1, 2, \dots, d_2\}$ is an arbitrary subset of column indices of cardinality at most s .

Remarks: This result follows directly by specializing Theorem 1 to the columnwise $(2,1)$ -norm, previously discussed in Example 5. Its dual norm is the columnwise $(2,\infty)$ -norm, and we have $\kappa_d = \sqrt{d_2}$. As discussed in Section 3.1, the $(2,1)$ -norm is decomposable with respect to subspaces of the type $\mathcal{M}(C)$, as defined in equation (17), where $C \subseteq \{1, 2, \dots, d_2\}$ is an arbitrary subset of column indices. For any such subset C of cardinality s , it can be calculated that $\Psi^2(\mathcal{M}(C)) = s$, and moreover, that $\|\Pi_{\mathcal{M}^\perp}(\Gamma^*)\|_{2,1} = \sum_{k \notin C} \|\Gamma_k^*\|_2$. Consequently, the bound (33) follows from Theorem 1.

As before, if we assume that Θ^* has exactly rank r and Γ^* has at most s non-zero columns, then both approximation error terms in the bound (33) vanish, and we recover an upper bound of the form $\|\hat{\Theta} - \Theta^*\|_{\text{F}}^2 + \|\hat{\Gamma} - \Gamma^*\|_{\text{F}}^2 \lesssim \lambda_d^2 r + \mu_d^2 s$. If we further specialize to the case of exact observations ($W = 0$), then Corollary 4 guarantees that

$$\|\hat{\Theta} - \Theta^*\|_{\text{F}}^2 + \|\hat{\Gamma} - \Gamma^*\|_{\text{F}}^2 \lesssim \alpha^2 \frac{s}{d_2}.$$

The following example shows, that given our conditions, even in the noiseless setting, no method can recover the matrices to precision more accurate than $\alpha^2 s / d_2$.

Example 7 (Unimprovability for columnwise sparse model). In order to demonstrate that the term $\alpha^2 s / d_2$ is unavoidable, it suffices to consider a slight modification of Example 6. In particular, let us define the matrix

$$\Theta^* := \frac{\alpha}{\sqrt{d_1 d_2}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \underbrace{[1 \ 1 \ 1 \ \dots \ 0 \ \dots \ 0]}_{f^T}, \quad (34)$$

where again the vector $f \in \mathbb{R}^{d_2}$ has s non-zeros. Note that the matrix Θ^* is rank one, has s non-zero columns, and moreover $\|\Theta^*\|_{2,\infty} = \frac{\alpha}{\sqrt{d_2}}$. Consequently, the matrix Θ^* is covered by Corollary 4. Since s columns of the matrix Γ^* can be chosen in an arbitrary manner, it is possible that $\Gamma^* = -\Theta^*$, in which case the observation matrix $Y = 0$. This fact can be exploited to show that *no method* can achieve squared Frobenius error much smaller than $\approx \frac{\alpha^2 s}{d_2}$; see Section 3.5 for the precise statement. Finally, we note that it is difficult to compare directly to the results of Xu et al. [28], since their results do not guarantee exact recovery of the pair (Θ^*, Γ^*) . ♣

As with the case of elementwise ℓ_1 -norm, more concrete results can be obtained when the noise matrix W is stochastic.

Corollary 5. *Suppose Θ^* has rank at most r , and Γ^* has at most s non-zero columns. If the noise matrix W has i.i.d. $N(0, \nu^2/(d_1 d_2))$ entries, and we solve the convex program (14) with regularization parameters $\lambda_d = \frac{8\nu}{\sqrt{d_1}} + \frac{8\nu}{\sqrt{d_2}}$ and*

$$\mu_d = 8\nu \sqrt{\frac{1}{d_2}} + \sqrt{\frac{\log d_2}{d_1 d_2}} + \frac{4\alpha}{\sqrt{d_2}},$$

then with probability greater than $1 - \exp(-2 \log(d_2))$, any optimal solution $(\hat{\Theta}, \hat{\Gamma})$ satisfies

$$e^2(\hat{\Theta}, \hat{\Gamma}) \leq c_1 \underbrace{\nu^2 \frac{r(d_1 + d_2)}{d_1 d_2}}_{\mathcal{K}_{\Theta}^*} + \nu^2 \underbrace{\left\{ \frac{sd_1}{d_1 d_2} + \frac{s \log d_2}{d_1 d_2} \right\}}_{\mathcal{K}_{\Gamma}^*} + c_2 \frac{\alpha^2 s}{d_2}. \quad (35)$$

Remarks: Note that the setting of λ_d is the same as in Corollary 2, whereas the parameter μ_d is chosen based on upper bounding $\|W\|_{2,\infty}$, corresponding to the dual norm of the columnwise $(2,1)$ -norm. With a slightly modified argument, the bound (35) can be sharpened slightly by reducing the logarithmic term to $\log(\frac{d_2}{s})$. As shown in Theorem 2 to follow in Section 3.5, this sharpened bound is minimax-optimal.

As with Corollary 2, both terms in the bound (35) are readily interpreted. The term \mathcal{K}_{Θ}^* has the same interpretation, as a combination of the number of degrees of freedom in a rank r matrix (that is, of the order $r(d_1 + d_2)$) scaled by the noise variance $\frac{\nu^2}{d_1 d_2}$. The second term \mathcal{K}_{Γ}^* has a somewhat more subtle interpretation. The problem of estimating s non-zero columns embedded within a $d_1 \times d_2$ matrix can be split into two sub-problems: first, the problem of estimating the sd_1 non-zero parameters (in Frobenius norm), and second, the problem of column subset selection—i.e., determining the location of the s non-zero parameters. The estimation sub-problem yields the term $\frac{\nu^2 sd_1}{d_1 d_2}$, whereas the column subset selection sub-problem incurs a penalty involving $\log \binom{d_2}{s} \approx s \log d_2$, multiplied by the usual noise variance. The final term $\alpha^2 s/d_2$ arises from the non-identifiability of the model. As discussed in Example 7, it is unavoidable without further restrictions.

We now turn to some consequences for the problem of robust covariance estimation formulated in Example 3. As seen from equation (4), the disturbance matrix in this setting can be written as a sum $(\Gamma^*)^T + \Gamma^*$, where Γ^* is a column-wise sparse matrix. Consequently, we can use a variant of the estimator (14), in which the loss function is given by $\|Y - \{\Theta^* + (\Gamma^*)^T + \Gamma^*\}\|_F^2$. The following result summarizes the consequences of Theorem 1 in this setting:

Corollary 6. *Consider the problem of robust covariance estimation with $n \geq d$ samples, based on a matrix Θ^* with rank at most r , and a corrupting matrix Γ^* with at most s rows and columns corrupted. If we solve SDP (14) with regularization parameters*

$$\lambda_d^2 = 8\|\Theta^*\|_{\text{op}} \frac{d}{n}, \quad \text{and} \quad \mu_d^2 = 8\|\Theta^*\|_{\text{op}} \frac{d}{n} + \frac{16\alpha^2}{d}, \quad (36)$$

then with probability greater than $1 - c_2 \exp(-c_3 \log(d))$, any optimal solution $(\hat{\Theta}, \hat{\Gamma})$ satisfies

$$e^2(\hat{\Theta}, \hat{\Gamma}) \leq c_1 \|\Theta^*\|_{\text{op}} \left\{ \frac{rd}{n} + \frac{sd}{n} \right\} + c_1 \frac{\alpha^2 s}{d}.$$

Some comments about this result: with the motivation of being concrete, we have given an explicit choice (36) of the regularization parameters, involving the operator norm $\|\Theta^*\|_{\text{op}}$, but any upper bound would suffice. As with the noise variance in Corollary 5, a typical strategy would choose this pre-factor by cross-validation. We also note that the terms appearing in our bounds have the natural intuition: in particular, the term rd corresponds (roughly) to the degrees of freedom of a rank r matrix in d dimensions, whereas sd corresponds to degrees of freedom in a symmetric column-sparse matrix with at most s non-zero rows.

3.5 Lower bounds

For the case of i.i.d Gaussian noise matrices, Corollaries 2 and 5 provide results of an achievable nature, namely in guaranteeing that our estimators achieve certain Frobenius errors. In this section, we turn to the complementary question: what are the fundamental (algorithmic-independent) limits of accuracy in noisy matrix decomposition? One way in which to address such a question is by analyzing statistical minimax rates.

More formally, given some family \mathcal{F} of matrices, the associated minimax error is given by

$$\mathfrak{M}(\mathcal{F}) := \inf_{(\tilde{\Theta}, \tilde{\Gamma})} \sup_{(\Theta^*, \Gamma^*)} \mathbb{E}[\|\tilde{\Theta} - \Theta^*\|_{\text{F}}^2 + \|\tilde{\Gamma} - \Gamma^*\|_{\text{F}}^2], \quad (37)$$

where the infimum ranges over all estimators $(\tilde{\Theta}, \tilde{\Gamma})$ that are (measurable) functions of the data Y , and the supremum ranges over all pairs $(\Theta^*, \Gamma^*) \in \mathcal{F}$. Here the expectation is taken over the Gaussian noise matrix W , under the linear observation model (1).

Given a matrix Γ^* , we define its support set $\text{supp}(\Gamma^*) := \{(j, k) \mid \Gamma_{jk}^* \neq 0\}$, as well as its column support $\text{colsupp}(\Gamma^*) := \{k \mid \Gamma_k^* \neq 0\}$, where Γ_k^* denotes the k^{th} column. Using this notation, our interest centers on the following two matrix families:

$$\mathcal{F}_{\text{sp}}(r, s, \alpha) := \left\{ (\Theta^*, \Gamma^*) \mid \text{rank}(\Theta^*) \leq r, |\text{supp}(\Gamma^*)| \leq s, \|\Theta^*\|_{\infty} \leq \frac{\alpha}{\sqrt{d_1 d_2}} \right\}, \quad \text{and} \quad (38a)$$

$$\mathcal{F}_{\text{col}}(r, s, \alpha) := \left\{ (\Theta^*, \Gamma^*) \mid \text{rank}(\Theta^*) \leq r, |\text{colsupp}(\Gamma^*)| \leq s, \|\Theta^*\|_{2, \infty} \leq \frac{\alpha}{\sqrt{d_2}} \right\}. \quad (38b)$$

By construction, Corollaries 2 and 5 apply to the families \mathcal{F}_{sp} and \mathcal{F}_{col} respectively.

The following theorem establishes lower bounds on the minimax risks (in squared Frobenius norm) over these two families:

Theorem 2. *There is a universal constant $c_0 > 0$ such that for all $\alpha \geq 32\sqrt{\log(d_1 d_2)}$, we have*

$$\mathfrak{M}(\mathcal{F}_{\text{sp}}(r, s, \alpha)) \geq c_0 \nu^2 \left\{ \frac{r(d_1 + d_2)}{d_1 d_2} + \frac{s \log(\frac{d_1 d_2 - s}{s/2})}{d_1 d_2} \right\} + c_0 \frac{\alpha^2 s}{d_1 d_2}, \quad (39)$$

and

$$\mathfrak{M}(\mathcal{F}_{\text{col}}(r, s, \alpha)) \geq c_0 \nu^2 \left(\frac{r(d_1 + d_2)}{d_1 d_2} + \frac{s}{d_2} + \frac{s \log(\frac{d_2 - s}{s/2})}{d_1 d_2} \right) + c_0 \frac{\alpha^2 s}{d_2}. \quad (40)$$

Note the agreement with the achievable rates guaranteed in Corollaries 2 and 5 respectively. (As discussed in the remarks following these corollaries, the sharpened forms of the logarithmic factors follow by a more careful analysis.) Theorem 2 shows that in terms of squared Frobenius error, the convex relaxations (10) and (14) are minimax optimal up to constant factors.

4 Simulation results

We have implemented the estimators (10) and (14). Doing so involves solving a convex program, and we have adapted simple first-order methods due to Nesterov [22] for this purpose. In this section, we report simulation results that demonstrate the excellent agreement between our theoretical predictions and the behavior in practice. In all cases, we used square matrices ($d = d_1 = d_2$), and a stochastic noise matrix W with i.i.d. $N(0, \frac{\nu^2}{d^2})$ entries, with $\nu^2 = 1$. For any given rank r , we generated Θ^* by randomly choosing the spaces of left and right singular vectors. We formed random sparse (elementwise or columnwise) matrices by choosing the positions of the non-zeros (entries or columns) uniformly at random.

Recall the estimator (10) from Example 4. It is based on a combination of the nuclear norm with the elementwise ℓ_1 -norm, and is motivated problem of recovering a low-rank matrix Θ^* corrupted by an arbitrary sparse matrix Γ^* . In our first set of experiments, we fixed the matrix dimension $d = 100$, and then studied a range of ranks r for Θ^* , as well as a range of sparsity indices s for Γ^* . More specifically, we studied linear scalings of the form $r = \gamma d$ for a constant $\gamma \in (0, 1)$, and $s = \beta d^2$ for a second constant $\beta \in (0, 1)$.

Note that under this scaling, Corollary 2 predicts that the squared Frobenius error should be upper bounded as $c_1 \gamma + c_2 \beta \log(1/\beta)$, for some universal constants c_1, c_2 . Figure 1(a) provides experimental confirmation of the accuracy of these theoretical predictions: varying γ (with β fixed) produces linear growth of the squared error as a function of γ . In Figure 1(b), we study the complementary scaling, with the rank ratio γ fixed and the sparsity ratio β varying in the interval $[.01, .1]$. Since $\beta \log(1/\beta) \approx \Theta(\beta)$ over this interval, we should expect to see roughly linear scaling. Again, the plot shows good agreement with the theoretical predictions.

Now recall the estimator (14) from Example 5, designed for estimating a low-rank matrix plus a columnwise sparse matrix. We have observed similar linear dependence on the analogs of the parameters γ and β , as predicted by our theory. In the interests of exhibiting a different phenomenon, here we report its performance for matrices of varying dimension, in all cases with Γ^* having $s = 3r$ non-zero columns. Figure 2(a) shows plots of squared Frobenius error versus the dimension for two choices of the rank ($r = 10$ and $r = 15$), and the matrix dimension varying in the range $d \in \{100 : 25 : 300\}$. As predicted by our theory, these plots decrease at the rate $1/d$. Indeed, this scaling is revealed by replotting the inverse squared error versus d , which produces the roughly linear plots shown in panel (b). Moreover, by

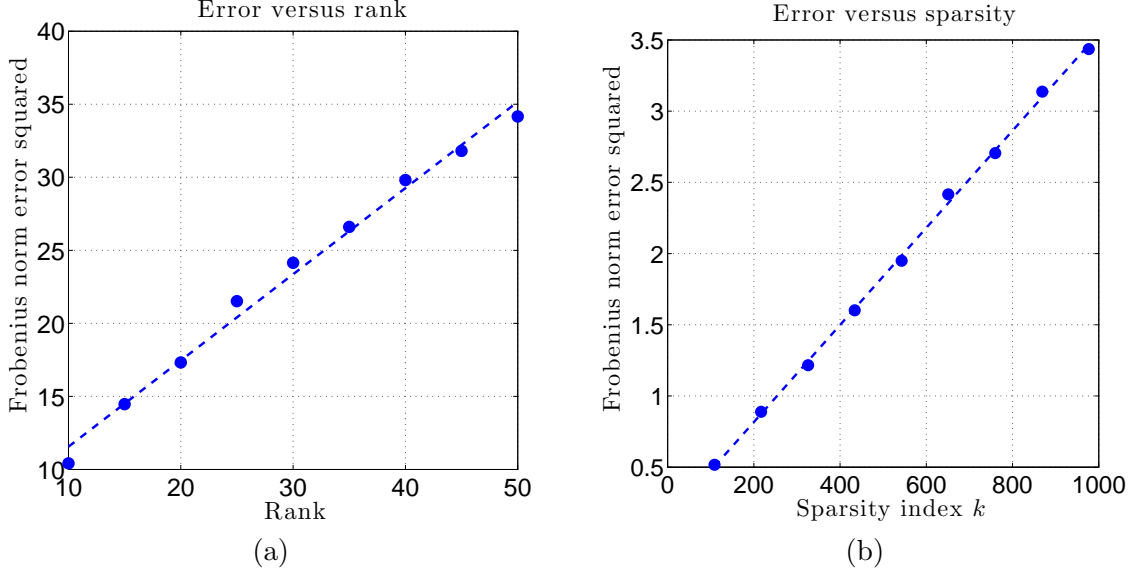


Figure 1. Behavior of the estimator (10). (a) Plot of the squared Frobenius error $e^2(\hat{\Theta}, \hat{\Gamma})$ versus the rank ratio $\gamma \in \{0.05 : 0.05 : 0.50\}$, for matrices of size 100×100 and $s = 2171$ corrupted entries. The growth of the squared error is linear in γ , as predicted by the theory. (b) Plot of the squared Frobenius error $e^2(\hat{\Theta}, \hat{\Gamma})$ versus the sparsity parameter $\beta \in [0.01, 0.1]$ for matrices of size 100×100 and rank $r = 10$. Consistent with the theory, the squared error scales approximately linearly in β in a neighborhood around zero.

comparing the relative slopes of these two curves, we see that the problem with rank $r = 15$ requires roughly a dimension that is roughly $\frac{3}{2}$ larger than the problem with $r = 10$ to achieve the same error. Again, this linear scaling in rank is consistent with Corollary 5.

5 Proofs

In this section, we provide the proofs of our main results, with the proofs of some more technical lemmas deferred to the appendices.

5.1 Proof of Theorem 1

For the reader's reference, we recall here the two assumptions on the regularization parameters:

$$\mu_d \geq 4\mathcal{R}^*(W) + \frac{4\alpha}{\kappa_d} > 0, \quad \text{and} \quad (41a)$$

$$\lambda_d \geq 4\|W\|_{\text{op}}. \quad (41b)$$

Furthermore, so as to simplify notation, let us define the error matrices $\hat{\Delta}^\Theta := \hat{\Theta} - \Theta^*$ and $\hat{\Delta}^\Gamma := \hat{\Gamma} - \Gamma^*$. Let $(\mathcal{M}, \mathcal{M}^\perp)$ denote an arbitrary subspace pair for which the regularizer \mathcal{R} is decomposable. Throughout these proofs, we adopt the convenient shorthand notation $\hat{\Delta}_{\mathcal{M}}^\Gamma := \Pi_{\mathcal{M}}(\hat{\Delta}^\Gamma)$ and $\hat{\Delta}_{\mathcal{M}^\perp}^\Gamma = \Pi_{\mathcal{M}^\perp}(\hat{\Delta}^\Gamma)$, with similar definitions for $\Gamma_{\mathcal{M}}^*$ and $\Gamma_{\mathcal{M}^\perp}^*$.

Lemma 1 (ICB). *Under condition (41a) on μ_d , then for any \mathcal{R} -decomposable pair $(\mathcal{M}, \mathcal{M}^\perp)$, we have*

$$\mathcal{R}(\hat{\Delta}_{\mathcal{M}^\perp}^\Gamma) \leq 3\mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) + 4\mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*). \quad (42)$$

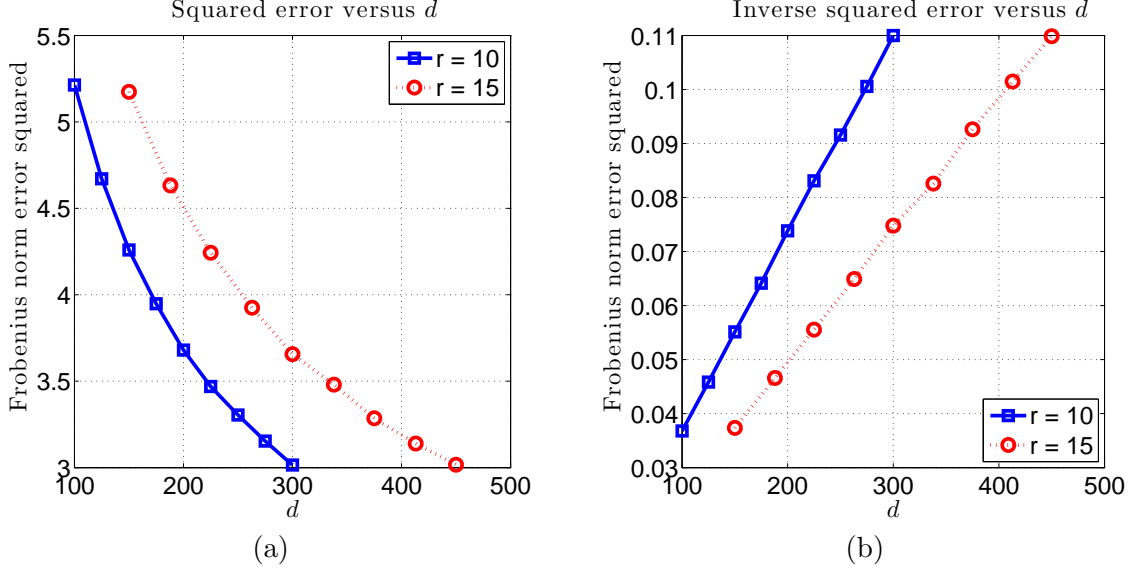


Figure 2. Behavior of the estimator (14). (a) Plot of the squared Frobenius error $e^2(\hat{\Theta}, \hat{\Gamma})$ versus the dimension $d \in \{100 : 25 : 300\}$, for two different choices of the rank ($r = 10$ and $r = 15$). (b) Plot of the *inverse* squared Frobenius error versus the dimension, confirming the linear scaling in d predicted by theory. In addition, the curve for $r = 15$ requires a matrix dimension that is $3/2$ times larger to reach the same error as the curve for $r = 10$, consistent with theory.

We now turn to a lemma that deals with the behavior of the error matrices $(\hat{\Delta}^\Theta, \hat{\Delta}^\Gamma)$ when measured together using a weighted sum of the nuclear norm and regularizer \mathcal{R} . In order to state the following lemma, let us define, for any positive (μ_d, λ_d) , the weighted norm

$$\Phi(\Theta, \Gamma) := \|\Theta\|_N + \frac{\mu_d}{\lambda_d} \mathcal{R}(\Gamma). \quad (43)$$

With this notation, we have the following:

Lemma 2. For any $r = 1, 2, \dots, d$, there is a decomposition $\hat{\Delta}^\Theta = \hat{\Delta}_A^\Theta + \hat{\Delta}_B^\Theta$ such that:

(a) The matrix satisfies the constraints $\text{rank}(\hat{\Delta}_A^\Theta) \leq 2r$, and

$$(\hat{\Delta}_A^\Theta)^T \hat{\Delta}_B^\Theta = (\hat{\Delta}_B^\Theta)^T \hat{\Delta}_A^\Theta = 0.$$

(b) We have the inequality

$$\Phi(\Theta^*, \Gamma^*) - \Phi(\Theta^* + \hat{\Delta}^\Theta, \Gamma^* + \hat{\Delta}^\Gamma) \leq \Phi(\hat{\Delta}_A^\Theta, \hat{\Delta}_M^\Gamma) + \sum_{j=r+1}^d \sigma_j(\Theta^*) + \frac{\mu_d}{\lambda_d} \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*). \quad (44)$$

(c) Under conditions (41a) and (41b) on μ_d and λ_d respectively, the error matrices $\hat{\Delta}^\Theta$ and $\hat{\Delta}^\Gamma$ satisfy

$$\Phi(\hat{\Delta}_B^\Theta, \hat{\Delta}_{\mathcal{M}^\perp}^\Gamma) \leq 3\Phi(\hat{\Delta}_A^\Theta, \hat{\Delta}_M^\Gamma) + 4\left\{ \sum_{j=r+1}^d \sigma_j(\Theta^*) + \frac{\mu_d}{\lambda_d} \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*) \right\}. \quad (45)$$

for any \mathcal{R} -decomposable pair $(\mathcal{M}, \mathcal{M}^\perp)$.

Our third lemma guarantees that the cost function $\mathcal{L}(\Theta, \Gamma) = \frac{1}{2}\|Y - (\Theta + \Gamma)\|_F^2$ is strongly convex in a restricted set of directions. In particular, if we let $\delta\mathcal{L}(\hat{\Delta}^\Theta, \hat{\Delta}^\Gamma)$ denote the error in the first-order Taylor series expansion around (Θ^*, Γ^*) , then some algebra shows that

$$\delta\mathcal{L}(\hat{\Delta}^\Theta, \hat{\Delta}^\Gamma) = \frac{1}{2}\|\hat{\Delta}^\Theta + \hat{\Delta}^\Gamma\|_F^2.$$

The following lemma shows that (up to a slack term) this Taylor error is lower bounded by the squared Frobenius norm.

Lemma 3 (Restricted strong convexity). *Under condition (41a) on μ_d , the error matrices $\hat{\Delta}^\Theta$ and $\hat{\Delta}^\Gamma$ satisfy*

$$\delta\mathcal{L}(\hat{\Delta}^\Theta, \hat{\Delta}^\Gamma) = \frac{1}{2}\|\hat{\Delta}^\Theta + \hat{\Delta}^\Gamma\|_F^2 \geq \frac{1}{2}(\|\hat{\Delta}^\Theta\|_F^2 + \|\hat{\Delta}^\Gamma\|_F^2) - \frac{\mu_d}{2}\mathcal{R}(\hat{\Delta}^\Gamma). \quad (46)$$

Using these three lemmas, we can now complete the proof of Theorem 1. We start with deriving bounds on the error in recovery of Γ^* , which will then be used to show the result of Theorem 1. Since $(\hat{\Theta}, \Gamma^*)$ is feasible for the problem (7), we can use the optimality of $(\hat{\Theta}, \hat{\Gamma})$ to conclude that

$$\frac{1}{2}\|Y - (\hat{\Theta} + \hat{\Gamma})\|_F^2 + \lambda_d\|\hat{\Theta}\|_N + \mu_d\mathcal{R}(\hat{\Gamma}) \leq \frac{1}{2}\|Y - (\hat{\Theta} + \Gamma^*)\|_F^2 + \lambda_d\|\hat{\Theta}\|_N + \mu_d\mathcal{R}(\Gamma^*).$$

Recalling that $Y = \Theta^* + \Gamma^* + W$, and re-arranging in terms of the errors $\hat{\Delta}^\Theta = \hat{\Theta} - \Theta^*$ and $\hat{\Delta}^\Gamma = \hat{\Gamma} - \Gamma^*$, we obtain

$$\frac{1}{2}\|\hat{\Delta}^\Gamma\|_F^2 - \langle \hat{\Delta}^\Gamma, W \rangle + \langle \hat{\Delta}^\Gamma, \hat{\Delta}^\Theta \rangle + \mu_d\mathcal{R}(\hat{\Gamma}) - \mu_d\mathcal{R}(\Gamma^*) \leq 0.$$

Using triangle inequality and the duality of the pair $\mathcal{R}, \mathcal{R}^*$, we have

$$\frac{1}{2}\|\hat{\Delta}^\Gamma\|_F^2 \leq \mathcal{R}(\hat{\Delta}^\Gamma)\mathcal{R}^*(W) + \mathcal{R}(\hat{\Delta}^\Gamma)\mathcal{R}^*(\hat{\Delta}^\Theta) + \mu_d\mathcal{R}(\hat{\Delta}^\Gamma).$$

We use Lemma 1 and the setting of μ_d to simplify the RHS further

$$\frac{1}{2}\|\hat{\Delta}^\Gamma\|_F^2 \leq \frac{3}{2}\mu_d\mathcal{R}(\hat{\Delta}^\Gamma) \leq \frac{3}{2}\mu_d(4\mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) + 4\mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*)) \leq \frac{3}{2}\mu_d(4\Psi(\mathcal{M})\|\hat{\Delta}^\Gamma\|_F + 4\mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*)),$$

where the last inequality follows from the definition of the model compatibility (18). Solving the quadratic inequality gives the bound

$$\|\hat{\Delta}^\Gamma\|_F \leq 18\Psi(\mathcal{M})\mu_d + 7\sqrt{\mu_d\mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*)}. \quad (47)$$

We now follow similar steps to bound the combined error in estimating Θ^* and Γ^* . Since $(\hat{\Theta}, \hat{\Gamma})$ and (Θ^*, Γ^*) are optimal and feasible, respectively, for the convex program (7), we have

$$\frac{1}{2}\|Y - (\hat{\Theta} + \hat{\Gamma})\|_F^2 + \lambda_d\|\hat{\Theta}\|_N + \mu_d\mathcal{R}(\hat{\Gamma}) \leq \frac{1}{2}\|Y - (\Theta^* + \Gamma^*)\|_F^2 + \lambda_d\|\Theta^*\|_N + \mu_d\mathcal{R}(\Gamma^*).$$

Simplifying as before, we obtain

$$\frac{1}{2} \|\hat{\Delta}^\Theta + \hat{\Delta}^\Gamma\|_F^2 \leq \langle\langle W, \hat{\Delta}^\Theta + \hat{\Delta}^\Gamma \rangle\rangle + \lambda_d \{\Phi(\Theta^* + \hat{\Delta}^\Theta, \Gamma^* + \hat{\Delta}^\Gamma) - \Phi(\Theta^*, \Gamma^*)\},$$

where we have used the definition (43) of the weighted regularizer Φ . Applying the lower bound from Lemma 3 to the left-hand side yields

$$\frac{1}{2} (\|\hat{\Delta}^\Theta\|_F^2 + \|\hat{\Delta}^\Gamma\|_F^2) \leq T_1 + T_2 + \frac{\mu_d}{2} \mathcal{R}(\hat{\Delta}^\Gamma), \quad (48)$$

where $T_1 := \langle\langle W, \hat{\Delta}^\Theta + \hat{\Delta}^\Gamma \rangle\rangle$, and $T_2 := \lambda_d \{\Phi(\Theta^* + \hat{\Delta}^\Theta, \Gamma^* + \hat{\Delta}^\Gamma) - \Phi(\Theta^*, \Gamma^*)\}$. It remains to upper bound T_1 and T_2 . On one hand, the bound (44) from Lemma 2 yields that

$$T_2 \leq \lambda_d \Phi(\hat{\Delta}_A^\Theta, \hat{\Delta}_S^\Gamma) + \lambda_d \sum_{j=r+1}^d \sigma_j(\Theta^*) + \mu_d \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*). \quad (49)$$

Turning to the noise term T_1 , we have

$$\begin{aligned} |T_1| &= |\langle\langle W, \hat{\Delta}^\Theta + \hat{\Delta}^\Gamma \rangle\rangle| \stackrel{(i)}{\leq} |\langle\langle W, \hat{\Delta}^\Theta \rangle\rangle| + |\langle\langle W, \hat{\Delta}^\Gamma \rangle\rangle| \\ &\stackrel{(ii)}{\leq} \|W\|_{\text{op}} \|\hat{\Delta}^\Theta\|_N + \mathcal{R}^*(W) \mathcal{R}(\hat{\Delta}^\Gamma), \end{aligned}$$

where step (i) is based on the triangle inequality, and step (ii) uses the duality between the operator and nuclear norms for the first term, and the duality between the norms \mathcal{R} and \mathcal{R}^* for the second term. Applying conditions (41a) and (41b) on the regularizers, we obtain

$$|T_1| \leq \frac{\lambda_d}{2} \|\hat{\Delta}^\Theta\|_N + \frac{\mu_d}{2} \mathcal{R}(\hat{\Delta}^\Gamma) = \frac{\lambda_d}{2} \Phi(\hat{\Delta}^\Theta, \hat{\Delta}^\Gamma),$$

where we have used the definition (43) of the composite regularizer Φ . By triangle inequality, we have

$$\begin{aligned} |T_1| &\leq \frac{\lambda_d}{2} \{\Phi(\hat{\Delta}_A^\Theta, \hat{\Delta}_{\mathcal{M}}^\Gamma) + \Phi(\hat{\Delta}_B^\Theta, \hat{\Delta}_{\mathcal{M}^\perp}^\Gamma)\} \\ &\stackrel{(i)}{\leq} \Phi(\hat{\Delta}_A^\Theta, \hat{\Delta}_{\mathcal{M}}^\Gamma) + \frac{\lambda_d}{2} \left\{ 3\Phi(\hat{\Delta}_A^\Theta, \hat{\Delta}_{\mathcal{M}}^\Gamma) + 4 \sum_{j=r+1}^d \sigma_j(\Theta^*) + \frac{4\mu_d}{\lambda_d} \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*) \right\}, \\ &= 2\lambda_d \Phi(\hat{\Delta}_A^\Theta, \hat{\Delta}_{\mathcal{M}}^\Gamma) + 2\lambda_d \sum_{j=r+1}^d \sigma_j(\Theta^*) + 2\mu_d \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*), \end{aligned} \quad (50)$$

where inequality (i) follows from the bound (45) in Lemma 2. Combining the earlier bound (49) with the inequality (50) and substituting into the inequality (48), we obtain

$$\begin{aligned} \frac{1}{2} (\|\hat{\Delta}^\Theta\|_F^2 + \|\hat{\Delta}^\Gamma\|_F^2) &\leq T_1 + T_2 + \frac{\mu_d}{2} \mathcal{R}(\hat{\Delta}^\Gamma) \\ &\leq 3\lambda_d \Phi(\hat{\Delta}_A^\Theta, \hat{\Delta}_{\mathcal{M}}^\Gamma) + 3\lambda_d \sum_{j=r+1}^d \sigma_j(\Theta^*) + 3\mu_d \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*) + \frac{\mu_d}{2} \mathcal{R}(\hat{\Delta}^\Gamma). \end{aligned} \quad (51)$$

From the decomposability of \mathcal{R} and Lemma 1, we have

$$\mathcal{R}(\hat{\Delta}^\Gamma) = \mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) + \mathcal{R}(\hat{\Delta}_{\mathcal{M}^\perp}^\Gamma) \leq 4\{\mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) + \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*)\}.$$

Combining with the inequality (51), we obtain

$$\begin{aligned} \frac{1}{2}(\|\hat{\Delta}^\Theta\|_F^2 + \|\hat{\Delta}^\Gamma\|_F^2) &\leq 3\lambda_d \Phi(\hat{\Delta}_A^\Theta, \hat{\Delta}_S^\Gamma) + 3\lambda_d \sum_{j=r+1}^d \sigma_j(\Theta^*) + 5\mu_d \|\Gamma_{\mathcal{M}^\perp}^*\|_1 + 2\mu_d \|\hat{\Delta}_S^\Gamma\|_1 \\ &= 3\lambda_d \|\hat{\Delta}_A^\Theta\|_N + 5\mu_d \mathcal{R}(\hat{\Delta}_{\mathcal{M}}) + 3\lambda_d \sum_{j=r+1}^d \sigma_j(\Theta^*) + 5\mu_d \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*), \end{aligned} \quad (52)$$

where we have used the definition (43) of the composite regularizer Φ .

From Lemma 2(a), we have $\text{rank}(\hat{\Delta}_A^\Theta) \leq 2r$, and hence

$$\|\hat{\Delta}_A^\Theta\|_N \leq \sqrt{2r} \|\hat{\Delta}_A^\Theta\|_F \leq \sqrt{2r} \|\hat{\Delta}^\Theta\|_F,$$

since $(\hat{\Delta}_A^\Theta)^T \hat{\Delta}_B^\Theta = 0$. On the other hand, by the definition of the subspace compatibility (18), we have

$$\mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) \leq \Psi(\mathcal{M}) \|\hat{\Delta}_S^\Gamma\|_F \leq \Psi(\mathcal{M}) \|\hat{\Delta}^\Gamma\|_F.$$

Substituting these inequalities into the upper bound (52) yields that $\frac{1}{2}(\|\hat{\Delta}^\Theta\|_F^2 + \|\hat{\Delta}^\Gamma\|_F^2)$ is at most

$$3\lambda_d(\sqrt{2r} \|\hat{\Delta}^\Theta\|_F) + 5\mu_d \Psi(\mathcal{M}) \|\hat{\Delta}\|_F + 3\lambda_d \sum_{j=r+1}^d \sigma_j(\Theta^*) + 5\mu_d \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*).$$

This inequality along with equation (47) implies that

$$\|\hat{\Delta}^\Theta\|_F^2 + \|\hat{\Delta}^\Gamma\|_F^2 \leq c \max \left\{ \lambda_d^2 r + \lambda_d \sum_{j=r+1}^d \sigma_j(\Theta^*), \mu_d^2 \Psi^2(\mathcal{M}) + \mu_d \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*) \right\},$$

for a universal constant c , as claimed in Theorem 1.

5.2 Proof of Corollary 2

Recall that this corollary is based on the specific choices $\mathcal{R}(\cdot) = \|\cdot\|_1$ and $\mathcal{R}^*(\cdot) = \|\cdot\|_\infty$. In order to prove this result, we need to verify that the stated choice of (λ_d, μ_d) satisfies the requirement (23) given in Corollary 1. By known results on the singular values of Gaussian random matrices [10], for the given Gaussian random matrix, we have

$$\mathbb{P} \left[\|W\|_{\text{op}} \geq 4\nu \left\{ \frac{1}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2}} \right\} \right] \leq 2 \exp(-c(d_1 + d_2)).$$

Consequently, setting $\lambda_d \geq 16\nu \left\{ \frac{1}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2}} \right\}$ ensures that the requirement (20) is satisfied. As for the associated requirement for μ_d , it suffices to upper bound the elementwise ℓ_∞ norm of W . Since the entries of W are i.i.d. and Gaussian with parameter $\nu/\sqrt{d_1 d_2}$, the standard Gaussian tail bound combined with union bound yields

$$\mathbb{P}[\|W\|_\infty \geq 4 \frac{\nu}{\sqrt{d_1 d_2}} \log(d_1 d_2)] \leq \exp(-\log d_1 d_2),$$

from which the statement of Corollary 2 follows.

In order to obtain the sharper result (in which $\log(d_1 d_2)$ is replaced by $\log(\frac{d_1 d_2}{s})$), we need to be more careful in upper bounding the noise term $\langle W, \hat{\Delta}^\Gamma \rangle$. We refer the reader to Appendix D.1 for details of this argument.

5.3 Proof of Corollary 3

For this model, the noise matrix is recentered Wishart noise—namely, $W = \frac{1}{n} \sum_{i=1}^n Z_i Z_i^T - \Sigma$, where each $Z_i \sim N(0, \Sigma)$. Letting $U_i \sim N(0, I_{d \times d})$ be i.i.d. Gaussian random vectors, we have

$$\begin{aligned} \|W\|_{\text{op}} &= \|\sqrt{\Sigma} \left(\frac{1}{n} \sum_{i=1}^n U_i U_i^T - I_{d \times d} \right) \sqrt{\Sigma}\|_{\text{op}} \\ &\leq \|\Sigma\|_{\text{op}} \left\| \frac{1}{n} \sum_{i=1}^n U_i U_i^T - I_{d \times d} \right\|_{\text{op}} \\ &\leq 4\|\Sigma\|_{\text{op}} \sqrt{\frac{d}{n}}, \end{aligned}$$

where the final bound holds with probability greater than $1 - 2\exp(-c_1 d)$, using standard tail bounds on Gaussian random matrices [10]. Thus, we see that the specified choice (30) of λ_d is valid for Theorem 1 with high probability.

We now turn to the choice of μ_d . The entries of W are products of Gaussian variables, and hence have sub-exponential tails (e.g., [4]). Therefore, for any entry (i, j) , we have the tail bound $\mathbb{P}[|W_{ij}| > \rho(\Sigma)t] \leq 2\exp(-nt^2/20)$, valid for all $t \in (0, 1]$. By union bound over all d^2 entries, we conclude that

$$\mathbb{P}[\|W\|_{\infty} \geq 8\rho(\Sigma)\sqrt{\frac{\log d}{n}}] \leq 2\exp(-c_2 \log d),$$

which shows that the specified choice of μ_d is also valid with high probability.

5.4 Proof of Corollary 5

For this corollary, we have $\mathcal{R}(\cdot) = \|\cdot\|_{2,1}$ and $\mathcal{R}^*(\cdot) = \|\cdot\|_{2,\infty}$. In order to establish the claim, we need to show that the conditions of Corollary 4 on the regularization pair (λ_d, μ_d) hold with high probability. The setting of λ_d is the same as Corollary 2, and is valid by our earlier argument. Hence, in order to complete the proof, it remains to establish an upper bound on $\|W\|_{2,\infty}$.

Let W_k be the k^{th} column of the matrix. Noting that the function $W_k \mapsto \|W_k\|_2$ is Lipschitz, by concentration of measure for Gaussian Lipschitz functions [16], we have

$$\mathbb{P}\left[\|W_k\|_2 \geq \mathbb{E}\|W_k\|_2 + t\right] \leq \exp\left(-\frac{t^2 d_1 d_2}{\nu^2}\right) \quad \text{for all } t \in [0, 1/4].$$

Using the Gaussianity of W_k , we have $\mathbb{E}\|W_k\|_2 \leq \frac{\nu}{\sqrt{d_1 d_2}} \sqrt{d_1} = \frac{\nu}{\sqrt{d_2}}$. Applying union bound over all d_2 columns, we conclude that with probability greater than $1 - \exp(-\frac{t^2 d_1 d_2}{\nu^2} + \log d_2)$, we have $\max_k \|W_k\|_2 \leq \frac{\nu}{\sqrt{d_2}} + t$. Setting $t = 2\nu \sqrt{\frac{\log d_2}{d_1 d_2}}$ yields

$$\mathbb{P}\left[\|W\|_{2,\infty} \leq \frac{\nu}{\sqrt{d_2}} + 2\nu \sqrt{\frac{\log d_2}{d_1 d_2}}\right] \leq \exp(-3 \log d_2),$$

from which the claim follows.

As before, a sharper bound (with $\log d_2$ replaced by $\log(d_2/s)$) can be obtained by a refined argument; we refer the reader to Appendix D.2 for the details.

5.5 Proof of Corollary 6

For this model, the noise matrix takes the form $W := \frac{1}{n} \sum_{i=1}^n U_i U_i^T - \Theta^*$, where $U_i \sim N(0, \Theta^*)$. The same argument as in the proof of Corollary 3 shows that the specified choice of λ_d is valid with high probability. It remains to bound the quantity $\|W\|_{2,\infty}$. By known matrix norm bounds [13], we have $\|W\|_{2,\infty} \leq \|W\|_{\text{op}}$, so that the claim follows by the previous argument.

5.6 Proof of Theorem 2

Our lower bound proofs are based on a standard reduction [12, 30, 29] from estimation to a multiway hypothesis testing problem over a packing set of matrix pairs. In particular, given a collection $\{(\Theta^j, \Gamma^j), j = 1, 2, \dots, M\}$ of matrix pairs contained in some family \mathcal{F} , we say that it forms a δ -packing in Frobenius norm if, for all distinct pairs $i, j \in \{1, 2, \dots, M\}$, we have

$$\|\Theta^i - \Theta^j\|_F^2 + \|\Gamma^i - \Gamma^j\|_F^2 \geq \delta^2.$$

Given such a packing set, it is a straightforward consequence of Fano's inequality that the minimax error over \mathcal{F} satisfies the lower bound

$$\mathbb{P}[\mathfrak{M}(\mathcal{F}) \geq \frac{\delta^2}{8}] \geq 1 - \frac{I(Y; J) + \log 2}{\log M}, \quad (53)$$

where $I(Y; J)$ is the mutual information between the observation matrix $Y \in \mathbb{R}^{d_1 \times d_2}$, and J is an index uniformly distributed over $\{1, 2, \dots, M\}$. In order to obtain different components of our bound, we make different choices of the packing set, and use different bounding techniques for the mutual information.

5.6.1 Lower bounds for elementwise sparsity

We begin by proving the lower bound (39) for matrix decompositions over the family $\mathcal{F}_{\text{sp}}(r, s, \alpha)$.

Packing for radius of non-identifiability Let us first establish the lower bound involving the radius of non-identifiability, namely the term scaling as $\frac{\alpha^2 s}{d_1 d_2}$ in the case of s -sparsity for Θ^* . Recall from Example 6 the “bad” matrix (27), which we denote here by B^* . By construction, we have $\|B^*\|_F^2 = \frac{\alpha^2 s}{d_1 d_2}$. Using this matrix, we construct a very simple packing set with $M = 4$ matrix pairs (Θ, Γ) :

$$\{(B^*, -B^*), (-B^*, B^*), (\frac{1}{\sqrt{2}}B^*, -\frac{1}{\sqrt{2}}B^*), (0, 0)\} \quad (54)$$

Each one of these matrix pairs (Θ, Γ) belongs to the set $\mathcal{F}_{\text{sp}}(1, s, \alpha)$, so it can be used to establish a lower bound over this set. (Moreover, it also yields a lower bound over the sets $\mathcal{F}_{\text{sp}}(r, s, \alpha)$ for $r > 1$, since they are supersets.) It can also be verified that for any two distinct pairs of matrices in the set (54), they differ in squared Frobenius norm by at least $\delta^2 = \frac{1}{2}\|B^*\|_F^2 = \frac{1}{2}\frac{\alpha^2 s}{d_1 d_2}$. Let J be a random index uniformly distributed over the four possible models in our packing set (54). By construction, for any matrix pair (Θ, Γ) in the packing set, we have $\Theta + \Gamma = 0$. Consequently, for any one of these models, the observation matrix Y is simply equal to pure noise W , and hence $I(Y; J) = 0$. Putting together the pieces, the Fano bound (53) implies that

$$\mathbb{P}[\mathfrak{M}(\mathcal{F}_{\text{sp}}(1, s, \alpha)) \geq \frac{1}{16} \frac{\alpha^2 s}{d_1 d_2}] \geq 1 - \frac{I(Y; J) + \log 2}{\log 4} = \frac{1}{2}.$$

Packing for estimation error: We now describe the construction of a packing set for lower bounding the estimation error. In this case, our construction is more subtle, based on the Cartesian product of two components, one for the low rank matrices, and the other for the sparse matrices. For the low rank component, we re-state a slightly modified form (adapted to the setting of non-square matrices) of Lemma 2 from the paper [20]:

Lemma 4. *For $d_1, d_2 \geq 10$, a tolerance $\delta > 0$, and for each $r = 1, 2, \dots, d$, there exists a set of $d_1 \times d_2$ -dimensional matrices $\{\Theta^1, \dots, \Theta^M\}$ with cardinality $M \geq \frac{1}{4} \exp\left(\frac{rd_1}{256} + \frac{rd_2}{256}\right)$ such that each matrix has rank r , and moreover*

$$\|\Theta^\ell\|_F^2 = \delta^2 \quad \text{for all } \ell = 1, 2, \dots, M, \quad (55a)$$

$$\|\Theta^\ell - \Theta^k\|_F^2 \geq \delta^2 \quad \text{for all } \ell \neq k, \quad (55b)$$

$$\|\Theta^\ell\|_\infty \leq \delta \sqrt{\frac{32 \log(d_1 d_2)}{d_1 d_2}} \quad \text{for all } \ell = 1, 2, \dots, M. \quad (55c)$$

Consequently, as long as $\delta \leq 1$, we are guaranteed that the matrices Θ^ℓ belong to the set $\mathcal{F}_{\text{sp}}(r, s, \alpha)$ for all $\alpha \geq 32\sqrt{\log(d_1 d_2)}$.

As for the sparse matrices, the following result is a modification, so as to apply to the matrix setting of interest here, of Lemma 5 from the paper [24]:

Lemma 5 (Sparse matrix packing). *For any $\delta > 0$, and for each integer $s < d_1 d_2$, there exists a set of matrices $\{\Gamma^1, \dots, \Gamma^N\}$ with cardinality $N \geq \exp\left(\frac{s}{2} \log \frac{d_1 d_2 - s}{s/2}\right)$ such that*

$$\|\Gamma^j - \Gamma^k\|_F^2 \geq \delta^2, \quad \text{and} \quad (56a)$$

$$\|\Gamma^j\|_F^2 \leq 8\delta^2, \quad (56b)$$

and such that each Γ^j has at most s non-zero entries.

We now have the necessary ingredients to prove the lower bound (39). By combining Lemmas 4 and 5, we conclude that there exists a set of matrices with cardinality

$$M N \geq \frac{1}{4} \exp\left\{\frac{s}{2} \log \frac{d_1 d_2 - s}{s/2} + \frac{rd_1}{256} + \frac{rd_2}{256}\right\} \quad (57)$$

such that

$$\|(\Theta^\ell, \Gamma^k) - (\Theta^{\ell'}, \Gamma^{k'})\|_F^2 \geq \delta^2 \quad \text{for all pairs such that } \ell \neq \ell' \text{ or } k \neq k', \text{ and} \quad (58a)$$

$$\|(\Theta^\ell, \Gamma^k)\|_F^2 \leq 9\delta^2 \quad \text{for all } (\ell, k). \quad (58b)$$

Let $\mathbb{P}^{\ell, k}$ denote the distribution of the observation matrix Y when Θ^ℓ and Γ^k are the underlying parameters. We apply the Fano construction over the class of MN such distributions, thereby obtaining that in order to show that the minimax error is lower bounded by $c_0 \delta^2$ (for some universal constant $c_0 > 0$), it suffices to show that

$$\frac{\frac{1}{\binom{MN}{2}} \sum_{(\ell, k) \neq (\ell', k')} \mathbb{D}(\mathbb{P}^{\ell, k} \parallel \mathbb{P}^{\ell', k'}) + \log 2}{\log(MN)} \leq \frac{1}{2}, \quad (59)$$

where $\mathbb{D}(\mathbb{P}^{\ell,k} \parallel \mathbb{P}^{\ell',k'})$ denotes the Kullback-Leibler divergence between the distributions $\mathbb{P}^{\ell,k}$ and $\mathbb{P}^{\ell',k'}$. Given the assumption of Gaussian noise with variance $\nu^2/(d_1 d_2)$, we have

$$\mathbb{D}(\mathbb{P}^j \parallel \mathbb{P}^k) = \frac{d_1 d_2}{2\nu^2} \left\| (\Theta^\ell, \Gamma^k) - (\Theta^{\ell'}, \Gamma^{k'}) \right\|_F^2 \stackrel{(i)}{\leq} \frac{18d_1 d_2 \delta^2}{\nu^2}, \quad (60)$$

where the bound (i) follows from the condition (58b). Combined with lower bound (57), we see that it suffices to choose δ such that

$$\frac{\frac{18d_1 d_2 \delta^2}{\nu^2} + \log 2}{\log \frac{1}{4} + \left\{ \frac{s}{2} \log \frac{d_1 d_2 - s}{s/2} + \frac{r d_1}{256} + \frac{r d_2}{256} \right\}} \leq \frac{1}{2}.$$

For d_1, d_2 larger than a finite constant (to exclude degenerate cases), we see that the choice

$$\delta^2 = c_0 \nu^2 \left\{ \frac{r}{d_1} + \frac{r}{d_2} + \frac{s \log \frac{d_1 d_2 - s}{s/2}}{d_1 d_2} \right\},$$

for a suitably small constant $c_0 > 0$ is sufficient, thereby establishing the lower bound (39).

5.6.2 Lower bounds for columnwise sparsity

The lower bound (40) for columnwise follows from a similar argument. The only modifications are in the packing sets.

Packing for radius of non-identifiability In order to establish a lower bound of order $\frac{\alpha^2 s}{d_2}$, recall the “bad” matrix (34) from Example 7, which we denote by B^* . By construction, it has squared Frobenius norm $\|B^*\|_F^2 = \frac{\alpha^2 s}{d_2}$. We use it to form the packing set

$$\{(B^*, -B^*), (-B^*, B^*), (\frac{1}{\sqrt{2}}B^*, -\frac{1}{\sqrt{2}}B^*), (0, 0)\} \quad (61)$$

Each one of these matrix pairs (Θ, Γ) belongs to the set $\mathcal{F}_{\text{col}}(1, s, \alpha)$, so it can be used to establish a lower bound over this set. (Moreover, it also yields a lower bound over the sets $\mathcal{F}_{\text{col}}(r, s, \alpha)$ for $r > 1$, since they are supersets.) It can also be verified that for any two distinct pairs of matrices in the set (61), they differ in squared Frobenius norm by at least $\delta^2 = \frac{1}{2}\|B^*\|_F^2 = \frac{1}{2}\frac{\alpha^2 s}{d_2}$. Consequently, the same argument as before shows that

$$\mathbb{P}[\mathfrak{M}(\mathcal{F}_{\text{col}}(1, s, \alpha)) \geq \frac{1}{16} \frac{\alpha^2 s}{d_2}] \geq 1 - \frac{I(Y; J) + \log 2}{\log 4} = \frac{1}{2}.$$

Packing for estimation error: We now describe packings for the estimation error terms. For the low-rank packing set, we need to ensure that the $(2, \infty)$ -norm is controlled. From the bound (55c), we have the guarantee

$$\|\Theta^\ell\|_{2,\infty} \leq \delta \sqrt{\frac{32 \log(d_1 d_2)}{d_2}} \quad \text{for all } \ell = 1, 2, \dots, M, \quad (62)$$

so that, as long as $\delta \leq 1$, the matrices Θ^ℓ belong to the set $\mathcal{F}_{\text{col}}(r, s, \alpha)$ for all $\alpha \geq 32\sqrt{\log(d_1 d_2)}$.

The following lemma characterizes a suitable packing set for the columnwise sparse component:

Lemma 6 (Columnwise sparse matrix packing). *For all $d_2 \geq 10$ and integers s in the set $\{1, 2, \dots, d_2 - 1\}$, there exists a family $d_1 \times d_2$ matrices $\{\Gamma^k, k = 1, 2, \dots, N\}$ with cardinality*

$$N \geq \exp\left(\frac{s}{8} \log \frac{d_2 - s}{s/2} + \frac{sd_1}{8}\right),$$

satisfying the inequalities

$$\|\Gamma^j - \Gamma^k\|_F^2 \geq \delta^2, \quad \text{for all } j \neq k, \text{ and} \quad (63a)$$

$$\|\Gamma^j\|_F^2 \leq 64\delta^2, \quad (63b)$$

and such that each Γ^j has at most s non-zero columns.

This claim follows by suitably adapting Lemma 5(b) in the paper by Raskutti et al. [23] on minimax rates for kernel classes. In particular, we view column j of a matrix Γ as defining a linear function in dimension \mathbb{R}^{d_1} ; for each $j = 1, 2, \dots, d_1$, this defines a Hilbert space \mathcal{H}_j of functions. By known results on metric entropy of Euclidean balls [17], this function class has logarithmic metric entropy, so that part (b) of the above lemma applies, and yields the stated result.

Using this lemma and the packing set for the low-rank component and following through the Fano construction yields the claimed lower bound (39) on the minimax error for the class $\mathcal{F}_{\text{col}}(r, s, \alpha)$, which completes the proof of Theorem 2.

6 Discussion

In this paper, we analyzed a class of convex relaxations for solving a general class of matrix decomposition problems, in which the goal is recover a pair of matrices, based on observing a noisy contaminated version of their sum. Since the problem is ill-posed in general, it is essential to impose structure, and this paper focuses on the setting in which one matrix is approximately low-rank, and the second has a complementary form of low-dimensional structure enforced by a decomposable regularizer. Particular cases include matrices that are elementwise sparse, or columnwise sparse, and the associated matrix decomposition problems have various applications, including robust PCA, robustness in collaborative filtering, and model selection in Gauss-Markov random fields. We provided a general non-asymptotic bound on the Frobenius error of a convex relaxation based on a regularizing the least-squares loss with a combination of the nuclear norm with a decomposable regularizer. When specialized to the case of elementwise and columnwise sparsity, these estimators yield rates that are minimax-optimal up to constant factors.

Various extensions of this work are possible. We have not discussed here how our estimator would behave under a partial observation model, in which only a fraction of the entries are observed. This problem is very closely related to matrix completion, a problem for which recent work by Negahban and Wainwright [20] shows that a form of restricted strong convexity holds with high probability. This property could be adapted to the current setting, and would allow for proving Frobenius norm error bounds on the low rank component. Finally, although this paper has focused on the case in which the first matrix component is approximately low rank, much of our theory could be applied to a more general class of matrix decomposition problems, in which the first component is penalized by a decomposable regularizer that is “complementary” to the second matrix component. It remains to explore the properties and applications of these different forms of matrix decomposition.

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A Proof of Lemma 1

By the standard conditions for optimality over a convex set [3], we are guaranteed that

$$\left\langle \left[\begin{array}{c} \hat{\Delta}^\Theta + \hat{\Delta}^\Gamma - W \\ \hat{\Delta}^\Theta + \hat{\Delta}^\Gamma - W \end{array} \right] + \left[\begin{array}{c} \lambda_d \partial \|\hat{\Theta}\|_N \\ \mu_d \partial \mathcal{R}(\hat{\Gamma}) \end{array} \right], \left[\begin{array}{c} \Theta - \hat{\Theta} \\ \Gamma - \hat{\Gamma} \end{array} \right] \right\rangle \geq 0.$$

for all feasible pairs (Θ, Γ) . In particular, since $(\hat{\Theta}, \Gamma^*)$ is feasible, we obtain

$$\langle \hat{\Delta}^\Theta + \hat{\Delta}^\Gamma - W + \mu_d \partial \mathcal{R}(\hat{\Gamma}), \Gamma^* - \hat{\Gamma} \rangle \geq 0.$$

Re-arranging yields the inequality

$$\begin{aligned} \mu_d \langle \partial \mathcal{R}(\hat{\Gamma}), \hat{\Gamma} - \Gamma^* \rangle &\leq -\langle W, \hat{\Delta}^\Gamma \rangle - \|\hat{\Delta}^\Gamma\|_F^2 - \langle \hat{\Delta}^\Theta, \hat{\Delta}^\Gamma \rangle \\ &\leq |\langle W, \hat{\Delta}^\Gamma \rangle| + |\langle \hat{\Delta}^\Theta, \hat{\Delta}^\Gamma \rangle|. \end{aligned}$$

By convexity of the regularizer and definition of the subdifferential, we have

$$\mathcal{R}(\Gamma^*) \geq \mathcal{R}(\hat{\Gamma}) + \langle \partial \mathcal{R}(\hat{\Gamma}), \Gamma^* - \hat{\Gamma} \rangle,$$

so that the left-hand side is lower bounded by $\mu_d \{\mathcal{R}(\hat{\Gamma}) - \mathcal{R}(\Gamma^*)\}$. Noting that $\hat{\Gamma} = \Gamma^* + \hat{\Delta}^\Gamma$, we conclude that

$$\mu_d \{\mathcal{R}(\Gamma^* + \hat{\Delta}^\Gamma) - \mathcal{R}(\Gamma^*)\} \leq |\langle W, \hat{\Delta}^\Gamma \rangle| + |\langle \hat{\Delta}^\Theta, \hat{\Delta}^\Gamma \rangle|. \quad (64)$$

Exploiting duality between the pair $(\mathcal{R}, \mathcal{R}^*)$ yields the upper bound

$$\mu_d \{\mathcal{R}(\Gamma^* + \hat{\Delta}^\Gamma) - \mathcal{R}(\Gamma^*)\} \leq \{\mathcal{R}^*(W) + \mathcal{R}^*(\hat{\Delta}^\Theta)\} \mathcal{R}(\hat{\Delta}^\Gamma).$$

Now since both $\hat{\Theta}$ and Θ^* are feasible, we have $\mathcal{R}^*(\hat{\Delta}^\Theta) \leq \mathcal{R}^*(\hat{\Theta}) + \mathcal{R}^*(\Theta^*) \leq \frac{2\alpha}{\kappa_d}$, and hence

$$\mu_d \{\mathcal{R}(\Gamma^* + \hat{\Delta}^\Gamma) - \mathcal{R}(\Gamma^*)\} \leq \frac{\mu_d}{2} \mathcal{R}(\hat{\Delta}^\Gamma), \quad (65)$$

using our assumed choice of μ_d .

We now lower the left-hand side of equation (65). By decomposability of the regularizer \mathcal{R} , we have

$$\begin{aligned} \mathcal{R}(\Gamma^* + \hat{\Delta}^\Gamma) - \mathcal{R}(\Gamma^*) &= \mathcal{R}(\Gamma_{\mathcal{M}}^* + \hat{\Delta}_{\mathcal{M}}^\Gamma) + \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^* + \hat{\Delta}_{\mathcal{M}^\perp}^\Gamma) - \mathcal{R}(\Gamma_{\mathcal{M}}^*) - \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*) \\ &\geq \{\mathcal{R}(\Gamma_{\mathcal{M}}^*) - \mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma)\} + \{\mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) - \mathcal{R}(\Gamma_{\mathcal{M}}^*)\} - \mathcal{R}(\Gamma_{\mathcal{M}}^*) - \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*) \\ &= \mathcal{R}(\hat{\Delta}_{\mathcal{M}^\perp}^\Gamma) - \mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) - 2\mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*). \end{aligned}$$

Combining this lower bound with the earlier upper bound (65) and canceling out $\mu_d > 0$, we obtain

$$\mathcal{R}(\hat{\Delta}_{\mathcal{M}^\perp}^\Gamma) - \mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) - 2\mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*) \leq \frac{1}{2} \mathcal{R}(\hat{\Delta}^\Gamma) = \frac{1}{2} \mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) + \frac{1}{2} \mathcal{R}(\hat{\Delta}_{\mathcal{M}^\perp}^\Gamma),$$

which implies that $\mathcal{R}(\hat{\Delta}_{\mathcal{M}^\perp}^\Gamma) \leq 3 \mathcal{R}(\hat{\Delta}_{\mathcal{M}}^\Gamma) + 4 \mathcal{R}(\Gamma_{\mathcal{M}^\perp}^*)$, as claimed.

B Proof of Lemma 2

The decomposition described in part (a) was established by Recht et al. [25], so that it remains to prove part (b). With the appropriate definitions, part (b) can be recovered by exploiting Lemma 1 from Negahban et al. [19]. Their lemma applies to optimization problems of the general form

$$\min_{\theta \in \mathbb{R}^p} \{ \mathcal{L}(\theta) + \gamma_n r(\theta) \},$$

where \mathcal{L} is a loss function on the parameter space, and r is norm-based regularizer that satisfies a property known as decomposability. The elementwise ℓ_1 -norm as well as the nuclear norm are both instances of decomposable regularizers. Their lemma requires that the regularization parameter γ_n be chosen such that $\gamma_n \geq 2r^*(\nabla \mathcal{L}(\theta^*))$, where r^* is the dual norm, and $\nabla \mathcal{L}(\theta^*)$ is the gradient of the loss evaluated at the true parameter.

We now discuss how this lemma can be applied in our special case. Here the relevant parameters are of the form $\theta = (\Theta, \Gamma)$, and the loss function is given by

$$\mathcal{L}(\Theta, \Gamma) = \frac{1}{2} \|Y - (\Theta + \Gamma)\|_F^2.$$

The sample size $n = d^2$, since we make one observation for each entry of the matrix. On the other hand, the regularizer is given by the function

$$r(\theta) = \Phi(\Theta, \Gamma) := \|\Theta\|_N + \frac{\mu_d}{\lambda_d} \mathcal{R}(\Gamma),$$

coupled with the regularization parameter $\gamma_n = \lambda_d$. By assumption, the regularizer \mathcal{R} is decomposable, and as shown in the paper [19], the nuclear norm is also decomposable. Since Φ is simply a sum of these decomposable regularizers over separate matrices, it is also decomposable.

It remains to compute the gradient $\nabla \mathcal{L}(\Theta^*, \Gamma^*)$, and evaluate the dual norm. A straightforward calculation yields that $\nabla \mathcal{L}(\Theta^*, \Gamma^*) = [W \ W]^T$. In addition, it can be verified by standard properties of dual norms

$$\Phi^*(U, V) = \|U\|_{\text{op}} + \frac{\lambda_d}{\mu_d} \mathcal{R}^*(V).$$

Thus, it suffices to choose the regularization parameter such that

$$\lambda_d \geq 2\Phi(W, W) = 2\|W\|_{\text{op}} + \frac{2\lambda_d}{\mu_d} \mathcal{R}^*(W).$$

Given our condition (41a), we have

$$2\|W\|_{\text{op}} + \frac{2\lambda_d}{\mu_d} \mathcal{R}^*(W) \leq 2\|W\|_{\text{op}} + \frac{\lambda_d}{2},$$

meaning that it suffices to have $\lambda_d \geq 4\|W\|_{\text{op}}$, as stated in condition (41b).

C Proof of Lemma 3

We begin by observing that

$$\frac{1}{2} (\|\hat{\Delta}^\Theta\|_F^2 + \|\hat{\Delta}^\Gamma\|_F^2) - \frac{1}{2} \|\hat{\Delta}^\Theta + \hat{\Delta}^\Gamma\|_F^2 = -\langle \hat{\Delta}^\Theta, \hat{\Delta}^\Gamma \rangle,$$

so that it suffices to upper bound $|\langle\langle \hat{\Delta}^\Theta, \hat{\Delta}^\Gamma \rangle\rangle|$. By the duality of the pair $(\mathcal{R}, \mathcal{R}^*)$, we have

$$|\langle\langle \hat{\Delta}^\Theta, \hat{\Delta}^\Gamma \rangle\rangle| \leq \mathcal{R}^*(\hat{\Delta}^\Theta) \mathcal{R}(\hat{\Delta}^\Gamma).$$

Now since $\hat{\Theta}$ and Θ^* are both feasible for the program (7), we have

$$\mathcal{R}^*(\hat{\Delta}^\Theta) \leq \mathcal{R}^*(\hat{\Theta}) + \mathcal{R}^*(\Theta^*) \leq \frac{2\alpha}{\kappa_d} \stackrel{(i)}{\leq} \frac{\mu_d}{2},$$

where inequality (i) follows from our choice of μ_d .

D Refinement of achievability results

In this appendix, we provide refined arguments that yield sharpened forms of Corollaries 2 and 5. These refinements yield achievable bounds that match the minimax lower bounds in Theorem 2 up to constant factors. We note that these refinements are significantly different *only when* the sparsity index s scales as $\Theta(d_1 d_2)$ for Corollary 2, or as $\Theta(d_2)$ for Corollary 5.

D.1 Refinement of Corollary 2

In the proof of Theorem 1, when specialized to the ℓ_1 -norm, the noise term $|\langle\langle W, \hat{\Delta}^\Gamma \rangle\rangle|$ is simply upper bounded by $\|W\|_\infty \|\hat{\Delta}^\Gamma\|_1$. Here we use a more careful argument to control this noise term. Throughout the proof, we assume that the regularization parameter λ_d is set in the usual way, whereas we choose

$$\mu_d = 16\nu \sqrt{\frac{\log \frac{d_1 d_2}{s}}{d_1 d_2}} + \frac{4\alpha}{\sqrt{d_1 d_2}}. \quad (66)$$

We split our analysis into two cases.

Case 1: First, suppose that $\|\hat{\Delta}^\Gamma\|_1 \leq \sqrt{s} \|\hat{\Delta}^\Gamma\|_F$. In this case, we have the upper bound

$$|\langle\langle W, \hat{\Delta}^\Gamma \rangle\rangle| \leq \sup_{\substack{\|\Delta\|_1 \leq \sqrt{s} \|\hat{\Delta}^\Gamma\|_F \\ \|\Delta\|_F \leq \|\hat{\Delta}^\Gamma\|_F}} |\langle\langle W, \Delta \rangle\rangle| = \|\hat{\Delta}^\Gamma\|_F \underbrace{\sup_{\substack{\|\Delta\|_1 \leq \sqrt{s} \\ \|\Delta\|_F \leq 1}} |\langle\langle W, \Delta \rangle\rangle|}_{Z(s)}$$

It remains to upper bound the random variable $Z(s)$. Viewed as a function of W , it is a Lipschitz function with parameter $\frac{\nu}{\sqrt{d_1 d_2}}$, so that

$$\mathbb{P}[Z(s) \geq \mathbb{E}[Z(s)] + \delta] \leq \exp\left(-\frac{d_1 d_2 \delta^2}{2\nu^2}\right).$$

Setting $\delta^2 = \frac{4s\nu^2}{d_1 d_2} \log\left(\frac{d_1 d_2}{s}\right)$, we have

$$Z(s) \leq \mathbb{E}[Z(s)] + \frac{2s\nu}{d_1 d_2} \left[\log\left(\frac{d_1 d_2}{s}\right) \right]$$

with probability greater than $1 - \exp\left(-2s \log\left(\frac{d_1 d_2}{s}\right)\right)$.

It remains to upper bound the expected value. In order to do so, we apply Theorem 5.1(ii) from Gordon et al. [11] with $(q_0, q_1) = (1, 2)$, $n = d_1 d_2$ and $t = \sqrt{s}$, thereby obtaining

$$\mathbb{E}[Z(t)] \leq c' \frac{\nu}{\sqrt{d_1 d_2}} \sqrt{s} \sqrt{2 + \log\left(\frac{2d_1 d_2}{s}\right)} \leq c \frac{\nu}{\sqrt{d_1 d_2}} \sqrt{s \log\left(\frac{d_1 d_2}{s}\right)}.$$

With this bound, proceeding through the remainder of the proof yields the claimed rate.

Case 2: Alternatively, we must have $\|\hat{\Delta}^\Gamma\|_1 > \sqrt{s} \|\hat{\Delta}^\Gamma\|_F$. In this case, we need to show that the stated choice (66) of μ_d satisfies $\mu_d \|\hat{\Delta}^\Gamma\|_1 \geq 2|\langle W, \hat{\Delta}^\Gamma \rangle|$ with high probability. As can be seen from examining the proofs, this condition is sufficient to ensure that Lemma 1, Lemma 2 and Lemma 3 all hold, as required for our analysis.

We have the upper bound

$$|\langle W, \hat{\Delta}^\Gamma \rangle| \leq \sup_{\substack{\|\Delta\|_1 \leq \|\hat{\Delta}^\Gamma\|_1 \\ \|\Delta\|_F \leq \|\hat{\Delta}^\Gamma\|_F}} |\langle W, \Delta \rangle| = \|\hat{\Delta}^\Gamma\|_F Z\left(\frac{\|\hat{\Delta}^\Gamma\|_1}{\|\hat{\Delta}^\Gamma\|_F}\right),$$

where for any radius $t > 0$, we define the random variable

$$Z(t) := \sup_{\substack{\|\Delta\|_1 \leq t \\ \|\Delta\|_F \leq 1}} |\langle W, \Delta \rangle|.$$

For each fixed t , the same argument as before shows that $Z(t)$ is concentrated around its expectation, and Theorem 5.1(ii) from Gordon et al. [11] with $(q_0, q_1) = (1, 2)$, $n = d_1 d_2$ yields

$$\mathbb{E}[Z(t)] \leq c \frac{\nu}{\sqrt{d_1 d_2}} t \sqrt{\log\left(\frac{d_1 d_2}{t^2}\right)}.$$

Setting $\delta^2 = \frac{4t^2\nu^2}{d_1 d_2} \log\left(\frac{d_1 d_2}{s}\right)$ in the concentration bound, we conclude that

$$Z(t) \leq c' t \frac{\nu}{\sqrt{d_1 d_2}} \left\{ \sqrt{\log\left(\frac{d_1 d_2}{s}\right)} + \sqrt{\log\left(\frac{d_1 d_2}{t^2}\right)} \right\}.$$

with high probability. A standard peeling argument (e.g., [27]) can be used to extend this bound to a uniform one over the choice of radii t , so that it applies to the random one $t = \frac{\|\hat{\Delta}^\Gamma\|_1}{\|\hat{\Delta}^\Gamma\|_F}$ of interest. (The only changes in doing such a peeling are in constant terms.) We thus conclude that

$$Z\left(\frac{\|\hat{\Delta}^\Gamma\|_1}{\|\hat{\Delta}^\Gamma\|_F}\right) \leq c' \frac{\|\hat{\Delta}^\Gamma\|_1}{\|\hat{\Delta}^\Gamma\|_F} \frac{\nu}{\sqrt{d_1 d_2}} \left\{ \sqrt{\log\left(\frac{d_1 d_2}{s}\right)} + \sqrt{\log\left(\frac{d_1 d_2}{\|\hat{\Delta}^\Gamma\|_1^2 / \|\hat{\Delta}^\Gamma\|_F^2}\right)} \right\}$$

with high probability. Since $\|\hat{\Delta}^\Gamma\|_1 > \sqrt{s} \|\hat{\Delta}^\Gamma\|_F$, we have $\frac{1}{\|\hat{\Delta}^\Gamma\|_1^2 / \|\hat{\Delta}^\Gamma\|_F^2} \leq \frac{1}{s}$, and hence

$$|\langle W, \hat{\Delta}^\Gamma \rangle| \leq \|\hat{\Delta}^\Gamma\|_F Z\left(\frac{\|\hat{\Delta}^\Gamma\|_1}{\|\hat{\Delta}^\Gamma\|_F}\right) \leq c'' \|\hat{\Delta}^\Gamma\|_1 \frac{\nu}{\sqrt{d_1 d_2}} \sqrt{\log\left(\frac{d_1 d_2}{s}\right)}$$

with high probability. With this bound, the remainder of the proof proceeds as before. In particular, the refined choice (66) of μ_d is adequate.

D.2 Refinement of Corollary 5

As in the refinement of Corollary 2 from Appendix D.1, we need to be more careful in controlling the noise term $\langle\langle W, \hat{\Delta}^\Gamma \rangle\rangle$. For this corollary, we make the refined choice of regularizer

$$\mu_d = 16\nu\sqrt{\frac{1}{d_2}} + 16\nu\sqrt{\frac{\log(d_2/s)}{d_1d_2}} + \frac{4\alpha}{\sqrt{d_2}} \quad (67)$$

As in Appendix D.1, we split our analysis into two cases.

Case 1: First, suppose that $\|\hat{\Delta}^\Gamma\|_{2,1} \leq \sqrt{s}\|\hat{\Delta}^\Gamma\|_F$. In this case, we have

$$|\langle\langle W, \hat{\Delta}^\Gamma \rangle\rangle| \leq \sup_{\substack{\|\Delta\|_{2,1} \leq \sqrt{s}\|\hat{\Delta}^\Gamma\|_F \\ \|\Delta\|_F \leq \|\hat{\Delta}^\Gamma\|_F}} |\langle\langle W, \Delta \rangle\rangle| = \|\hat{\Delta}^\Gamma\|_F \underbrace{\sup_{\substack{\|\Delta\|_{2,1} \leq \sqrt{s} \\ \|\Delta\|_F \leq 1}} |\langle\langle W, \Delta \rangle\rangle|}_{\tilde{Z}(s)}$$

The function $W \mapsto \tilde{Z}(s)$ is a Lipschitz function with parameter $\frac{\nu}{\sqrt{d_1d_2}}$, so that by concentration of measure for Gaussian Lipschitz functions [16], it satisfies the upper tail bound $\mathbb{P}[\tilde{Z}(s) \geq \mathbb{E}[\tilde{Z}(s)] + \delta] \leq \exp(-\frac{d_1d_2\delta^2}{2\nu^2})$. Setting $\delta^2 = \frac{4s\nu^2}{d_1d_2} \log(\frac{d_2}{s})$ yields

$$\tilde{Z}(s) \leq \mathbb{E}[\tilde{Z}(s)] + 2\nu\sqrt{\frac{s \log(\frac{d_2}{s})}{d_1d_2}} \quad (68)$$

with probability greater than $1 - \exp(-2s \log(\frac{d_2}{s}))$.

It remains to upper bound the expectation. Applying the Cauchy-Schwarz inequality to each column, we have

$$\begin{aligned} \mathbb{E}[\tilde{Z}(s)] &\leq \mathbb{E}\left[\sup_{\substack{\|\Delta\|_{2,1} \leq \sqrt{s} \\ \|\Delta\|_F \leq 1}} \sum_{k=1}^{d_2} \|W_k\|_2 \|\Delta_k\|_2\right] \\ &= \mathbb{E}\left[\sup_{\substack{\|\Delta\|_{2,1} \leq \sqrt{s} \\ \|\Delta\|_F \leq 1}} \sum_{k=1}^{d_2} (\|W_k\|_2 - \mathbb{E}[\|W_k\|_2]) \|\Delta_k\|_2\right] + \sup_{\|\Delta\|_{2,1} \leq \sqrt{s}} \left(\sum_{k=1}^d \|\Delta_k\|_2\right) \mathbb{E}[\|W_1\|_2] \\ &\leq \mathbb{E}\left[\sup_{\substack{\|\Delta\|_{2,1} \leq \sqrt{s} \\ \|\Delta\|_F \leq 1}} \sum_{k=1}^{d_2} \underbrace{(\|W_k\|_2 - \mathbb{E}[\|W_k\|_2])}_{V_k} \|\Delta_k\|_2\right] + 4\nu\sqrt{\frac{s}{d_2}}, \end{aligned}$$

using the fact that $\mathbb{E}[\|W_1\|_2] \leq \nu\sqrt{\frac{d_1}{d_2d_2}} = \frac{\nu}{\sqrt{d_2}}$.

Now the variable V_k is zero-mean, and sub-Gaussian with parameter $\frac{\nu}{\sqrt{d_1d_2}}$, again using concentration of measure for Lipschitz functions of Gaussians [16]. Consequently, by setting $\delta_k = \|\Delta_k\|_2$, we can write

$$\mathbb{E}[\tilde{Z}(s)] \leq \mathbb{E}\left[\sup_{\substack{\|\delta\|_1 \leq 4\sqrt{s} \\ \|\delta\|_2 \leq 1}} \sum_{k=1}^{d_2} V_k \delta_k\right] + 4\nu\sqrt{\frac{s}{d_2}},$$

Applying Theorem 5.1(ii) from Gordon et al. [11] with $(q_0, q_1) = (1, 2)$, $n = d_2$ and $t = 4\sqrt{s}$ then yields

$$\mathbb{E}[\tilde{Z}(s)] \leq c \frac{\nu}{\sqrt{d_1 d_2}} \sqrt{s} \sqrt{2 + \log\left(\frac{2d_2}{16s}\right)} + 4\nu \sqrt{\frac{s}{d_2}},$$

which combined with the concentration bound (68) yields the refined claim.

Case 2: Alternatively, we may assume that $\|\hat{\Delta}^\Gamma\|_{2,1} > \sqrt{s}\|\hat{\Delta}^\Gamma\|_F$. In this case, we need to verify that the choice (67) μ_d satisfies $\mu_d\|\hat{\Delta}^\Gamma\|_{2,1} \geq 2|\langle W, \hat{\Delta}^\Gamma \rangle|$ with high probability. We have the upper bound

$$|\langle W, \hat{\Delta}^\Gamma \rangle| \leq \sup_{\substack{\|\Delta\|_{2,1} \leq \|\hat{\Delta}^\Gamma\|_{2,1} \\ \|\Delta\|_F \leq \|\hat{\Delta}^\Gamma\|_F}} |\langle W, \Delta \rangle| = \|\hat{\Delta}^\Gamma\|_F \tilde{Z}\left(\frac{\|\hat{\Delta}^\Gamma\|_{2,1}}{\|\hat{\Delta}^\Gamma\|_F}\right),$$

where for any radius $t > 0$, we define the random variable

$$\tilde{Z}(t) := \sup_{\substack{\|\Delta\|_{2,1} \leq t \\ \|\Delta\|_F \leq 1}} |\langle W, \Delta \rangle|.$$

Following through the same argument as in Case 2 of Appendix D.1 yields that for any fixed $t > 0$, we have

$$\tilde{Z}(t) \leq c \frac{\nu}{\sqrt{d_1 d_2}} t \sqrt{2 + \log\left(\frac{2d_2}{t^2}\right)} + 4\nu \frac{t}{\sqrt{d_2}} + 2\nu t \sqrt{\frac{\log(\frac{d_2}{s})}{d_1 d_2}}$$

with high probability. As before, this can be extended to a uniform bound over t by a peeling argument, and we conclude that

$$\begin{aligned} |\langle W, \hat{\Delta}^\Gamma \rangle| &\leq \|\hat{\Delta}^\Gamma\|_F \tilde{Z}\left(\frac{\|\hat{\Delta}^\Gamma\|_{2,1}}{\|\hat{\Delta}^\Gamma\|_F}\right) \\ &\leq c\|\hat{\Delta}^\Gamma\|_{2,1} \left\{ \frac{\nu}{\sqrt{d_1 d_2}} \sqrt{2 + \log\left(\frac{2d_2}{\|\hat{\Delta}^\Gamma\|_{2,1}^2 / \|\hat{\Delta}^\Gamma\|_F^2}\right)} + 4\nu \frac{1}{\sqrt{d_2}} + 2\nu \sqrt{\frac{\log(\frac{d_2}{s})}{d_1 d_2}} \right\} \end{aligned}$$

with high probability. Since $\frac{1}{\|\hat{\Delta}^\Gamma\|_{2,1}^2 / \|\hat{\Delta}^\Gamma\|_F^2} \leq \frac{1}{s}$ by assumption, the claim follows.

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