Computational Physics HW 7 Alek Hutson

1. $\nabla \cdot \vec{u} = 0$; $\dot{\vec{u}} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \frac{1}{Re} \nabla^2 \vec{u}$

$\vec{u} = (u_x, u_y, 0)$; $u(y=1) = u(y=-1) = 0$

w/ periodic BC's in x-direction

a) $p = p_0 - \alpha x$; $u_x = \alpha Re \frac{(1-y^2)}{2}$; $u_y = 0$

This is a steady state solution. Therefore,

$\nabla \cdot \vec{u} = 0$; $(\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \frac{1}{Re} \nabla^2 \vec{u}$

$$\nabla \cdot \vec{u} = \frac{\partial}{\partial x} u_x + \frac{\partial}{\partial y} u_x = \frac{\partial}{\partial x}\left[\frac{\alpha Re (1-y^2)}{2}\right] + \cancel{\frac{\partial}{\partial y}[0]} = 0$$

$$\Rightarrow \cancel{\frac{\alpha Re}{2}} \frac{\partial}{\partial x}(1-y^2) = 0 \Rightarrow \underbrace{1 - y^2 = 0}$$

This is only true for $y = \pm 1$

Problem 1 continued...

b) $\dot{\vec{u}} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\nabla p + \frac{1}{Re}\nabla^2\vec{u}$

$$\nabla\times[\dot{\vec{u}} + (\vec{u}\cdot\vec{\nabla})\vec{u}] = \nabla\times[-\nabla p + \frac{1}{Re}\nabla^2\vec{u}]$$

$$= \frac{d}{dt}\underbrace{\nabla\times\vec{u}}_{\vec{w}} + (\vec{u}\cdot\vec{\nabla})\underbrace{\nabla\times\vec{u}}_{\vec{w}} = -\underbrace{\nabla\times\nabla p}_{0} + \frac{1}{Re}\nabla\times\nabla^2\vec{u}$$

$$\Rightarrow \dot{\vec{w}} + (\vec{u}\cdot\vec{\nabla})\vec{w} = \frac{1}{Re}\nabla\times(\nabla\cdot\nabla)\vec{u}$$

$$= \frac{1}{Re}\underbrace{(\nabla\cdot\nabla)}_{\nabla^2}\underbrace{\nabla\times\vec{u}}_{\vec{w}}$$

$$\therefore\ \dot{\vec{w}} + (\vec{u}\cdot\vec{\nabla})\vec{w} = \frac{1}{Re}\nabla^2\vec{w}$$

Problem 1 continued...

c) No, I don't believe we can solve this using methods described in class. This is because the boundary conditions on the vorticity are not defined.

Perhaps this issue could be resolved by guessing the vorticity at the boundary and then updating using an Euler-like approach.

Problem 2

a) $A[n(r)] = \int \frac{n(r')n(r'')}{|r'-r''|} dr'dr''$

$$\frac{\delta A[n(r)]}{\delta n(r)} = \lim_{\varepsilon \to 0} \frac{A[n(r)+\varepsilon\delta(r-r')] - A[n(r)]}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\int \frac{[n(r')+\varepsilon\delta(r'-r)][n(r'')+\varepsilon\delta(r''-r)]}{|r'-r''|} dr'dr'' - \int \frac{n(r')n(r'')}{|r'-r''|} dr'dr''}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\cancel{\varepsilon}} \int dr'dr'' \frac{\cancel{n(r')n(r'')} + n(r')\cancel{\varepsilon}\delta(r''-r) + \cancel{\varepsilon}\delta(r'-r)n(r'') + \varepsilon^{\cancel{2}}\delta(r'-r)\delta(r''-r) - \cancel{n(r')n(r'')}}{|r'-r''|}$$

$$= \cancel{\lim_{\varepsilon \to 0}} \int \frac{n(r')\delta(r''-r) + \delta(r'-r)n(r'') + \cancel{\varepsilon\delta(r'-r)\delta(r''-r)}}{|r'-r''|} dr'dr''$$

$$= \int \left[\frac{n(r')}{|r'-r|} + \frac{n(r'')}{|r-r''|}\right] dr'dr''$$

$$= \int 2\frac{n(r')}{|r-r'|} dr' = \boxed{2\int \frac{n(r')}{|r-r'|} dr' = \frac{\delta A[n(r)]}{\delta n(r)}}$$

Problem 2 continued...

b) $A[n(r)] = \int |\nabla n(r')|^2 dr'$

$$\frac{\delta A[n(r)]}{\delta n(r)} = \lim_{\varepsilon \to 0} \frac{A[n(r) + \varepsilon \delta(r-r')] - A[n(r)]}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\int \left[|\nabla[n(r') + \varepsilon \delta(r'-r)]|^2 - |\nabla n(r)|^2 \right] dr'}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\int \left[|\nabla^2 [\cancel{n(r')^2} + 2n(r')\cancel{\varepsilon}\delta(r'-r) + \varepsilon^{\cancel{2}} \delta(r'-r)^2]| - \cancel{|\nabla n(r)|^2} \right] dr}{\cancel{\varepsilon}}$$

$$= \lim_{\cancel{\varepsilon \to 0}} \int \left[|\nabla^2 (2n(r')\delta(r'-r) + \cancel{\varepsilon \delta(r'-r)^2})| \right] dr'$$

$$= \int |\nabla^2 (2n(r')\delta(r'-r))| dr'$$

$$= \boxed{2 \int |\nabla^2 n(r')| dr' = \frac{\delta A[n(r)]}{n(r)}}$$