Visualization and (MultiQubit) Gates

Pagonis Alexandros

QSilver (QGreece)

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Overview

1. State representation
   - Global and Local Phase
   - A simple expression
2. Bloch's Sphere
   - Definition
   - Rotation Gates on Bloch's Sphere
   - Phase Gates in Bloch's Sphere
   - U gate
3. MultiQubit Gates
   - Tensor Products

Yesterday

$e^{i\phi} = \cos(\phi) + i\sin(\phi)$	Euler's Formula
$z = a + ib = re^{i\phi}$	Complex Numbers
$\lvert z\rvert^2 = z^* z = r^2$	Length of Complex Number
$\lvert\psi\rangle = z_1\lvert 0\rangle + z_2\lvert 1\rangle$	General state 1
$\lvert\psi\rangle = r_1 e^{i\phi_1}\lvert 0\rangle + r_2 e^{i\phi_2}\lvert 1\rangle$	General state 2
$\langle\alpha\lvert\beta\rangle = \sum_i \alpha_i^* \beta_i$	Inner product
$A^{-1} = A^\dagger \equiv \overline{A^T}$	Unitary matrices
$A = A^\dagger$	(Hermitian) Matrix for Observables

State representation

A state is usually represented as

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

But it can also be represented as

$$|\psi\rangle = e^{i\delta} \left(\cos(\frac{\theta}{2}) |0\rangle + e^{i\phi} \sin(\frac{\theta}{2}) |1\rangle \right)$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi, \delta < 2\pi$

And it's normalized since: $\cos^2(x) + \sin^2(x) = 1$ and $|e^{ix}| = 1$.
Global phase: $e^{i\delta}$ / Local phase: $e^{i\phi}$

Measuring Phases

$$|\psi\rangle = e^{i\delta}\left(\cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle\right)$$

Try and measure states $|0\rangle$ and $|1\rangle$:

$$Pr(0) = \cancel{|e^{i\delta}|^2}|\cos(\frac{\theta}{2})|^2 = \cos^2(\frac{\theta}{2})$$

$$Pr(1) = \cancel{|e^{i\delta}|^2}\cancel{|e^{i\phi}|^2}|\sin(\frac{\theta}{2})|^2 = \sin^2(\frac{\theta}{2})$$

So, phases don't matter?...

Phases do matter

They do matter. Example. Consider these two states:

$$|\psi\rangle = \cos(\frac{\theta}{2})\,|0\rangle + e^{i0}\sin(\frac{\theta}{2})\,|1\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$$

$$|\phi\rangle = \cos(\frac{\theta}{2})\,|0\rangle + e^{i\pi}\sin(\frac{\theta}{2})\,|1\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle$$

If I apply H gate onto them:

$$H\,|\psi\rangle = |0\rangle$$

$$H\,|\phi\rangle = |1\rangle$$

Conversions

Both representations are equivalent

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = e^{i\delta}\left(\cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle\right)$$

We can easily find θ: $\frac{\theta}{2} = \arcsin(\sqrt{|\beta|^2}) = \arcsin(|\beta|)$ with $0 \leq \theta \leq \pi \implies 0 \leq \arcsin(|\beta|) \leq \frac{\pi}{2} \implies \boxed{\theta = 2\arcsin(|\beta|)}$

$\phi =$?? Well... a bit more complicated! Just forget it!

How many variables do we need?

The most general state is: $|\psi\rangle = \alpha e^{i\chi}|0\rangle + \beta e^{i\phi}|1\rangle$ with $\alpha, \beta, \chi, \phi \in R^{+}$.
It seems we need 4 variables... But, wait:

$$\alpha^2 + \beta^2 = 1 \implies \boxed{\beta = +\sqrt{1-\alpha^2}}$$

So we only need 3, right? Well, look at this: Expectation values (which is all we have) in QM are of the form $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$ with $A = A^{\dagger}$.
Thus, we find:

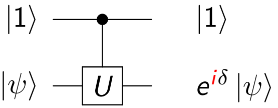
$$\langle A\rangle \overset{(A_{ij}=A^*_{ji})}{=\!=} \alpha^2 A_{00} + \beta^2 A_{11} + 2\alpha\beta Re(A_{10}e^{i(\chi-\phi)})$$

Result: only two real variables are needed!!!!!!!

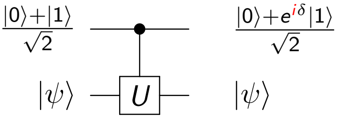
$$|\psi\rangle = \cancel{e^{i\delta}}\left(\cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle\right)$$

Spoiler from QPE: Fetching global phase :)

Global phase will not change the probabilities in a qubit, no matter what gate we apply on it. But suppose that a gate adds a phase to a state. We can fetch it with this trick:

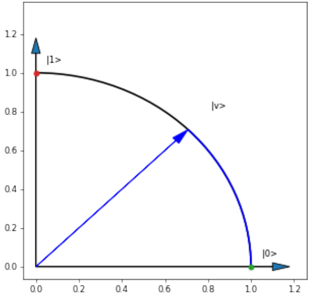


$$\Longrightarrow \; |1\rangle\,(e^{i\delta}|\psi\rangle) = (e^{i\delta}|1\rangle)\,|\psi\rangle$$

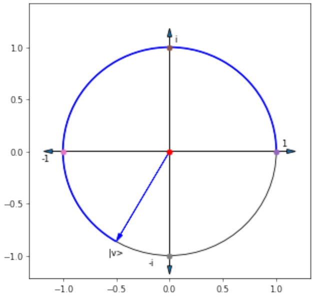


A naive visualization of angles

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



$$\theta = \frac{\pi}{2}$$



$$\phi = \frac{4\pi}{3}$$

Which representation seems suitable?

Our states have:

- 2 degrees of freedom (two real variables)
- Norm (length) $= 1$

What 2-D shape has these properties?

Bloch's Sphere

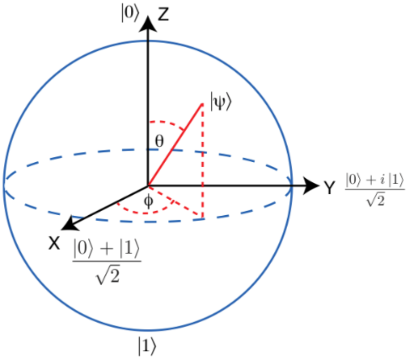
Instead of plotting the two angles separately, we can combine them into one shape.
This shape is 2-D and every point (state) has a length of 1:
So, it's a spherical shell called **Bloch's Sphere**.

$$x = \sin(\theta)\cos(\phi)$$

$$y = \sin(\theta)\sin(\phi)$$

$$z = \cos(\theta)$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



Pauli Gates on Bloch's Sphere

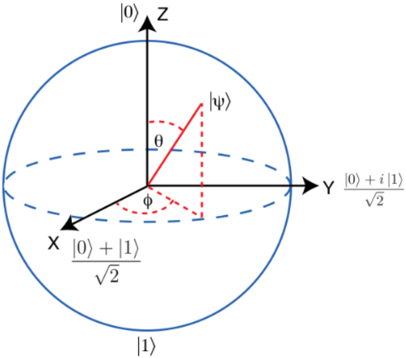
Pauli X,Y,Z gates do a 180° rotation around each axis.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



Rotation Gates on Bloch's Sphere

Rotation gates do θ rotation around each axis:

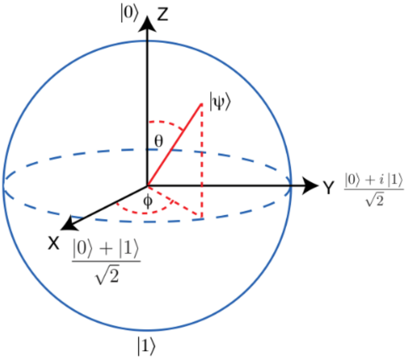
$$R_X(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

$$R_Y(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

$$R_Z(\theta) = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$$

Note:
$R_{\{X,Y,Z\}}(\pi) = -i\{X, Y, Z\}$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



Phase Gates in Bloch's Sphere

We define the square root G of gate F if:

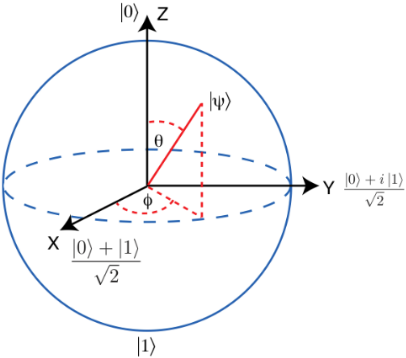
$$G^2 = F \implies G = \sqrt{F}$$

This way we have:

$$S = \sqrt{Z} \implies S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$T = \sqrt{S} \implies T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

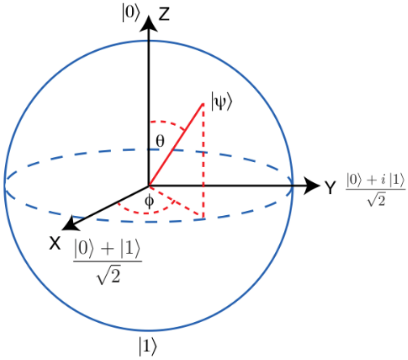


General Phase Gate in Bloch's Sphere

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

The general phase gate is known as a P gate:

$$P(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix}$$



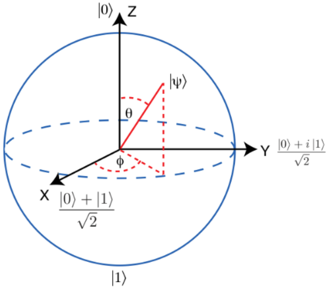
The U gate is the most general single-qubit transformation.

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i\lambda+i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

All other gates can be written as U gates:

$$T = U\left(0, 0, \frac{\pi}{4}\right)$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



Tensor Product

A single state represents two (or many) qubits. The whole state is the tensor product of the two. Tensor product is defined as:

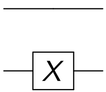
$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

But, sometimes the reverse process can not be made:

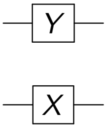
$$\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} \neq \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix}, \qquad \forall a, b, c, d \in C$$

These states are called entangled!

Not entangled qubits

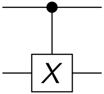


$$I \otimes X = \begin{pmatrix} X & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix}$$

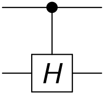


$$Y \otimes X = \begin{pmatrix} y_{11}X & y_{12}X \\ y_{21}X & y_{22}X \end{pmatrix}$$

Entangled qubits - Control gate

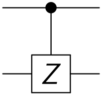


$$I \oplus X = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

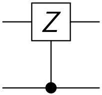


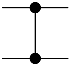
$$I \oplus H = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & H \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Special Example: CZ Gate



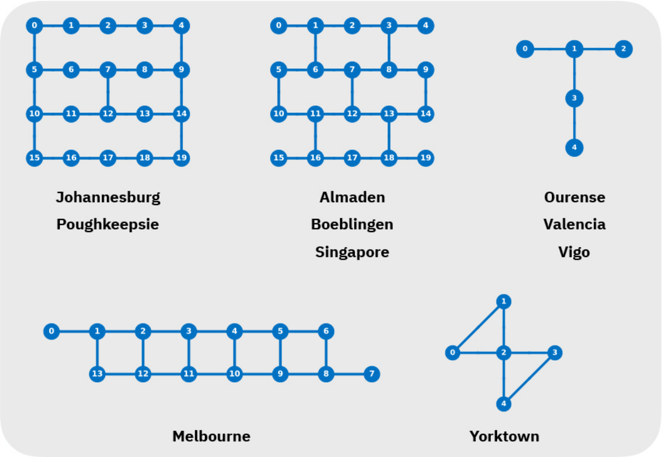
$$I \oplus Z = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



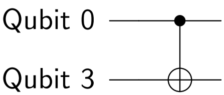


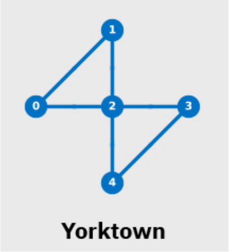
Topologies

IBM's 10 Quantum Device Lineup

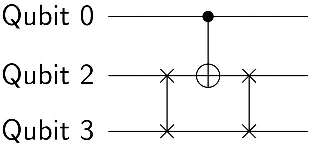


Unconnected qubits





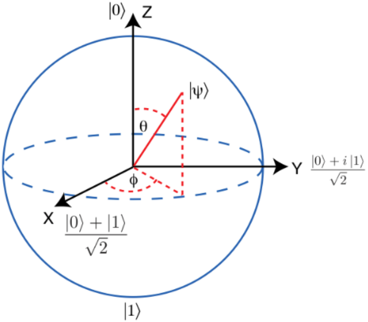
Unconnected qubits 2



$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Summary

- General State Representation:
  $|\psi\rangle = e^{i\delta}\left(\cos(\frac{\theta}{2})\,|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})\,|1\rangle\right)$
- Every Gate is a Rotation on that Sphere.
- Phase is rotation around Z-axis.
- Not-entangled: $X_1 \otimes X_2 \otimes \ldots \otimes X_n$
- Entangled: Control gate $I \oplus X$
- Watch out for the topologies



Questions?

The End