Mathematical Notation of Quantum Computing

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Overview

1. Complex Numbers
   - Definition
   - Representation in a plane
   - Euler's Formula
   - Complex Numbers in Python

2. Dirac's Notation
   - Inner Product
   - Unitary Matrices
   - Observables (Optional)

Complex Numbers

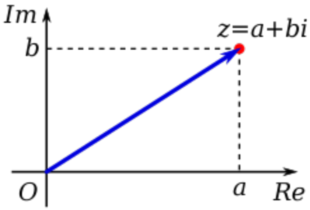
- Number Sets: $N \subset Q \subset R \subset C$
- Eg: $2 - 5 \notin N$
- $\sqrt{-5} \notin R$, but $\sqrt{-5} \in C$
- $\boxed{i = \sqrt{-1}}$
- $\sqrt{-4} = \sqrt{4(-1)} = 2\sqrt{-1} = 2i$
- $\sqrt{-5} = \sqrt{5}i$
- Complex Number: $\boxed{\alpha + \beta i \in C, \quad \alpha, \beta \in R}$

Basic Operations

Assuming $i^2 = -1$ all operations follow naturally:

- $(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$
- $(a + bi)(c + di) = ... = (ac - bd) + (ad + bc)i$
- Define conjugate: $z = a + bi \implies \bar{z} = z^* = a - bi$
- $\frac{a+bi}{c+di} = \frac{(a+bi)\overline{(c+di)}}{(c+di)\overline{(c+di)}} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

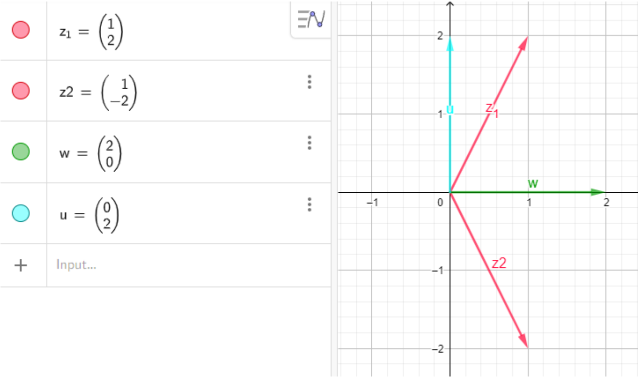
We also define:
$|a + bi| \equiv \sqrt{a^2 + b^2}$.
Note that: $z^* z = |z|^2$



Basic Operations - Examples

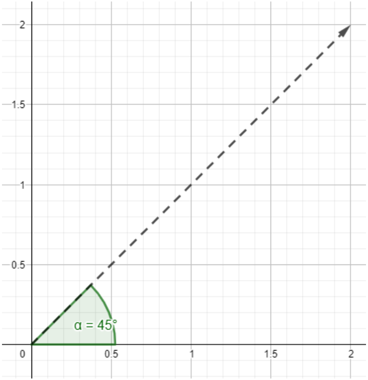
- $(1+2i)+(3+4i)=4+6i$
- $z=1+2i \implies \bar{z}=1-2i$
- $|1+2i|=\sqrt{1^2+2^2}=\sqrt{5}$

Geometric Representation



$$z = a + b{\color{red}i} \implies \vec{v} = (a, b)$$

Polar Form



Length and angle (from Re-axis)

Conversions

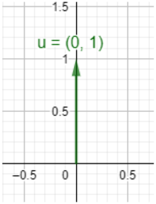
Orthogonal to Polar:

$$r = \sqrt{a^2 + b^2}$$
$$\phi = \arctan(\frac{b}{a}), a > 0$$

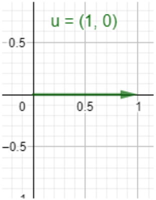
Polar to Orthogonal:

$$a = r\cos(\phi)$$
$$b = r\sin(\phi)$$

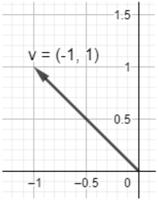
Conversion examples



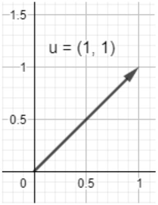
$$z = 0 + i1$$



$$z = 1 + i0$$



$$z = -1 + i1$$



$$z = 1 + i1$$

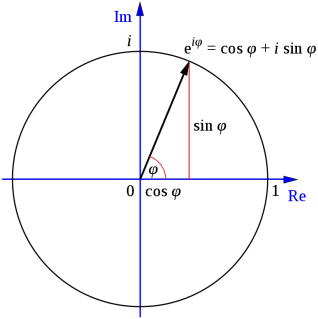
Euler's formula

Theorem (Euler's formula)

$e^{i\phi} = \cos(\phi) + i\sin(\phi)$

Usefull tool indeed:

$$(r_1 e^{ia_1})(r_2 e^{ia_2}) = r_1 r_2 e^{i(a_1+a_2)}$$
$$(re^{ia})^k = r^k e^{i(ka)}$$
$$z = re^{ia} \implies z^* = re^{-ia}$$
$$|z|^2 = z^* z = (re^{-ia})(re^{ia}) = r^2$$



Complex Numbers in Python

```python
z1 = 3+2j
print(z1)
z2 = complex(3,2)
print(z2)
z3 = 5j
print(z3)
z4 = complex(0,5)
print(z4)
```

Output:
(3+2j)
(3+2j)
5j
5j

Basic Complex Operations in Python

```python
z1 = 3+2j
z2 = 4+5j
print('z1=',z1,'z2=',z2)
print('z1+z2=',z1+z2)
print('z1-z2=',z1-z2)
print('z1*z2=',z1*z2)
print('z1/z2=',z1/z2)
```

Output:

z1= (3+2j) z2= (4+5j)
z1+z2= (7+7j)
z1-z2= (-1-3j)
z1*z2= (2+23j)
z1/z2= (0.5365853658536587-0.17073170731707318j)

Complex Operations in Python

```python
z = 3+2j
print(z.real)
print(z.imag)
print(z.conjugate())
print(abs(z))
print(pow(z,3))
```

Output:
3.0
2.0
(3-2j)
3.605551275463989 (-9+46j)

Review of States

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha |0\rangle + \beta |1\rangle$$

As of now you knew that $\alpha, \beta \in R$ and the probabilities for 0 and 1 were:

$$Pr(0) = \alpha^2, \qquad Pr(1) = \beta^2$$

And of course: $Pr(0) + Pr(1) = 1$

But, what if $|\psi\rangle = i\,|0\rangle \implies \boxed{Pr(0) = i^2 = -1}$????

Also: $(\psi, \psi) = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha^2 + \beta^2 = 1$

States with Complex Numbers

$$|\psi\rangle = \begin{pmatrix} \frac{1+i}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1+i}{\sqrt{3}}|0\rangle - \frac{1}{\sqrt{3}}|1\rangle$$

$$|a+bi| = \sqrt{a^2+b^2} \implies |a+bi|^2 = a^2+b^2$$

So the probabilities are:

$$Pr(0) = |\frac{1+i}{\sqrt{3}}|^2 = (\frac{1}{\sqrt{3}})^2 + (\frac{1}{\sqrt{3}})^2 = \frac{2}{3}$$
$$Pr(1) = |\frac{-1}{\sqrt{3}}|^2 = (\frac{-1}{\sqrt{3}})^2 = \frac{1}{3}$$

Of course: $\sum P(x) = 1$

Inner Product Necessity

Suppose we have two vectors: $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$.

Their inner product is:

$$(\vec{v}, \vec{u}) = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \sum_i v_i u_i$$

What if the elements are complex? What's the inner product then?

Constructing Inner Product

$$|\psi\rangle = \begin{pmatrix} \frac{1+i}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1+i}{\sqrt{3}} |0\rangle - \frac{1}{\sqrt{3}} |1\rangle$$

We know that:

$$Pr(0) = \left| \frac{1+i}{\sqrt{3}} \right|^2, \qquad Pr(1) = \left| \frac{-1}{\sqrt{3}} \right|^2 \text{ and } |z|^2 = z^* z.$$

We also know that:

$$(\vec{v}, \vec{v}) = \sum_i v_i v_i = 1 \text{ for } \vec{v}, \vec{u} \in R^n \text{ and } Pr(0) + Pr(1) = 1.$$

How should we change: $(\vec{v}, \vec{u}) = \sum_i v_i u_i$

Dirac's Notation

Let $|\psi\rangle$ denote a vector: $|\psi\rangle = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. This vector is called **ket**.

Now we define the transpose conjugate of this vector (called **bra**):
$\langle\psi| = \begin{pmatrix} c_1^* & \cdots & c_n^* \end{pmatrix}$ or with the dagger ($\dagger$) symbol: $\langle\psi| = |\psi\rangle^\dagger$

Together they form an inner product (called **bra-ket**...):

$$\langle\alpha|\beta\rangle \equiv \langle\alpha| \cdot |\beta\rangle = \begin{pmatrix} \alpha_1^* & \cdots & \alpha_n^* \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \sum_i \alpha_i^* \beta_i$$

What does inner product mean?

But what does $|\langle\psi|\phi\rangle|$ mean?
Hint:

$$\langle 0|0\rangle = \langle 1|1\rangle = 1$$

$$\langle 1|0\rangle = \langle 0|1\rangle = 0$$

Measurement

Given a state $|\psi\rangle$, the number $0 \leq |\langle\psi|\phi\rangle|^2 \leq 1$ shows the probability that state $|\phi\rangle$ will be observed after the measurement.
It is a metric of how "close" two states are.

Norm

Norm is defined as $\| \, |\phi\rangle \, \| = \sqrt{\langle\psi|\psi\rangle} = \sqrt{\sum_i c_i^* c_i} = \sqrt{\sum_i |c_i|^2}$.

In Quantum Computing all states are normalized, so: $\| \, |\phi\rangle \, \| = 1$.

From State to State

One state is useless.
How can I move from state to state? With a matrix!
Any matrix??? What about this one:

$$G = \begin{pmatrix} i & 1 \\ 1 & 1 \end{pmatrix}$$

A non acceptible matrix

$$G|0\rangle = \begin{pmatrix} i & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Well, look at this:

$$\left\| \begin{pmatrix} i \\ 1 \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} -i & 1 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix}} = \sqrt{|i|^2 + |1|^2} = \sqrt{1+1} = \sqrt{2} \neq 1$$

Unitary Matrices

Let's apply a matrix to a state: $\hat{A}\,|\psi\rangle = |\phi\rangle$.
We want the result $|\phi\rangle$ to be a valid quantum state, hence normalized!
That means $\|\,|\phi\rangle\,\| = 1$.
These matrices that leave the norm unchanged are called **Unitary Matrices** (or length preserving matrices) and for them it holds that:
$\overline{A^T} = A^{-1}$
We remind you that: $\boxed{\overline{A^T} \equiv A^{\dagger}}$

They are also reversible: $A^{\dagger}A = I$

Unitary Matrices - Examples

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \implies A^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$$

Unitary Matrices - Examples

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \implies A^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$AA^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Observables - Optional

We can't "see" a state. We can only make certain measurements of a state called Observables. An observable corresponds to a matrix A and all we can predict is its expectation value: $\langle A \rangle = \langle \psi | \hat{A} | \psi \rangle$.
This matrix must have real eigenvalues so it must be Hermitian: $A = A^\dagger$.
Example - Pauli gates:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Summary

$e^{i\phi} = \cos(\phi) + i\sin(\phi)$	Euler's Formula
$z = a + ib = re^{i\phi}$	Complex Numbers
$\|z\|^2 = z^* z = r^2$	Length of Complex Number
$\|\psi\rangle = z_1 \|0\rangle + z_2 \|1\rangle$	General state 1
$\|\psi\rangle = r_1 e^{i\phi_1} \|0\rangle + r_2 e^{i\phi_2} \|1\rangle$	General state 2
$\langle\alpha\|\beta\rangle = \sum_i \alpha_i^* \beta_i$	Inner product
$A^{-1} = A^\dagger \equiv \overline{A^T}$	Unitary matrices
$A = A^\dagger$	(Hermitian) Matrix for Observables

Questions?

The End