CS132 Homework 11

6(1.) $x \cdot w = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 6 \cdot 3 + (-2)(-1) + 3(-5) = 18 + 2 - 15 = 5$

$x \cdot x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = 6 \cdot 6 + (-2)(-2) + 3 \cdot 3 = 36 + 4 + 9 = 49$

$\frac{x \cdot w}{x \cdot x} = \frac{5}{49}$

$\left(\frac{x \cdot w}{x \cdot x}\right) x = \frac{5}{49} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{30}{49} \\ \frac{-10}{49} \\ \frac{15}{49} \end{bmatrix}$

8(1.) $\|x\| = \sqrt{x \cdot x} = \sqrt{6^2 + (-2)^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7$

10(1.) $\|x\| = \sqrt{(-6)^2 + 4^2 + (-3)^2} = \sqrt{36 + 16 + 9} = \sqrt{61}$

$\frac{1}{\|x\|} x = \frac{1}{61} \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{-6}{61} \\ \frac{4}{61} \\ \frac{-3}{61} \end{bmatrix}$

12(1.) $\|x\| = \sqrt{(8/3)^2 + 2^2} = \sqrt{(64/9) + 4} = \sqrt{(100/9)} = \frac{10}{3}$

$\frac{1}{\|x\|} x = \frac{1}{\frac{10}{3}} \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

14(1.)) $u - v = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}$

$\|u - v\| = \sqrt{4^2 + (-4)^2 + (-6)^2} = \sqrt{16 + 16 + 36} = \sqrt{68} = \underline{\underline{2\sqrt{17}}}$

16(1.)) $u \cdot v = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 12 \cdot 2 + 3(-3) + (-5) \cdot 3 = 24 - 9 - 15 = \underline{0}$

They are orthogonal!

18(1.)) $u \cdot v = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = (-3) \cdot 1 + 7 \cdot (-8) + 4 \cdot 15 + 0 \cdot (-7) = -3 - 56 + 60 = \underline{1}$

They are not orthogonal!

20(1.)) a) The dot product minus the dot product yields 0, naturally. TRUE

b) That's not even the same thing! FALSE

c) That's the def of $W^\perp$. TRUE

d) TRUE

e) $(\text{row } A)^\perp = \text{null } A$, TRUE

28(1.))

?

2(2)) $u \cdot v = \begin{bmatrix}1\\-2\\1\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix} = 1\cdot 0 + (-2)\cdot 1 + 1\cdot 2 = -2+2 = \underline{0}$

$u \cdot w = \begin{bmatrix}1\\-2\\1\end{bmatrix}\begin{bmatrix}-5\\-2\\1\end{bmatrix} = 1\cdot(-5) + (-2)(-2) + 1\cdot 1 = -5+4+1 = \underline{0}$

$v \cdot w = \begin{bmatrix}0\\1\\2\end{bmatrix}\begin{bmatrix}-5\\-2\\1\end{bmatrix} = 0(-5) + 1(-2) + 2\cdot 1 = -2+2 = \underline{0}$

They are orthogonal!

4(2.)) $u \cdot v = \begin{bmatrix}2\\-5\\-3\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix} = 2\cdot 0 + (-5)0 + (-3)0 = \underline{0}$

$u \cdot w = \begin{bmatrix}2\\-5\\-3\end{bmatrix}\cdot\begin{bmatrix}4\\-2\\6\end{bmatrix} = 2\cdot 4 + (-5)(-2) + (-3)6 = 8+10-18 = \underline{0}$

$v \cdot w = \begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}4\\-2\\6\end{bmatrix} = 0\cdot 4 + 0\cdot(-2) + 0\cdot 6 = \underline{0}$

They are orthologanal!

6(2)) $u \cdot v = \begin{bmatrix}5\\-4\\0\\3\end{bmatrix}\cdot\begin{bmatrix}-4\\1\\-3\\8\end{bmatrix} = 5\cdot(-4) + (-4)\cdot 1 + 0(-3) + 3\cdot 8 = -20-4+24 = \underline{0}$

$u \cdot w = \begin{bmatrix}5\\-4\\0\\3\end{bmatrix}\cdot\begin{bmatrix}3\\3\\5\\-1\end{bmatrix} = 5\cdot 3 + (-4)\,3 + 0\cdot 5 + 3\,(-1) = 15-12-3 = \underline{0}$

$v \cdot w = \begin{bmatrix}-4\\1\\-3\\8\end{bmatrix}\cdot\begin{bmatrix}3\\3\\5\\-1\end{bmatrix} = (-4)\cdot 3 + 1\cdot 3 + (-3)\cdot 5 + 8\cdot(-1) = -12+3-15-8 = \underline{-32}$

They are not orthologanal!

8(2.)) $u_1 \cdot u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 6 \end{bmatrix} = 3 \cdot (-2) + 1 \cdot 6 = -6 + 6 = 0$

It is orthogonal

$$\begin{bmatrix} 3 & -2 \\ 1 & 6 \end{bmatrix} \cdot -3$$

$$\downarrow$$

$$\begin{bmatrix} 0 & -20 \\ 1 & 6 \end{bmatrix} /-20$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 6 \end{bmatrix} \leftarrow \text{2 pivot columns, so spans } \mathbb{R}^2$$

$$\begin{bmatrix} 3 & -2 & -6 \\ 1 & 6 & 3 \end{bmatrix} \cdot -3$$

$$\downarrow$$

$$\begin{bmatrix} 0 & -20 & -15 \\ 1 & 6 & 3 \end{bmatrix} /-20$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 & 3/4 \\ 1 & 6 & 3 \end{bmatrix}$$

$x_2 = \frac{3}{4}$

$x_1 + 6x_2 = 3$

$x_1 = 3 - \frac{6 \cdot 3}{4} = -\frac{3}{2}$

$$x = -\frac{3}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

10(a))
$$v_1 \cdot v_2 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 3 \cdot 2 + (-3) \cdot 2 + 0 \cdot (-1) = 6 - 6 = \underline{0}$$

$$v_1 \cdot v_3 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 3 \cdot 1 + (-3) \cdot 1 + 0 \cdot 4 = 3 - 3 = \underline{0}$$

$$v_2 \cdot v_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 2 \cdot 1 + 2 \cdot 1 + (-1) \cdot 4 = 2 + 2 - 4 = \underline{0}$$

They are orthogonal!

$$\begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} +1 \to \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & -1 & 4 \end{bmatrix} +4 \to \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 18 \\ 0 & -1 & 4 \end{bmatrix}$$

← Three pivot columns so it spans $\mathbb{R}^3$!

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{bmatrix} +1 \to \begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & 4 & 2 & 2 \\ 0 & -1 & 4 & 1 \end{bmatrix} +4 \to \begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & 0 & 18 & 6 \\ 0 & -1 & 4 & 1 \end{bmatrix} \begin{matrix} /3 \\ /18 \\ /-1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2/3 & 1/3 & 5/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 1 & -4 & -1 \end{bmatrix}$$

$$x_3 = \underline{1/3}$$

$$x_2 - 4x_3 = -1$$

$$x_2 = -1 + 4\frac{1}{3} = \underline{\frac{1}{3}}$$

$$x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 = \frac{5}{3}$$

$$x_1 = -\frac{2}{3} \cdot \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{3} + \frac{5}{3} = \frac{4}{3}$$

$$x = \frac{4}{3}\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

12(2.))

$$\frac{u \cdot v}{v \cdot v} \cdot v$$

$$(u \cdot v) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 1(-1) + (-1)3 = -1 - 3 = -4$$

$$(v \cdot v) = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = (-1)(-1) + 3 \cdot 3 = 1 + 9 = 10$$

$$\frac{u \cdot v}{v \cdot v} = \frac{-4}{10} = \frac{-2}{5}$$

$$\frac{u \cdot v}{v \cdot v} \cdot v = -\frac{2}{5} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix}$$

14(2.))

$$(y \cdot u) = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = 2 \cdot 7 + 6 \cdot 1 = 14 + 6 = 20$$

$$(u \cdot u) = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = 7 \cdot 7 + 1 \cdot 1 = 49 + 1 = 50$$

$$\frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

$$y = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

16(2.)

$y \cdot u = \begin{bmatrix} -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-3) \cdot 1 + 9 \cdot 2 = -3 + 18 = 15$

$u \cdot u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot 1 + 2 \cdot 2 = 1 + 4 = 5$

$\frac{15}{5} u = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\| y - \hat{y} \| = \| y - \frac{y \cdot u}{u \cdot u} \cdot u \| = \left\| \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\| =$

$\left\| \begin{bmatrix} -6 \\ 3 \end{bmatrix} \right\| = \sqrt{(-6)^2 + 3^2} = \sqrt{36 + 9} = \sqrt{45} = \underline{\underline{3\sqrt{5}}}$

24(2.)

a) They have to to be orthogonal... FALSE

b) FALSE

c) TRUE

d) TRUE

e) It's square and linearly independent, so TRUE

28(2.) If U is orthogonal, it also has n orthonormal vectors. These are all linearly independent, so they form basis for $\mathbb{R}^n$.

30(2.) See explanation for 28.

If you interchange some of the columns, they will still be orthogonal because it is not affected by position/movement