CS132 Homework 13
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8) a) Least square fit model: $y = X\beta + \epsilon$

$$X = \begin{bmatrix} x_1 & x_1^2 & x_1^3 \\ x_2 & x_2^2 & x_2^3 \\ \vdots & \vdots & \vdots \\ x_n & x_n^2 & x_n^3 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

b)
$$X = \begin{bmatrix} 4 & 16 & 64 \\ 6 & 36 & 216 \\ 8 & 64 & 512 \\ 10 & 100 & 1000 \\ 12 & 144 & 1728 \\ 14 & 196 & 2744 \\ 16 & 256 & 4096 \\ 18 & 324 & 5832 \end{bmatrix}, \quad y = \begin{bmatrix} 1.58 \\ 2.08 \\ 2.5 \\ 2.8 \\ 3.1 \\ 3.4 \\ 3.8 \\ 4.32 \end{bmatrix}$$

1. Enter the matricies into Python
2. Solve for $\hat{\beta}$ by using formula

$(X^TX)^{-1}X^Ty$ (using np.transpose, np.dot, np.linalg.inv)

3. Get result matrix (success)

$$\begin{bmatrix} 0.5132 \\ -0.03348 \\ 0.001016 \end{bmatrix} \rightarrow y = 0.5132x - 0.03348x^2 + 0.001016x^3$$

(see bottom for curve, from WolframAlpha)

10) a) Model is $y = X\beta + \epsilon$

$$\begin{bmatrix} e^{-0.02(10)} & e^{-0.07(10)} \\ e^{-0.02(11)} & e^{-0.07(11)} \\ e^{-0.02(12)} & e^{-0.07(12)} \\ e^{-0.02(14)} & e^{-0.07(14)} \\ e^{-0.02(15)} & e^{-0.07(10)} \end{bmatrix}, \quad y = \begin{bmatrix} 21.34 \\ 20.68 \\ 20.05 \\ 18.87 \\ 18.30 \end{bmatrix}, \quad \beta = \begin{bmatrix} M_a \\ M_b \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

b)

1. Enter matrices X and y into python
2. Solve for $\hat{\beta}$ using formula

$(X^T X)^{-1} X^T y$ (using np.transpose, np.dot, np.linalg.inv)

3. Get result matrix

$$\begin{bmatrix} 19.94 \\ 10.10 \end{bmatrix} \rightarrow y = 19.94e^{-0.02t} + 10.10e^{-0.07t}$$

12) Model is: $y = X\beta + \epsilon$

$$X = \begin{bmatrix} 1 & 3.78 \\ 1 & 4.11 \\ 1 & 4.41 \\ 1 & 4.73 \\ 1 & 4.88 \end{bmatrix}, \quad y = \begin{bmatrix} 97 \\ 98 \\ 103 \\ 110 \\ 112 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

1. Input matrices X and y into Python
2. Solve for $\hat{\beta}$ using formula

$(X^T X)^{-1} X^T y$ (using np.transpose, np.dot, np.linalg.inv)

3. Got result

$$\begin{bmatrix} 18.56 \\ 19.24 \end{bmatrix} \rightarrow p = 18.56 + 19.24 \ln w$$

when w is 100, p is $\underline{\underline{107.16}}$ mm of mercury

8(7.1)) Checking if orthogonal

$$A^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 + 1/2 & 1/2 - 1/2 \\ 1/2 - 1/2 & 1/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It is the identity matrix, so A is orthogonal QED

$$A^{-1} = A^T = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \quad (\text{because } AA^T = I)$$

10(7.1)) Checking if orthogonal

$$A^T = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

It is not the identity matrix, so A is not orthogonal

12(7.7)) Checking if orthogonal

$$A^T = \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix}^T = \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is the identity matrix, so A is orthogonal <u>QED</u>

$A^{-1} = A^T =$ because $A^T A = I$

14) $\det(A - \lambda I) = (1-\lambda)(1-\lambda) - 25 = 0$

$\lambda^2 - 2\lambda + 1 - 25 = 0$

$\lambda^2 - 2\lambda - 24 = 0$

$\underline{\lambda = 6} \vee \underline{\lambda = -4}$

Eigenvector for $\lambda = 6 \rightarrow A - 6I = 0$

$$\begin{bmatrix} 1-6 & 5 & 0 \\ 5 & 1-6 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 0 \\ 5 & -5 & 0 \end{bmatrix} +1$$

$$\begin{bmatrix} -5 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \rightarrow -5x_1 + 5x_2 = 0 \rightarrow x_1 = x_2 \\ \rightarrow \end{matrix} \Bigg\} x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Eigenvector for $\lambda = -4 \rightarrow A + 4I = 0$

$$\begin{bmatrix} 1+4 & 5 & 0 \\ 5 & 1+4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 0 \\ 5 & 5 & 0 \end{bmatrix} -1$$

$$\begin{bmatrix} 5 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \rightarrow x_1 + x_2 = 0 \rightarrow x_1 = -x_2 \\ \rightarrow \end{matrix} \Bigg\} x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

That means $\begin{bmatrix}1\\1\end{bmatrix}$ normalize $= \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$, $\begin{bmatrix}-1\\1\end{bmatrix}$ norm: $\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$$D = \begin{bmatrix}6 & 0\\0 & -4\end{bmatrix}, \quad P = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

16(7.1))

$$\det(A-\lambda I) = (-7-\lambda)(7-\lambda) - 576 = 0$$
$$\lambda^2 - 49 - 576 = 0$$
$$\lambda^2 - 625 = 0$$
$$\lambda = \pm 25$$

Eigenvector for $\lambda = 25 \rightarrow A - 25I = 0$

$$\begin{bmatrix}-7-25 & 24 & 0\\24 & 7-25 & 0\end{bmatrix} = \begin{bmatrix}-32 & 24 & 0\\24 & -18 & 0\end{bmatrix} \begin{matrix}/-8\\/6\end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix}4 & -3 & 0\\4 & -3 & 0\end{bmatrix} \cdot -1$$

$$\downarrow$$

$$\begin{bmatrix}4 & -3 & 0\\0 & 0 & 0\end{bmatrix} \begin{matrix}\rightarrow 4x_1 - 3x_2 = 0 \rightarrow x_1 = \frac{3}{4}x_2\\ \rightarrow\end{matrix} \Big\} \; x = \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\begin{bmatrix}3/4\\1\end{bmatrix}$$

Eigenvector for $\lambda = -25 \rightarrow A + 25I = 0$

$$\begin{bmatrix}-7+25 & 24 & 0\\24 & 7+25 & 0\end{bmatrix} = \begin{bmatrix}18 & 24 & 0\\24 & 36 & 0\end{bmatrix} \begin{matrix}18\\16\end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix}4 & 3 & 0\\4 & 3 & 0\end{bmatrix} \cdot -1$$

$$\downarrow$$

$$\begin{bmatrix}4 & 3 & 0\\0 & 0 & 0\end{bmatrix} \begin{matrix}\rightarrow 3x_1 + 4x_2 = 0 \rightarrow x_1 = -\frac{4}{3}x_2\\ \rightarrow\end{matrix} \Big\} \; x = x_2\begin{bmatrix}-4/3\\1\end{bmatrix}$$

That means $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ normalize $= \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ normalize $= \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$

$$D = \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix} \quad P = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

18(7.2)) Eigenvector for $\lambda = 25 \rightarrow A - 25I = 0$

$$\begin{bmatrix} -2-25 & -36 & 0 & 0 \\ -36 & -23-25 & 0 & 0 \\ 0 & 0 & 3-25 & 0 \end{bmatrix} = \begin{bmatrix} -27 & -36 & 0 & 0 \\ -36 & -48 & 0 & 0 \\ 0 & 0 & -22 & 0 \end{bmatrix} \begin{array}{l} 1 \cdot -9 \\ 1 \cdot -6 \\ \end{array}$$

$$\downarrow$$

$$\begin{bmatrix} 3 & 4 & 0 & 0 \\ 6 & 8 & 0 & 0 \\ 0 & 0 & -22 & 0 \end{bmatrix} \cdot -2$$

$$\downarrow$$

$$\begin{bmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -22 & 0 \end{bmatrix} \begin{array}{l} \rightarrow 3x_1 + 4x_2 = 0 \rightarrow x_1 = -\frac{4}{3}x_2 \\ \rightarrow \\ \rightarrow x_3 = 0 \end{array} \Bigg\} \; x = x_2 \begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvector for $\lambda = 3 \rightarrow A - 3I = 0$

$$\begin{bmatrix} -2-3 & -36 & 0 & 0 \\ -36 & -23-3 & 0 & 0 \\ 0 & 0 & 3-3 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -36 & 0 & 0 \\ -36 & -26 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot -36/5$$

$$\downarrow$$

$$\begin{bmatrix} -5 & -36 & 0 & 0 \\ 0 & 1166/5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow x_1 = 0 \\ \rightarrow x_2 = 0 \\ \rightarrow \end{array} \Bigg\} \; x = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvector for $\lambda = -50 \rightarrow A + 50I = 0$

$$\begin{bmatrix} -2+50 & -36 & 0 & 0 \\ -36 & -23+50 & 0 & 0 \\ 0 & 0 & 3+50 & 0 \end{bmatrix} = \begin{bmatrix} 48 & -36 & 0 & 0 \\ -36 & 27 & 0 & 0 \\ 0 & 0 & 53 & 0 \end{bmatrix} \begin{array}{l} /6 \\ /9 \\ \end{array}$$

$$\downarrow$$

$$\begin{bmatrix} 8 & -6 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 53 & 0 \end{bmatrix} \begin{array}{l} -2R_2 \\ \\ \end{array}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow -4x_1 + 3x_2 = 0 \rightarrow x_1 = \frac{3}{4}x_2 \Big\} \; x = x_2 \begin{bmatrix} 3/4 \\ 1 \\ 0 \end{bmatrix}$$

That means

$$\begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix} \text{normalize} = \begin{bmatrix} -4/5 \\ 3/5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{normalize} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3/4 \\ 1 \\ 0 \end{bmatrix} \text{normalize} = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -50 \end{bmatrix}, \quad P = \begin{bmatrix} -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \end{bmatrix}$$

26(7.1)) a) As defined in the theorem, TRUE

b) I think I saw this in the text somewhere TRUE

c) Needs to be symmetric, FALSE

d) As defined in the theorem, TRUE

2(7.2)) a) $x^T A x = [x_1\ x_2\ x_3]\begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$

$$[x_1\ x_2\ x_3]\begin{bmatrix} 4x_1 + 3x_2 \\ 3x_1 + 2x_2 + x_3 \\ x_2 + x_3 \end{bmatrix} = 4x_1^2 + 6x_1x_2 + 2x_2^2 + 2x_2x_3 + x_3^2 =$$

$$\underline{\underline{4x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 2x_2x_3}}$$

b) $[2\ -1\ 5]\begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 8-3 \\ 6-2+5 \\ -1+5 \end{bmatrix}[2\ -1\ 5] = \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix}[2\ -1\ 5] =$

$5 \cdot 2 - 9 + 4 \cdot 5 = \underline{\underline{21}}$

c) $[1/\sqrt{3}\ \ 1/\sqrt{3}\ \ 1/\sqrt{3}]\begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 4/\sqrt{3} + 3/\sqrt{3} \\ 3/\sqrt{3} + 2/\sqrt{3} + 1/\sqrt{3} \\ 1/\sqrt{3} + 1/\sqrt{3} \end{bmatrix}[1/\sqrt{3}\ 1/\sqrt{3}\ 1/\sqrt{3}]$

$(7/\sqrt{3})(1/\sqrt{3}) + (6/\sqrt{3})(1/\sqrt{3}) + (2/\sqrt{3})(1/\sqrt{3}) = \underline{\underline{5}}$

4(7.2)) a) $20x_1^2 + 15x_1x_2 - 10x_2^2$

$$\downarrow$$

$$\left(20x_1^2 + \frac{15}{2}x_1x_2\right) + \left(\frac{15}{2}x_1x_2 - 10x_2^2\right)$$

$$\downarrow$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \underline{\underline{\begin{bmatrix} 20 & \frac{15}{2} \\ \frac{15}{2} & -10 \end{bmatrix}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) x_1x_2

$$\downarrow$$

$$\left(0x_1^2 + \frac{1}{2}x_1x_2\right) + \left(\frac{1}{2}x_1x_2 + 0x_2^2\right)$$

$$\downarrow$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \underline{\underline{\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

6(7.2)) a) $5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$

$$\downarrow$$

$$\left(5x_1^2 + \frac{5}{2}x_1x_2 - \frac{3}{2}x_1x_3\right) + \left(\frac{5}{2}x_1x_2 - x_2^2 + 0x_2x_3\right) + \left(-\frac{3}{2}x_1x_3 + 0x_2x_3 + 7x_3^2\right.$$

$$\downarrow$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \underline{\underline{\begin{bmatrix} 5 & \frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & -1 & 0 \\ -\frac{3}{2} & 0 & 7 \end{bmatrix}}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

b) $x_3^2 - 4x_1x_2 + 4x_2x_3$

$\downarrow$

$(0x_1^2 - \frac{4}{2}x_1x_2 + 0x_1x_3) + (-\frac{4}{2}x_1x_2 + 0x_2^2 + \frac{4}{2}x_2x_3) + (0x_1x_3 + \frac{4}{2}x_2x_3 + x_3^2)$

$\downarrow$

$$[x_1\ x_2\ x_3]\underline{\underline{\begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

22(7.2)) a) This is literally exaclty what the book says, TRUE

b) P must be orthogonal and P^TAP diagonal, FALSE

c) Could be a single point, no point, etc. too, FALSE

d) It is both positive and negative, FALSE

e) That's what the theorem says, TRUE

10(7.3)) $-3x_1^2 + 5x_2^2 - 2x_1x_2$

$\downarrow$

$(-3x_1^2 - \frac{2}{2}x_1x_2) + (-\frac{2}{2}x_1x_2 + 5x_2^2$

$\downarrow$

$$[x_1\ x_2]\begin{bmatrix} -3 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda^2 - 2\lambda - 16$

$\downarrow$

$\lambda = 1 - \sqrt{17} \ \vee \ \lambda = \underline{\underline{1 + \sqrt{17}}}$

constrained max val

$$A - \lambda I = \begin{bmatrix} -3-\lambda & -1 \\ -1 & 5-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = (-3-\lambda)(5-\lambda) - 1 = -15 + 3\lambda + \lambda^2 - 5\lambda - 1 =$
$\underline{\lambda^2 - 2\lambda - 16}$

6(7.3)) a) $7x_1^2 + 3x_2^2 + 3x_1x_2$

$$\downarrow$$

$$\left(7x_1^2 + \frac{3}{2}x_1x_2\right) + \left(\frac{3}{2}x_1x_2 + 3x_2^2\right)$$

$$\downarrow$$

$$A = \begin{bmatrix} 7 & \frac{3}{2} \\ \frac{3}{2} & 3 \end{bmatrix}$$

$$\downarrow$$

(Got them from Wolfram Alpha. Didn't have time to calculate it myself due to finals. Hope you understand)

Eigenvalues are $\underline{\underline{\lambda = \frac{15}{2}}}$ v $\lambda = \frac{5}{2}$

Max val would be the greatest eigenvalue

b) $\lambda = \frac{15}{2}$

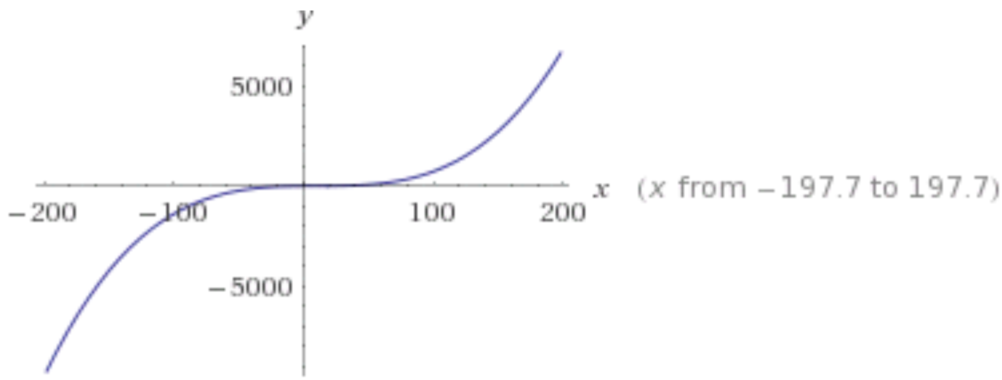
(Also Wolfram Alpha)

$$\downarrow$$

Eigenvector: $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Which is $\underline{\underline{\begin{bmatrix} 3\sqrt{10} \\ 1\sqrt{10} \end{bmatrix}}}$ when normalized

Curve from first problem from WolframAlpha:



(It’s finals week and I don’t have the time or effort to put a lot into this homework right now, I hope you understand that and that it’s okay I used WolframAlpha to solve some of the parts of the problems when I’ve solved similar problems by hand before (both in this homework and in earlier homeworks))