CS132 Homework 12
Aleksander Skjølsvik

2(a)) The vector in span $\{u_1\}$ is $\frac{v \cdot u_1}{u_1 \cdot u_1} u_1$

$$v \cdot u_1 = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = 4 \cdot 1 + 5 \cdot 2 + (-3) \cdot 1 + 3 \cdot 1 = \underline{14}$$

$$u_1 \cdot u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 + 1 \cdot 1 = 1 + 4 + 1 + 1 = \underline{7}$$

$$\frac{14}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}}}$$

? $\{u_2, u_3, u_4\}$

4(a.) $$u_1 \cdot u_2 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = 3(-4) + 4 \cdot 3 + 0 \cdot 0 = -12 + 12 = \underline{0}$$

$\{u_1, u_2\}$ is an orthogonal set!

$$\frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \hat{y}$$

$$y \cdot u_1 = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = 6 \cdot 3 + 3 \cdot 4 + (-2) \cdot 0 = 18 + 12 = \underline{30}$$

$$u_1 \cdot u_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = 3 \cdot 3 + 4 \cdot 4 + 0 \cdot 0 = 9 + 16 = \underline{25}$$

$$y \cdot u_2 = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = 6(-4) + 3 \cdot 3 + (-2) \cdot 0 = -24 + 9 = \underline{-15}$$

$$u_2 \cdot u_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = (-4)(-4) + 3 \cdot 3 = 16 + 9 = \underline{25}$$

$$\frac{30}{25} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \frac{15}{25} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.6 \\ 4.8 \\ 0 \end{bmatrix} - \begin{bmatrix} -2.4 \\ 1.8 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.6 + 2.4 \\ 4.8 - 1.8 \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}}$$

6(2.) $u_1 \cdot u_2 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (-4) \cdot 0 + (-1) \cdot 1 + 1 \cdot 1 = -1 + 1 = \underline{0}$

$\{u_1, u_2\}$ is an orthogonal set

$$\frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \hat{y}$$

$$y \cdot u_1 = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} = 6 \cdot (-4) + 4 \cdot (-1) + 1 \cdot 1 = -24 - 4 + 1 = \underline{-27}$$

$$u_1 \cdot u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} = (-4)(-4) + (-1)(-1) + 1 \cdot 1 = 16 + 1 + 1 = \underline{18}$$

$$y \cdot u_2 = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 6 \cdot 0 + 4 \cdot 1 + 1 \cdot 1 = 4 + 1 = \underline{5}$$

$$u_2 \cdot u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 1 + 1 = \underline{2}$$

$$-\frac{27}{18} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1.5 \\ -1.5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2.5 \\ 2.5 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}}}$$

c(2.) $\frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$

$$y \cdot u_1 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 + 4 + 3 = \underline{6}$$

$$u_1 \cdot u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 + 1 + 1 = \underline{3}$$

$$y \cdot u_2 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = (-1)(-1) + 4 \cdot 3 + 3(-2) = 1 + 12 - 6 = \underline{7}$$

$$u_2 \cdot u_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = (-1)(-1) + 3 \cdot 3 + (-2)(-2) = 1 + 9 + 4 = \underline{14}$$

$$\frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{14} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -0.5 \\ 1.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3.5 \\ 1 \end{bmatrix}$$

$$y - \hat{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 3.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.5 \\ 0.5 \\ 2 \end{bmatrix}$$

$$\underline{\underline{y = \begin{bmatrix} 1.5 \\ 3.5 \\ 1 \end{bmatrix} + \begin{bmatrix} -2.5 \\ 0.5 \\ 2 \end{bmatrix}}}$$

10 (2.)

$$\frac{y \cdot u_1}{u_1 u_1} u_1 + \frac{y \cdot u_2}{u_2 u_2} u_2 + \frac{y \cdot u_3}{u_3 u_3} u_3$$

$$y \cdot u_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3 \cdot 1 + 4 \cdot 1 + 5 \cdot 0 + 6(-1) = 3 + 4 - 6 = \underline{1}$$

$$u_1 \cdot u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 1 + 1 = \underline{3}$$

$$y \cdot u_2 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3 + 5 + 6 = \underline{14}$$

$$u_2 u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 1 + 1 + 1 = \underline{3}$$

$$y \cdot u_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 3 \cdot 0 + 4(-1) + 5 \cdot 1 + 6(-1) = -4 + 5 - 6 = \underline{-5}$$

$$u_3 u_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = (-1)(-1) + 1 \cdot 1 + (-1)(-1) = 1 + 1 + 1 = \underline{3}$$

$$\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5/3 \\ -5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

$$y - \hat{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

12(2.) $\frac{y \cdot v_1}{v_1 v_2} v_1 + \frac{y \cdot v_2}{v_2 v_2} v_2$

$$y \cdot v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} = 3 \cdot 1 + (-1)(-2) + 1(-1) + 13 \cdot 2 = 3 + 2 - 1 + 26 = \underline{30}$$

$$v_1 v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} = 1 \cdot 1 + (-2)(-2) + (-1)(-1) + 2 \cdot 2 = 1 + 4 + 1 + 4 = \underline{10}$$

$$y \cdot v_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = 3 \cdot (-4) + (-1) \cdot 1 + 1 \cdot 0 + 13 \cdot 3 = -12 - 1 + 39 = \underline{26}$$

$$v_2 v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = (-4)(-4) + 1 \cdot 1 + 0 \cdot 0 + 3 \cdot 3 = 16 + 1 + 9 = \underline{26}$$

$$\frac{30}{10} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{26}{26} \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}}}$$

This is the closest point to y in W

14(2.) $\frac{z \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{z \cdot v_2}{v_2 \cdot v_2} v_2$

$$z \cdot v_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} = 2 \cdot 2 + 4 \cdot 0 + 0(-1) + (-1)(-3) = 4 + 3 = \underline{7}$$

$$v_1 v_1 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ -1 \\ -3 \end{bmatrix} = (-2)(-2) + (-1)(-1) + (-3)(-3) = 4 + 1 + 9 = \underline{14}$$

$$z v_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} = 2 \cdot 5 + 4(-2) + 0 \cdot 4 + (-1)2 = 10 - 8 - 2 = \underline{0}$$

$$v_2 v_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} = 5 \cdot 5 + (-2)(-2) + 4 \cdot 4 + 2 \cdot 2 = 25 + 4 + 16 + 4 = \underline{49}$$

$$\frac{7}{14}\begin{bmatrix}2\\0\\-1\\-3\end{bmatrix}+\frac{0}{49}\begin{bmatrix}5\\-2\\4\\2\end{bmatrix}=\begin{bmatrix}1\\0\\-0.5\\-1.5\end{bmatrix}$$ Best apprx to z by $c_1v_1+c_2v_2$

16(2.) $\hat{y}=\begin{bmatrix}-1\\-5\\-3\\9\end{bmatrix}$ (from problem 12)

$$y-\hat{y}=\begin{bmatrix}3\\-1\\1\\13\end{bmatrix}-\begin{bmatrix}-1\\-5\\-3\\9\end{bmatrix}=\begin{bmatrix}4\\4\\4\\4\end{bmatrix}$$

$$d=\|y-\hat{y}\|^2=\left\|\begin{bmatrix}4\\4\\4\\4\end{bmatrix}\right\|^2=\sqrt[2]{4^2+4^2+4^2+4^2}^{\,2}=\underline{\underline{64}}$$

4(4.) a) $A^TAx=A^Tb$

$$A^TA=\begin{bmatrix}1&1&1\\3&-1&1\end{bmatrix}\cdot\begin{bmatrix}1&3\\1&-1\\1&1\end{bmatrix}=\begin{bmatrix}1\cdot1+1\cdot1+1\cdot1 & 1\cdot3+1(-1)+1\cdot1\\3\cdot1+(-1)1+1\cdot1 & 3\cdot3+(-1)(-1)+1\cdot1\end{bmatrix}=$$

$$\begin{bmatrix}3&3\\3&11\end{bmatrix}$$

$$A^Tb=\begin{bmatrix}1&1&1\\3&-1&1\end{bmatrix}\begin{bmatrix}5\\1\\0\end{bmatrix}=\begin{bmatrix}1\cdot5+1\cdot1+1\cdot0\\3\cdot5+(-1)(1)+1\cdot0\end{bmatrix}=\begin{bmatrix}6\\14\end{bmatrix}$$

$A^TAx=A^Tb$

$$\begin{bmatrix}3&3\\3&11\end{bmatrix}x=\begin{bmatrix}6\\14\end{bmatrix}$$

b) $\begin{bmatrix} 3 & 3 & 6 \\ 3 & 11 & 14 \end{bmatrix} -1$

$\downarrow$

$\begin{bmatrix} 3 & 3 & 6 \\ 0 & 8 & 8 \end{bmatrix} \begin{matrix} /3 \\ /8 \end{matrix}$

$\downarrow$

$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} -1$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \left.\begin{matrix} \to x_1 = 1 \\ \to x_2 = 1 \end{matrix}\right\} \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

6(u.) $A^T A x = A^T b$

(Used a calculator)

$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$

(calculator)

$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$

$A^T A x = A^T b$

$\begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} x = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 6 & 3 & 3 & 27 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{array}\right] \quad -1$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 6 & 3 & 3 & 27 \\ 0 & 3 & -3 & -3 \\ 3 & 0 & 3 & 15 \end{array}\right] \quad -1$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 6 & 30 \\ 0 & 3 & -3 & -3 \\ 3 & 0 & 3 & 15 \end{array}\right] \begin{array}{l} /6 \\ /3 \\ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 3 & 15 \end{array}\right] \begin{array}{l} \\ \\ -3 \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left.\begin{array}{l} x_1 = 5 - x_3 \\ x_2 = -1 + x_3 \\ x_3 \text{ is free} \end{array}\right\} \quad x = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + x \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

8(4.) $\|b - A\hat{x}\|$

$$b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$A\hat{x} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 3\cdot 1 \\ 1\cdot 1 + (-1)1 \\ 1\cdot 1 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$b - A\hat{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\|b - A\hat{x}\| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1+1+4} = \underline{\underline{\sqrt{6}}}$$

10(4.) a) $\dfrac{b\,a_1}{a_1 a_1} a_1 + \dfrac{b\,a_2}{a_2 a_2} a_2$

$$b a_1 = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 3\cdot 1 + (-1)(-1) + 5\cdot 1 = 3 + 1 + 5 = \underline{9}$$

$$a_1 a_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1\cdot 1 + (-1)(-1) + 1\cdot 1 = 1 + 1 + 1 = \underline{3}$$

$$b a_2 = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 3\cdot 2 + (-1)4 + 5\cdot 2 = 6 - 4 + 10 = \underline{12}$$

$$a_2 a_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 2\cdot 2 + 4\cdot 4 + 2\cdot 2 = 4 + 16 + 4 = \underline{24}$$

$$\frac{9}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{12}{24} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}}}$$

b) $A\hat{x} = \hat{b}$

We know from a) what to multiply with A to get $\hat{b}$, so this is therefore also $\hat{x}$:

$$\begin{bmatrix} \frac{9}{3} & \frac{12}{24} \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 & \frac{1}{2} \end{bmatrix}}}$$

14(4.) $$Au = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 4 + 1 \cdot (-5) \\ (-3) \cdot 4 + (-4)(-5) \\ 3 \cdot 4 + 2 \cdot (-5) \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$$

$$Av = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 6 + 1(-5) \\ (-3) \cdot 6 + (-4)(-5) \\ 3 \cdot 6 + 2(-5) \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix}$$

No, because the number of rows in Au / Av needs to be equal to the number of columns in A

18(4.) a) True, this is as defined in the text

b) $A\hat{x}$ would be ~~the~~ point closest to b, FALSE

c) This is as defined, and like i said in 18b, TRUE

d) Only when linearly independent, FALSE

e) ?

f) FALSE

19(4.) a) If $Ax = 0 \rightarrow A^T A x = A^T 0 = 0$ QED

x b) If $A^T A x = 0 \rightarrow x^T A^T A x = x^T 0 = 0$

20(4.) If $Ax = 0$, then $A^T A x = 0$ (as shown)
$A^T A$ is invertible, and the x must therefore be 0.
From that we can conclude the columns of
A are invertible

24(4.) $A^T A = A^T b$, so $\hat{x} = A^T b$

35(2.) $A \cdot A^T$ = matrix

If all off diagonal elements are equal, which they are, the columns are orthogonal. See image further down.

36(2.) a) $U = A/\text{norm}$

$U^T U = U^T \cdot U$
$U U^T = U \cdot U^T$

They are of different size and don't look like each other

b) y = random matrix
$p = U \cdot U^T \cdot y$
$z = y - p$

p is in Col A because U is just a scaled version of A, and $p = U(U^T y)$ which means it's in Col U.

c)

?

d) From c we can assume z is orthogonal to each column in A, therefore z is orthogonal to each vector in A, and therefore z is in Col $A^\perp$

25 (3.) Closest point to y is $\hat{y}$ on Col A

$$\hat{y} = UU^T y$$

26 (3.) The distance is $\|b - \hat{b}\|$, and $\hat{b} = UU^T b$

```python
def problem35():
    A = np.array([[-6,-3, 6, 1],
                  [-1, 2, 1,-6],
                  [ 3, 6, 3,-2],
                  [ 6,-3, 6,-1],
                  [ 2,-1, 2, 3],
                  [-3, 6, 3, 2],
                  [-2,-1, 2,-3],
                  [ 1, 2, 1, 6]])
    transpose = np.transpose(A)
    result = np.dot(A, transpose)
    print "Problem 35:"
    print "The columns are orthogonal if the off diagonal elements are equal"
    print result
```

```python
def problem36():
    A = np.array([[-6,-3, 6, 1],
                  [-1, 2, 1,-6],
                  [ 3, 6, 3,-2],
                  [ 6,-3, 6,-1],
                  [ 2,-1, 2, 3],
                  [-3, 6, 3, 2],
                  [-2,-1, 2,-3],
                  [ 1, 2, 1, 6]])
    U = np.divide(A, np.linalg.norm(A))
    transpose = np.transpose(U)
    UTU = np.dot(transpose, U)
    UUT = np.dot(U, transpose)
    print "Problem 36a:"
    print "UTU:\n" + str(UTU)
    print "UUT:\n" + str(UUT)

    y = np.random.randint(-10, 10, size=(8, 4))
    p = np.dot(U, transpose)
    p = np.dot(p, y)
    z = np.subtract(y, p)
    print "\nProblem 36b:"
    print "p:\n" + str(p)
    print "z:\n" + str(z)
```

```python
def problem25():
    A = np.array([[-6,-3, 6, 1],
                  [-1, 2, 1,-6],
                  [ 3, 6, 3,-2],
                  [ 6,-3, 6,-1],
                  [ 2,-1, 2, 3],
                  [-3, 6, 3, 2],
                  [-2,-1, 2,-3],
                  [ 1, 2, 1, 6]])
    U = np.divide(A, np.linalg.norm(A))
    transpose = np.transpose(U)
    UUT = np.dot(U, transpose)
    print(UUT)
    y = np.array([1, 1, 1, 1, 1, 1, 1, 1])
    yHat = np.dot(UUT, y)
    print "Problem 25:"
    print "The closest point is:\n" + str(yHat)
```

```
In [54]: problem35()
Problem 35:
The columns are orthogonal if the off diagonal elements are equal
[[ 82   0 -20   8   6  20  24   0]
 [  0  42  24   0 -20   6  20 -32]
 [-20  24  58  20   0  32   0   6]
 [  8   0  20  82  24 -20   6   0]
 [  6 -20   0  24  18   0  -8  20]
 [ 20   6  32 -20   0  58   0  24]
 [ 24  20   0   6  -8   0  18 -20]
 [  0 -32   6   0  20  24 -20  42]]

In [55]: problem36()
Problem 36a:
UTU:
[[  2.50000000e-01  -6.93889390e-18  -8.67361738e-18   0.00000000e+00]
 [ -6.93889390e-18   2.50000000e-01   0.00000000e+00  -6.93889390e-18]
 [ -8.67361738e-18   0.00000000e+00   2.50000000e-01   0.00000000e+00]
 [  0.00000000e+00  -6.93889390e-18   0.00000000e+00   2.50000000e-01]]
UUT:
[[ 0.205  0.    -0.05   0.02   0.015  0.05   0.06   0.   ]
 [ 0.     0.105  0.06   0.    -0.05   0.015  0.05  -0.08 ]
 [-0.05   0.06   0.145  0.05   0.     0.08   0.     0.015]
 [ 0.02   0.     0.05   0.205  0.06  -0.05   0.015  0.   ]
 [ 0.015 -0.05   0.     0.06   0.045  0.    -0.02   0.05 ]
 [ 0.05   0.015  0.08  -0.05   0.     0.145  0.     0.06 ]
 [ 0.06   0.05   0.     0.015 -0.02   0.     0.045 -0.05 ]
 [ 0.    -0.08   0.015  0.     0.05   0.06  -0.05   0.105]]
```

```
Problem 36b:
p:
[[ 1.455 -0.195 -1.155  1.595]
 [-1.215 -1.165 -0.575 -1.7  ]
 [-1.38  -1.305 -1.35  -0.85 ]
 [ 1.395 -0.405  0.105 -1.295]
 [ 0.915  0.315 -0.015  0.685]
 [-0.945 -0.795 -2.025  1.6  ]
 [ 0.035 -0.515 -0.335 -0.585]
 [ 0.44   0.465 -0.55   1.95 ]]
z:
[[ 5.545 -0.805 -2.845  6.405]
 [-6.785 -8.835  5.575 -3.3  ]
 [-2.62  -3.695 -6.65  -1.15 ]
 [ 4.605  1.405  0.895 -5.705]
 [ 1.085 -6.315 -0.985  6.315]
 [-2.055 -0.205 -7.975  3.4  ]
 [-3.035 -1.485 -3.665 -5.415]
 [-2.44  -0.465  4.55   4.05 ]]

In [58]: problem25()
Problem 25:
The closest point is:
[ 0.3  0.1  0.3  0.3  0.1  0.3  0.1  0.1]
```