CS132 HW10
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4(1.) $A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}$

$$\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}^k = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}^k \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} (-3)^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} 3(-3)^k & -2(-2)^k \\ 2(-3)^k & -(-2)^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -3(-3)^k + 4(-2)^k & 6(-3)^k - 6(-2)^k \\ -2(-3)^k + 2(-2)^k & 4(-3)^k - 3(-2)^k \end{bmatrix} =$$

$$(-3)^k \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix} + (-2)^k \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix}$$

6(1.) According to the theorem, eigenvalues are entries of diag matrix D, $\lambda = 3, 4$

$$B_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

8(1) $\lambda = 3 \rightarrow A - 3I = 0$

$$\begin{bmatrix} 3-3 & 2 & 0 \\ 0 & 3-3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \left. \begin{array}{l} x_2 = 0 \\ x_1 \text{ is free} \end{array} \right\} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

eigenvector for $\lambda = 3$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

It is not diagonizable because the eigenvector does not span $\mathbb{R}^2$

10(1) $\det(A - \lambda I)$

$(1-\lambda)(2-\lambda)-12=0$

$2-\lambda-2\lambda+\lambda^2-12=0$

$\lambda^2-3\lambda-10=0$

$\lambda=5 \vee \lambda=-2$

$A-5I=0$

$$\begin{bmatrix}1-5 & 3 & 0\\ 4 & 2-5 & 0\end{bmatrix} = \begin{bmatrix}-4 & 3\\ 4 & -3\end{bmatrix} +1$$

$$\begin{bmatrix}-4 & 3 & 0\\ 0 & 0 & 0\end{bmatrix}$$ & x_2 is free

$-4x_1+3x_2=0$

$x_1 = \frac{3}{4}x_2$

$x = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = x_2\begin{bmatrix}3/4\\ 1\end{bmatrix} \rightarrow \begin{bmatrix}3/4\\ 1\end{bmatrix}$ is eigenvector for $\lambda=5$

$A+2I=0$

$$\begin{bmatrix}1+2 & 3 & 0\\ 4 & 2+2 & 0\end{bmatrix} = \begin{bmatrix}3 & 3 & 0\\ 4 & 4 & 0\end{bmatrix} \begin{matrix}1/3\\ 1/4\end{matrix}$$

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\end{bmatrix} -1$$

$$\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$ $\rightarrow x_2$ is free

$x_1 = -x_2$

$\rightarrow x = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = x_2\begin{bmatrix}-1\\ 1\end{bmatrix} \rightarrow \begin{bmatrix}-1\\ 1\end{bmatrix}$ is eigenvector for $\lambda=-2$

That means:

$P = \begin{bmatrix}3/4 & -1\\ 1 & 1\end{bmatrix}$

$D = \begin{bmatrix}5 & 0\\ 0 & -2\end{bmatrix}$

12(a.) $\lambda = 2$

$\downarrow$

$A - 2I = 0$

$$\begin{bmatrix} 3-2 & 1 & 1 & 0 \\ 1 & 3-2 & 1 & 0 \\ 1 & 1 & 3-2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix}$$

$\downarrow$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} x_1 = -x_2 - x_3 \\ \rightarrow x_2 \text{ is free} \\ \rightarrow x_3 \text{ is free} \end{matrix} \Bigg\} \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

eigenvectors for λ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 5$

$A - 5I = 0$

$$\begin{bmatrix} 3-5 & 1 & 1 & 0 \\ 1 & 3-5 & 1 & 0 \\ 1 & 1 & 3-5 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \begin{matrix} \\ -1 \\ +2 \end{matrix}$$

$\downarrow$

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -3 & 0 & 0 \\ -3 & 3 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ +1 \end{matrix}$$

$\downarrow$

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \rightarrow 3x_1 - 3x_2 = 0 \rightarrow x_1 = x_2 \\ \rightarrow x_3 \text{ is free} \end{matrix}$$

$$\left. \begin{matrix} -2x_1 + x_2 + x_3 = 0 \\ -2x_1 + x_1 + x_3 = \\ x_1 = x_3 = x_2 \end{matrix} \right\} \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

eigenvector for $\lambda = 5$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

That means:

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

M(1.)

$\underline{\lambda = 3}$

$\downarrow$

$A - 3I = 0$

$$\left[\begin{array}{ccc|c} 2-3 & 0 & -2 & 0 \\ 1 & 3-3 & 2 & 0 \\ 0 & 0 & 3-3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] +1$$

$\downarrow$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left.\begin{array}{l} x_1 = -2x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{array}\right\} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

eigenvectors for $\lambda = 3$ are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\underline{\lambda = 2}$

$\downarrow$

$A - 2I = 0$

$$\left[\begin{array}{ccc|c} 2-2 & 0 & -2 & 0 \\ 1 & 3-2 & 2 & 0 \\ 0 & 0 & 3-2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \left.\begin{array}{l} \rightarrow x_1 = -x_2 \\ \rightarrow x_3 = 0 \end{array}\right\}$$

eigenvector for $\lambda = 2$ is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

That means

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

22(1.) a) The eigenvectors must be lin independent, FALSE

b)

c) TRUE

d) They're not related ? FALSE

24(1.) Two one-dimensional eigenspace sums up to 2, but the matrix is a 3x3 mtrx. So it's not diagonalizable

34(2.) Eigenvalues from Python : -2, 5

$\lambda = -2 \to A + 2I_n = 0$

$$\begin{bmatrix} 4+2 & -9 & -7 & 8 & 2 & 0 \\ -7 & -9+2 & 0 & 7 & 14 & 0 \\ 5 & 10 & 5+2 & -5 & -10 & 0 \\ -2 & 3 & 7 & 0+2 & 4 & 0 \\ -3 & -13 & -7 & 10 & 11+2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -9 & -7 & 8 & 2 & 0 \\ -7 & -7 & 0 & 7 & 14 & 0 \\ 5 & 10 & 7 & -5 & -10 & 0 \\ -2 & 3 & 7 & 2 & 4 & 0 \\ -3 & -13 & -7 & 10 & 13 & 0 \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} 1 & 0 & 0 & 2/5 & -3/5 & 0 \\ 0 & 1 & 0 & -7/5 & -7/5 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

eigenvector for $\lambda = -2$ is $\begin{bmatrix} 0 \\ 1 \\ -5/7 \\ 3/7 \\ 2/7 \end{bmatrix}$

$\lambda = 5 \rightarrow A - 5I = 0$

$$\begin{bmatrix} 4-5 & -9 & -7 & 8 & 2 & 0 \\ -7 & -9-5 & 0 & 7 & 14 & 0 \\ 5 & 10 & 5-5 & -5 & -10 & 0 \\ -2 & 3 & 7 & 9-5 & 4 & 0 \\ -3 & -13 & -7 & 10 & 11-5 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -9 & -7 & 8 & 2 & 0 \\ -7 & -14 & 0 & 7 & 14 & 0 \\ 5 & 10 & 0 & -5 & -10 & 0 \\ -2 & 3 & 7 & -5 & 4 & 0 \\ -3 & -13 & -7 & 10 & 6 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

eigenvector: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

That means:

$$P = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ -5/7 & -6/7 & 0 & 0 & 0 \\ 3/7 & 3/7 & -1 & -1 & -1 \\ 2/7 & 2/7 & 1 & 1 & 1 \end{bmatrix}$$

3 a) $$\begin{bmatrix} 0 & 1/3 & 0 & 1/2 & 1/2 \\ 1/3 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 0 & 1/2 & 0 \\ 1/3 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix}$$

b) $$\begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$$

4 a)

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \end{bmatrix} \rightarrow P' = \begin{bmatrix} 0 & 0 & 1 & 0 & 1/5 \\ 1/3 & 0 & 0 & 1/2 & 1/5 \\ 1/3 & 0 & 0 & 1/2 & 1/5 \\ 1/3 & 1/2 & 0 & 0 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/5 \end{bmatrix}$$

$$P'' = \begin{bmatrix} 1/500 & 1/500 & 124/155 & 1/500 & 1/5 \\ 83/250 & 1/500 & 1/500 & 497/000 & 1/5 \\ 83/250 & 1/500 & 1/500 & 497/1000 & 1/5 \\ 83/250 & 497/1000 & 1/500 & 1/500 & 1/5 \\ 1/500 & 497/000 & 1/500 & 1/500 & 1/5 \end{bmatrix}$$

4(b)

$$P = \begin{bmatrix} 0 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 1/2 & 0 \\ 0 & 0 & 1/4 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

$$P' = \begin{bmatrix} 1/6 & 1/2 & 1/4 & 0 & 0 & 1/6 \\ 1/6 & 0 & 1/4 & 0 & 0 & 1/6 \\ 1/6 & 1/2 & 0 & 1/2 & 0 & 1/6 \\ 1/6 & 0 & 1/4 & 0 & 1/2 & 1/6 \\ 1/6 & 0 & 1/4 & 1/2 & 0 & 1/6 \\ 1/6 & 0 & 0 & 0 & 1/2 & 1/6 \end{bmatrix}$$

$$P'' = \begin{bmatrix} 1/6 & 149/300 & 299/1200 & 1/600 & 1/600 & 1/6 \\ 1/6 & 1/600 & 299/200 & 1/600 & 1/600 & 1/6 \\ 1/6 & 149/300 & 1/600 & 149/300 & 1/600 & 1/6 \\ 1/6 & 1/600 & 299/200 & 1/600 & 149/300 & 1/6 \\ 1/6 & 1/600 & 299/200 & 149/300 & 1/600 & 1/6 \\ 1/6 & 1/600 & 1/600 & 1/600 & 149/300 & 1/6 \end{bmatrix}$$

Results for problem 4 (see hw10Code.py for code):

```
arrray =
[[ 0.002  0.002  0.992  0.002  0.2   ]
 [ 0.332  0.002  0.002  0.497  0.2   ]
 [ 0.332  0.002  0.002  0.497  0.2   ]
 [ 0.332  0.497  0.002  0.002  0.2   ]
 [ 0.002  0.497  0.002  0.002  0.2   ]]
final order = [1 4 3 2 5]
importance = [ 0.23648889  0.21038811  0.21038811  0.21038811  0.13234678]

arrray =
[[ 0.16666667  0.49666667  0.24916667  0.00166667  0.00166667  0.16666667]
 [ 0.16666667  0.00166667  0.24916667  0.00166667  0.00166667  0.16666667]
 [ 0.16666667  0.49666667  0.00166667  0.49666667  0.00166667  0.16666667]
 [ 0.16666667  0.00166667  0.24916667  0.00166667  0.49666667  0.16666667]
 [ 0.16666667  0.00166667  0.24916667  0.49666667  0.00166667  0.16666667]
 [ 0.16666667  0.00166667  0.00166667  0.00166667  0.49666667  0.16666667]]
final order = [3 5 4 1 6 2]
importance = [ 0.19983853-0.j  0.19944284-0.j  0.19944284-0.j  0.15057436-0.j
  0.14998280-0.j  0.10071863-0.j]
```