2 Cases of Non-Stationarity

Case I: deterministic trends

Case II: unit roots ("stochastic trends")

a) "Pure Random Walk"

$$Y_t = Y_{t-1} + u_t$$

$\mathrm{lsp500}_t$ $\mathrm{lsp500}_{t-1}$

Last class

$$\Delta Y_t = \boxed{u_t}$$

$[Y_t - Y_{t-1}]$

$\longrightarrow Y_t$ is non-stationary!

ΔY_t is stationary!

AR(1) $|\rho|<1$

$$\left[Y_t = \underset{\substack{\| \\ 1}}{\rho} Y_{t-1} + \underset{\text{iid}}{u_t} \right]$$

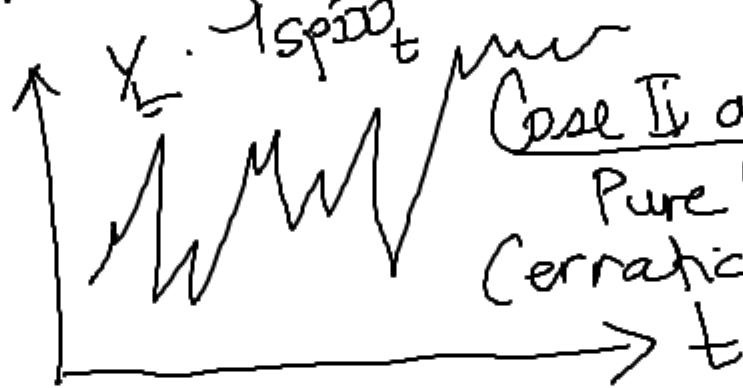
Case II b) Random Walk with a Drift

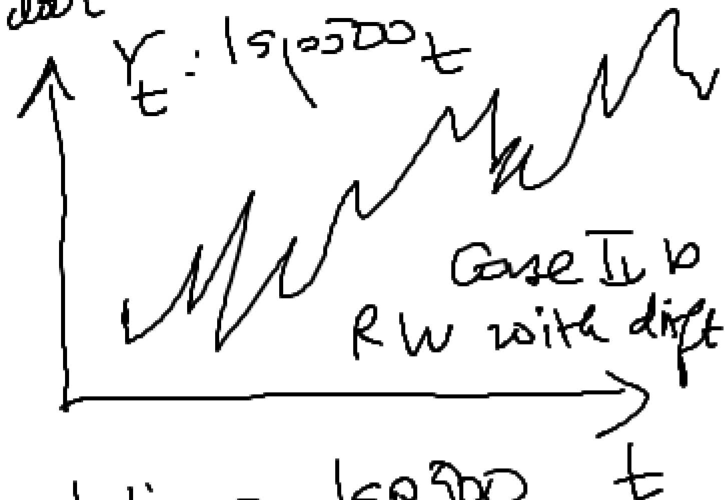
$$Y_t = \alpha + Y_{t-1} + u_t$$

α → Drift

Y_{t-1} → unit root

Visually





· tsline lsp500

· tsline lsp500

$\underbrace{\text{Case II (b)}}$ $Y_t = \alpha + Y_{t-1} + u_t$

Random walk with a drift $\hookrightarrow$ non-stationary!

$E(Y_t) = ?$ ✓

$Var(Y_t) = ?$ ✓

Rewrite: $Y_t = \alpha + (\alpha + Y_{t-2} + u_{t-1}) + u_t$

$= \alpha + \alpha + (\alpha + Y_{t-3} + u_{t-2}) + u_{t-1} + u_t$

$\vdots$

$$Y_t = \underbrace{\alpha . t}_{\text{cons.}} + \underbrace{Y_0}_{\text{cons.}} + u_1 + u_2 + \cdots + u_{t-1} + u_t$$

$$E(Y_t) = E(\alpha t + Y_0) + E[u_1 + u_2 + \cdots + u_t]$$

$$= \alpha t + Y_0$$

$\therefore\ E(Y_t) = \alpha . t + Y_0 \longrightarrow$ Time-variant mean!

$$Var(Y_t) = Var(\underbrace{\alpha t + Y_0}_{\substack{\text{cons.}\\ \text{non-R.V}}} + u_1 + u_2 + \dots + u_{t-1} + u_t)$$

$$= Var(u_1 + u_2 + \dots + u_{t-1} + u_t)$$

$$= \underbrace{Var(u_1)}_{\sigma_u^2} + \underbrace{Var(u_2)}_{\sigma_u^2} + \dots + \overbrace{Var(u_{t-1})}^{\sigma_u^2} + \underbrace{Var(u_t)}_{\sigma_u^2} + \cancel{2Cov(u_1, u_2)}^{\,0} + \dots + \dots + \cancel{2Cov(u_{t-1}, u_t)}^{\,0}$$

Under u_t is $\underline{\underline{iid}}$ $\sim N(0, \sigma_u^2)$ (like ε_t)

$$= t . \sigma_u^2 \rightarrow \text{Time-variant variance!}$$

<u>Cure:</u> Case II b of non-stationarity!

Again; differencing to transform the Y_t into a stationary form!!

$$Y_t = \alpha + Y_{t-1} + u_t$$

$$\Delta Y_t = \underbrace{Y_t - Y_{t-1}}_{} = \boxed{\alpha} + \boxed{u_t}$$

$$\underbrace{E(\Delta Y_t)}_{} = E(\alpha + u_t)$$

$$= \underbrace{\alpha}_{} \checkmark \text{ (time invariant)}$$

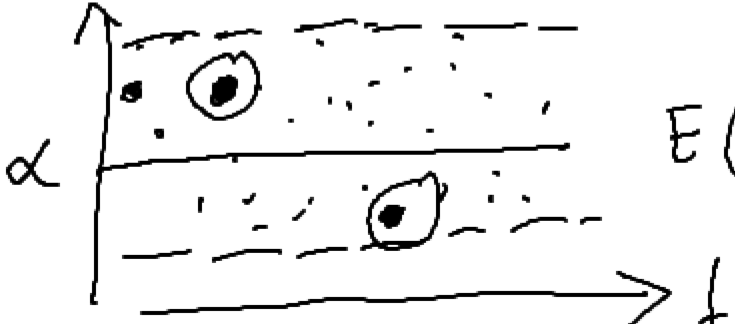
Y_t : $lsp500_t$

ΔY_t : $\Delta lsp500_t$

: growth rate of $sp500_t$

: rate of return on sp500 index

$\boxed{\Delta Y_t}$: d.lsp500 $= \Delta lsp500_t$



$E(\Delta Y_t) = \alpha$

"good"

$\Delta Y_t > \underbrace{E(\Delta Y_t)}_{\alpha} \rightarrow u_t > 0$

when

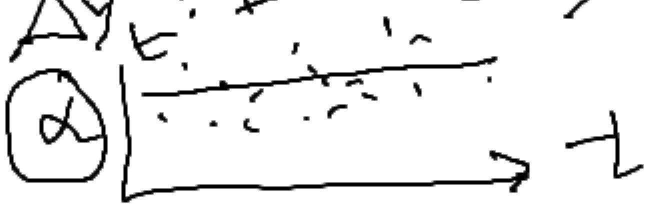
$\Delta Y_t < \underbrace{E(\Delta Y_t)}_{\alpha} \rightarrow u_t < 0$

"bads"

Case IIb: Y_t: $lsp500_t$

$\rightarrow$ non-stationary or inft

ΔY_t: $\boxed{\text{d.lsp500}}$ $\rightarrow$ stationary



$$
\begin{aligned}
Var(\Delta Y_t) &= Var(\alpha + u_t) \\
&= Var(u_t) \\
&= \sigma_u^2 \rightarrow \text{time invariant}
\end{aligned}
$$

$\therefore$ Case II b:

$$
\begin{tikzcd}
Y_t \arrow[rr] \arrow[d] & & \Delta Y_t \arrow[d] \\
\text{non-stationary} & & \text{stationary}
\end{tikzcd}
$$

Use ΔY_t for regressions, rather than Y_t!

Dickey-Fuller Tests : $r3_t$, $r6_t$
"UNIT ROOTS"

[. dfuller Y, trend
regress] 3m & 6m T-Bills maturity!

D-Fuller Equation:

$$\left(Y_t = \alpha + \rho Y_{t-1} + u_t \right) \begin{cases} \alpha = 0 \rightarrow \text{Pure R.W} \\ \alpha \neq 0 \rightarrow \text{R.W with drift} \end{cases}$$

testing for unit root $\Rightarrow \boxed{\rho = 1}$ or $\rho \neq 1$

$H_0: \rho = 1 \Rightarrow \exists$ unit root!

$H_A: \rho \neq 1 \Rightarrow \exists$ no unit root!

D-F Equation

$$Y_t = \alpha + \boxed{\rho} Y_{t-1} + \overset{\text{iid}}{u_t}$$

$=1$

$H_0: \rho = 1$
$H_A: \rho \neq 1$?

$$\underbrace{Y_t - Y_{t-1}}_{} = \alpha + \rho Y_{t-1} - Y_{t-1} + u_t$$

$$\Delta Y_t = \underset{\substack{\shortparallel \\ 0}}{\alpha} + \underbrace{(\rho - 1)}_{\delta = 0} Y_{t-1} + u_t$$

Test $H_0: \rho = 1 \approx$ Test $H_0: \delta = 0$

(insign. δ)

D-F Equation: Estimate

$\hat{\delta} = \hat{\rho} - 1$

. reg d.Y l.Y

ΔY_t $\hat{\alpha}$ $\hat{\delta}$ Y_{t-1}

insign insign

If $\hat{\delta}$ is insignificant,

$H_0: \delta = 0$

$(\rho = 1)$

Case II a

$\hat{\alpha}$ insign $\hat{\delta}$ insign

Case II b

$\hat{\alpha}$ sign $\hat{\delta}$ insign

$\boxed{\rho = 1}$ in the original reg.

D-F also assessed "normal t-tests of significance"

does NOT work when $H_0: \rho=1$ $(\delta=0)$

INSTEAD

own critical values → d-fuller

D-F tests

• dfuller Y, trend regress