FINA 4320: Investment Management
Capital Asset Pricing and Arbitrage Pricing Theory Part III

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Road map

CHAPTER 7: Capital Asset Pricing and Arbitrage Pricing Theory

- The **Capital Asset Pricing Model**
  - Assumptions
  - Implications
  - Applications
- Implementing the CAPM: **Index models**
- **Multifactor models** and the CAPM
- **Arbitrage Pricing Theory**

Arbitrage Pricing Theory

The Arbitrage Pricing Theory

- Developed by Ross (1976)
- Why? The assumptions of the CAPM are **unrealistic**. Is possible to arrive at the same mean-beta equation with different assumptions?

The Arbitrage Pricing Theory (APT) derives the asset pricing equation by assuming:

- A **factor structure** of returns
- An **arbitrage-free** market

- **Assumption 1**:
  The returns follow a **factor-structure model**, e.g. the single-index model (can be extended to multifactor models)

  $$R_i = \alpha_i + \beta_i R_m + e_i$$

  where:
  - $R_i = r_i - r_f$ is the excess return of security i
  - $R_m = r_m - r_f$ is the excess return of a tradable **systematic factor** (e.g. the market index)
  - e_i is the **non-systematic part** of the return which is specific to firm i
  - e_i has an average of zero and it is uncorrelated with R_m and uncorrelated across securities

The Arbitrage Pricing Theory

- The non-systematic portion of the portfolio return is

$$e_P = \sum_{i=1}^{n} w_i e_i$$

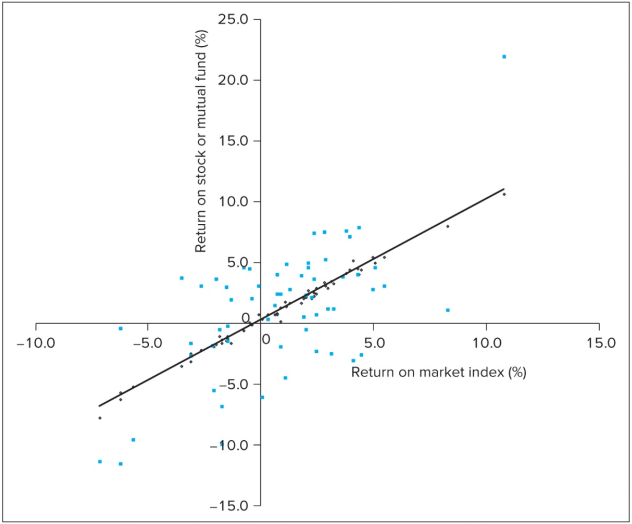
and the portfolio non-systematic variance is

$$\sigma^2(e_P) = \sum_{i=1}^{n} w_i^2 \sigma^2(e_i)$$

which goes to zero as the number of securities increase $\longrightarrow$ **diversification benefits**

The Arbitrage Pricing Theory

Diversification benefits. The blue dots are the returns of a single stock, while the black dots are the returns of a diversified mutual fund.



The Arbitrage Pricing Theory

- **Assumption 2**:
  The market is **arbitrage-free**
  - What is an **arbitrage**? It is the act of exploiting mispricing of two or more securities to achieve risk-free profits
- **APT Theorem**:
  If the returns follow a factor-structure and there are no arbitrage opportunities then for a **well-diversified** portfolio

$$E[r_p] - r_f \quad = \quad \beta_p(E[r_m] - r_f)$$

The Arbitrage Pricing Theory

Intuition:

- For a **well-diversified** portfolio

$$r_p - r_f \quad = \quad \alpha_P + \beta_p(r_m - r_f)$$

  because e_P is almost zero (it has zero mean and almost zero standard deviation, so it is virtually zero).

- Assume that $\beta_P = 0.7$, and α_P is strictly positive $\longrightarrow$ A **positive alpha** gives rise to **arbitrage opportunities**
- How to construct the arbitrage:
  - Consider the *mimicking* portfolio with weights 0.7 in the market index and 0.3 in the T-bill. The beta of this portfolio is 0.7
  - Invest $1 in portfolio P (long) and sell-short $1 dollar in the mimicking portfolio (short)

The **net investment is zero**, but the proceeds are equal to α. It is an **arbitrage**.

Long \$1 in portfolio P	$\$1 \times \left[r_f + \alpha_P + .7\left(r_M - r_f\right)\right]$
Short \$1 in mimicking portfolio	$-\$1 \times \left[r_f + .7\left(r_M - r_f\right)\right]$
Net profit	$\$1 \times \alpha_P$

The Arbitrage Pricing Theory

- Investors would pursue this strategy (long portfolio P and short the mimicking portfolio) at an infinitely large scale and make infinitely large profit
- The buying pressure on portfolio P would increase the price until alpha is driven to zero
- **To conclude**: In $R_p = \alpha_p + \beta_p R_m$, we need $\alpha_p = 0$ by **no arbitrage**
- We therefore have the CAPM relationship

$$\begin{aligned} r_p - r_f &= \beta_p(r_m - r_f) \\ E[r_p] - r_f &= \beta_p(E[r_m] - r_f) \end{aligned}$$

The Arbitrage Pricing Theory

What are the **differences with the CAPM**?

- We **do not need restrictions** on preferences/capital markets
- No discussion about **“market portfolio”**
- APT applies to **well-diversified** portfolios, not individual securities

The Arbitrage Pricing Theory: Applications

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  Answer: The risk-free rate, otherwise there is an arbitrage opportunity

- Is possible to have two well diversified portfolios with different returns but the same beta?

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- Is possible to have two well diversified portfolios with different returns but the same beta?

  Answer: no, otherwise you can construct a long-short strategy which delivers a return of alpha with a zero-net investment

APT: Examples

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  - Borrow at 6% and invest at 7%
- Case 2: The riskfree rate is 6%. A well diversified zero-beta portfolio earns a sure return of 5%. How can you exploit this?
  - Sell short the zero-beta portfolio and lend at 6%

APT: Examples

- In the example, the portfolio has zero beta. What if it doesn't?
- One can always construct a zero-beta portfolio
- Consider the beta of a portfolio of two assets with betas β_v and β_u and use the following weights

$$\frac{-\beta_u}{\beta_v - \beta_u};\ \frac{\beta_v}{\beta_v - \beta_u}$$

- We get

$$\beta_{u+v} = \beta_v \frac{-\beta_u}{\beta_v - \beta_u} + \beta_u \frac{\beta_v}{\beta_v - \beta_u} = 0$$

- If the return on the resulting portfolio is not equal to the risk-free rate, there is an arbitrage opportunity

APT: Examples

- Suppose the risk-free rate is 7%. There is a well-diversified portfolio V with a beta of 1.3 and an alpha of 2%, and another well-diversified portfolio U with a beta of 0.8 and an alpha of 1%
- Use weights

$$w_v = \frac{-0.8}{1.3 - 0.8} = -1.6; \;\; w_u = \frac{1.3}{1.3 - 0.8} = 2.6$$

- The beta of the resulting portfolio is

$$(-1.6 \times 1.3) + (2.6 \times 0.8) = 0$$

- The alpha of the portfolio is

$$(-1.6 \times 2\%) + (2.6 \times 1\%) = -0.6\%$$

- The arbitrage consists of going short on the portfolio and investing the proceeds at 7%. The gain is 60 bps.

APT: Short-Long Strategies

- An important insight associated with the APT is the use of **long-short strategies** to eliminate factor risk (like the previous example).
- A positive alpha indicates an "abnormal" expected return to buying (going long). A negative alpha indicates a negative expected (extra) return to buying, which means a positive return to going short.
- If we find well-diversified portfolios with the same amount of factor risk, we can get rid of the factor risk by establishing a short/long position. This guarantees a profit.
- This is a strategy which is increasingly used by hedge funds. What about risk? This does NOT apply to individual assets

EXERCISE 1.
Consider the following data for a one-factor economy. All portfolios are well-diversified

Portfolio	$E(r)$	Beta
A	10%	1.0
F	4	0

Suppose another portfolio E is well-diversified with a beta of 2/3 and expected return of 9%. Would an arbitrage opportunity exist? If so, what would be the arbitrage strategy?

APT: EXERCISES

ANSWER.

- Since the beta of portfolio F is zero, the expected return for Portfolio F equals the risk-free rate.
- The weights of a zero-beta portfolio are $w_A = \frac{-\beta_E}{\beta_A - \beta_E} = -2$ and $w_E = \frac{\beta_A}{\beta_A - \beta_E} = 3$
- The expected return of the zero-beta strategy is $-2 \times 10 + 3 \times 9 = 7$ and the alpha is 3.
- The arbitrage consists in going long the zero-beta portfolio and borrow at the risk-free rate.

APT: EXERCISES

EXERCISE 2.
Assume both portfolios A and B are well-diversified, that $E[r_A] = 14\%$ and $E[r_B] = 14.8\%$. If the economy has only one factor, and $\beta_A = 1$ while $\beta_B = 1.1$, what must be the risk free rate?

ANSWER.
$E[r_m] = E[r_A] = 14\%$ because $\beta_A = 1$. Then we can find r_f by solving the equation $E[r_B] = r_f + \beta_B(E[r_m] - r_f)$. The solution is $r_f = 6\%$.

EXERCISE 3.

Assume a well-diversified portfolios A has an expected return of 10% and a beta of 0.9. Assume the economy has only one factor, the market index. If the return of the market index turns out to be 2% more than you expected, how would you revise your expectation on the return of portfolio A?

ANSWER.

The new expected return is equal to the old expected return plus $\beta \times \Delta r_m$, so the new expected return is $10 + 0.9 \times 2 = 11.8$