The Brachistochrone Problem for Constant and Inverse-Square Forces

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1. Introduction

1.1. Infinite-dimensional optimization

In the simplest optimization problems, familiar from single-variable calculus, we must find a point (or points) on the real line where a function is maximized or minimized. More complicated optimization problems may involve multiple variables—rather than searching for a point on the real line, we now search for a point in n-dimensional space. But some natural optimization problems are not of even this more general type; for example, consider the problems of

(a) finding the shortest path between two points on a surface,

(b) finding the surface of minimal area with a given boundary, or

(c) finding the state of a wave or force field that minimizes its energy.

Instead of finding a *point* (a number or several numbers) which extremizes a *function*, we must now find a *function* which extremizes a *functional* (a function of a function).

Such problems are the domain of the *calculus of variations*. Intuitively, we can think of calculus of variations as a mathematical theory with which we can solve "infinite-dimensional" optimization problems, just as multi-variable calculus is a mathematical theory with which we can solve finite-dimensional optimization problems. Clearly the method of solving finite-dimensional optimization problems (modulo some details, just find where the derivative of the function vanishes) does not trivially carry over to the infinite-dimensional case—what is the "derivative" of a functional? Fortunately, answering this question will allow us to reduce such optimization problems to solving one or more differential equations. Thus, the seemingly intractable problem of finding the "best" function out of an enormous space of possibilities yields to a powerful general procedure.

1.2. A brief history of the brachistochrone problem

One of the earliest and most famous problems in the calculus of variations is the so-called *brachistochrone problem* (from Ancient Greek βράχιστος χρόνος, meaning "shortest time"). The problem was posed by Johann Bernoulli in the journal *Acta Eruditorum* in June 1696, as a challenge to other mathematicians:

> *Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?*

More concretely: if a bead slides down a wire between two fixed points, what shape of the wire will minimize the travel time?

Galileo had already studied the brachistchrone problem in his 1638 work *Two New Sciences*, and correctly deduced that a straight line would not give the shortest possible time of descent. However, he did not find the actual brachistochrone curve, incorrectly concluding that it is a circular arc. Johann Bernoulli, who had already arrived at his own solution, received four solutions in response to his challenge, from Newton, Leibniz, l'Hôpital, and his brother Jacob Bernoulli.

These original solutions used various clever geometric methods to find the brachistochrone curve. In 1744, Euler generalized these geometric methods in his *Method for finding plane curves that show some property of maxima and minima*, deriving what we now know as the Euler-Lagrange equation for extrema of a functional. Lagrange put these results in an analytic framework in his 1760 paper *Essay on a new method of determining the maxima and minima of indefinite integral formulas*, developing the basic theory still used to study problems of extremizing functionals today. [3]

1.3. Generalizations of the problem

The classical brachistochrone problem assumes a constant force acting on the particle. This is perfectly reasonable when considering gravity near the Earth's surface. But if, say, we wanted to construct a sufficiently large brachistochrone in space near Earth, it would be inappropriate to assume that gravity is approximately constant, so the more correct inverse-square law (universally applicable in non-relativistic physics) would be required to reach a reasonably accurate solution.

The variational methods we use to solve the constant force brachistochrone problem readily yield solutions, albeit not explicit ones, to the brachistochrone problem for an arbitrary central force. Thus we can obtain an (integral) equation for the brachistochrone curve under an attractive inverse-square force, like the electric force (in case of opposite charges) or gravitational force. The resulting elliptic integral, while impossible to express in terms of elementary functions, can be approximated numerically with a computer.

2. Calculus of variations

2.1. Functionals and their extrema

First, some terminology: a *functional* is simply a map from some function space to the real numbers. One can also speak of functionals with values in the complex numbers or some other field, but since we are interested in optimization problems, we will only deal with real-valued functionals.

We say that a functional I has a *maximum* at a function y if $I[\tilde{y}] - I[y] \leqslant 0$ for all $\tilde{y}$ in a sufficiently small neighborhood of y; likewise, we say that I has a *minimum* at y if $I[\tilde{y}] - I[y] \geqslant 0$ for all $\tilde{y}$ in a sufficiently small neighborhood of y. (By "$\tilde{y}$ in a sufficiently small neighborhood of y," we mean $|\tilde{y} - y| < \varepsilon$ for some $\varepsilon > 0$; the choice of norm on the function space does not concern us here.) As usual, the maxima and minima of I are both called *extrema* of I.

2.2. The Euler-Lagrange equation

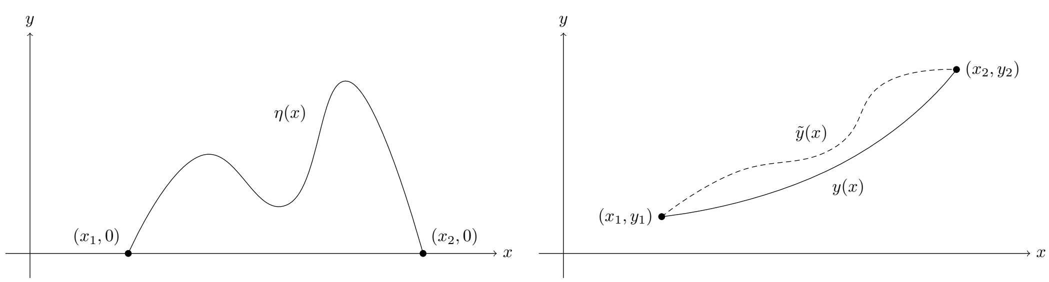
To derive a useful condition to determine where a functional has extrema, we will need to narrow our scope slightly, from general functionals to functionals of the form

$$I[y] = \int_{x_1}^{x_2} \mathcal{L}(x, y(x), y'(x))\, dx \qquad \left(\text{where } y' = \frac{dy}{dx}\right). \tag{1}$$

This integrand $\mathcal{L}$ is called the *Lagrangian* of the functional I. From now on, we also assume that the argument y is always in $\mathcal{C}^2$, meaning it has continuous second derivatives—otherwise, other methods are required.

Suppose that y is an extremum of such a functional. Let η be some arbitrary function (also with continuous second derivatives) such that $\eta(x_1) = \eta(x_2) = 0$, and let $\tilde{y} = y + \varepsilon\eta$. Now note that the value of I over these curves can be considered a function of the parameter ε alone:

$$\tilde{I}(\varepsilon) = I[y + \varepsilon\eta].$$



Since I has an extremum at y, the associated function $\tilde{I}$ must have an extremum at $\varepsilon = 0$. Therefore

$$\left.\frac{d\tilde{I}}{d\varepsilon}\right|_{\varepsilon=0} = 0.$$

Now we finally have an equation suitable for further calculation. First, we bring the derivative inside the integral and apply the chain rule:

$$\frac{d\tilde{I}}{d\varepsilon} = \int_{x_1}^{x_2} \frac{d}{d\varepsilon}\mathcal{L}(x, \tilde{y}, \tilde{y}')\, dx = \int_{x_1}^{x_2} \left(\frac{\partial\mathcal{L}}{\partial\tilde{y}}\frac{\partial\tilde{y}}{\partial\varepsilon} + \frac{\partial\mathcal{L}}{\partial\tilde{y}'}\frac{\partial\tilde{y}'}{\partial\varepsilon}\right) dx.$$

Clearly $\partial\tilde{y}/\partial\varepsilon = \eta$ and $\partial\tilde{y}'/\partial\varepsilon = \eta'$. Furthermore, $\varepsilon = 0$ implies $\tilde{y} = y$ and $\tilde{y}' = y'$, so we have

$$0 = \int_{x_1}^{x_2} \left(\frac{\partial\mathcal{L}}{\partial y}\eta(x) + \frac{\partial\mathcal{L}}{\partial y'}\eta'(x)\right) dx.$$

Integrating the second term by parts yields

$$0 = \frac{\partial \mathcal{L}}{\partial y'}\eta(x)\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{\partial \mathcal{L}}{\partial y}\eta(x) - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'}\eta(x)\right) dx,$$

but since η vanishes at the endpoints x_1 and x_2, this simplifies to

$$0 = \int_{x_1}^{x_2} \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'}\right) \eta(x)\, dx.$$

Remember that this holds for *any* choice of η. To complete the derivation, we need to make use of one more fact:

> **The fundamental lemma of calculus of variations (weak version).** *Let f be a continuous real-valued function on $[a, b]$. If $\int_a^b f(x)g(x)\,dx = 0$ for all $g \in \mathcal{C}^2$ such that $g(a) = g(b) = 0$, then $f(x) \equiv 0$.*
>
> *Proof.* Suppose $f(x_0) \neq 0$ at some point $x_0 \in [a, b]$. Without loss of generality, we can assume that $f(x_0) > 0$. Since f is continuous, there exists a neighborhood U of x_0 such that $f(x) > 0$ for all $x \in U$. Take g to be a $\mathcal{C}^2$ function which vanishes outside of U and is strictly positive inside U. (The existence of such smooth functions with compact support, or "bump" functions, is easily shown by construction.) Then $\int_a^b f(x)g(x)\,dx = \int_U f(x)g(x)\,dx > 0$, a contradiction. Thus f must be identically zero on $[a, b]$. ■

By the fundamental lemma, we finally obtain

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = 0. \tag{2}$$

This is the *Euler-Lagrange equation*. It gives a necessary condition for $y \in \mathcal{C}^2$ to be an extremum of $I = \int \mathcal{L}\,dx$. [1]

2.3. Euler-Lagrange equations for multiple functions or variables

The same reasoning used to derive the Euler-Lagrange equation yields similar equations for the extrema of functionals dependent on multiple functions of one variable, one function of multiple variables, or multiple functions of multiple variables. Here we state the results, which will be used in some examples later.

First, suppose our functional depends on multiple functions $\mathbf{y} = (y_1, \ldots, y_n)$ of one independent variable x, so that

$$I[\mathbf{y}] = \int_{x_1}^{x_2} \mathcal{L}(x, \mathbf{y}(x), \mathbf{y}'(x))\, dx. \tag{3}$$

Then the extrema satisfy the Euler-Lagrange *equations*

$$\frac{\partial \mathcal{L}}{\partial y_1} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y_1'} = 0,\ \ldots\ ,\ \frac{\partial \mathcal{L}}{\partial y_n} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y_n'} = 0. \tag{4}$$

We can summarize these neatly with vector notation as

$$\frac{\partial \mathcal{L}}{\partial \mathbf{y}} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \mathbf{y}'} = 0. \tag{5}$$

Next, suppose our functional depends on one function y of multiple independent variables $\mathbf{x} = (x_1, \ldots, x_n)$, so that

$$I[y] = \int_\Omega \mathcal{L}(\mathbf{x}, y(\mathbf{x}), \nabla y(\mathbf{x}))\, d\mathbf{x} \qquad (\text{where } D \subset \mathbf{R}^n)\,. \tag{6}$$

Then the extrema satisfy the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx_1}\frac{\partial \mathcal{L}}{\partial y_{x_1}} - \cdots - \frac{d}{dx_n}\frac{\partial \mathcal{L}}{\partial y_{x_n}} = 0, \tag{7}$$

where $y_{x_i} = \partial y / \partial x_i$. As before, this equation can be rewritten compactly as

$$\frac{\partial \mathcal{L}}{\partial y} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla y} = 0. \tag{8}$$

Finally, suppose our functional depends on multiple functions $\mathbf{y} = (y_1, \ldots, y_n)$ of multiple independent variables $\mathbf{x} = (x_1, \ldots, x_m)$, so that

$$I[\mathbf{y}] = \int_\Omega \mathcal{L}(\mathbf{x}, \mathbf{y}(\mathbf{x}), \nabla \mathbf{y}(\mathbf{x}))\, d\mathbf{x} \qquad (\text{where } \nabla \mathbf{y} \text{ is the Jacobian of } \mathbf{y})\,. \tag{9}$$

Then the extrema satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial y_i} - \frac{d}{dx_1}\frac{\partial \mathcal{L}}{\partial y_{i,x_1}} - \cdots - \frac{d}{dx_m}\frac{\partial \mathcal{L}}{\partial y_{i,x_m}} = 0 \quad \text{for} \quad i = 1, \ldots, n, \tag{10}$$

so for this most general form of the Euler-Lagrange equations, we have the vector equation

$$\frac{\partial \mathcal{L}}{\partial \mathbf{y}} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla \mathbf{y}} = 0. \tag{11}$$

With these multi-variable versions of the Euler-Lagrange equation, we can find not just extremal curves in the plane, but extremal curves or extremal surfaces (or their higher-dimensional analogues) in n-dimensional space. [1]

3. Examples of applications of calculus of variations

3.1. Geodesics on the plane

Before applying the Euler-Lagrange formalism to the brachistochrone problem, let us first test it on some other classic optimization problems. We begin with the most familiar of them: given two points in the plane, what is the shortest curve connecting them? That is, we must find the function y which minimizes the arc length functional

$$s = \int \sqrt{1 + y'^2}\, dx.$$

The Lagrangian and its partial derivatives for this problem are

$$\mathcal{L} = \sqrt{1+y'^2}, \quad \frac{\partial \mathcal{L}}{\partial y} = 0, \quad \frac{\partial \mathcal{L}}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

so the Euler-Lagrange equation for one function of one variable is

$$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0.$$

Thus $y'/\sqrt{1+y'^2} = C$ for some constant C, yielding $y' = \pm\sqrt{C^2/(1-C^2)}$. That is, y' is simply a constant, call it a, so

$$y(x) = ax + b.$$

As expected, the shortest path is in fact a line.

3.2. Dirichlet's principle for harmonic functions

Recall that a harmonic function u is one that satisfies the Laplace equation $\Delta u = 0$. The functional $E[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 \, d\mathbf{x}$ is called the *Dirichlet energy* of u. It is not hard to show (using Green's first identity) that harmonic functions minimize the Dirichlet energy for a given boundary condition. It turns out that the converse is also true: functions that minimize the Dirichlet energy are harmonic. Thus Laplace's equation is actually equivalent to a variational problem: "what function (or, in physics, potential) minimizes the energy?" Let us show this using calculus of variations. Our functional is

$$E = \iiint_\Omega \frac{1}{2}(u_x^2 + u_y^2 + u_z^2)\, dx\, dy\, dz.$$

The Lagrangian and its partial derivatives are therefore

$$\mathcal{L} = \frac{1}{2}(u_x^2 + u_y^2 + u_z^2), \quad \frac{\partial \mathcal{L}}{\partial u} = 0, \quad \frac{\partial \mathcal{L}}{\partial u_x} = u_x, \quad \frac{\partial \mathcal{L}}{\partial u_y} = u_y, \quad \frac{\partial \mathcal{L}}{\partial u_z} = u_z,$$

so the Euler-Lagrange equation for one function of multiple variables becomes

$$-u_{xx} - u_{yy} - u_{zz} = 0.$$

That is, $\Delta u = 0$. [4]

3.3. The principle of least action

Just as the Laplace equation can be restated as a minimization problem, so too can the equations of motion in classical mechanics. For a particle whose position at time t is $\mathbf{x}$, we define the Lagrangian as

$$\mathcal{L} = T(\mathbf{x}') - U(t, \mathbf{x}, \mathbf{x}'), \tag{12}$$

where T is its kinetic energy and U its potential energy. The functional obtained by integrating this Lagrangian over time is known as the *action*:

$$S = \int_{t_1}^{t_2} \mathcal{L}(t, \mathbf{x}, \mathbf{x}')\, dt. \tag{13}$$

Now classical mechanics can be rephrased thusly: *a particle moves in a way that makes the action stationary*. (That is, the trajectory of the particle is either a minimum, maximum, or saddle point of the action. In any case, the Euler-Lagrange equations are applicable.)

To see how the Lagrangian theory relates to the Newtonian one, suppose we have a particle with kinetic energy $T(\mathbf{x}') = \frac{1}{2}m|\mathbf{x}'|^2$ moving under the influence of a conservative force, so the potential energy U only depends on $\mathbf{x}$. Then

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = -\nabla U, \quad \frac{\partial \mathcal{L}}{\partial \mathbf{x}'} = m\mathbf{x}'.$$

Then the Euler-Lagrange equation for multiple functions of one variable gives

$$-\nabla U = m\mathbf{x}''. \tag{14}$$

Notice that the left-hand side is just the net force on the particle—so we have actually rederived Newton's Second Law from the principle of least action!

4. Brachistochrone curves for a constant force

4.1. Deriving the time functional

Having developed the requisite theory, we now turn to our primary object of study, the brachistochrone curve. Let the y axis point down, in the direction of gravity. Without loss of generality, suppose the particle starts at the origin, and must reach the point (x_1, y_1) in minimal time.

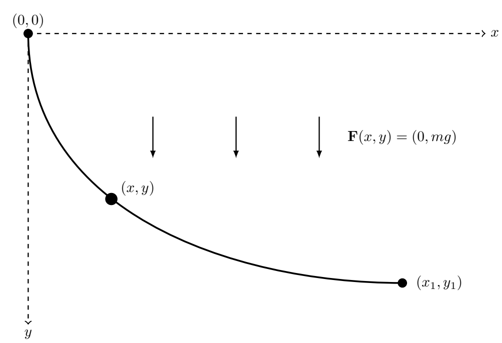


Figure 1: Illustration of the constant-force problem

Conservation of energy implies $\frac{1}{2}mv^2 - mgy = 0$ at all points along the descent, so the velocity is given by $v = \sqrt{2gy}$. The time differential and arc length differential are

$$dt = \frac{ds}{v}, \quad ds = \sqrt{1 + x'^2}\, dy \quad \left(\text{where } x' = \frac{dx}{dy}\right).$$

Therefore the total time t, obtained by integrating dt over the whole path, is

$$t = \frac{1}{\sqrt{2g}} \int_0^{y_1} \sqrt{\frac{1 + x'^2}{y}}\, dy. \tag{15}$$

4.2. Applying the Euler-Lagrange equation

The Lagrangian and its partial derivatives are

$$\mathcal{L} = \sqrt{\frac{1+x'^2}{y}}, \quad \frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial x'} = \frac{x'}{\sqrt{y(1+x'^2)}},$$

so the Euler-Lagrange equation, which reduces to $d/dy\,(\partial\mathcal{L}/\partial x') = 0$, gives

$$\frac{x'}{\sqrt{y(1+x'^2)}} = \frac{1}{\sqrt{\alpha}} \quad \text{(where } \alpha \text{ is some constant).}$$

Solving for x', we find that $x' = \sqrt{y/(\alpha - y)}$. Integration then gives the desired expression for x as a function of y:

$$x(y_0) = \int_0^{y_0} \sqrt{\frac{y}{\alpha - y}}\, dy. \tag{16}$$

The constant α is simply whatever ensures that the curve passes through (x_1, y_1). Plots created with numerical integration in MATLAB for $\alpha = 2, 4, 6, 8, 10$ are shown below.

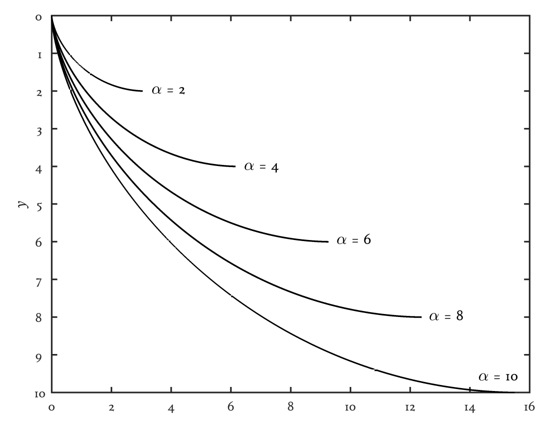


Figure 2: Computed constant-force brachistochrones

4.3. Identifying the resulting curve

Consider the change of variable

$$y = \alpha \sin^2 \frac{\theta}{2} = \frac{\alpha}{2}(1 - \cos\theta), \quad dy = \alpha \sin\frac{\theta}{2} \cos\frac{\theta}{2}\, d\theta.$$

Substituting this into our expression for x' gives

$$dx = \sqrt{\frac{\alpha \sin^2 \frac{\theta}{2}}{\alpha - \alpha \sin^2 \frac{\theta}{2}}} \cdot \alpha \sin\frac{\theta}{2} \cos\frac{\theta}{2}\, d\theta = \alpha \sin^2 \frac{\theta}{2}\, d\theta = \frac{\alpha}{2}(1 - \cos\theta)\, d\theta.$$

Therefore our curve has parametric equations

$$x(\theta) = \frac{\alpha}{2}(\theta - \sin\theta), \quad y(\theta) = \frac{\alpha}{2}(1 - \cos\theta). \tag{17}$$

These are the equations of a *cycloid*, the curve traced out by a point fixed to a circle of radius $\alpha/2$ rolling on level ground. As before, the parameter α is chosen so that the curve goes through the given point (x_1, y_1). [1]

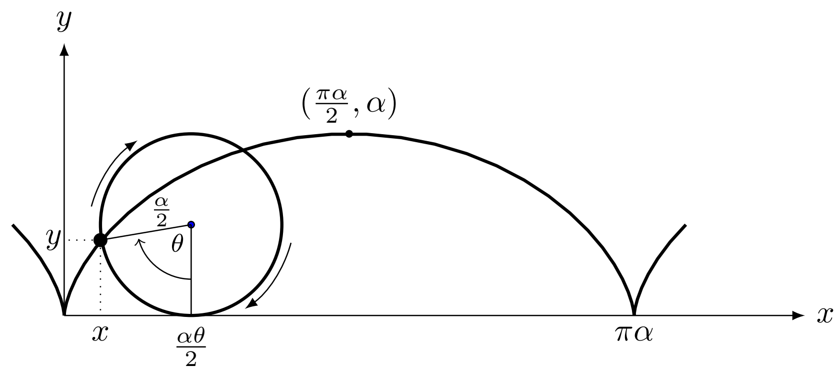


Figure 3: Construction of a cycloid

5. Brachistochrone curves for an inverse-square force

5.1. The problem for a general attractive central force

In this section, we exclusively use polar coordinates (r, θ). The particle is now placed in a potential ϕ which depends only on r and monotonically increases with r (so the force on the particle always points towards the central point $r = 0$). Without loss of generality, suppose the particle starts at $(R, 0)$, moves in the positive θ direction, and must reach (r_1, θ_1) in minimal time.

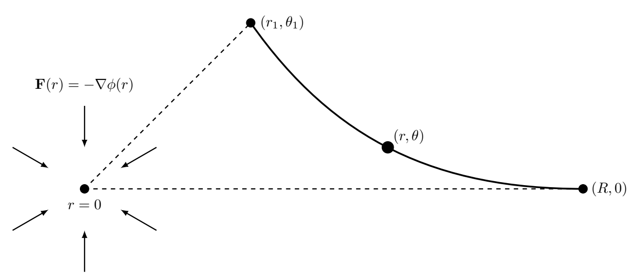


Figure 4: Illustration of the central-force problem

We proceed as in the constant-force case, first deriving the time functional, then applying the Euler-Lagrange equation to find the optimal path.

From conservation of energy, we have $\phi(R) = \frac{1}{2}mv^2 + \phi(r)$ at all points along the descent, so the velocity is given by $v = \sqrt{-(2/m)\Delta\phi(r)}$, where for convenience we let $\Delta\phi(r) = \phi(r) - \phi(R)$ denote the change in potential energy.

The time differential and arc length differential are

$$dt = \frac{ds}{v}, \quad ds = -\sqrt{1 + r^2\theta'^2}\, dr \quad \left(\text{where } \theta' = \frac{d\theta}{dr}\right).$$

Therefore the total time of descent t is

$$t = \sqrt{\frac{m}{2}} \int_{r_1}^{R} \sqrt{\frac{1 + r^2\theta'^2}{-\Delta\phi(r)}}\, dr. \tag{18}$$

The Lagrangian and its partial derivatives are

$$\mathcal{L} = \sqrt{\frac{1 + r^2\theta'^2}{-\Delta\phi(r)}}, \quad \frac{\partial \mathcal{L}}{\partial \theta} = 0, \quad \frac{\partial \mathcal{L}}{\partial \theta'} = \frac{r^2\theta'}{\sqrt{-\Delta\phi(r)(1 + r^2\theta'^2)}},$$

so the Euler-Lagrange equation, which reduces to $d/dr\,(\partial\mathcal{L}/\partial\theta') = 0$, becomes

$$\frac{r^2\theta'}{\sqrt{-\Delta\phi(r)(1 + r^2\theta'^2)}} = \frac{1}{\sqrt{\alpha}} \quad (\text{where } \alpha \text{ is some constant}).$$

Solving for θ', we find that $\theta' = -\sqrt{-\Delta\phi(r)/(\alpha r^2 + \Delta\phi(r))}/r$. Integration then gives the equation of our brachistochrone curve:

$$\theta(r_0) = \int_{r_0}^{R} \frac{1}{r} \sqrt{\frac{-\Delta\phi(r)}{\alpha r^2 + \Delta\phi(r)}}\, dr. \tag{19}$$

Any further simplification requires specific knowledge of the potential function ϕ.

5.2. The problem for an inverse-square force

For a force obeying an attractive inverse-square law, where $\mathbf{F}(r) = -(\mu/r^2)\hat{\mathbf{r}}$ for some constant μ, we have

$$\phi(r) = -\frac{\mu}{r}, \quad \Delta\phi(r) = \mu\left(\frac{1}{R} - \frac{1}{r}\right).$$

Substituting this into the formula we derived for θ, and letting $\beta = \alpha R/\mu$, we obtain

$$\theta(r_0) = \int_{r_0}^{R} \frac{1}{r} \sqrt{\frac{-(r - R)}{\beta r^3 + r - R}}\, dr. \tag{20}$$

This is an elliptic integral (and a nasty one at that), and cannot be expressed purely in terms of elementary functions. Thus we turn to computational methods. [2]

5.3. Numerical approximations of the curves

As with the constant-force case, we use numerical integration in MATLAB to plot the brachistochrone curves (only this time, there is no closed-form solution to fall back on). First, we compute and plot the function $\theta(r)$ using the integral formula we just derived, with the constants set to $R = 10$ and $\beta = 0.01, 0.1, 1, 10, 100$. Then we can use polar coordinates to plot the actual brachistochrone curves in the plane.

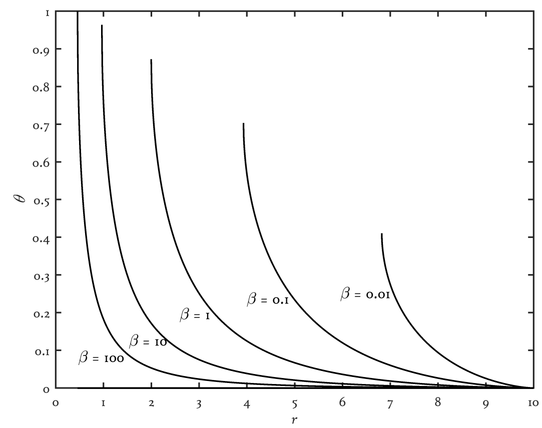


Figure 5: Computed $\theta(r)$ curves

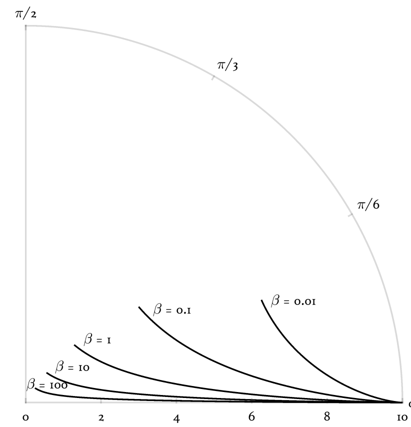


Figure 6: Computed central-force brachistochrones

Notice that for a constant force, the brachistochrone curves through any two points were similar (in the geometric sense), but here, for a variable force, they are not.

6. Further possible areas of investigation

We have considered the brachistochrone problem in the case of a constant force, a central force, and an inverse-square force. Naturally, one could try to find the brachistochrone curves for other forces as well. Using Cartesian coordinates, and assuming the particle starts at the origin and ends up at (x_1, y_1), the same calculations done earlier show that $v = \sqrt{-(2/m)\Delta\phi(x,y)}$ for a potential energy $\phi(x,y)$, and thus

$$t = \sqrt{\frac{m}{2}} \int_0^{y_1} \sqrt{\frac{1+x'^2}{-\Delta\phi(x,y)}}\, dy = \sqrt{\frac{m}{2}} \int_0^{x_1} \sqrt{\frac{1+y'^2}{-\Delta\phi(x,y)}}\, dx. \tag{21}$$

In general, the Lagrangian now depends on *three* variables (either (y, x, x') or (x, y, y'), respectively), so the Euler-Lagrange equation will not simplify like it did in the cases we examined. Thus, potentials with symmetries of some kind (those that can be written as a function of a *single* variable) are much more easily studied.

Also note that we only considered plane curves in our analysis, which suggests another interesting generalization of the classical problem: finding brachistochrones on surfaces. This involves replacing the arc length differential $ds^2 = dx^2 + dy^2$ with the more general $ds^2 = E\,du^2 + G\,dv^2$, where the metric coefficients E, G depend on the geometry of the surface parametrized by u, v. Once again, the general integral, taken at face value, is not *prima facie* especially enlightening; particular (symmetric) geometries are needed to find explicit brachistochrone curves.

Their additional complications notwithstanding, all these generalizations of the original problem can be analyzed with the same techniques from calculus of variations, demonstrating the theory's power and flexibility. Far from a mere 18th century curiosity, the brachistochrone problem has proved to be an important catalyst for the development of a branch of mathematics with far-reaching applications in geometry, analysis, and physics.

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