AST1100 - Oblig 8

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1 Problem 1

1. We know that the time inverval Δt measured on the wristwatch of an observer between two events happening at the same location as the observer, near a non-rotating, uncharged gravitational body is related to the time interval $\Delta t'$ measured by an observer far away, through the Schwarzschild metric. The wristwatch time measured by the shell-observer is what we call proper time and is always equal to the space-time line element, in this case given by the Schwarzschild metric. We get,

$$\Delta s^2 = \Delta\tau^2 = \Delta t^2 = \left(1 - \frac{2M}{r}\right)(\Delta t')^2 - \frac{\Delta r^2}{\left(1 - \frac{2M}{r}\right)} - r^2\Delta\phi^2$$

Where $\Delta r = 0 \wedge \Delta\phi = 0$

$$\Delta t^2 = \left(1 - \frac{2M}{r}\right)(\Delta t')^2$$

$$\Delta t = \sqrt{\left(1 - \frac{2M}{r}\right)}\Delta t' \Rightarrow \frac{\Delta t}{\sqrt{\left(1 - \frac{2M}{r}\right)}}$$

2. If we let Δt be the time interval between two peaks on the electromagnetic wave from the laser, and since we measure time and space in the same units, $\Delta t = \lambda$. From the previous result we get for the gravitational doppler shift,

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda}{\lambda} = \frac{\Delta t' - \Delta t}{\Delta t} = \frac{\frac{\Delta t}{\sqrt{\left(1 - \frac{2M}{r}\right)}} - \Delta t}{\Delta t} = \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} - 1$$

3. Since $\frac{2M}{r} \approx 0$ when $r \gg 2M$, we can expand $\frac{1}{\sqrt{1-x}}$ by a Taylor series where $x = \frac{2M}{r}$ around 0. We get,

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$$

Using this approximation we get,

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} - 1 \approx 1 + \frac{1}{2}\frac{2M}{r} - 1 = \frac{M}{r}$$

When $r \gg 2M$.

4.

(a) To get the mass of the Sun in meters, we just multiply it with the appropriate constants.

$$M_{Sun} = M_{Sun}\frac{G}{c^2} = 2.0 \cdot 10^{30} kg \frac{6.67 \cdot 10^{-11} \frac{m^3}{kgs^2}}{(3.0 \cdot 10^8 \frac{m}{2})^2} \approx 1482m$$

(b)

$$\frac{M_{Sun}}{r_{Sun}} = \frac{1482m}{7.0 \cdot 10^8 m} \approx 2.1 \cdot 10^{-6}$$

(c) In problem 1.3 we found that $\frac{\Delta\lambda}{\lambda} \approx \frac{M}{r}$ when $r \gg 2M$.

$$\frac{\Delta\lambda}{\lambda} \approx \frac{M_{Sun}}{r_{Sun}} = 2.1 \cdot 10^{-6}$$

(d) The human eye is barely able to distinguish between wavelenghts of 1nm difference. A difference of $\pm 2.1 \cdot 10^{-6}\lambda_0$ is way to small to perceive. Though in theory the color is changed someewhat.

(e)

$$M_{Earth} = 6.0 \cdot 10^{24} kg \frac{6.67 \cdot 10^{-11} \frac{m^3}{kgs^2}}{(3.0 \cdot 10^8 \frac{m}{2})^2} \approx 4.4 \cdot 10^{-3} m$$

$$\frac{M_{Earth}}{r_{Earth}} = \frac{4.4 \cdot 10^{-3} m}{6.4 \cdot 10^6 m} \approx 6.9 \cdot 10^{-10}$$

(f) Since the wave is comming from far away the wavelength is the time interval between two peaks on the wave, $\Delta t'$. When it reaches Earth the time between the two peaks has been reduced by a fator $\sqrt{\left(1 - \frac{2M}{r}\right)}$. In other words the wavelenght of the beam has been blueshifted and is now shorter. We have the relation $\Delta t = \sqrt{\left(1 - \frac{2M}{r}\right)}\Delta t'$ and we get,

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta t - \Delta t'}{\Delta t'} = \frac{\sqrt{\left(1 - \frac{2M}{r}\right)}\Delta t' - \Delta t'}{\Delta t'} = \sqrt{\left(1 - \frac{2M}{r}\right)} - 1$$

For the Earth we get,

$$\frac{\Delta\lambda}{\lambda} = \sqrt{\left(1 - \frac{2M_{Earth}}{r_{Earth}}\right)} - 1 = \sqrt{\left(1 - \frac{2 \cdot 4.4 \cdot 10^{-3}}{6.4 \cdot 10^{6}}\right)} - 1 \approx -6.9 \cdot 10^{-10}$$

The blueshift is too small to be perceived by the unaided human eye.

5. Since we assumed that the doppler shifts from the velocities cancel each other out, and that the doppler shift from the movement of the quasar is corrected for, and we still see a huge redshift it could mean that it is due to a gravitational doppler shift due to a black hole. We can find the radius of the black hole in terms of the black hole mass M, from the formula for the doppler shift we derived in problem 1.2.

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} - 1$$

$$\frac{2M}{r} = 1 - \frac{1}{\left(\frac{\Delta\lambda}{\lambda} + 1\right)^2}$$

$$\frac{r}{M} = \frac{2}{1 - \left(\frac{\Delta\lambda}{\lambda} + 1\right)^{-2}} = \frac{2}{1 - \left(\frac{2150-600}{600} + 1\right)^{-2}} \approx 2.17$$

This is a very small number compared to the Sun ($\approx 5.7 \cdot 10^5$), it seems likely that a black hole is at the center of quasars.

6. The gravitational doppler shift near a black hole at $r = 2.01M$ is,

$$\frac{\Delta\lambda}{\lambda} = 1 - \sqrt{\left(1 - \frac{2M}{2.01M}\right)} \approx 0.93$$

If a star radiated at $500nm$ wavelength the apperent wavelength near the horizon would be,

$$\frac{500nm - \lambda'}{500nm} = 0.93 \Rightarrow \lambda' = 500nm - 500nm \cdot 0.93 = 35nm$$

I guess you still could use optical telescopes to observe the star, but you wouldn't see the radiation at 500nm. All the radiation would be shifted to 7 % of it's original value.