AST1100 - Oblig 10

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1 Problem 1

1. We are going to find the constant, N that normalizes the Maxwell-Boltzmann distrubution,

$$n(v)dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}} 4\pi v^2 dv \tag{1}$$

We have,

$$n_{norm}(v) = \frac{1}{N} n(v) \tag{2}$$

and by definition,

$$\int_{-\infty}^{\infty} n_{norm}(v)dv = 1$$

Which means that by integrating eq. 2 on both sides we get,

$$1 = \frac{1}{N}\int_{-\infty}^{\infty} n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}} 4\pi v^2 dv$$

$$N = 4\pi n\left(\frac{m}{2\pi kT}\right)^{3/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{mv^2}{kT}} v^2 dv \tag{3}$$

We now need to solve the integral in eq. 3. We observe that the integral is symetric about zero because of v^2. So we have,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{mv^2}{kT}} v^2 dv = 2\int_{0}^{\infty} e^{-\frac{1}{2}\frac{mv^2}{kT}} v^2 dv \tag{4}$$

From Rottmann we have,

$$\int_{0}^{\infty} e^{-\lambda x^2} x^k dx = \frac{1}{2}\lambda^{-\frac{k+1}{2}}\Gamma\left(\frac{k+1}{2}\right), k > -1, \lambda > 0 \tag{5}$$

Which gives us,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{mv^2}{kT}} v^2 dv = \left(\frac{m}{2kT}\right)^{-3/2} \Gamma\left(\frac{3}{2}\right) \tag{6}$$

Plugging the solution from eq. 6 in to eq. 3 we get,

$$N = 4\pi n \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{m}{2kT}\right)^{-3/2} \Gamma\left(\frac{3}{2}\right)$$

$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, so almost everything cancel. We are left with,

$$N = n \tag{7}$$

2. We can now use eq. 7 to write the normalized Maxwell-Boltzmann distribution (eq. 2) as,

$$n_{norm}(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}} 4\pi v^2 \tag{8}$$

The mean of a function $f(x)$ is given by.

$$< f(x) >= \int f(x)P(x)dx \tag{9}$$

We want to find the mean kinetic energy of a particle in an ideal gas, which gives us $f(x) = K(v) = \frac{1}{2}mv^2$ and $P(x) = n_{norm}(v)$. If we now integrate this over all the speeds, we get for the kinetic energy,

$$\begin{aligned} < K >&= \int_0^{\infty} \frac{1}{2}mv^2 \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}} 4\pi v^2 dv \\ &= 2\pi m \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^{\infty} e^{-\frac{1}{2}\frac{mv^2}{kT}} v^4 dv \end{aligned} \tag{10}$$

We can solve the integral in eq. 10 as we did in eq. 4, by using eq. 5,

$$\int_0^{\infty} e^{-\frac{1}{2}\frac{mv^2}{kT}} v^4 dv = \frac{1}{2}\left(\frac{m}{2kT}\right)^{-5/2} \Gamma\left(\frac{5}{2}\right) \tag{11}$$

Using the solution from eq. 11 in eq. 10, we get for the mean kinetic energy,

$$< K >= 2\pi m \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{1}{2}\left(\frac{m}{2kT}\right)^{-5/2} \Gamma\left(\frac{5}{2}\right)$$

And we know that $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\frac{\sqrt{\pi}}{2} = \frac{3\pi}{4}$, which gives us,

$$< K >= \frac{3}{2}kT \tag{12}$$

3a. Comapring the Maxwell-Boltzmann distribution for $\vec{v}$,

$$n(\vec{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}}$$

with the gaussian distribution,

$$P(\vec{v}) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-(v_x^2+v_y^2+v_z^2)/(2\sigma^2)}$$

We see that since $v^2 = v_x^2 + v_y^2 + v_z^2$, σ is just: $\sigma = \sqrt{\frac{kT}{m}}$.

3b. The kinetic energy calculated analytically from eq. 12 for $T = 6000K$, using $k = 1.38 \cdot 10^{-23} m^2 kg s^{-2} K^{-1}$ for Boltzmann's constant is,

$$< K >= \frac{3}{2} \cdot 1.38 \cdot 10^{-23} \cdot 6000 = 1,242 \cdot 10^{-19} J$$

We see that the numerical result below is quite good, atleast for the higher number of particles.

```
from scitools.all import *
import random

k = 1.38e-23  # Boltzmann constant [m^2 kg s^-2 K^-1]
T = 6000      # Temperature [K]
m = 1.67e-27  # Weight of hydrogen atom [kg]
N = 10000     # Number of particles

sigma = sqrt(k*T/m)

# Random distribution of velocities for each particle
v = zeros((N,3))
for i in range(N):
    for j in range(3):
        v[i,j] = random.gauss(0,sigma)

# Calculating the kinetic energy of the particles and taking the mean
KE = zeros(N)
for i in range(N):
    KE[i] = 0.5*m*(v[i,0]**2 + v[i,1]**2 + v[i,2]**2)
mean_KE = sum(KE)/N
```

```
print 'Mean kinetic energy: %g J, with %d particles' % (mean_KE, N)

# Result
'''
Mean kinetic energy: 1.23652e-19 J, with 10000 particles
Mean kinetic energy: 1.24281e-19 J, with 100000 particles
'''
```