FYS3140 Mathematical Methods, Home exam 2013

Due Friday March 22 at 14:30h (strict deadline!)

12. mars 2013

Important – please read:

- **Mark your paper with your** *candidate number*, **not your name!**
- **Hand in at the front office, same place as for homeworks – do not use email**
- **Keep a copy of your paper!**

Good luck! :)

Problem 1: Residue theory

Calculate the following integrals. Briefly(!) explain your reasoning at the most important steps. (PV denotes principal value).

a)

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{x-3i} dx \tag{1}$$

b)

$$PV \int_{-\infty}^{\infty} \frac{\cos(2x)}{x^3+1} dx \tag{2}$$

Problem 2: Fourier series

a) Find the Fourier series of the following function, which is assumed to have period 2π. Start by drawing a sketch of the function, including several periods.

$$f(x) = \begin{cases} k, & -\pi/2 < x < \pi/2; \\ 0, & \pi/2 < x < 3\pi/2. \end{cases} \tag{3}$$

b) Use your result from **a)** to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4} \tag{4}$$

c) Find the complex Fourier series for $f(x) = e^x$ if $-\pi < x < \pi$, and $f(x+2\pi) = f(x)$. Again start by sketching the periodic function.

d) Obtain the equivalent $\sin - \cos$ series *directly* from your result in **c)**.

Problem 3: Fröbenius method

The aim of this problem is to use the Fröbenius method to solve the differential equation

$$4xy'' + 2y' + y = 0. \tag{5}$$

So we are looking for solutions of the form $y = x^s \sum_{n=0}^{\infty} a_n x^n$.

a) Show that the indicial equation is $s(s-1) + \frac{1}{2}s = 0$, and solve it to determine the two roots s_1 and s_2.

b) Show that the solution for the smaller value of s is $y_1 = a_0 \cos\sqrt{x}$, then find the solution y_2 corresponding to the larger value of s.

c) Check explicitly that the two solutions are linearly independent. Thus write down the full, general solution to (5).

Problem 4: Cauchy-Riemann equations

Consider the function

$$u(x, y) = \cos bx \cosh y.$$

Determine b such that $u(x, y)$ is harmonic, i.e. satisfies Laplace's equation. Then find a conjugate harmonic function $v(x, y)$. Derive the analytic function corresponding to u and v, expressed in terms of z.