VFI Toolkit Workshop, pt3B: Other Endogenous States

vfitoolkit.com/2025-workshop-lse

Robert Kirkby

robertdkirkby.com
Victoria University of Wellington

VFI Toolkit

- Time for some more 'advanced' features.
- Alternative endogenous states: human capital, riskyasset (portfolio choice).

Life-Cycle Model

- Two endogenous states.
- Assets (savings) and Housing.
- Both are standard endogenous states. We will use divide-and-conquer on the first endo state.
- Because Housing is just 4 points, this solves easy enough.

Using two endogenous states with 'large grids' in principle works, but as of 2025 you may or may not be able to solve depending on how powerful your gpu is.

Life-Cycle Model: Two Endogenous States

- Example: Workshop Model 13 (and WorkshopModel13_ReturnFn)
- a =assets, h =house, c =consumption, d=housing services.

$$V(a,h,z,j) = \max_{c,d,a',h'} \frac{[(\theta c^{\upsilon} + (1-\theta)d^{\upsilon}]^{1-\sigma}}{1-\sigma} + \beta s_j E_j[V(a',h',z',j+1)|z]$$

$$\text{subject to } \tau_{h,h'} = \begin{cases} 0 & \text{if } h' = h \\ \phi h & \text{if } h' \neq h \end{cases}$$

$$d = h' \quad \textit{(housing services come from home-owned)}$$

$$c + a' + h' = (1+r)a + w\kappa_j z + h - \tau_{h,h'}$$

$$a' \geq -(1-\gamma)h'$$

renter $(h' = 0)$:

$$c + a' + pd = (1+r)a + w\kappa_j z + h - \tau_{h,h'} + \mathbb{I}_{j \geq Jr} pension$$

$$a' \geq 0$$

Turns out we don't need to distinguish c from d for renters, as this part can be done analytically. You will see this in codes (Derivation). So no decision variables, just two endogenous states.

Life-Cycle Model: Endogenous States

- Endogenous state: *a* today, and *aprime* tomorrow.
- Standard endogenous state: *aprime* can be chosen directly.
- Let's look at some alternative endogenous states where *aprime* cannot be chosen directly.

Life-Cycle Model: Endogenous States

- Endogenous state: a today, and $aprime$ tomorrow.
- Standard endogenous state: $aprime$ can be chosen directly.
- Every model until now, the endogenous states were standard endogenous states.
  Not quite true, we saw an alternative 'residualasset' in Loss Aversion example. But I glossed over it.
- VFI Toolkit recognises alternative endogenous states, where $aprime$ cannot be chosen directly, but is instead a function of decisions.
- Alternative endogenous states:
  - experienceasset: $aprime(d, a)$
  - experienceassetu: $aprime(d, a, u)$
  - riskyasset: $aprime(d, u)$
- Let's do some examples.

Note: experienceasset could be rewritten as a standard endogenous state since inverse of this function gives $d(aprime, a)$. But often you can put way less grid points on d than $aprime$, so faster and less memory, and/or it is just much easier to write as $aprime(d, a)$.

Life-Cycle Model: Human Capital

- Alternative endogenous state: 'experienceasset'
- Example: Workshop Model 14 (and WorkshopModel14_ReturnFn)
- experienceasset: $aprime(d, a)$
- Two main changes in codes:
  - First is we need to define the function $aprime(d, a)$
    Call it $vfoptions.aprimeFn$
  - Second is action space no longer includes $aprime$.
    It cannot be chosen, so cannot be in action space.

Life-Cycle Model 42 shows 'savings and human capital' (one standard endogenous state and one experienceasset endogenous state).

Life-Cycle Model: Human Capital

- Example: Workshop Model 14 (and WorkshopModel14_ReturnFn)
- h =human capital, l =labor supply ($1-l$ =time studying).

$$V(h,z,j) = \max_{c,l} \frac{c^{1-\sigma}}{1-\sigma} + \beta E_j[V(h',z',j+1)|z]$$

subject to

$$c = w\, l\, h\, exp(z)$$

$$h' = ability(h(lscaler(1-l)))^{\alpha_h} + (1-\delta_h)h$$

Trade-off is 'work now to earn today' or 'study for higher future earnings'.
Note: h' is not a decision (look at subscripts of the *max*).
Note: the equation for h' is the 'human capital production fn' and is what in the codes will be *vfoptions.aprimeFn*

Life-Cycle Model: Uncertain Human Capital

- Alternative endogenous state: 'experienceassetu'
- Example: Workshop Model 15 (and reuse WorkshopModel14_ReturnFn)
- experienceasset: $aprime(d, a, u)$
  u is a between-period i.i.d. shock
- Two main changes in codes:
  - First is we need to define the function $aprime(d, a, u)$
    Call it *vfoptions.aprimeFn*
  - Second is action space no longer includes *aprime*.
    It cannot be chosen, so cannot be in action space.

Life-Cycle Model 43 shows 'savings and uncertain human capital' (one standard endogenous state and one experienceassetu endogenous state).

Life-Cycle Model: Uncertain Human Capital

- Example: Workshop Model 15 (and reuse WorkshopModel14_ReturnFn)
- h =human capital, l =labor supply ($1-l$ =time studying).
- u is a between-period i.i.d. shock.

$$V(h,z,j) = \max_{c,l} \frac{c^{1-\sigma}}{1-\sigma} + \beta E_j[V(h',z',j+1)|z]$$

subject to

$$c = w\, l\, h\, exp(z)$$

$$h' = exp(u)ability(h(lscaler(1-l)))^{\alpha_h} + (1-\delta_h)h$$

Trade-off is 'work now to earn today' or 'study for higher future earnings'.
Note: h' is not a decision (look at subscripts of the *max*).
Note: the equation for h' is the 'human capital production fn' and is what in the codes will be *vfoptions.aprimeFn*

Life-Cycle Model: Portfolio-Choice

- Alternative endogenous state: 'riskyasset'
- Example: Workshop Model 16
- experienceasset: $aprime(d, u)$
  u is a between-period i.i.d. shock
- Two main changes in codes:
  - First is we need to define the function $aprime(d, u)$
    Call it *vfoptions.aprimeFn*
  - Second is action space no longer includes *aprime*.
    It cannot be chosen, so cannot be in action space.

Life-Cycle Model 35 shows 'housing and portfolio-choice' (one standard endogenous state and one riskyasset endogenous state).

Life-Cycle Model: Portfolio-Choice

- Example: Workshop Model 16 (and WorkshopModel16_ReturnFn)
- Saving into a combination of safe and risky assets.

$$V(a,z,j) = \max_{c,riskyshare,savings} \frac{c^{1-\sigma}}{1-\sigma} + \beta s_j E_j[V(a',z',j+1)|z]$$

subject to

$$c = w\,\kappa_j\,exp(z) + a - savings$$

$$a' = (1 - riskyshare) * savings * (1 + r) + riskyshare * savings * (1 + r + u)$$

$$u \sim N(r + riskpremium, \sigma_u^2)$$

risky assets give higher average return than the safe assets, but come with risk.
Notice: $aprime(riskyshare, savings, u)$ will be $vfoptions.aprimeFn$

Life-Cycle Model: More Endogenous States

- Summary: can do two endogenous states.
  Some, but not all.
- Various alternative endogenous states: experienceasset, experienceassetu, riskyasset.
- You can use 'one standard and one alternative endogenous state'
- Life-Cycle Model 42: savings and human capital (standard and experienceasset).
- Life-Cycle Model 43: savings and uncertain human capital (standard and experienceassetu).
- Life-Cycle Model 35: housing and portfolio-choice (standard and riskyasset).
- Portfolio-Choice (riskyasset) supports Epstein-Zin preferences (Life-Cycle Models 31-35).

Divide-and-conquer works with all of these, always applies to the first endogenous state.

References I

Robert Kirkby. VFI toolkit, v2. *Zenodo*, 2022. doi: https://doi.org/10.5281/zenodo.8136790.