Fundamental Groups And Covering Spaces

What’s the shape of the Earth?

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?

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?

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What about…

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??

How would we tell?

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Topology is in the business of telling spaces apart

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$f: X \to Y$ and $g: Y \to X$ such that $g(f(x)) = x$ and $f(g(y)) = y$

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(for all x in X and y in Y)

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If X and Y are *NOT* homeomorphic, how do you show *that* ??

Ex: how do you know the real number line isn’t homeomorphic to the plane?

how do you know the torus isn’t homeomorphic to the sphere, or plane?

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Either quantities, or properties, or even other mathematical objects, that don’t change when you apply a homeomorphism.

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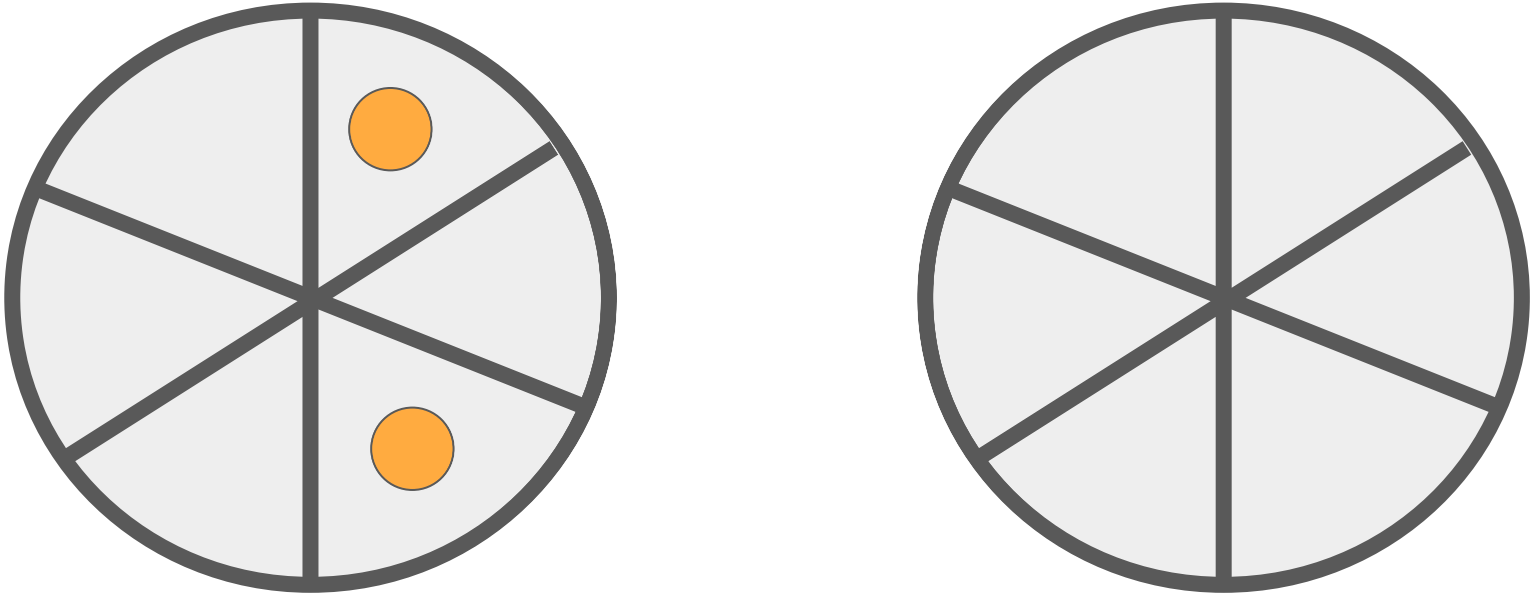
If you calculate this thing for two different spaces, and you get two *different* quantities/properties/objects, then what do you know?

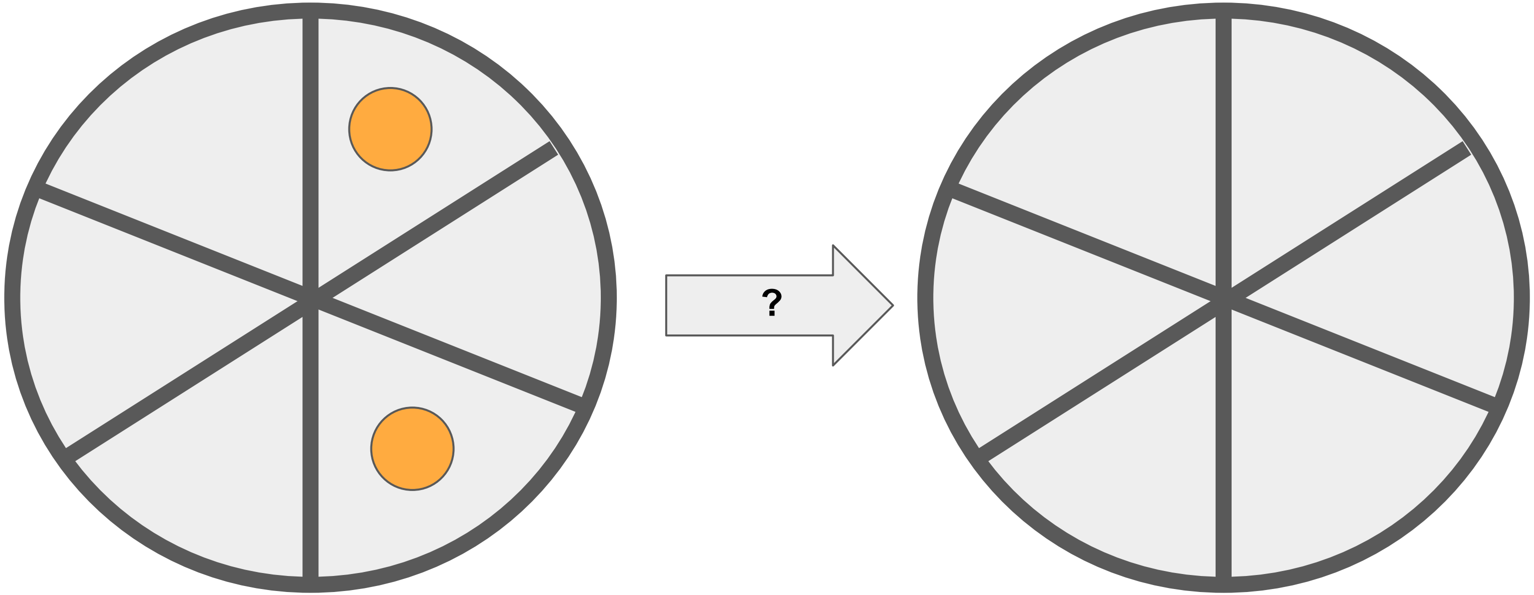
Answer: *They can't be homeomorphic!* Otherwise, the invariant would…
…um…be invariant.

Invariants as a problem solving strategy



Invariants as a problem solving strategy

Allowable moves: Adding (or removing) 2 pennies to NEIGHBORING sections



Can you get from this ^ setup to a position where you cleared all the pennies?

Yes? No? Let’s vote!

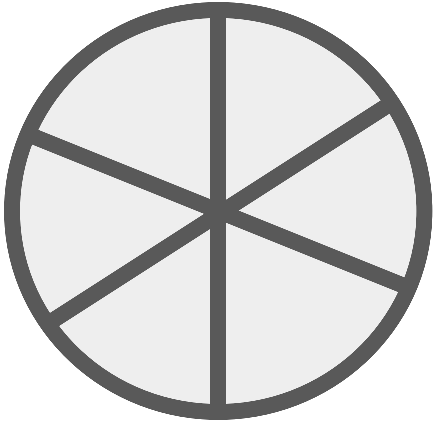
Yes? No? Let’s vote!

How many people guessed based on context?

Yes? No? Let’s vote!

It’s no. There’s an invariant property/quantity you can show. The allowable moves do not affect this quantity/property. The start and end configurations have differ on this property. So there cannot be a sequence of moves connecting the configurations.

Should I spoil the property? Can you guess what it is?



The start of algebraic topology

From a space, define this “algebraic object” (in this case, a set with an operation on it) that is “invariant” under homeomorphism. Called the “Fundamental Group”!

Intuitively, it’s the “possible ways of walking around in the space”

How would we tell?

Back to this nonsense

How might we perceive living in these worlds?

Walk around your world “leaving rope as you go”

Eventually, walk back to where you started. Tie the rope off.

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Can this loop of rope be reeled in to a point? If it can be, up to continuous deformation, it’s the same as the “do nothing walk”

Otherwise, it’s *fundamentally different!*

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Technical note: this relation is called homotopy. Ignoring the difference between homotopic loops fixes 2 issues: now concatenation is associative, and small deformations don’t matter (as they shouldn’t)

Historic Note

Topology got its start in part when people realized there were certain properties in other areas of math that didn’t depend on “exact shape”, so small deformations didn’t matter.

For example, there are certain path integrals (remember your multivariable calc??) you can define where the integral along a path only depends on the endpoints of that path. Or, the integral along a loop only depends on how many times that loop winds around the origin.

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What are the fundamentally different ways of walking a closed loop around a circular track? (note, the middle part is not actually a part of your space. Imagine a flagpole in the middle that your rope cannot pass through)

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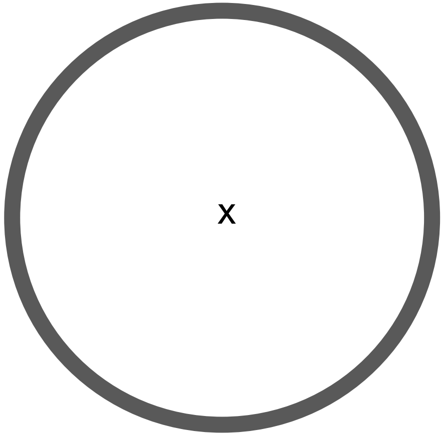

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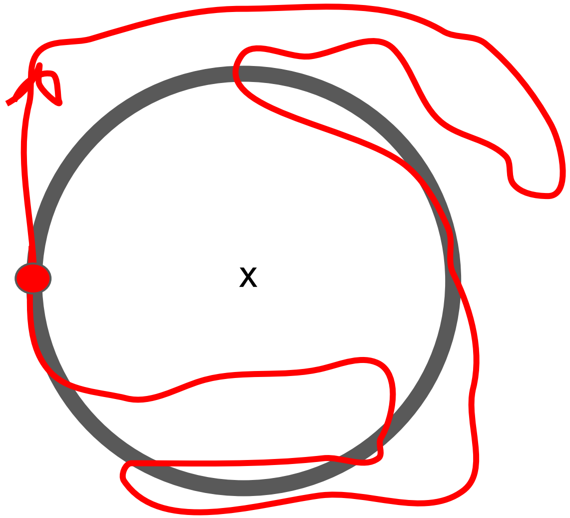
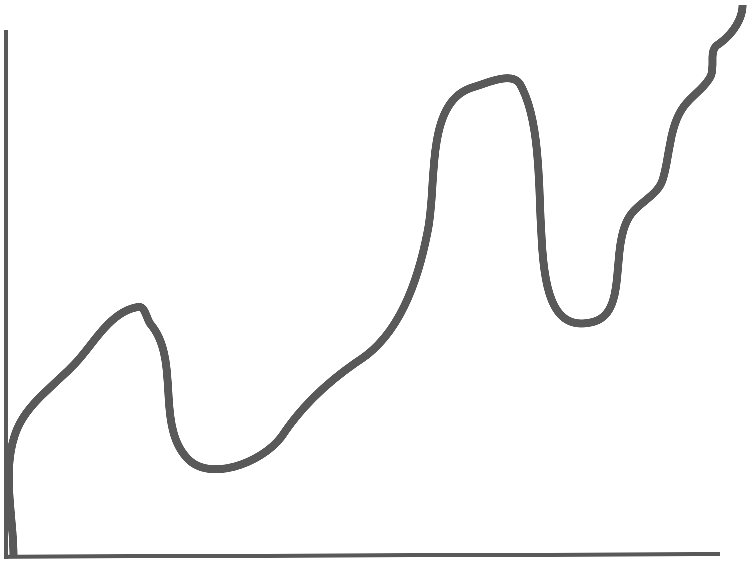


Plot angle over time



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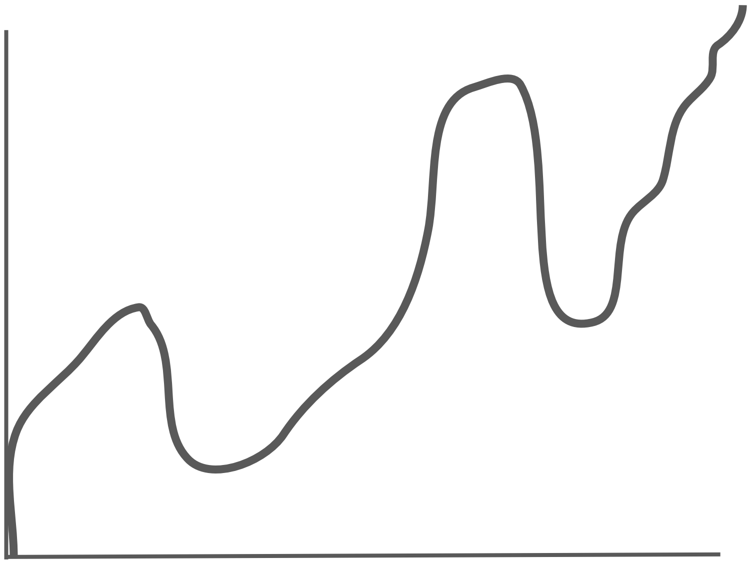

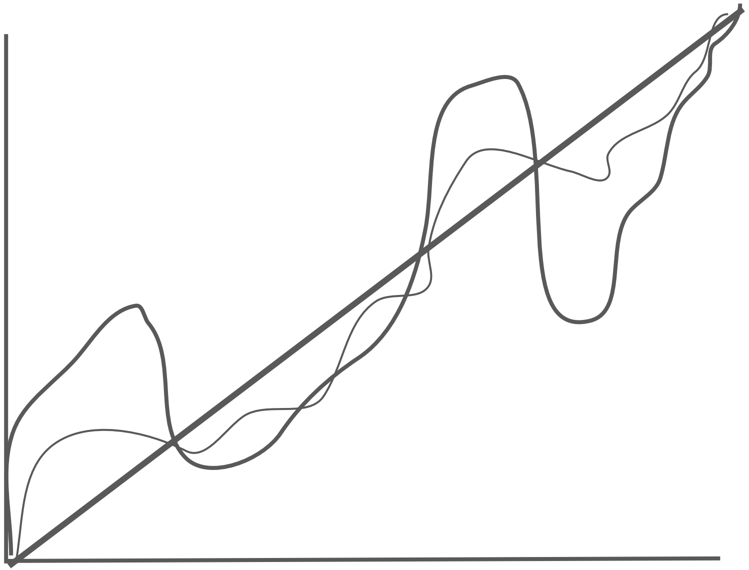

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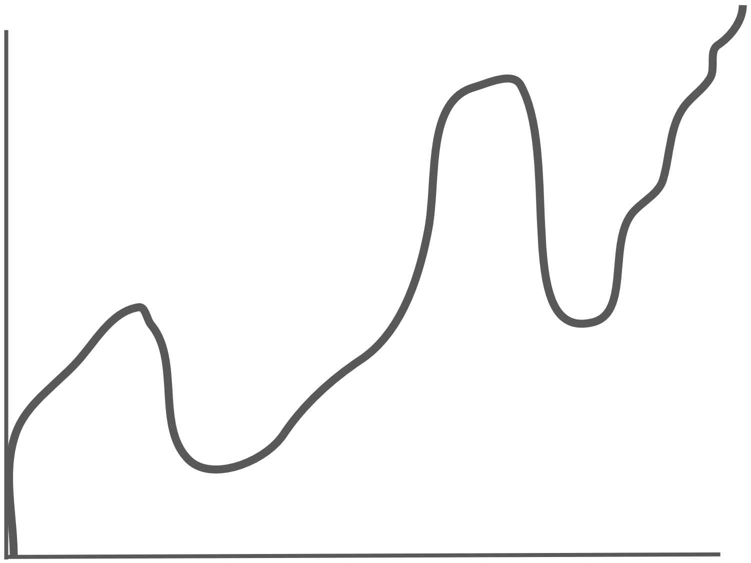

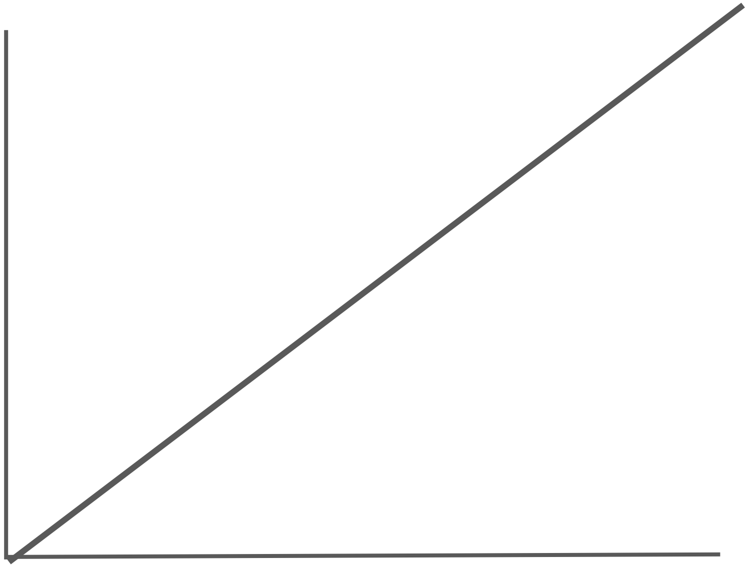

Fundamental group of the circle is the integers, Z

Two loops in the circle are fundamentally different if they wind a different net number of times around the origin. This includes orientation (negatives).

The possible winding numbers are all the integers, so $Pi_1(S^1) = Z$

I can’t resist: Brouwer Fixed Point Theorem!!

For any continuous function from the closed unit disk D = { (x,y) | x^2+y^2 <=1} to itself has a fixed point!

As in, for all continuous f: D -> D, there exists (x,y) in D such that f(x,y) = (x,y)

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This is why we define objects such as the fundamental group and establish some properties about it!

Suppose for contradiction, there is no fixed point

Then you can always draw the ray from p to f(p). Where that ray hits the circle, call it r(p)

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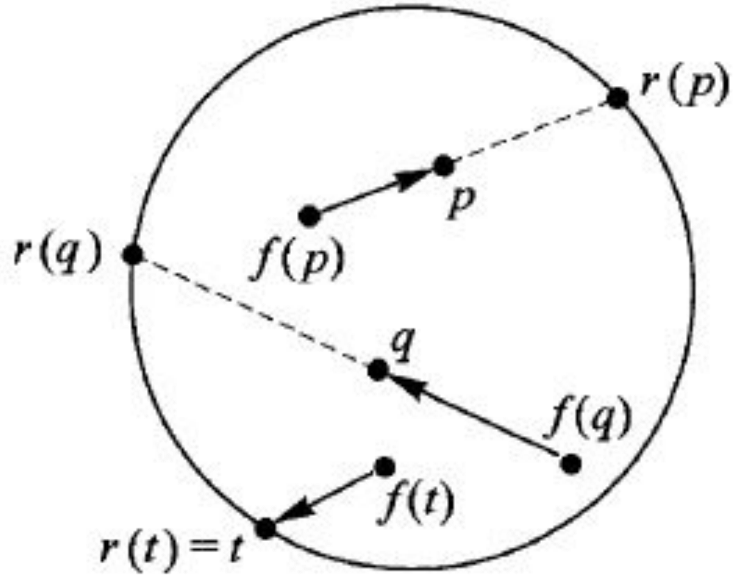

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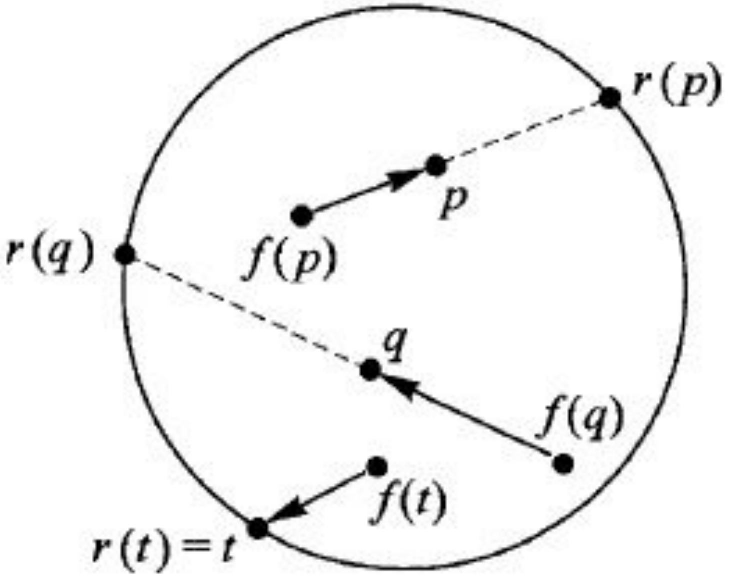
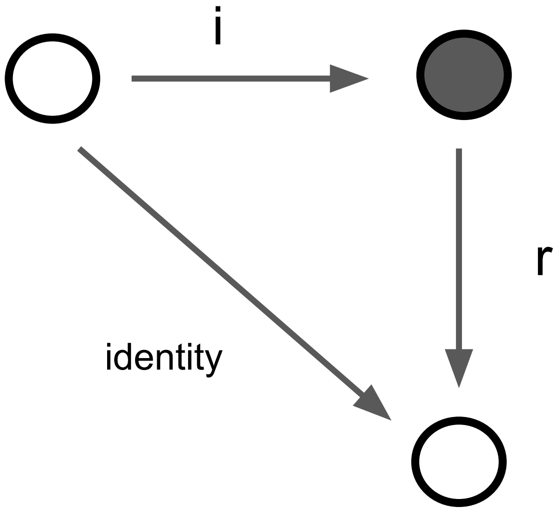


If p started off on the boundary, r(p) = p

So r is a continuous function from D to S^1.

Moreover, if i: S^1 -> D is the inclusion function,

then r of i is the identity on the circle.


i
r
identity

$$
\begin{tikzcd}
S^1 \arrow[r, "i"] \arrow[dr, "\text{identity}"'] & D \arrow[d, "r"] \\
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This diagram “commutes”...

$$
\begin{tikzcd}
\mathrm{Pi}_1\, S^1 \arrow[r, "i^*"] \arrow[dr, "\text{identity}"'] & \mathrm{Pi}_1\, D \arrow[d, "r^*"] \\
 & \mathrm{Pi}_1\, S^1
\end{tikzcd}
$$

…so therefore, so should this…

$$
\begin{tikzcd}
Z \arrow[r, "i^*"] \arrow[dr, "\text{identity}"'] & \{0\} \arrow[d, "r^*"] \\
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Clearly impossible for the identity function on the integers to be the composition of a map into the one element set.

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Clearly impossible for the identity function on the integers to be the composition of a map into the one element set.

Contradiction! So such an r cannot exist

The intuition for why r cannot exist:

The “continuous function”, r “rips a hole” in the disk.

Imagine a latex sheet stretched over a metal ring. Where it’s attached to the ring, you cannot move it.

Now continuously bend the latex so all of it is touching the ring. You can’t! That would rip a hole.

You can never get rid of all of the rubber “from the middle”.

Key point: the fundamental group invariant made this intuitive idea precise

Remember the original theorem: f has a fixed point.

Remember, assuming f:D -> D has no fixed points, then we CAN define such an r.

Therefore, since we know we can’t, f must have at least one fixed point.

The fundamental group is great

Summary: we study a space by studying *the possible loops in the space!*

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(technical note, it’s convenient to pick a basepoint in the space that all loops start and end at)

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…but how do we study the loops?

How do we tell loops apart?

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Sometimes you can use an idea like “total angle traversed”, but that’s a geometric property! What if your space doesn’t have an easy to describe quantity like that?

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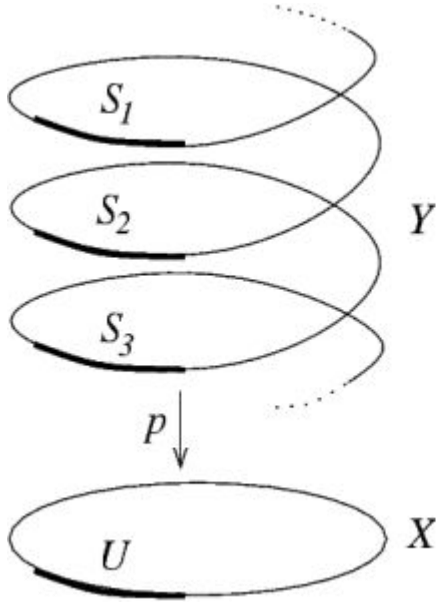

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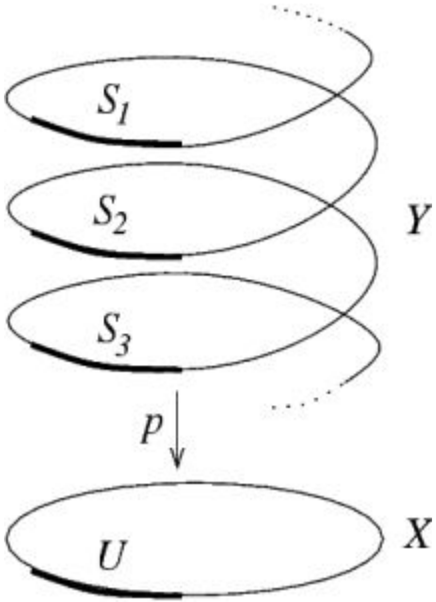
If B is the base space, a "covering space" for B is a space E and a continuous function p "projecting E down to B" satisfying this property:

Around any point b in B, if you restrict your attention narrow enough, you can find some open set U where the set of points in E mapping to U (aka, $p^{-1}U$), just looks like a bunch of exact copies of U that p maps homeomorphically to U.

Cue picture again

Here, p:Y -> X is a covering space for X.

We can say “Y covers X (by p)”



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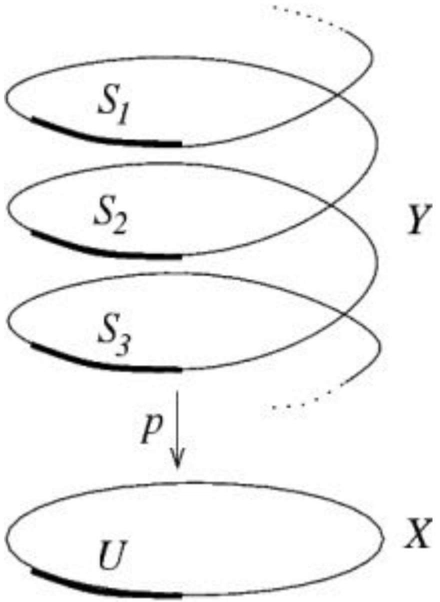
The S_i are called “sheets above U”.



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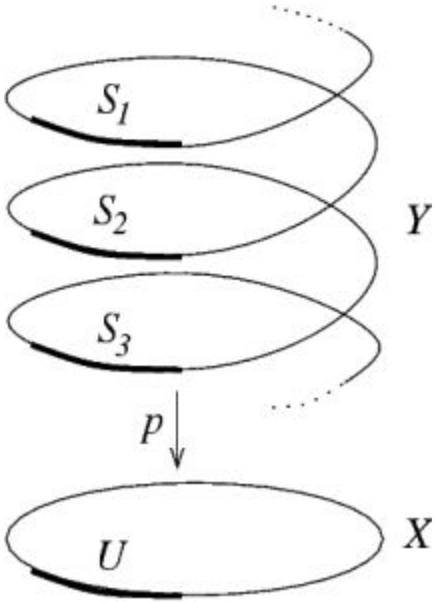
Notice how locally, the sheets are disjoint, but globally, they may be connected up in interesting ways.



Definition of a “covering space”, vs their main property

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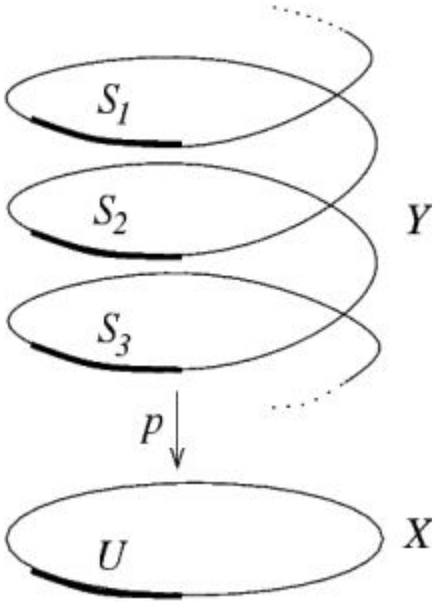
Loops downstairs no longer have to stay as loops!

p:Y -> X. Lets elaborate of how to lift paths from X to Y:

Let's say $f:[0,1] \to X$ is a path starting at x_0.

Let's say y_0 is some point in Y such that $p(y_0)=x_0$



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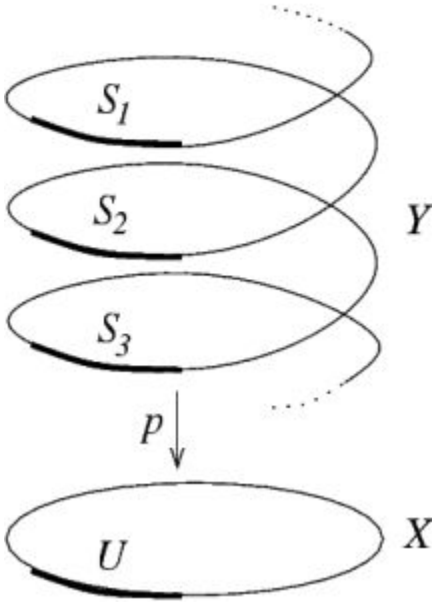

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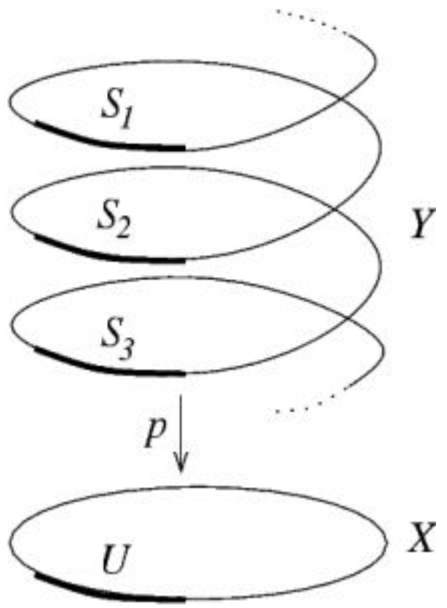
Starting at $g(0)=y_0$, always keep g(t) directly above f(t).



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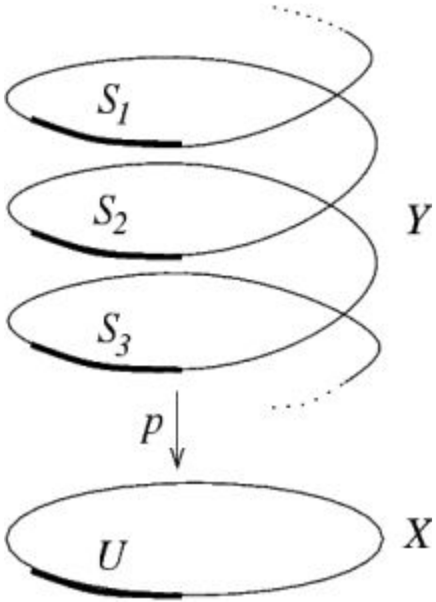
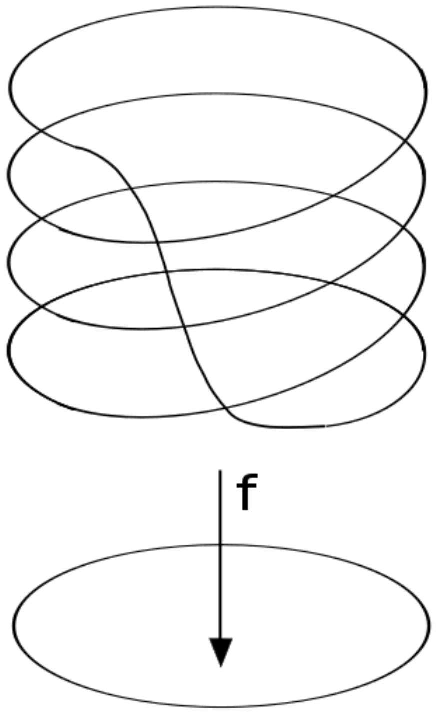


Intuition: as t varies from 0 to 1, f traces a path in X.

Starting at $g(0)=y_0$, always keep g(t) directly above f(t).

There isn’t any choice! g(t) is forced into position.

f is a covering space of the circle



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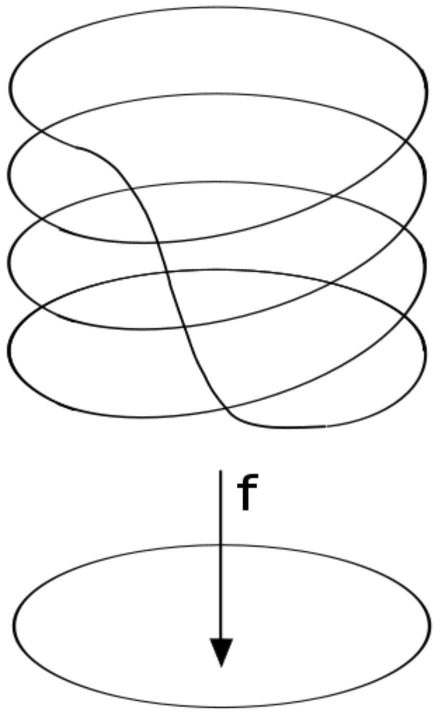
We can give f by the formula:

f(theta) = 4 * theta



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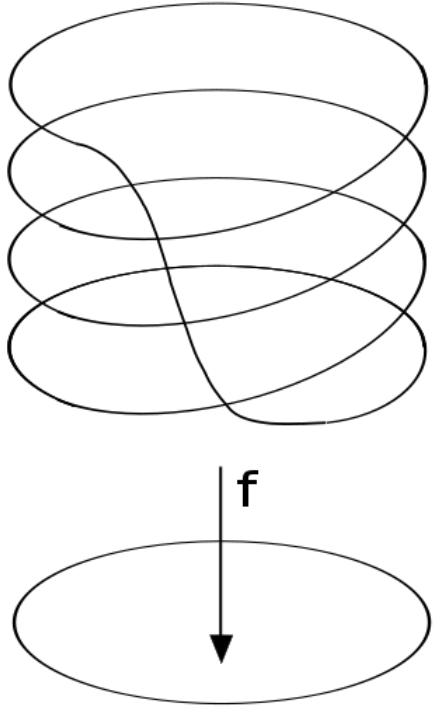

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Or, if you think of S^1 in the complex plane, then $f(z) = z^4$

They all look like winding the circle n times around itself

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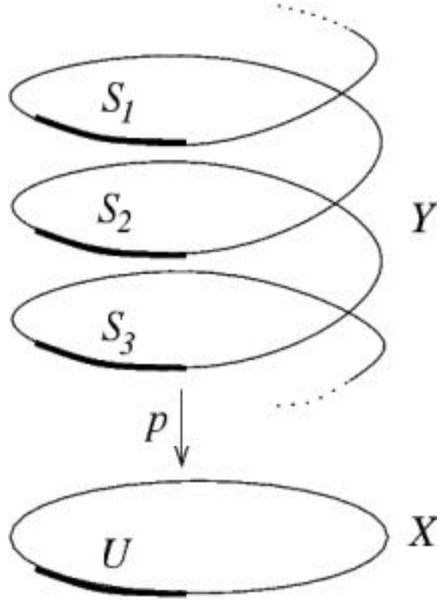
OR the real number line winding infinitely many times

p given by the formula:

$p(r) = (\cos(r), \sin(r))$



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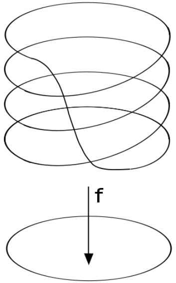

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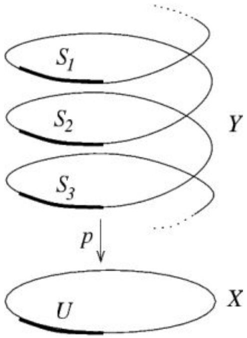

(a lot to say here)

Slightly more abstract nonsense

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then

f*: FG(X) -> FG(Y)

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If f: X -> Y **continuous**

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Key Fact: if f is a covering map, then f* will be one-to-one, and so FG(X) can be seen as a subgroup of FG(Y)

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Keyer Fact: all the subgroups of the fundamental group of a space X correspond to the possible covering spaces of X. This is the mainest point!

Worth repeating

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The actual correspondence comes from the following:

Given a covering space p:Y -> X, need a subgroup of Pi_1X

Take the subgroup of loops in X that when lifted to Y, *are still loops!*

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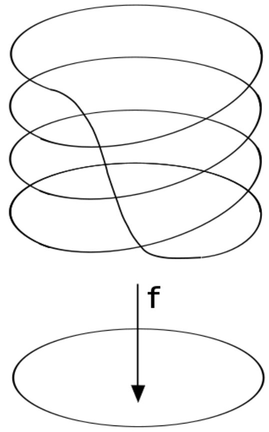
(and not just paths)

Given a subgroup of Pi$_1$X, need a covering space p:Y -> X

Call the subgroup H. Idea is we want to unravel *some* of the loops in X, but not the ones in H!

For example, unraveling the loops in the circle by a factor of 4 means every 4th loop will stay a loop, and the rest become merely paths.



Given a subgroup of Pi$_1$X, need a covering space p:Y -> X

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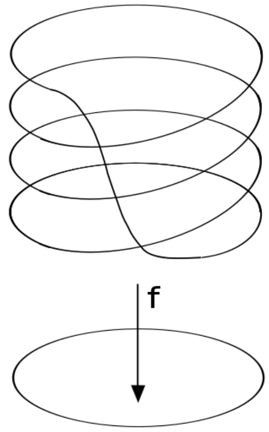
Formally, loops in the circle with a winding number 4n stay loops,

while those with a winding number 4n+1, 4n+2, and 4n+3

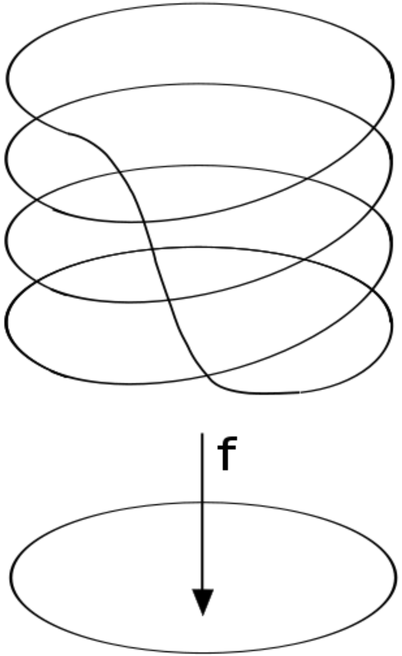
go to different levels.



The FG downstairs is Z. The subgroup is 4Z. There are 4 cosets of 4Z in Z

Notice, a loop upstairs with winding number 1 goes to a loop downstairs with winding number 4.

There are 4 levels.



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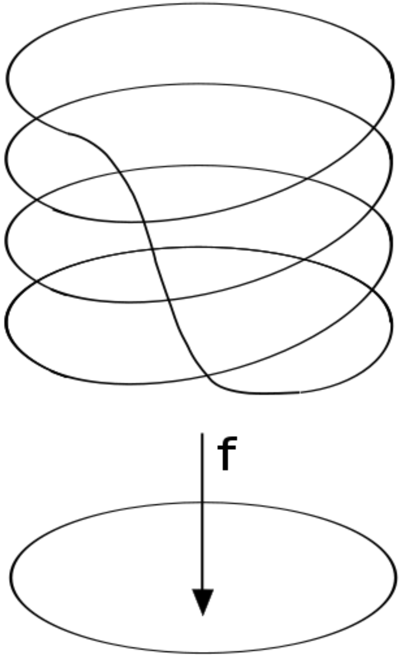

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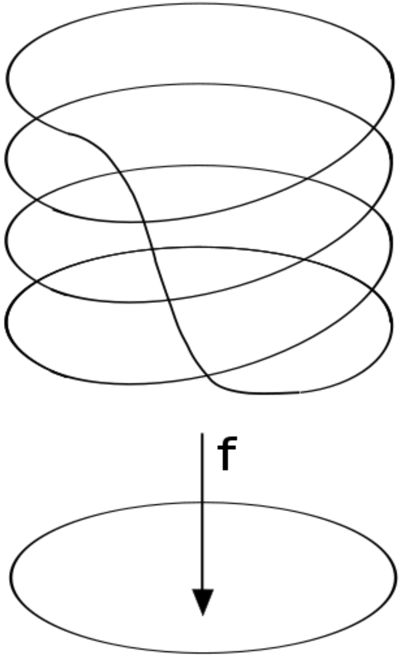

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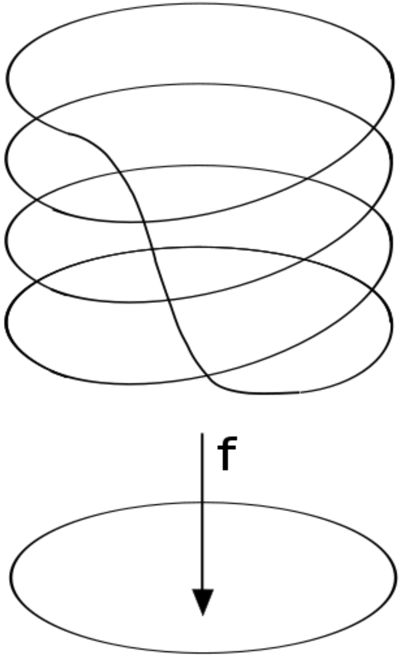
There are 4 levels.

Fact: the image of f* is exactly the subgroup of loops downstairs that lift to a loop (and not just a path) upstairs.

Fact: the cosets of this subgroup are in bijection with the levels



Universal Covers! When the fundamental group of the cover is trivial

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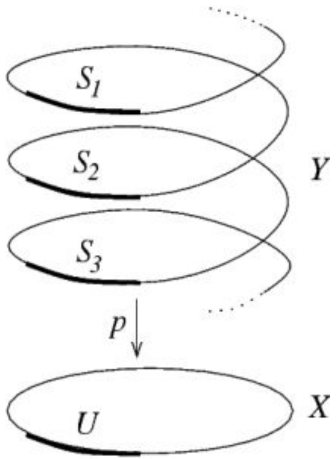
So if the cover has trivial Pi_1, then the levels are in bijection with the fundamental group of the original space!

Example: $Pi_1(S^1) = Z$. The real number line covers S^1

R has trivial Pi_1, and so we expect (and we get!) that the levels of the cover are in bijection with Z.



Intuition for universal covers: all loops downstairs are unwound! As opposed to other types of covers

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That's also why there are "no more loops left to unwind". Pi_1 of the universal cover is trivial.

If a cover *doesn't* have trivial Pi_1, then there are still loops left to unwind!

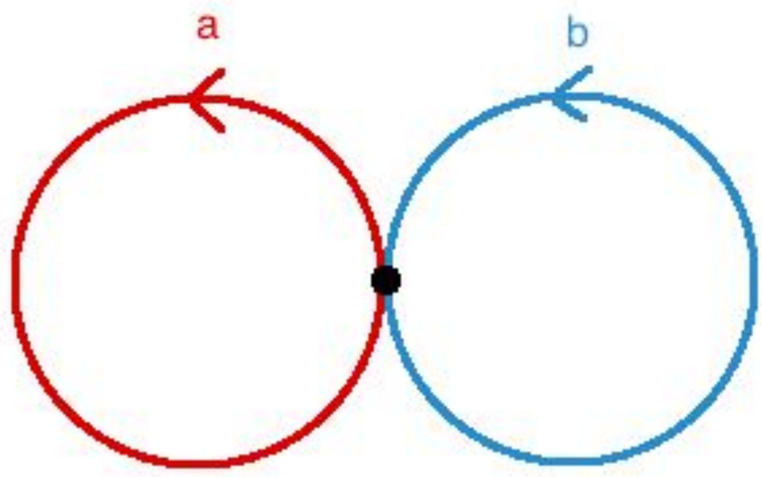
The group of these is H from before, and the cosets of H form the levels

(Ex: keeping every 4th loop a loop, you get 4 levels before wrapping back)

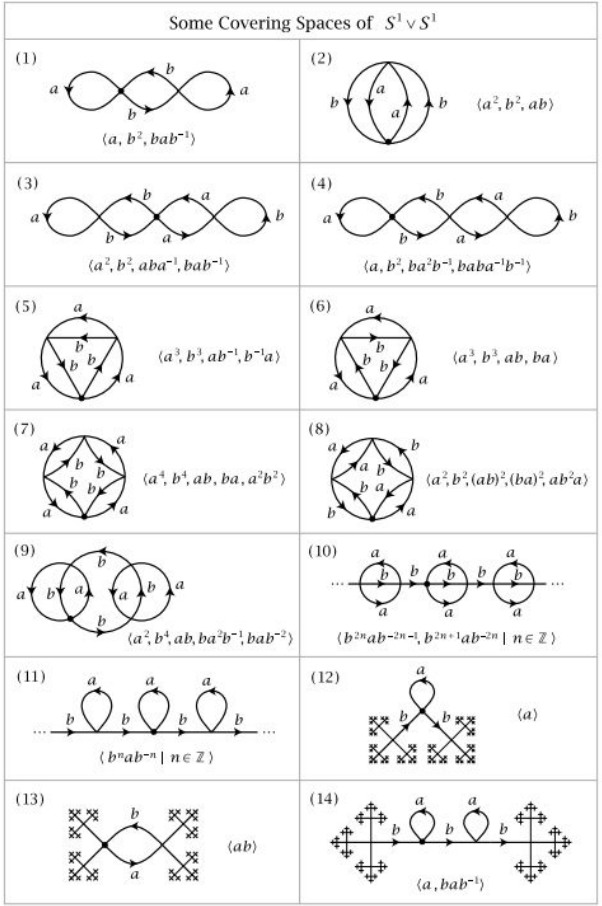
Galois and Grothendieck

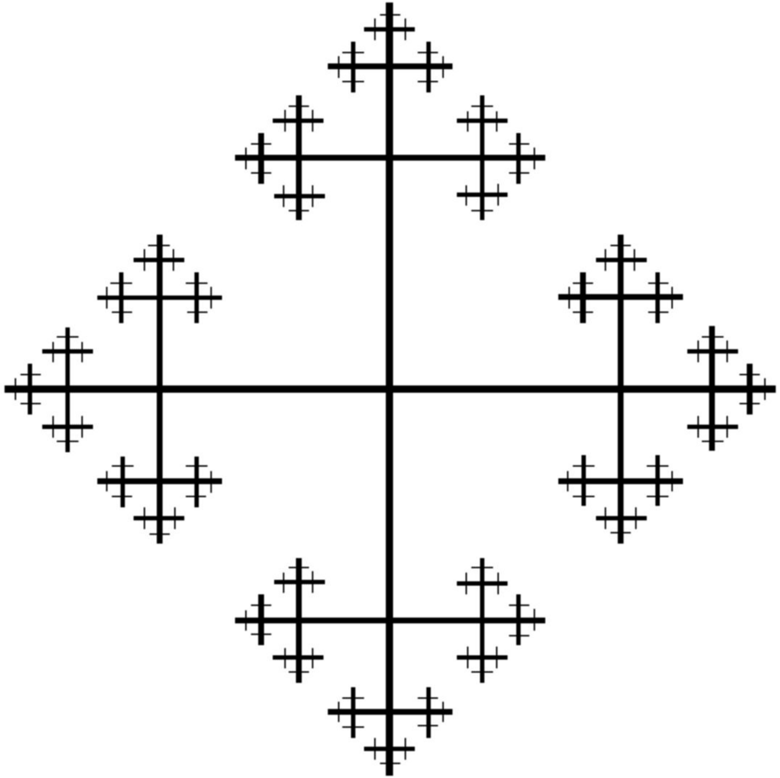
Galois and Grothendieck

How complicated can coverings of the figure 8 be…?



Covering Spaces of the figure 8





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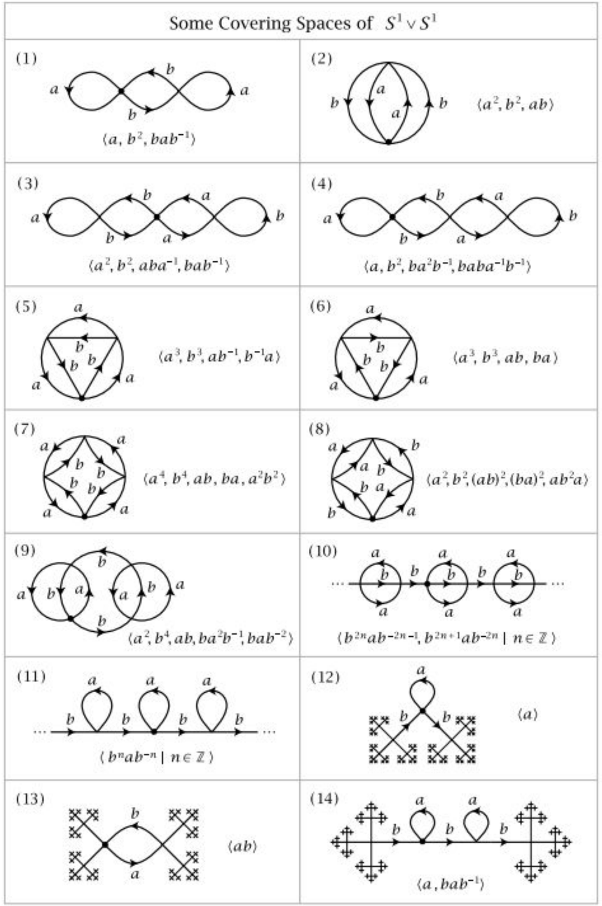

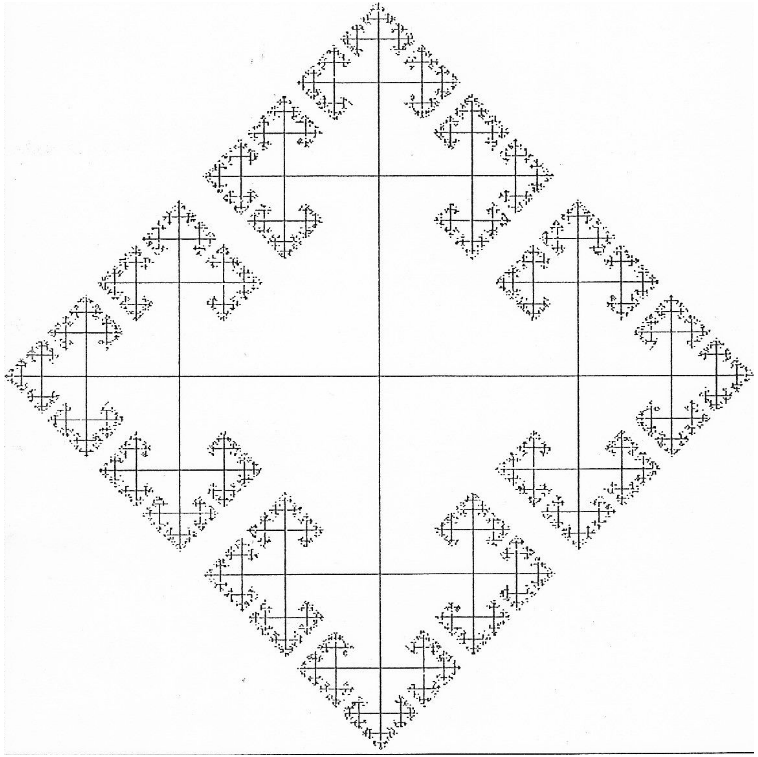

Some lightning fast examples of covering spaces:

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SU(2) double covers SO(3)

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More generally, the n-sphere double covers (real) projective n-space

The 11-holed donut covers the 4-holed donut

Picture time

The (mn+1)-holed donut covers the (n+1)-holed donut

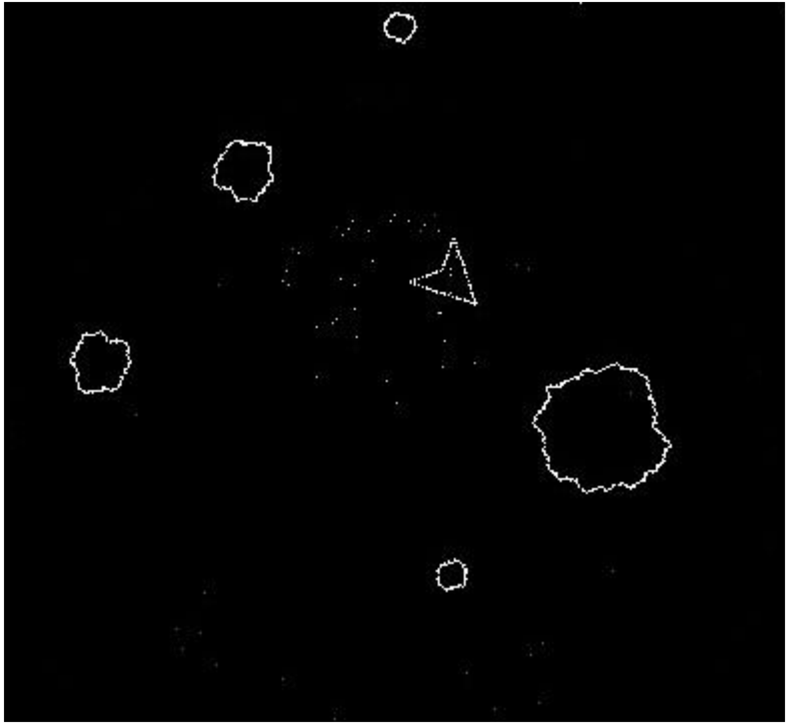
This is actually all of them. If a compact, closed surface covers another compact closed surface, then the number of holes will be of this form!

The universal cover of the donut is the plane!

Still picture time

The universal cover of the donut is the plane!

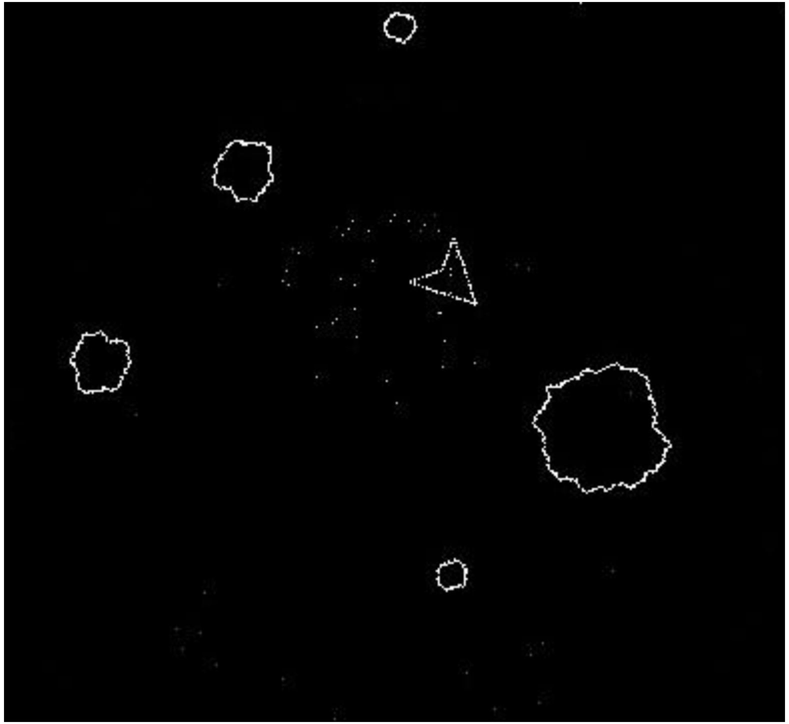
Ever played this game?



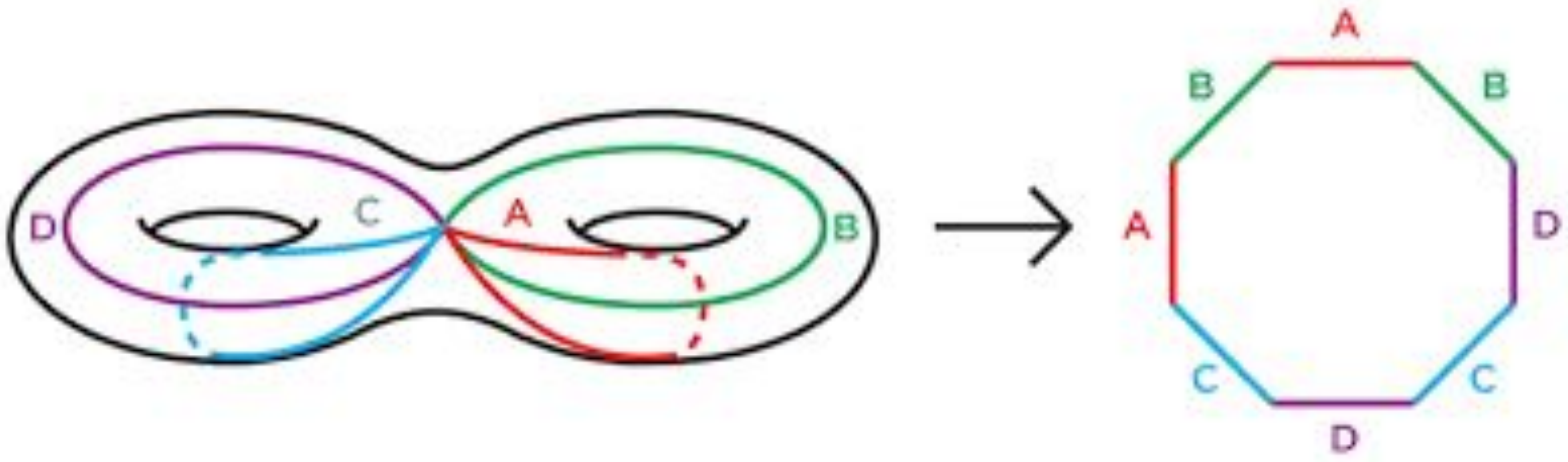
The universal cover of the donut is the plane!

Ever played this game?

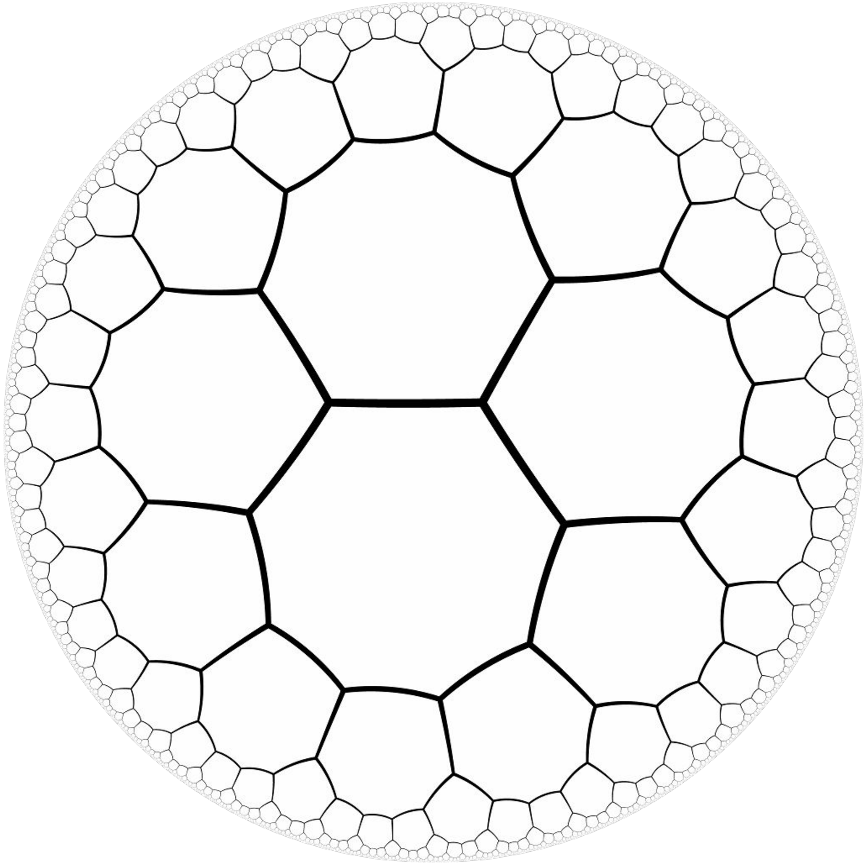
Then you understand lifts!



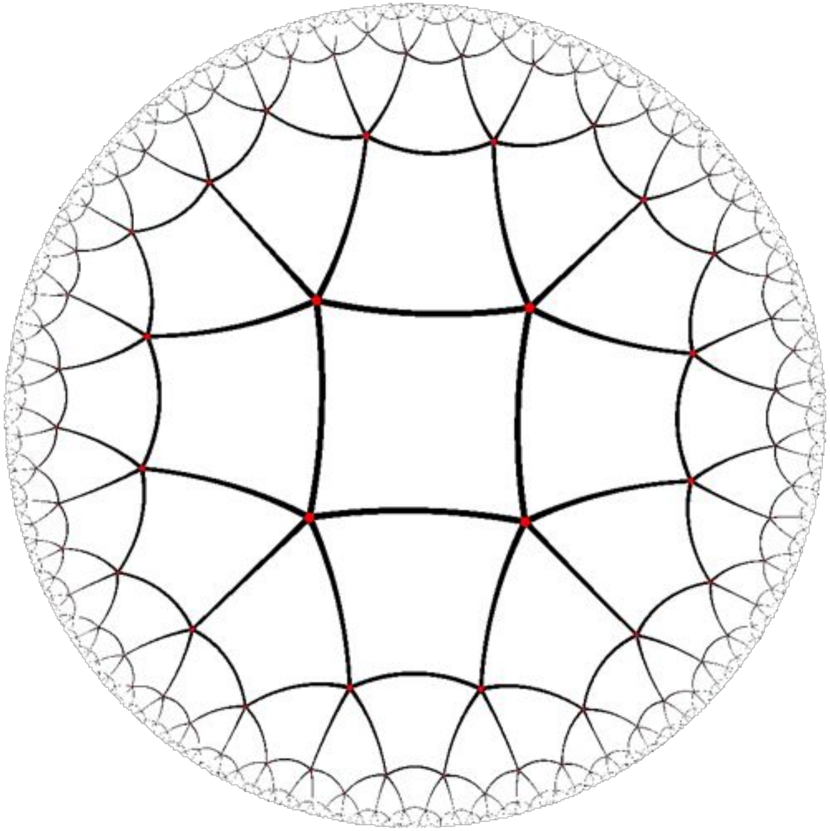
For more than 2 holes, the universal cover is…

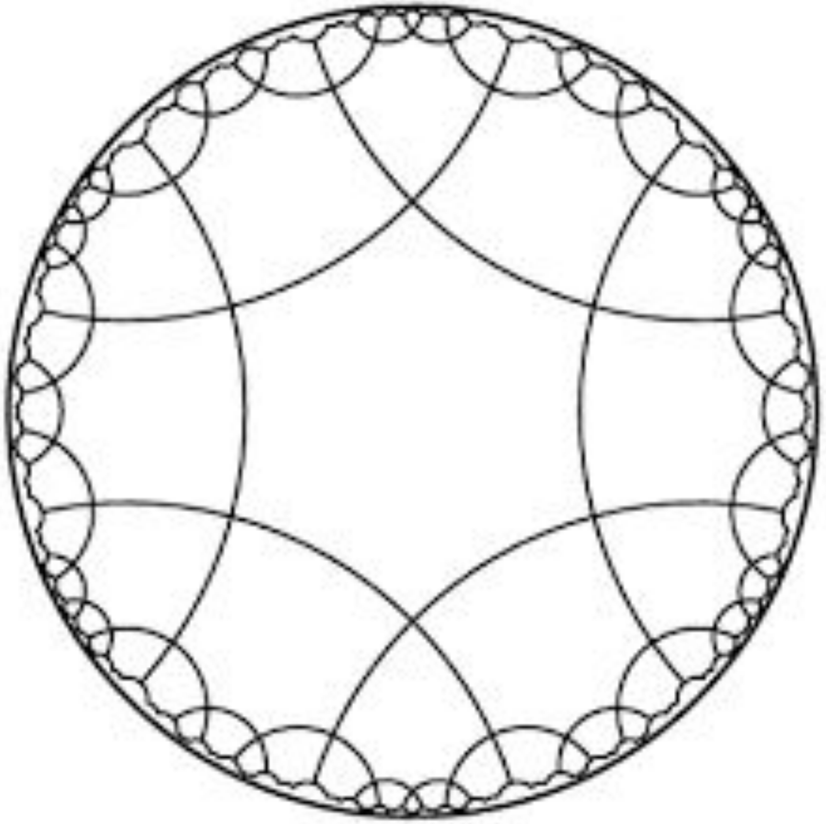


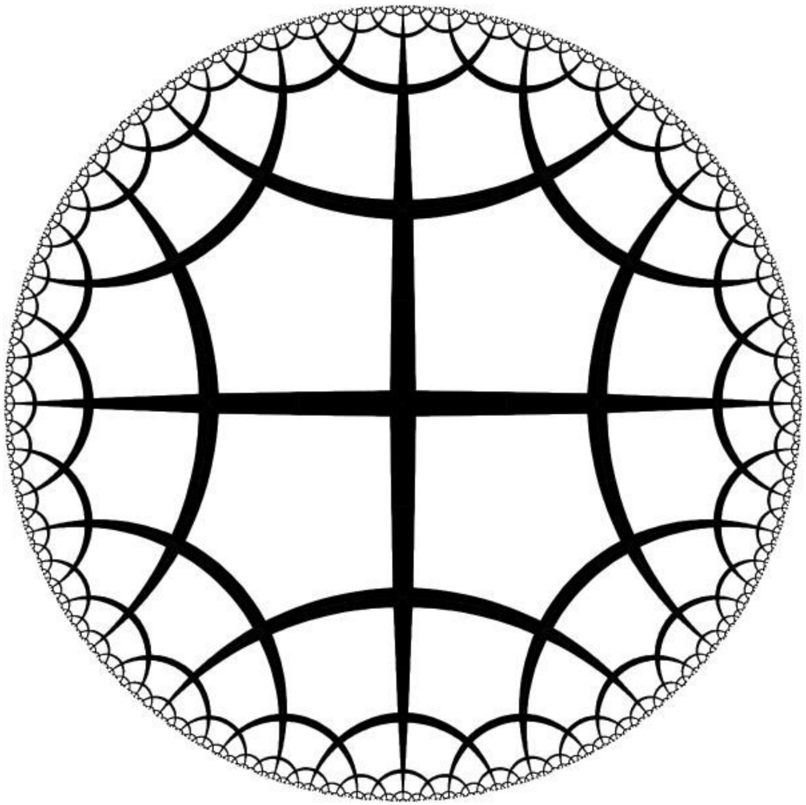
Hyperbolic Space!

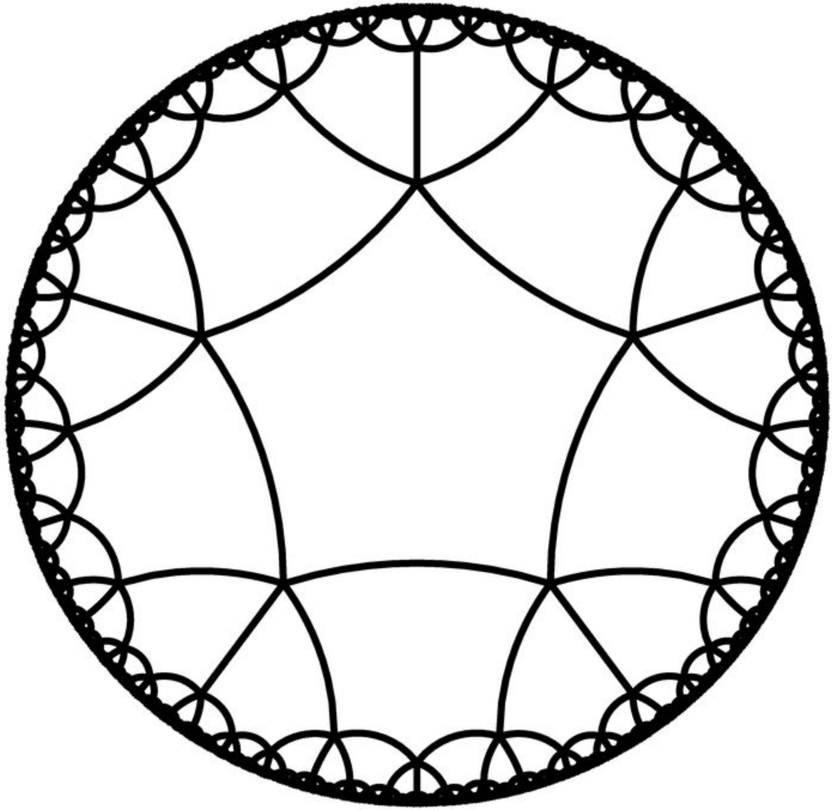


MC Escher Time



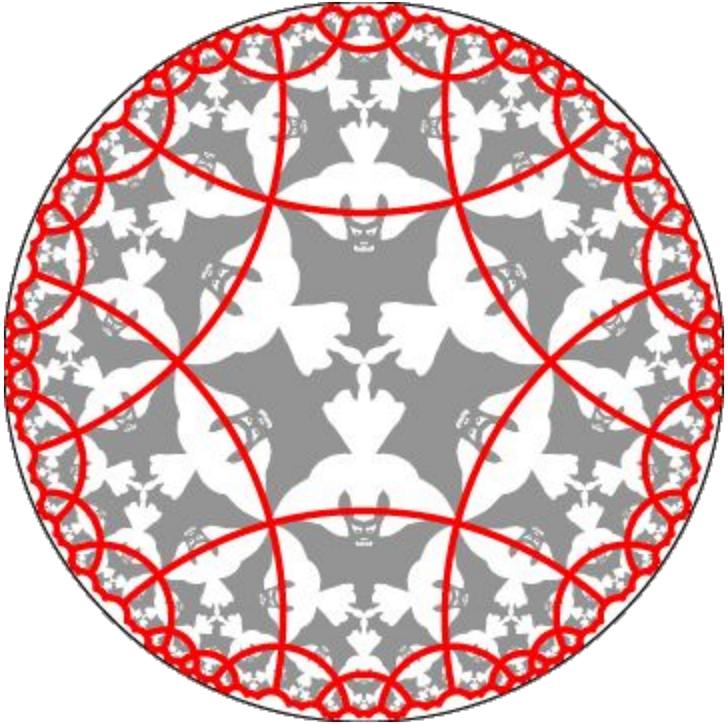






More art

https://www.josleys.com/show_gallery.php?galid=325



Thanks for coming!