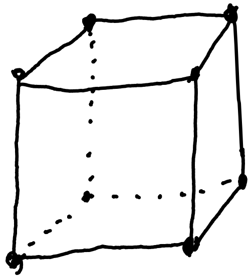
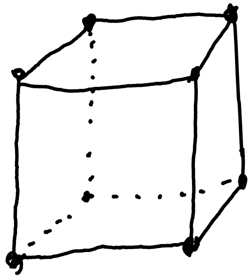


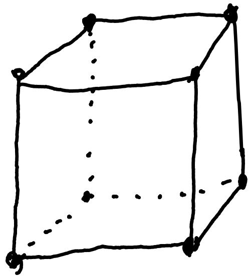
$V =$

$-E =$

$+F =$



$$V = 8$$
$$-\ E = 12$$
$$+\ F = 6$$



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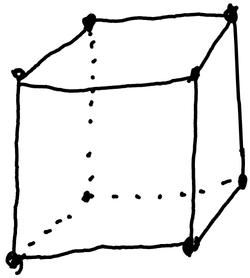
$$8 - 12 + 6 = 14 - 12 = 2$$



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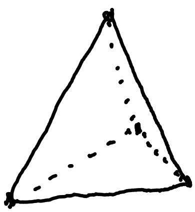
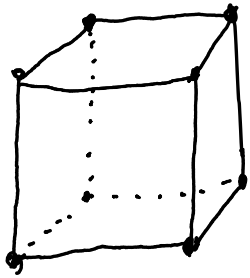


V

$-E$

$+F$



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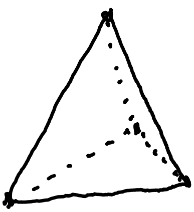

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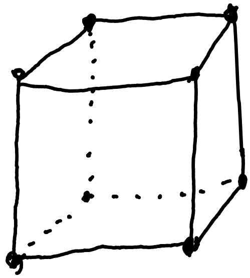

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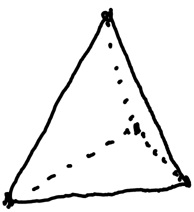

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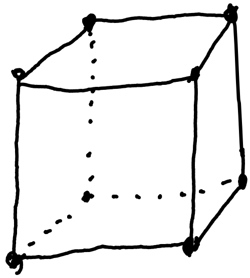
$$4 - 6 + 4 = 8 - 6 = 2$$



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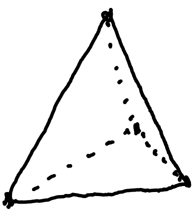

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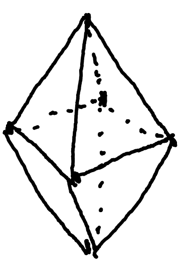

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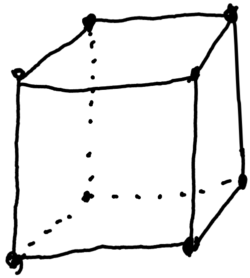

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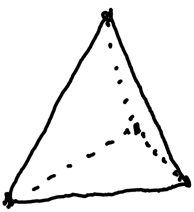

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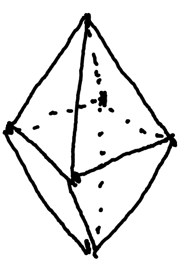

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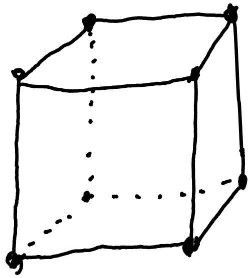

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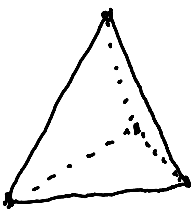
$8 - 12 + 6 = 14 - 12 = 2$



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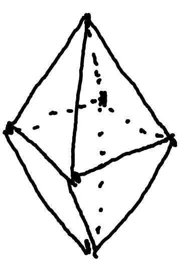


$$\begin{array}{r} V = 6 \\ -\,E = 12 \\ +\,F = 8 \\ \hline \end{array}$$

$6 - 12 + 8 = 2$

I cant draw it, but trust me, it works for the icosahedron and dodecahedron as well $(12-30+20)$ and $(20-30+12)$.

These are the regular (Platonic) solids, but there are more "semi regular" (Archemedian) solids!

Loosly, a regular 3D arrangement of regular n-gons, but allowing different types.

Eg: truncated regular polyhedra, snub polyhedra, etc. (Try EC here)

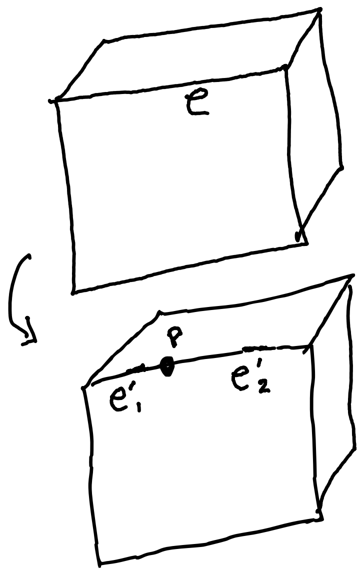
Even works for non regular faces!
(Look up rhombus one)

May not be suprising that the "exact shape of things" doesnt seem to change V-E+F.

We've changed the curvature, and thus, the shape of our edges and faces, but we havent changed what's connected to what, and so we havent changed the number of vertices edges or faces.

So V-E+F = 2 still.

By now, you're probably dying to know why! Here's some intuition:

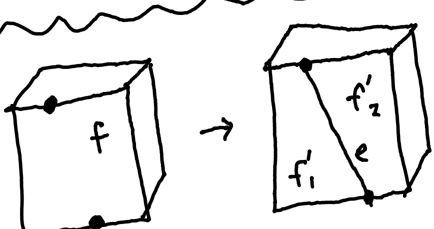


If I add a vertex on an existing edge, I've increased V by one, but I also split one edge into 2! So E increases by one as well

$$\text{So} \quad (V+1)-(E+1)+F$$

$$= V - E + F \text{ still.}$$ It doesn't change.

Likewise, if I add an edge, I end up splitting one face into 2. Both E and F go up by one.



$$V-(E+1)+(F+1) = V-E+F$$

To turn this idea into a proof that any polyhedron has $V-E+F=2$, you need to argue why you can get from any polyhedron to any other using these moves. Since these moves dont change $V-E+F$, we would then know this quantity would be the same for all polyhedra.

Then just calculate it for any one of them, like we did, and we see it is 2.

More technical than you'd think! But I believe the intuition is good enough.

Ok, so it works. Who cares?

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— algebraic topologists

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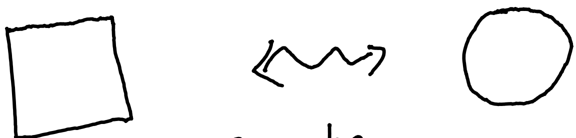
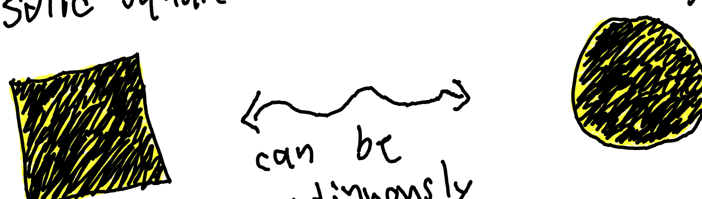
But really, theres something going on here. The quantity $V - E + F$ doesnt seem to depend on the exact shape of things, like distances, angles, or volume. The number $V - E + F$ doesnt "see" these geometric properties...

This brings us to topology.
It's "looser" than geometry because it "ignores" more. In topology:

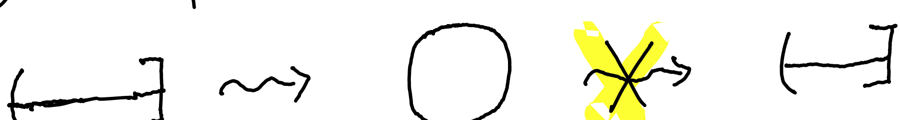
a square can be continuously deformed into a circle!



a solid square can be continuously deformed into a disk!



But topology doesn't ignore <u>everything</u>!

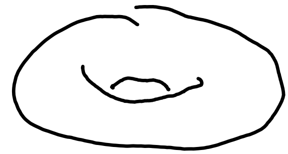


So, If we take some shape, put vertices edges and faces on it, and compute $V - E + F$, it ends up only depending on the topology of the shape!

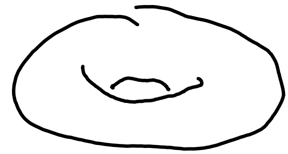
Every example of a polyhedron so far has basically been an (angular and pointy) sphere!

what if we do this with something that isn't topologically a sphere? Like ...

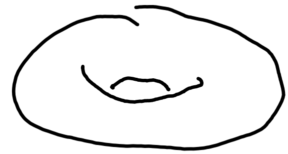
A donut!



Lets put some vertices and edges on this shape.

A donut! 

Lets put some vertices and edges on this (holesome) shape.

A donut! 

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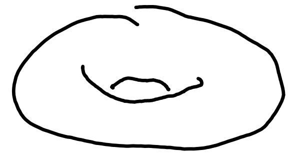

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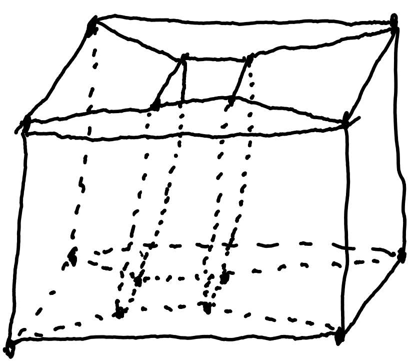

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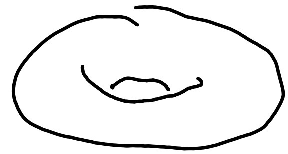

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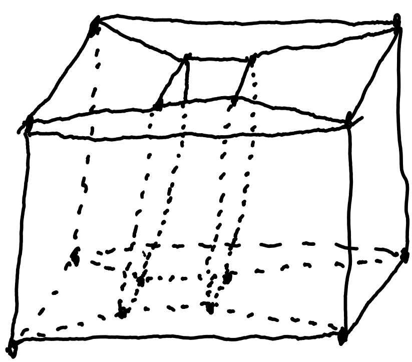
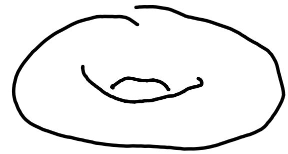


$V = 16$

A donut!



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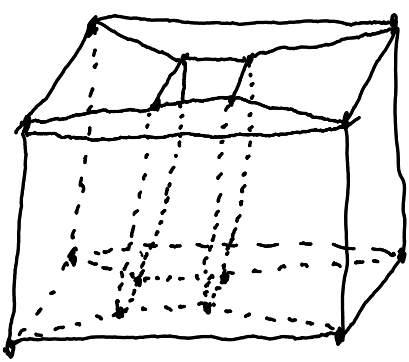
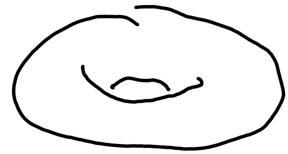


$V = 16$

$E = 32$

A donut! 

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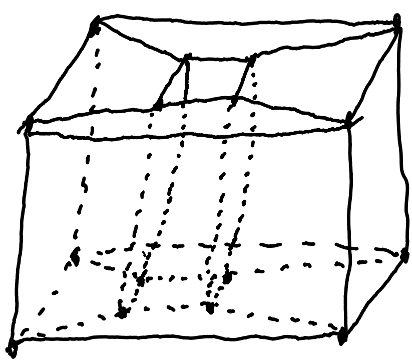

V = 16

E = 32

F = 16

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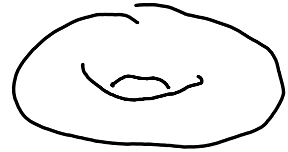

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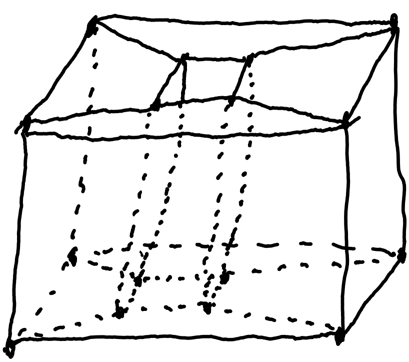

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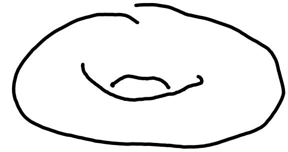
$$V - E + F = 16 - 32 + 16 = 0$$

Big Wow, it isn't 2!

A donut! 

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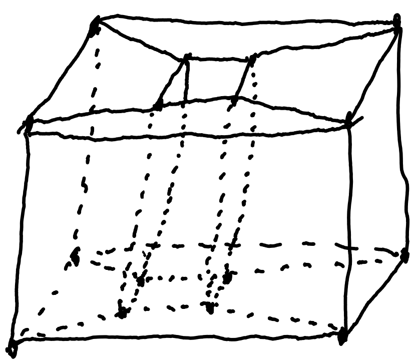

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* technically, this is contingent on Theorem 3.4.18.

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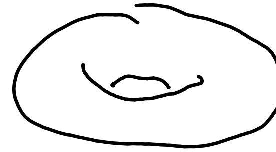

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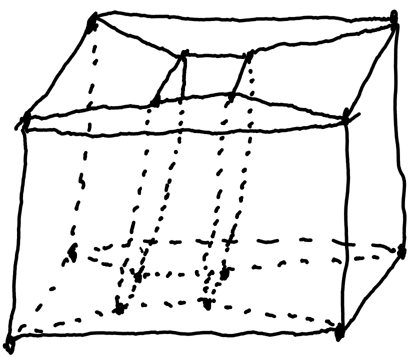

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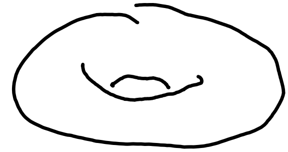
† Theorem 3.4.18: $0 \neq 2$

A donut! 

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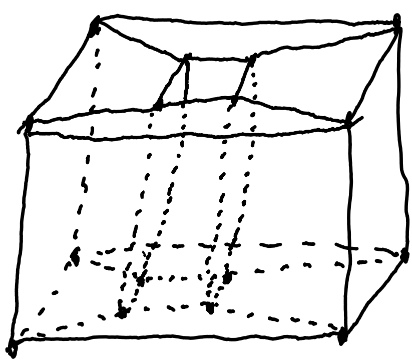

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Big WOW, it isn't 2!*

* technically, this is contingent on Theorem 3.4.18.†

(thats a joke, but it doesn't always feel that way)

† Theorem 3.4.18: $0 \neq 2$

So if we slap v's, e's, f's on a donut (torus), $V-E+F$ is different than with the sphere!

In general, $V-E+F$ of some shape is called that shape's Euler characteristic, and its denoted χ.

So $\chi(\text{sphere}) = 2$,

$\chi(\text{torus}) = 0$.

Its a fact that it actually doesnt depend on how you place the vertices edges and faces

χ only depends on the topology of the shape.

So if we slap v's, e's, f's on a donut (torus), V-E+F is different than with the sphere!

for any "triangulation", since they will all give the same number

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Deep connections to other areas of math:

The Euler characteristic, in its modern form, is one of the most powerful invariants used across geometry and topology. It connects to many different concepts, even though its fundamentally a topological property. 2 big connections:

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- to geometry (Gauss-Bonnet)

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Big Theorems relating Euler characteristic to something seemingly unrelated!

Deep connections to other areas of math:

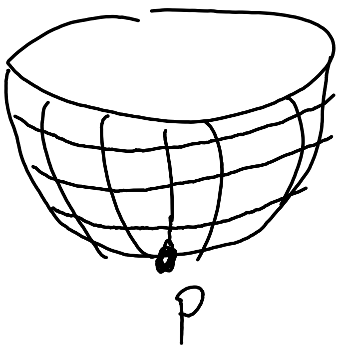
The Euler characteristic, in its modern form, is one of the most powerful invariants used across geometry and topology. It connects to many different concepts, even though its fundamentally a topological property. 2 big connections:

- to geometry (Gauss-Bonnet)
  (specifically, curvature)

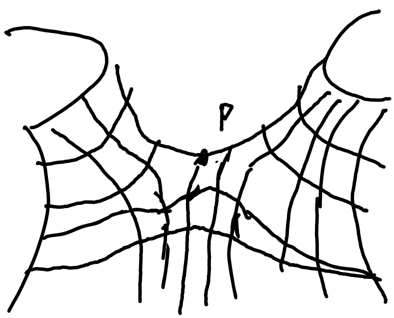
- to physics (Poincaré-Hopf)
  (specifically, vector fields)

Big Theorems relating Euler characteristic to something seemingly unrelated!

Gauss - Bonnet

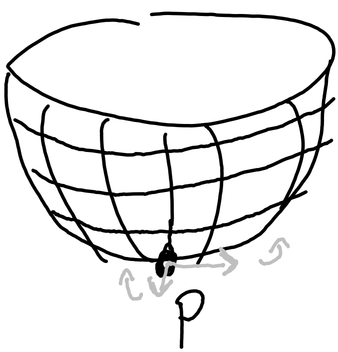


positive curvature at p



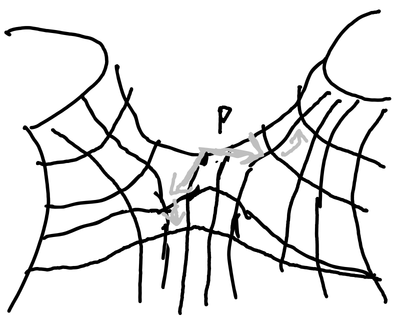
negatively curved at p.

Gauss - Bonnet



positive curvature at p

Tangent vectors in direction of greatest and least curvature curve in the same direction



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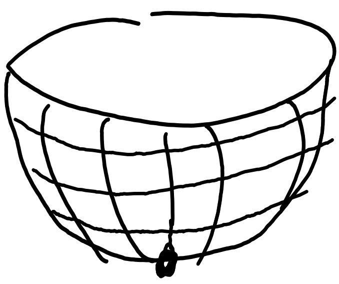

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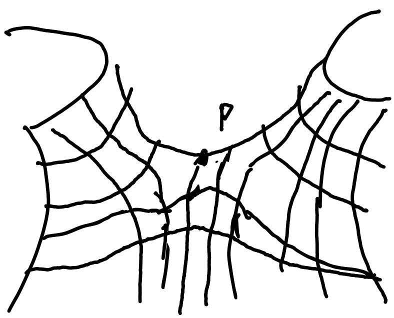

negatively curved at p.

Tangent vectors in direction of greatest and least curvature curve in different directions

analogy: $(+) \cdot (+) = +$

$(-) \cdot (-) = +$

$0 \cdot (+) = 0$

$(-) \cdot (+) = -$

$0 \cdot (-) = 0$

$(+) \cdot (-) = -$

The theorem:

"Add up" the measure of curvature at each point (the technical way to do this is to take an integral).

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Gauss-Bonnet: this total curvature across a surface S always equals $2\pi \cdot \chi(S)$.

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"Add up" the measure of curvature at each point (the technical way to do this is to take an integral).

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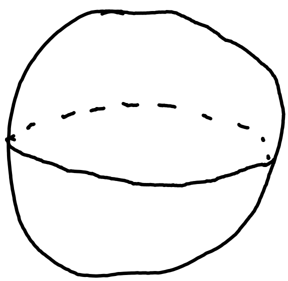
In symbols,

$$\underbrace{\int_S \kappa \, dA}_{\text{(In principle, could depend on the geometry)}} = \underbrace{2\pi \chi(S)}_{\text{(only depends on the topology!)}}$$

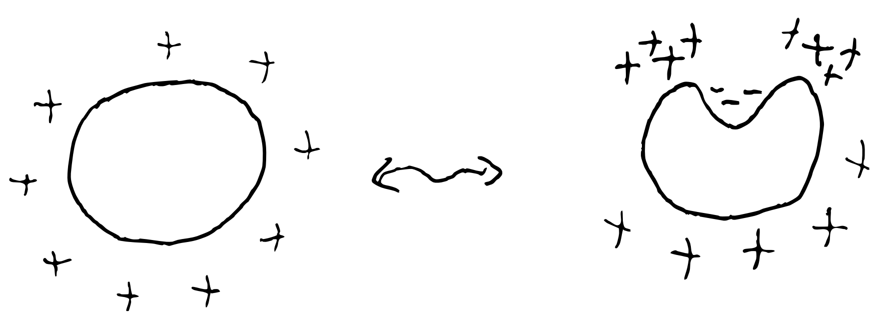
Illustration:

"geometric" sphere



still a topological sphere.

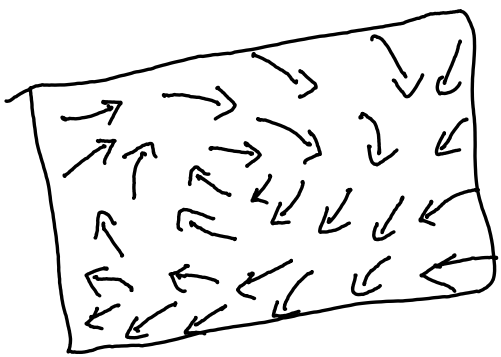
in 2 dimensions:



you can make a geometric distortion, but all you do is shift around where the curvature is. Total curvature is conserved, because (miraculously) it's topological!

Poincare - Hopf :

put a (smooth) vector field on your surface :



(Better pictures exist.)

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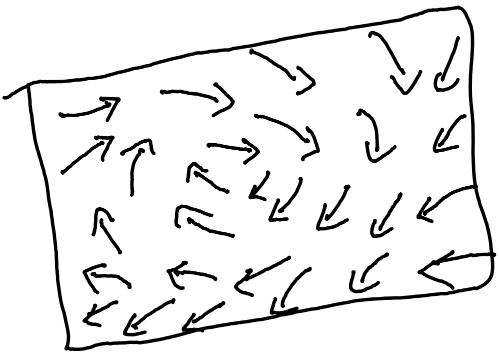

.

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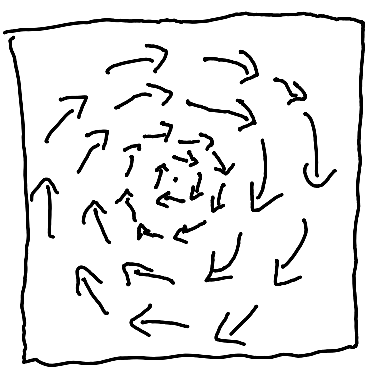
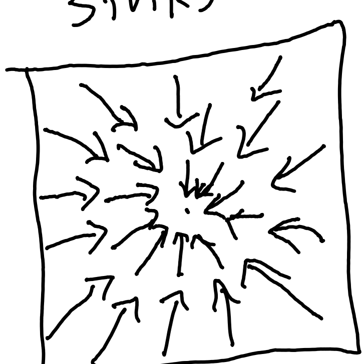
There could be "stationary points", aka, zero's of the vector field

spirals



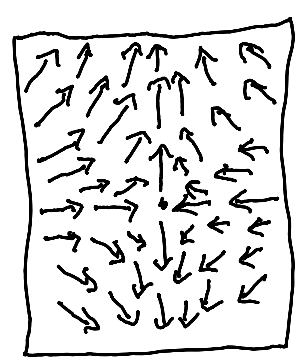
,

sinks



(and sources)

saddles

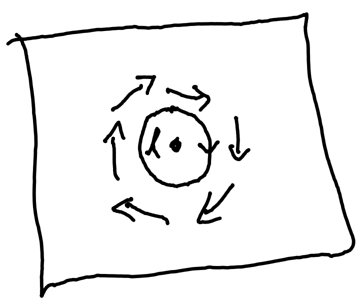


etc.

Lets define the <u>Index</u> of a stationary point:

As you take a small (clockwise) loop around the zero, how many times does the arrow turn clockwise?

spiral:

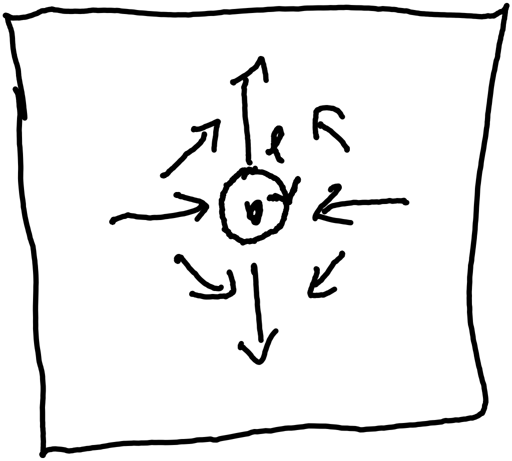


as we walk around ℓ,

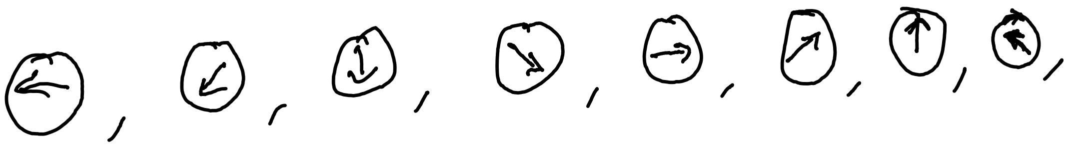
the arrows look like $\downarrow$, $\swarrow$, $\leftarrow$, $\nwarrow$, $\uparrow$, $\nearrow$, $\rightarrow$, $\searrow$, back to $\downarrow$.

Arrows net turn is one time clockwise. Index of this zero is 1!

what about a saddle?



as we traverse ℓ clockwise, the arrows do this:



then back to . Arrows go around <u>counter clockwise</u>!

we say this zero has index -1!

Big theorem (PH):

put any smooth vector field on a surface S having finitely many stationary points. Then:

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Worth stressing: this works for any vector field!

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Easy by-product:
If S is any surface that has $\chi(S) \neq 0$ (like how a sphere has $\chi = 2$),

then <u>any</u> vector field you put on S must have some stationary point!

Real life "application":
- There is always some spot on earth with zero wind!

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Real life "application":

- There is always some spot on earth with zero wind!

- You cant comb the hair on a hairy ball (welcome to higher math!)

Big theorems over now

Heres some fun stuff you can do with the Euler Characteristic :

Show that if P is a polyhedron with only pentagon and hexagon faces, and every vertex in P meets 3 edges, then P has exactly 12 pentagons!

(A "soccer ball" (truncated icosahedron) is an example of this)

2) give an alternative proof using χ(any polyhedron with no holes) $= 2$ that there are only 5 platonic solids!

3) The game Brussel Sprouts!
(like our last game Sprouts, but where the rules are actually right, and every vertex has 4 edges coming out!)

Show that its actually a joke game. No matter who does what, player 1 (resp 2) wins if we start with an odd (resp even) number of crosses

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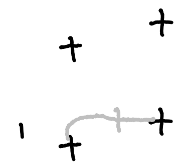

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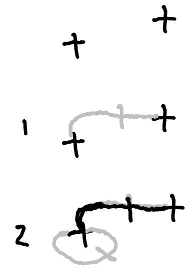

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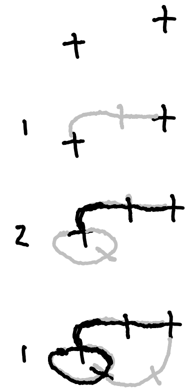

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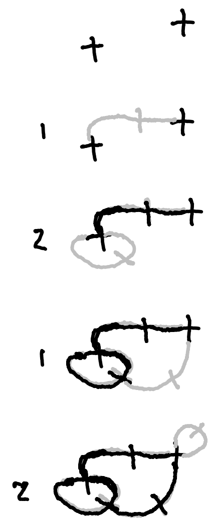

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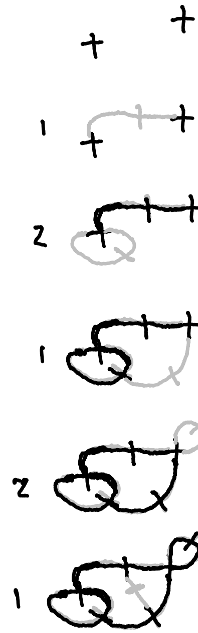

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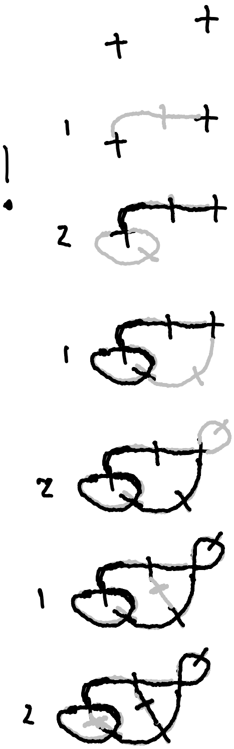

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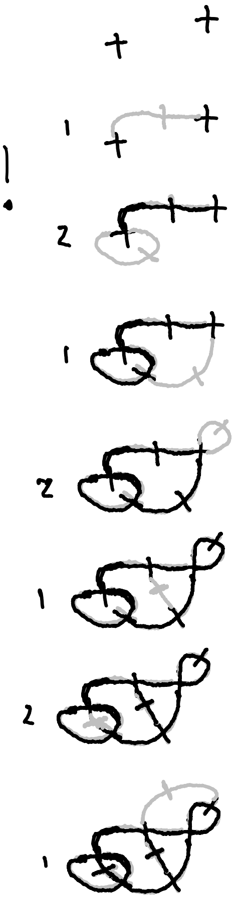

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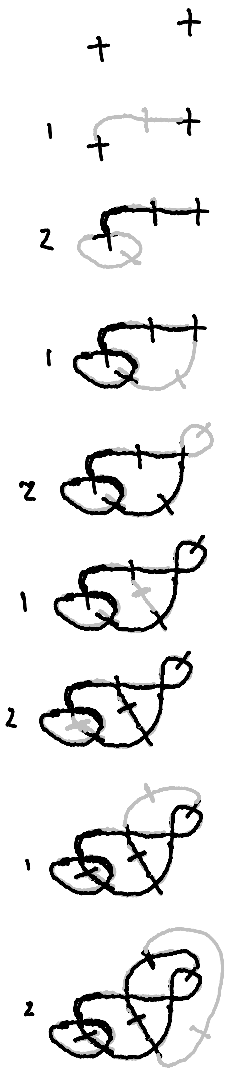

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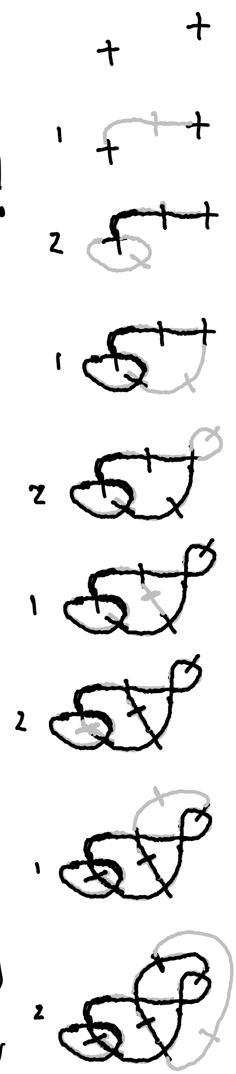

Player 2 wins, since 1 cant move.

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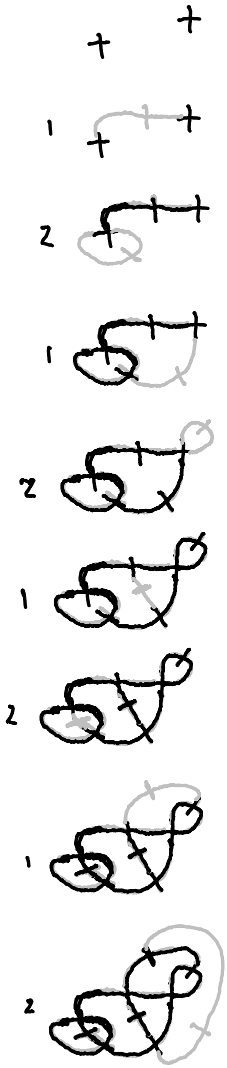

Thanks For Comming!

Have a great Thursday!

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