On Dr. Vaughan Jones’s life and work

A vaguely adequate account

/*picture with cigar quote*/

About this talk

To tell a story -- a rough narrative of what Dr. Jones worked on, and why he was one of the more creative fields medalists (and arguably the most accomplished Vanderbilt faculty members).

The story is a modern day mathematical wonder (like Moonshine, and Langlands).

It also plays out like a rom com. Or maybe Pulp Fiction. Several different characters' arcs come together at the end. It's unexpected. It's unbelievable. It's satisfying.

Teaser

Dr. Jones discovered a connection between two (seemingly) very different areas of math:

Quantum Mechanics / Operator Algebras on one hand...

...and Knot Theory / Low Dimensional Topology on the other.

Jones's work ties these two together in a spectacular way (pun intended).

The connection was so profound that it basically spawned off into a whole new active branch of math, called quantum algebra.

David Penneys Quote: A good functional definition of Quantum Algebra is 'the mathematics derived from the Jones polynomial.'

The most direct consequence of Vaughan's work (which was initially purely in OAs) is the most important knot invariant we have to this day: the above mentioned Jones polynomial.

Plot Overview

Vaughan studied operator algebras in grad school -- in particular, von Neumann algebras.

(a topic in functional analysis -- basically the math behind Quantum Mechanics)

He introduced / studied many definitions and constructions of central importance.

E.g, subfactors and their type (either I, II, or III), the Jones index, the Jones tower, etc.

His exploration of the tower yielded an algebra satisfying an interesting set of equations.

He asked around and discovered these equations are similar to the equations of “braids”

He adapted the braid picture, and gave a diagrammatic interpretation to his tower’s equations.

This naturally led to the definition of the Jones polynomial of a knot.

He received his fields medal wearing a rugby jersey.

Whirlwind Intro to Functional Analysis

Mathematicians think of it as “trickier infinite dimensional linear algebra”

Physicists think of it as “Quantum Mechanics but with extra steps”

It ties algebra, analysis, and topology together.

Banach and Hilbert Spaces

(Complete) Vector spaces with a norm, or an inner product space, respectively.

- Vector space means you can add vectors, and scale them by real (or complex) numbers
- Norm means length. If x is a vector, then ||x|| is a nonnegative real number.
  - Satisfies rules like ||x + y|| <= ||x|| + ||y||, etc. Look it up! Or take Aldroubi's class.
- Inner product is basically like "something that follows the usual dot product rules"
  - If x and y are vectors, then <x|y> is a scalar. Some rules hold.
  - Important: every inner product space is also a normed space:
    - Just define the norm of x as ||x|| = SquareRoot (<x|x>)

Completeness

One last definition (ties in analysis): A norm induced a metric, which means distance function

(just define distance from x to y as ||x - y||).

A very good property for metric spaces to have is called completeness.

Intuitively, it means there aren't any holes. Formally: Cauchy Sequences converge.

Example: the completion of he rational number line is the real number line

Q is super duper not complete. E.g, (3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592 …)

A normed space whose metric is complete is called a Banach space

An inner product space whose metric is complete is called a Hilbert space.

Functional Analysis

Key examples: given a finite list of numbers, (x1, x2, x3 … xn), if p is a real number 1 or greater:

The p-norm is the p-th root of the sum of the p-th powers of the coordinates.

If p = 2, this is just the dot product, which is an inner product, not just a norm.

Why is it “basically infinite dimensional linear algebra”?

Why is it “basically QM with extra steps”?

Operator Algebras

If X is a Banach or Hilbert space, it has both algebraic and analytic structure

The natural “morphisms” from X to itself should respect both of these.

Aka, continuous linear transformations (equivalently: bounded linear operators).

The collection of bounded linear operators B(X,Y) between Banach spaces X and Y has a lot of nice and mysterious algebraic structure.

Even more so if X = Y. Or if X and Y are not just Banach, but Hilbert spaces.

For instance, the ***spectral theorem*** is really about factorization in B(X,Y) (Hilbert spaces).

C* and Von Neumann Algebras

Wikipedia says about C*-algebras:

A **C*-algebra** (pronounced "C-star") is a Banach algebra together with an involution satisfying the properties of the adjoint.

- it is a topologically closed set in the norm topology of operators.
- it is closed under the operation of taking adjoints of operators.

And about von Neumann algebras:

von Neumann algebra is a *-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator.

More definitions

- A **factor** is a von Neumann algebra with trivial center, i.e. a center consisting only of scalar operators.

Center means the stuff that commutes with everything. Aka, x is in the center if xy = yx, for all y.

Subfactor -- a subalgebra that is a factor and contains 1.

Jones Index!

Jones index theorem: If N is a subfactor M, both of type II1, then the index [M:N] is of the form:

4cos(pi/n)^2 for n = 3,4,5,6...

Or……. is at least 4! And all of these values occur. https://en.wikipedia.org/wiki/Subfactor

Now for some Wikipedia surfing...