Finite Combinatorial Pre-geometries

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matroids

What is a Matroid?

Any one of these 6 easy definitions

- A pair (E,I) of a set E and a family, I, of subsets of E such that:
  - 1) I is nonempty, 2) (B in I and A a subset of B) => (A in I), and 3) for all A, B in I with |A|<|B|, there exists a b in B such that (A u {b}) in I.
- A pair (E,B) of a set E and a family, B, of subsets of E such that:
  - 1) B is nonempty, 2) for all X, Y in B, and for all x in X\Y, there exists a y in Y\X such that (X\{x}) u {y} is in B.
- A pair (E,r) of a set E and a function r:2^E -> N such that r satisfies:
  - 1) For all subsets A of E, 0 <= r(A) <= |A|, 2) For all subsets A and B of E, r(A)+r(b)>=r(AuB)+r(A intersect B), and 3) for all A subset of E, x in E, x not in A, r(A)<=r(Au{x})<=r(A)+1
- A pair (E,cl) of a set E and a function cl:2^E->2^E such that cl satisfies
  - 1) For all A in P(E), A contained in cl(A), 2) cl(A) = cl(cl(A)), 3) for all A, B in P(E), A in B implies cl(A) in cl(B), and 4) for all a and b in E, and all subsets Y of E, a in cl(Yu{b})\cl(Y) iff b in cl(Yu{a})\cl(Y)
- A pair (E,F) of a set E and a family, F, of subsets of E such that:
  - 1) E in F, 2) A and B in F implies (A intersect B) in F, and 3) if S in F, then the sets in F that cover S under the containment relationship form a partition of E\S.
- A finite atomic semimodular lattice

Got That? Let’s Proceed…

No JK. All of that is true, but what ***IS*** a matroid?

What even is math?

Big theme: idea becomes encapsulated by an object

- Linearity encapsulated by vector spaces
- Continuity encapsulated by topological spaces
- Symmetry encapsulated by groups
- Smoothness encapsulated by manifolds
- “Volume” encapsulated by measurable spaces (and sigma algebras)
- Connectivity encapsulated by graphs
- Relatedness encapsulated by posets and lattices
- Composition encapsulated by a category
- Things that might happen encapsulated by a probability space

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Secondary theme: “object” means set with structure

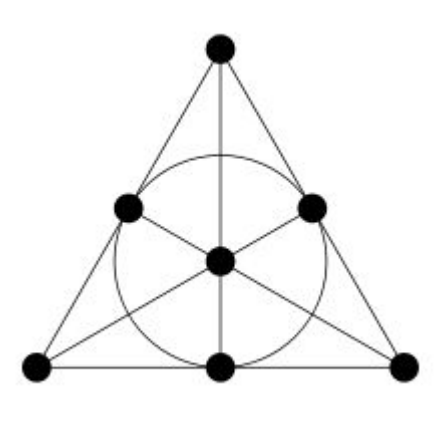
- Algebraic structure: operations, relations, what have you
- Structure could be like a function on the set (e.g. metric on a metric space, atlas of charts on a manifold)
- Sometimes, structure means certain distinguished subsets of some set based on the original, or the original itself (e.g. topological spaces, sigma algebras, relations, graphs)
- Then the axioms!

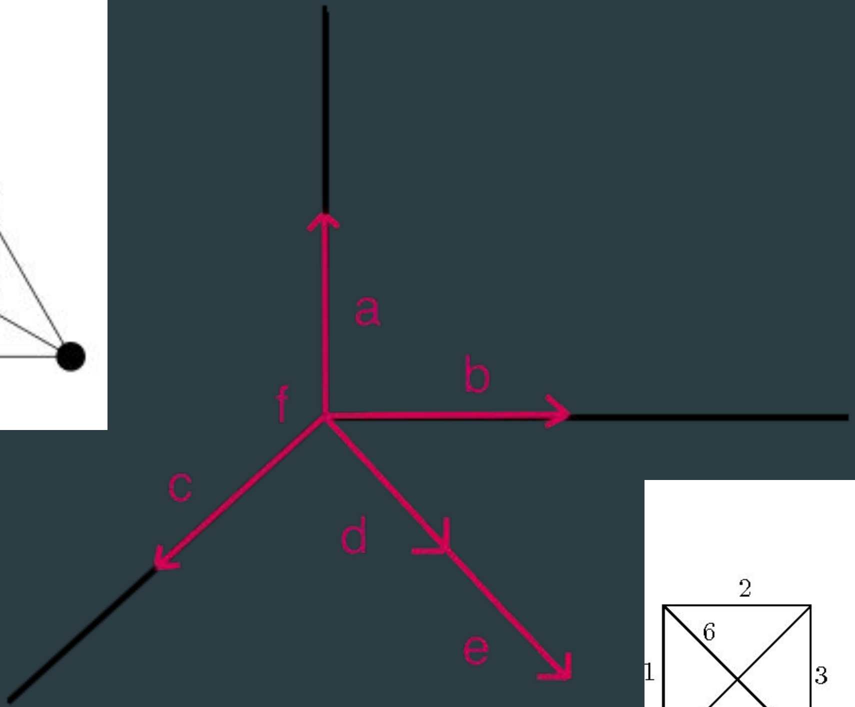
Matroids fit all these patterns

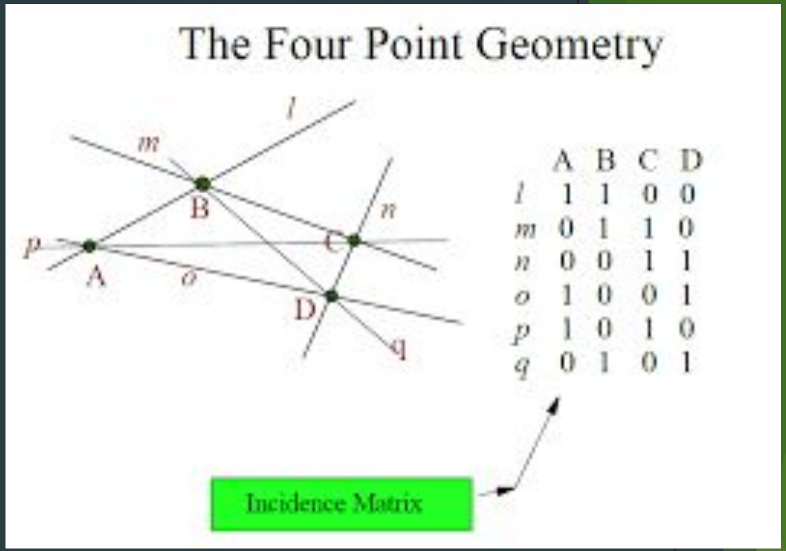
They also reveal deep similarities between

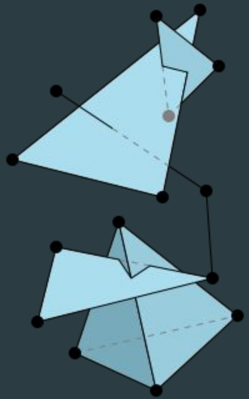
- Linear Algebra
- Graph Theory
- Finite Geometry

Pictures of Matroids

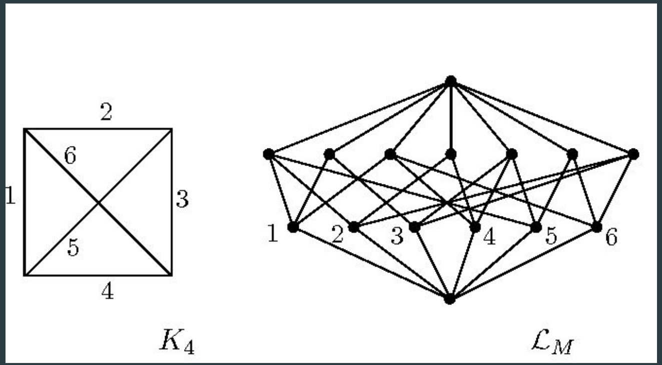


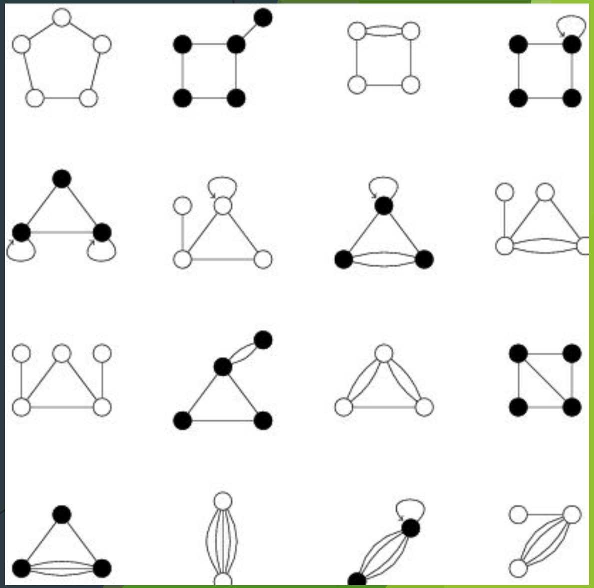






<- not quite actually





How exactly matroids do this

Idea: linear independence, or redundancy

Execution: set along with a collection of subsets

Some History

- Witney
- Rota
  - “It is as if one were to condense all trends of present-day mathematics onto a single finite structure, a feat that anyone would a priori deem impossible, were it not for the mere fact that matroids do exist.”
  - He really lobbied hard to call them Combinatorial Geometries
- Robbins

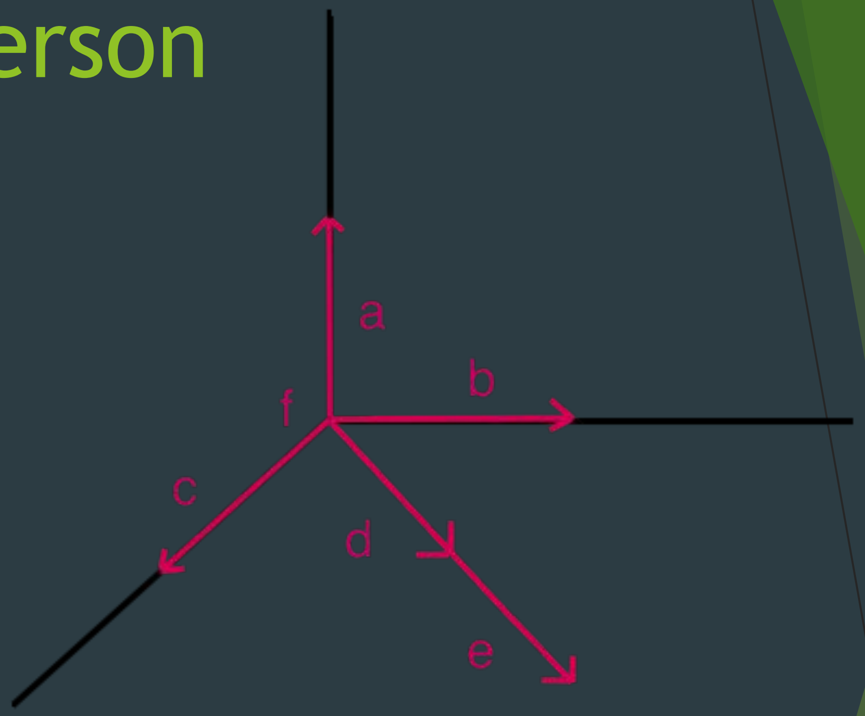
Example for the non math person

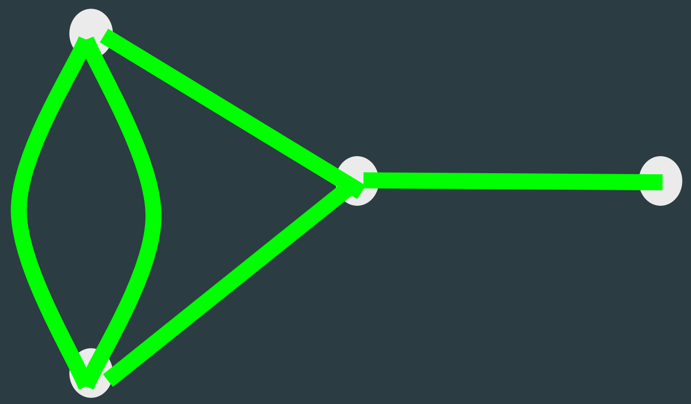
- Pages in a book
  - You have to read some history book chapter for a class
  - You’ve got a strong feeling that any 5 pages will suffice
  - Boom, Matroid

Examples for the math person

- Vectors in a vector space

- Edges in a graph





Definitions Revisited - Independent Sets

- A pair (E,I) of a set E and a family, I, of subsets of E such that:
  - 1) I is nonempty
  - 2) (B in I and A a subset of B) => (A in I),
  - 3) for all A, B in I with |A|<|B|, there exists a b in B such that (A u {b}) in I.

Definitions Revisited - Basis Sets

- A pair (E,B) of a set E and a family, B, of subsets of E such that:
  - 1) B is nonempty
  - 2) for all X, Y in B, and for all x in X\Y, there exists a y in Y\X such that (X\{x}) u {y} is in B.

Definitions Revisited - Rank Function

- A pair (E,r) of a set E and a function r:2^E -> N such that r satisfies:
  - 1) For all subsets A of E, 0 <= r(A) <= |A|
  - 2) For all subsets A and B of E, r(A)+r(b)>=r(AuB)+r(A intersect B)
  - 3) for all A subset of E, x in E, x not in A, r(A)<=r(Au{x})<=r(A)+1

Definitions Revisited - Closure Operator

- A pair (E,cl) of a set E and a function cl:2^E->2^E such that cl satisfies:
  - 1) For all A in P(E), A contained in cl(A)
  - 2) cl(A) = cl(cl(A))
  - 3) for all A, B in P(E), A in B implies cl(A) in cl(B)
  - 4) for all a and b in E, and all subsets Y of E, a in cl(Yu{b})\cl(Y) iff b in cl(Yu{a})\cl(Y)

Definitions Revisited - Flats (closed sets)

- A pair (E,F) of a set E and a family, F, of subsets of E such that:
  - 1) E in F
  - 2) A and B in F implies (A intersect B) in F
  - 3) if S in F, then the sets in F that cover S under the containment relationship form a partition of E\S.

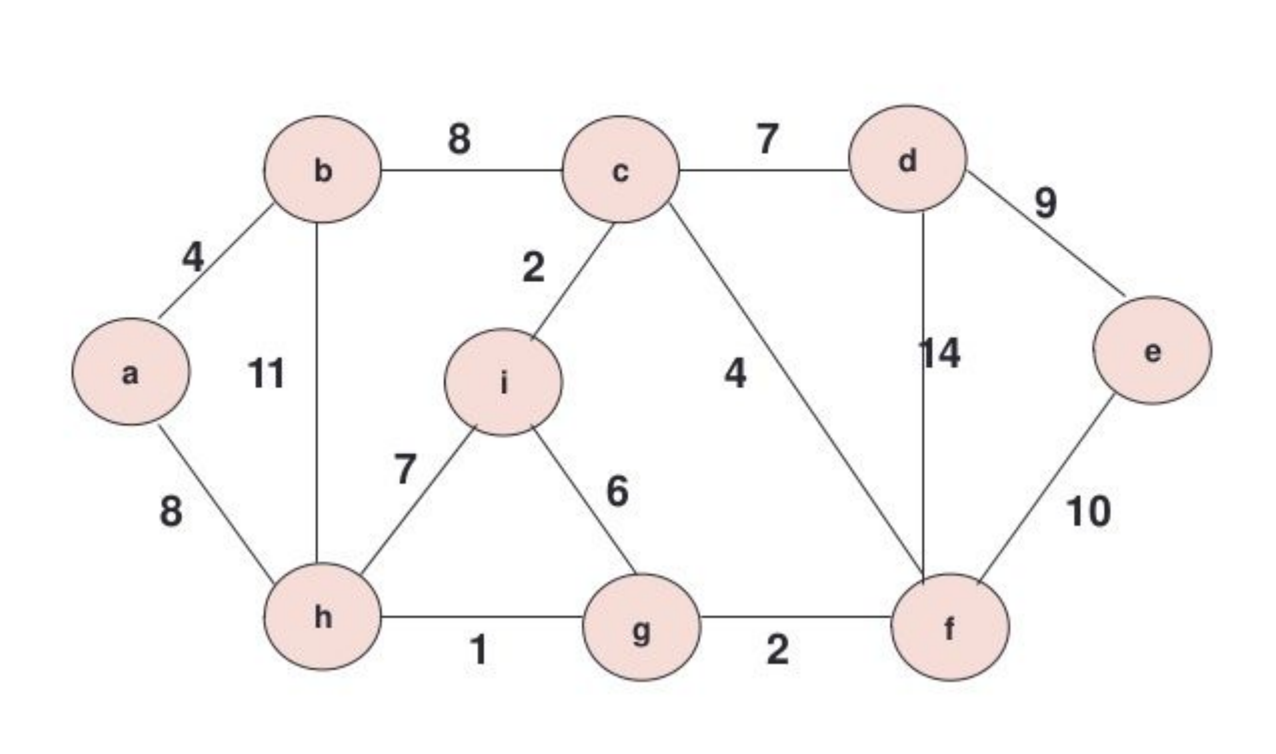
Why do we care

- Cliched answer – matroids are nice

Greedy algorithm

- How nice? Greedy algorithm!
  - if your elements have weights, you can always build a minimum weight basis set by adding the cheapest element that isn't dependent with anything else you've added so far.

Example of Greedy Alg on a (simple) Graphic Matroid - Kruskal's Algorithm



Further Reading

- James Oxley - Matroid Theory
- Gordon, McNulty - Matroids, a Geometric Introduction
- Wikipedia

Thanks for Coming