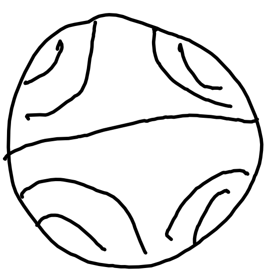
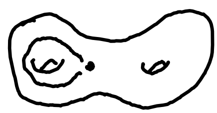
Surface $S_{g,b,p}$ genus, boundary, punctures.

other than ~~fen~~ fin many surfaces,

$\exists$ map : $\pi_1(S) \longrightarrow PSL(2,\mathbb{R})$

discrete injective $\cong$? $Isom^+(\mathbb{H}^2)$

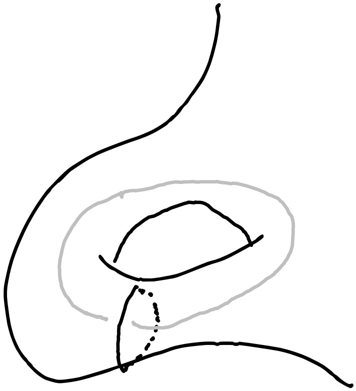


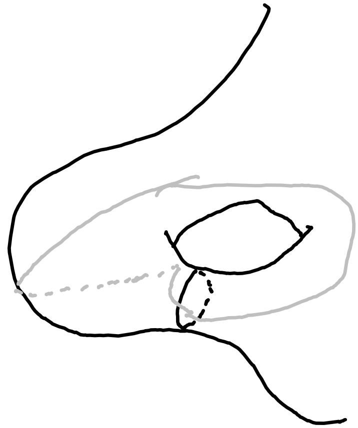


$MCG(S) := Homeo^+(S) \Big/ \text{homotopy}$ orientation pres

is fin presented.

finite generation can be nice (Dehn twist)





Nielsen Thurston Classification of $MCG(S)$

i) finite order

ii) reducible

iii) pseudo-anosov

$MCG \circlearrowleft$ Homology of surface

symplectic rep $MCG(S) \longrightarrow Aut(H_1 S)$

classifying space for surface bundles

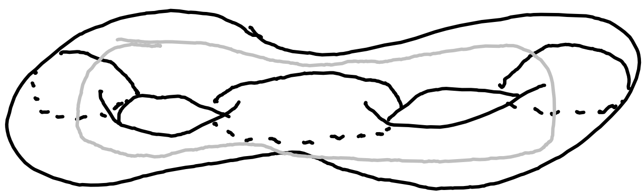
Hyperbolic 3 manifolds

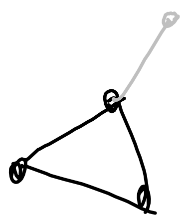
mapping torus of an element

virtual fibering thm

every closed hyperbolic 3 manifold
has a (finite sheeted) covering space
by the mapping torus of some
pseudo-anasov map on an S.

Curve complex.





Every 3 manifold w/o boundary is a 3-sphere with a knot deleted, and then glued back in somehow

(Dehn Surgery)

$MCG(S) \circlearrowright H_1(S, \mathbb{Z})$

$\Psi : MCG(S) \longrightarrow Aut(\overbrace{\mathbb{Z}^{2g}}^{H_1(S,\mathbb{Z})})$

$\ker(\Psi)$ called the Torelli group of S.

For torus, its trivial.

$i(\_,\_) : H_1 \times H_1 \to \mathbb{Z}$

algebraic intersection number of 2 1-chains.

$\Psi \; MCG(S) \to Sp(2g, \mathbb{Z})$

1) understand $H_1(T^2, \mathbb{Z})$

2) understand $MCG(T^2)$

i.e. action of Dehn Twists and compositions on simple closed curves.

3) understand $\Psi : MCG(T^2) \xrightarrow{\cong} SL(2,\mathbb{Z})$

(prove iso).

4) cor: $MCG(T^2)$ generated by Dehn twists

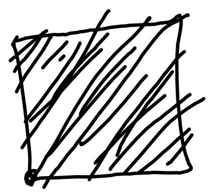
5) Higher Genus

$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$

eigenvalues : $\lambda, \frac{1}{\lambda}$

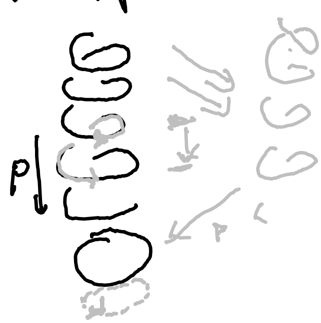
if irrational, $Ax = \lambda x$, then A fixes no curves.

Fellenson



MCG splits into 3 types of elements.

Sharp turn. The following thoughts and notes are about chapter 1 hatcher and the mapping class group of the torus in the primer. When I think of visualizations and explanations for relevant topics like why deck transformations of the universal cover of a space are isomorphic to the fundamental group.

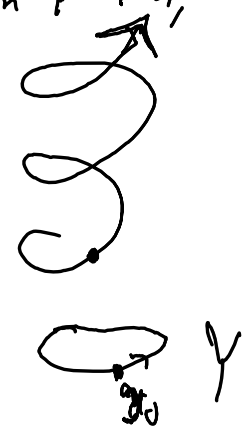


univ cover?

covering space def ✓

a path γ starting at the basepoint x_0 uniquely lifts to a path $\tilde{\gamma}$ in univ cover $\tilde{X}$ once the start of $\tilde{\gamma}$, which will be in $p^{pre}(x_0)$, is fixed.

$$\begin{array}{c} \tilde{X} \\ \downarrow p \\ X \end{array}$$



fix $\tilde{y}_0 \in p^{-1}(y_0)$

$$\begin{tikzcd}
 & \tilde{Y} \arrow[d, "p"] \\
I \arrow[r, "\gamma"] \arrow[ur, dashed, "\tilde{\gamma}"] & Y
\end{tikzcd}$$

$\exists! \tilde{\gamma}$ st $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{y}_0$

a deck transformation is like a homeomorph $\tilde{Y} \xrightarrow{f} \tilde{Y}$ commuting with $\begin{array}{c} \tilde{Y} \\ \downarrow p \\ Y \end{array}$. subgroup of automorphism group

It must send a point in $p^{pre}(y_0)$ to another, since they commute with p.
So this subgroup of $Aut(\tilde{Y})$ generates a group of permutations on $p^{pre}(y_0)$

A deck transform just looks like shifting sheets around. But like, by connecting various sheets to others globally using paths, we end up forcing a certain regularity to how we shuffle sheets, which algebraically ends up being exactly the fundamental group. Key idea is a loop at y_0 lifts to a path from $\tilde{y}_0$ to another point in $p^{pre}(y_0)$.

Really good to review #7,

Covering spaces and π_1.

Homotopy lifting property:

we say a map $\begin{array}{c} A \\ \downarrow p \\ X \end{array}$ satisfies the homotopy lifting prop

iff: given a homotopy $F : Y \times I \to X$, (notation: $F(y,t) = f_t(y) : Y \to X$), and a lift of f_0 along p, $\tilde{f}_0$, then:

$\exists!$ lift $\tilde{F} : Y \times I \to A$ st

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d] & A \arrow[d, "p"] \\ Y \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & X \end{tikzcd}$$

commutes

Thm: Covering spaces satisfy the homotopy lifting property.

What if Y a point?

$$\begin{tikzcd}
\{0\}\times\{0\} \arrow[r, "\tilde{x}_0"] \arrow[d] & A \arrow[d, "p"] \\
\{0\}\times I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & X
\end{tikzcd}$$

γ is just a path in X. $\tilde{x}_0(0,0) =: \tilde{x}_0 \in A$ is just a point. For this diagram to commute, $p(\tilde{x}_0) = \gamma(0)$. So this means there is a unique path $\tilde{\gamma}$ in A projecting down to the path γ in X starting at a given point in $p^{-1}(\gamma(0))$.

What if $Y = I$? call the variable s. s shall stand for "specific". As in, this is a special case. The standard time variable t is reserved for the homotopy itself, not the parameterization of any one specific path. ~~After~~ all, Y need not be I, so it makes sense not to change up variables:

$$\begin{tikzcd} I \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d] & A \arrow[d, "p"] \\ I^2 \arrow[r, "f_t(s)"] \arrow[ur, dashed, "\tilde{F}"] & X \end{tikzcd}$$

given a homotopy of free paths, $(s,t) \longmapsto f_t(s)$, and a lift $\tilde{f}_0$ of f_0 (In that $p \circ \tilde{f}_0 = f_0$), then $\exists!$ lift of the homotopy making this commute.

I.e, a homotopy $\tilde{F} : I^2 \to A$ s.t. $\tilde{F}(s,0) =: \tilde{f}_0(s)$ $\forall s \in I$, (upper triangle commutes) and (lower triangle:) $p(\tilde{F}(s,t)) = f_t(s)$ $\forall s,t \in I$

lifting correspondence follows

given a covering map
$$\begin{array}{c}(\tilde{X}, \tilde{x}_0)\\ \big\downarrow p\\ (X, x_0)\end{array},$$
then we can define

$\Phi : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$ as follows.

Take a loop in X based at x_0 and lift it to a path in $\tilde{X}$ based at $\tilde{x}_0$. This path may or may not be a loop, but the right endpoint will be some point in $p^{-1}(x_0)$. This is the associated point in $p^{-1}(x_0)$ to the original loop in X. I.e.,

$$\Phi([\gamma]) = \tilde{\gamma}(1)$$

Prop: Φ is well defined

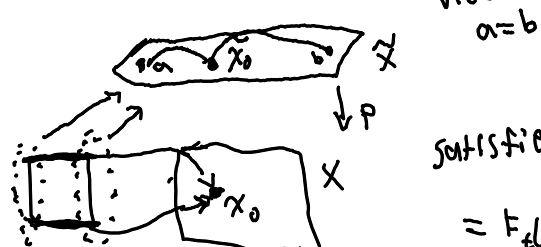
a) $\Phi : \pi_1(X, x_0) \to p^{-1}(x_0)$

by $\Phi([f]) = \tilde{f}(1)$ is well def since

if $[f] = [f']$ then there is a homotopy in X
$F_t(s)$ s.t. $F_0 = f$ and $F_1 = f'$, and
$\forall t, F_t(0) = x_0 = F_t(1)$. This homotopy lifts to
a path homotopy $\tilde{F}$ from $\tilde{f}$ to $\tilde{f}'$.
But then $\tilde{f}$ and $\tilde{f}'$ must have the same
endpoints, because $\tilde{F}_t(1)$ is a path from $\tilde{f}(1)$ to $\tilde{f}'(1)$,
but $p\tilde{F}_t(1) = F_t(1)$ is the constant path at x_0.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ I^2 \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & (X, x_0) \end{tikzcd}$$

So $\tilde{F}_t(1)$ is a lift of the constant path. By the lifting thm, this is unique. Meanwhile the constant path $\gamma_{\tilde{f}(1)}$ satisfies $p\gamma_{\tilde{f}(1)} = \gamma_{x_0} = F_t(1) = p\tilde{F}_t(1)$.



And since $\gamma_{\tilde{f}(1)}$ and $\tilde{F}_t(1)$ both start at $\tilde{f}(1)$, they're the same. So $\tilde{F}_1(1) = \tilde{f}'(1) = \tilde{f}(1)$ ■

In conclusion, if $[f]=[f']$, then we lift the homotopy $F: f\simeq f'$ to a homotopy $\tilde{F}: \tilde{f} \simeq \tilde{f}'$, and since the unique lift of a constant path is constant, and $F_t(1): x_0 \rightsquigarrow x_0$ is the constant path, then $\tilde{F}_t(1): \tilde{f}(1) \rightsquigarrow \tilde{f}'(1)$ must be the constant path, so then the map $\Phi: \pi_1(X, x_0) \to p^{-1}(x_0)$ is well defined since $\Phi([f]) = \tilde{f}(1) = \tilde{f}'(1) = \Phi([f'])$.

Prop: if $p:(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map, then $p^*: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective, and $\mathrm{im}(p^*) = \{[f] \in \pi_1(X, x_0) \mid \Phi[f] = \tilde{x}_0\}$

$= \Phi^{-1}(\tilde{x}_0)$

that is

$$p^*(\pi_1(\tilde{X}, \tilde{x}_0)) = \Phi^{-1}(\tilde{x}_0)$$

proof of

Injectivity: let $[f]$, $[g]$ be loops at $\tilde{x}_0$.

Suppose $p^*[f] = p^*[g]$. Then $[p \circ f] = p^*[f] = p^*[g] = [p \circ g]$

so $p \circ f \simeq p \circ g$ (rel $\{0,1\}$). Call

this homotopy F. Then by the lifting thm,

$\exists!$ homotopy F' s.t. $p \circ F' = F$ and

$F' : f \simeq g$. Thus $[f] = [g]$ and p^* is injective.

general lifting criterion

if Y path connected and locally path conn, then $f: (Y, y_0) \to (X, x_0)$ lifts to $\tilde{f} : (Y, y_0) \to (\tilde{X}, x_0)$

iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, x_0))$

proof $\Longleftarrow$

$$\begin{tikzcd} & (\tilde{X}, x_0) \arrow[d] \\ (Y, y_0) \arrow[ur] \arrow[r] & (X, x_0) \end{tikzcd}$$

For each $y \in Y$, choose a path $g_y : I \to Y$ from y_0 to y.

$f \circ g_y$ is a path in X starting at x_0, so lifts to $\widetilde{f \circ g_y}$ in $\tilde{X}$

define $\hat{f}(y) = \widetilde{f \circ g_y}(1)$. well def?

Let g_y' be another path from y_0 to y.

Then $h_0 = (f \circ g_y') \cdot (\overline{f \circ g_y})$ is a loop in X based at x_0

but $(f \circ g_y') \cdot (\overline{f \circ g_y}) = \underbrace{f \circ (g_y' \cdot \overline{g}_y)}_{h_0}$

So $h_0 \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, x_0))$

So $h_0 \overset{(\text{rel } \{0,1\})}{\simeq} p \circ k$, where k is a loop at $\tilde{x}_0$ in $\tilde{X}$

by uniqueness of lifts, we have

$\tilde{h}_0 = (\widetilde{f \circ g'}) \cdot (\widetilde{\overline{f \circ g}})$. But also $\tilde{h}_0 = k$, a loop.

means these paths must be compatible.

$\widetilde{f \circ g'}(1) = \widetilde{\overline{f \circ g}}(0) = \widetilde{f \circ g}(1)$. So well def ✓