What is FT about?

Rep theoretically, about decomp of algebra $L(G)$, via spans of characters.

What remains in NC case?

Orthonormal decomp into pieces

What changes? No longer 1 dimensional

Characters replaced with irreps.

$g, h \in G \qquad g + ih \in \mathbb{C}[G]$

$\boxed{f : G \to \mathbb{C}}$

$$(g^2 - h)(ig + 3h)$$
$$= ig^3 - ihg + 3g^2h - 3h^2$$

where

$\widehat{f * g}(\xi) =$

$\sum_{x \in G} f * g(x) \rho_\xi(x)$

$= \sum_{x \in G} \left(\sum_{\substack{a,b \in G \\ ab = x}} f(a) g(b) \right) \rho_\xi(x)$

$\shortparallel$

$\left(\sum_{x \in G} f(x) \rho_\xi(x) \right) \left(\sum_{x \in G} g(x) \rho_\xi(x) \right)$

$\hat{f}(\xi) = \sum_{x \in G} f(x) \rho_\xi(x)$

$\widehat{f * g}(\xi) = \hat{f}(\xi) \hat{g}(\xi)$

fourier trans

$$\hat{f}(\xi) = \sum_{x \in G} f(x)\rho_{\xi}(x)$$

inv fourier trans

$$f(x) = \frac{1}{|G|}\sum_{\xi \in \hat{G}} \dim V_{\xi}\ \mathrm{Trace}\left(\hat{f}(\xi)\rho_{\xi}(x^{-1})\right)$$

$G = S_3$ $\qquad$ $\hat{G} = \{\xi_{triv}, \xi_{sign}\ \xi_{RP}\}$

$$\xi_{triv} \longrightarrow \rho^{(triv)}(\sigma) = 1 \quad \forall x \in S_3$$

$\xi_{sign} \longrightarrow \rho^{(sign)}(\sigma) = \pm 1$, depends on even/odd

$\xi_{RP} \longrightarrow \rho^{RP}\sigma = ?$ $\quad \rho^{RP}(e) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\rho^{RP}(12) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\rho^{RP}(13) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\rho^{RP}(23) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho^{RP}(123) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\rho^{RP}(132) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

basis 1-2 , 1-3

Let $f = \begin{pmatrix} e \\ 12 \\ 13 \\ 23 \\ 123 \\ 132 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$

$$\hat{f}(\xi_{triv}) = \sum_{x \in G} f(x) \underbrace{\rho_{triv}(x)}_{=1}$$

$$= 1 + 2 + 3 + 4 + 5 + 6 = \boxed{21 = \hat{f}(\xi_{triv})}$$

$$\hat{f}(\xi_{sign}) = 1 - 2 - 3 - 4 + 5 + 6$$

$$= 12 - 9 = \boxed{3 = \hat{f}(\xi_{sign})}$$

$$\hat{f}(\xi_{RP}) = \sum_{x \in G} \hat{f}(x) \rho_{RP}(x)$$

$\begin{bmatrix} \bullet & \\ & \end{bmatrix}$ $1 - 2 + 3 - 5 = -3$

$\begin{bmatrix} & \bullet \\ & \end{bmatrix}$ $-2 + 4 - 5 + 6 = 3$

$\begin{bmatrix} & \\ \bullet & \end{bmatrix}$ $-3 + 4 + 5 - 6 = 0$

$\begin{bmatrix} & \\ & \bullet \end{bmatrix}$ $1 + 2 - 3 - 6$

$$= 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} + 4 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} +$$

$$5 \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} + 6 \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -3 & 3 \\ 0 & -6 \end{bmatrix} = \hat{f}(\xi_{RP})}$$

so if $f\begin{pmatrix} 0\\ 12\\ 13\\ 23\\ 123\\ 132\end{pmatrix} = \begin{pmatrix}1\\ 2\\ 3\\ 4\\ 5\\ 6\end{pmatrix}$ then

$\hat{f}(\xi_{triv}) = 21$

$\hat{f}(\xi_{sign}) = 3$

$\hat{f}(\xi_{RP}) = \begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}$

Check:

$$f(x) = \frac{1}{|G|}\sum_{\xi\in\hat{G}} \dim V_\xi \; \mathrm{Trace}\left(\hat{f}(\xi)\rho_\xi(x^{-1})\right)$$

$$f(e) \overset{?}{=} \frac{1}{|S_3|}\Big(1\cdot \mathrm{trace}(21\times 1) + 1\cdot \mathrm{Trace}(3\cdot 1) + 2\cdot \mathrm{trace}\Big(\begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Big)\Big)$$

$$= \frac{1}{6}\left(21 + 3 + 2\cdot(-9)\right) = \frac{1}{6}\left(24 - 18\right) = 6/6 = 1$$

$$f(123) = \frac{1}{6}\Big(1\cdot 5 + 1\cdot 5 + 2\,\mathrm{trace}\Big(\begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}\cdot\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}^{-1}\Big)\Big)$$

$\overset{?}{=} 5$

$$= \frac{1}{6}\left(5 + 5 + 2\cdot 3\right) = \frac{1}{6}16$$

$$= \begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix} = \begin{bmatrix}-3 & \cdot\\ \cdot & \boxed{6}\end{bmatrix}$$

$$|\mathrm{Supp}(f)| \cdot |\mathrm{supp}\, \hat{f}| \geq |G|$$

$$|\mathrm{supp}(f)| \cdot \sum_{\rho_\xi \in \mathrm{supp}(\hat{f})} (\dim V_\xi)^2 \geq |G|$$

uncertainty principle

$\rho_\xi \in \mathrm{supp}(\hat{f})$ means

$\hat{f}(\xi)$ not the 0 transformation

$$\hat{f}(\xi) = \sum_{x \in G} f(x) \rho_\xi(x)$$

$$f(12) = \frac{1}{|G|} \sum_{\xi \in \hat{G}} \dim(V_\xi)\, \text{trace}\left(\hat{f}(\xi)\, \rho_\xi(x^{-1})\right)$$

$$= \frac{1}{6}\left(1 \times (21 \cdot 1) + 1 \times (3 \cdot (-1)) + 2\left(\begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}\begin{bmatrix}-1 & -1\\ 0 & 1\end{bmatrix}\right)\right)$$

$\hat{f}(\xi_{triv}) = 21$

$\hat{f}(\xi_{sign}) = 3$

$\hat{f}(\xi_{RP}) = \begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}$

$$= \frac{1}{6}\left(21 - 3 + 2\,\text{Trace}\begin{bmatrix}3 & 6\\ 0 & -6\end{bmatrix}\right)$$

$$= \frac{1}{6}\left(18 + 2 \cdot (-3)\right) = \frac{1}{6}(18 - 6) = \frac{12}{6} = \boxed{2 = f(12)}$$

$$f(13) = \frac{1}{6}\left(1 \times (21 \cdot 1) + 1 \times (3 \cdot (-1)) + 2\,\text{Trace}\left(\begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix}\right)\right)$$

$$= \frac{1}{6}\left(21 - 3 + 2\,\text{Trace}\begin{bmatrix}-6 & -3\\ 6 & 6\end{bmatrix}\right) = \frac{1}{6}(18) = \boxed{3 = f(13)}$$

$$f(23) = \frac{1}{6}\left(21 - 3 + 2\,\text{Trace}\begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right) = \frac{1}{6}\left(18 + 2\,\text{trace}\begin{bmatrix}3 & -3\\ -6 & 0\end{bmatrix}\right) = \frac{1}{6}\,24 = \boxed{4 = f(23)}$$

$$f(123) = \frac{1}{6}\left(21 + 3 + 2\,\text{trace}\begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}^{-1}\right) = \frac{1}{6}\left(21 + 3 + 2\,\text{Trace}\begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\right)$$

$$= \frac{1}{6}\left(24 + 2\,\text{trace}\begin{bmatrix}-3 & -6\\ 6 & 6\end{bmatrix}\right) = \frac{1}{6}(24 + 6) = 30/6 = \boxed{5 = f(123)}$$

$$f(132) = \frac{1}{6}\left(21 + 3 + 2\,\text{Trace}\begin{bmatrix}-3 & 3\\ 0 & -6\end{bmatrix}\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}\right) = \frac{24}{6} + \frac{2\,\text{trace}\begin{bmatrix}6 & \sim\\ - & 0\end{bmatrix}}{6} = \boxed{6 = f(132)}$$

uncertainty principle:

$$|supp(f)| \cdot \sum_{\xi \in supp(\hat{f})} (\dim V_\xi)^2 \geq |G|$$

test this on $f\begin{pmatrix} e \\ 12 \\ 13 \\ 23 \\ 123 \\ 132 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\hat{f}(\xi_{triv}) = \sum_{x \in G} f(x)\rho_{triv}(x) = f(23)\rho_{triv}(x) = 1 : \mathbb{C} \to \mathbb{C}$$

$$\hat{f}(\xi_{sign}) = \sum_{x \in G} f(x)\rho_{sign}(x) = f(23)\rho_{sign}(23) = -1 : \mathbb{C} \to \mathbb{C}$$

$$\hat{f}(\xi_{RP}) = f(23)\rho_{sign}(23) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$$

these are all nonzero transformations, so all contribute to $\sum_{\xi \in supp \hat{f}} (\dim V_\xi)^2 = 1^2 + 1^2 + 2^2$

$$= 6$$

$1 \cdot 6 \geq 6$ ✓ so UP doesn't fail.

all f with supp(f) clearly satisfy this in this example.

what if $\rho_{\xi_{RP}}$ is the 0 matrix?

$$\hat{f}(\xi_{RP}) = \sum_{f(x)} f(x)\,\rho_{\xi}(x). \quad \text{if } f\begin{pmatrix} e \\ 12 \\ 13 \\ 23 \\ 123 \\ 132 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$+ e\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} a+c-(b+e) & d+f-(b+e) \\ d+e-(c+f) & a+b-(c+f) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

a b c d e f a b c d e f

$$a+c = b+e$$
$$b+e = d+f$$

so $a+c = d+f$,

$a+b = d+e$

is it possible to have only one or 2 of these be nonzero?

$$a+b = c+f$$
$$d+e = c+f$$

not possible at all for only 1 nonzero

only 2 nonzero: if one is nonzero, forces at least 2 others!!!

what if ξ_{triv} or ξ_{sgn} are 0 ?

$$\rho_{\text{triv}}(x) = a+b+c+d+e+f : \mathbb{C} \to \mathbb{C} \quad (\forall x)$$

$$\rho_{\text{sgn}} \equiv a-b-c-d+e+f : \mathbb{C} \to \mathbb{C}.$$

can these be 0 ?

suppose $a+b+c+d+e+f=0$, $\quad a+e+f = b+c+d$

so $0 = a+e+f = -(b+c+d)$

then impossible for $a+c = b+e = d+f$

$$3(a+c) = a+b+c+d+e+f = 0, \quad \text{so} \quad a+c=0 = b+e = d+f$$

$$3(a+c) = 3(b+e) = 3(d+f)$$