Universal Algebra Notes

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Contents

1 Partially Ordered Sets

A *partially ordered set* or a *poset* is a system $\mathbf{P} = (P, \leq)$ consisting of a non empty set P and a binary relation $\leq$ on P such that the following conditions are satisfied for all $x, y, z \in P$:

P_1: $x \leq x$ *(reflexivity)*

P_2: If $x \leq y$ and $y \leq x$, then $x = y$ *(antisymmetry)*

P_3: If $x \leq y$ and $y \leq z$, then $x \leq z$ *(transitivity)*

A binary relation which satisfies P_1, P_2, P_3 is said to be a *partial ordering* on P and P is said to be a *partially ordered* by the relation $\leq$.

If $\mathbf{P} = (P, \leq)$ is a poset, we shall often identify $\mathbf{P}$ with the set P, and thus speak of P being a poset (with respect to $\leq$).

As usual, we shall write "$x \leq y$" to mean "$x \leq y$ and $x \neq y$"; we also write "$x \geq y$" instead of "$y \leq x$, and $x \geq y$" instead of "$y < x$". A partial ordering on P is called *total* (other terms: linear, full) if any two elements are *comparable*, i.e.,

P_4: $x \leq y$ or $y \leq x$ for all $x, y \in P$.

In this case, P is said to be a *chain* (or a totally ordered set, linearly ordered set, etc).

At the other extreme, an abstract set may be regarded as an *antichain* (totally unordered set), in which $x \leq y$ holds only in case $x = y$; in other words, any two distinct elements x and y are *incomparable* (in symbols, $x \parallel y$).

If Q is a non-empty subset of a poset P, then the ordering of P, restricted to Q, is a partial ordering of Q. If Q is, with respect to the restricted ordering, a *chain (antichain)*, then Q is said to be a *chain (anticking)* in P.

Examples of posets include the following. The set $\mathbb{R}$ of all real numbers is a chain with respect to the "natural ordering" defined by $x \leq y \leftrightarrow y - x$ is nonnegative. The set $\mathcal{P}(X)$ of all subsets of a set with respect to set inclusion. The set $\mathbb{N}$ of all natural numbers with respect to the relation $\leq$ defined by $x \leq y$ iff x divides y.

Let P and Q be posets. A map $\phi : P \to Q$ is said to be an *order homomorphism* (or an *order preserving map*) from P to Q provided that $\phi(x) \leq \phi(y)$ whenever $x \leq y$. A surjective (onto) order homomorphism is called an *order epimorphism.* A map $\phi : P \to Q$ is called an *order monomorphism* (or an *order embedding*) provided that $x \leq y$ iff $\phi(x) \leq \phi(y)$ for all $x, y \in P$. It is clear that an order monomorphism is an injective order homomorphism, but the converse need not be true. An *order isomorphism* is a bijection $\phi : P \to Q$ such that both ϕ and ϕ^{-1} are order homomorphisms. If there is an order isomorphism ϕ from p to Q, then we write $\phi : P \simeq Q$ or simply $P \simeq Q$, and say P is *order isomorphic* to Q. Finally, an order homomorphism (isomorphism) from P to P is called an *order endomorphism (automorphism)* of P.

Finite posets can be represented diagrammatically by using the following "covering" relation. Let us say that y covers x in P, and write $x \prec y$ or $y \succ x$, if $x < y$ and there is no element $z \in P$ such that $x < z < y$.

Consider now a finite poset and construct the following figure. Draw a small circle for each element in P in such a way that the circle associated with x lies above the circle associated with y whenever $x > y$. Draw a straight segment joining x and y whenever $x \prec y$. The resulting figure is called a *(Hasse) diagram* of P.

Some examples of diagrams are shown below.

Note that figure 1(c) represents the set of all subsets of $\{1, 2, 3\}$ partially ordered by set inclusion. Also, figure 1(d) represents the set of integers $\{1, 2, \ldots, 12\}$ with $\leq$ defined by $x \leq y$ iff x divides y.

Note that in a finite poset $x \leq y$ iff there is a sequence of elements $z_0, z_1, \ldots, z_m$ in P such that $x = z_0 \succ z_1 \succ \cdots \succ z_m = y$. In view of this fact it should be clear that *two finite posets can be represented by the same diagram iff they are order isomorphic.*

By a *least* element of a subset X of a poset P, we mean an element $a \in X$ such that $a \leq x$, for all $x \in X$. Note that if such an element exists, it is unique. Indeed, if a_0 and a_1 are both least elements of X, then by defini-

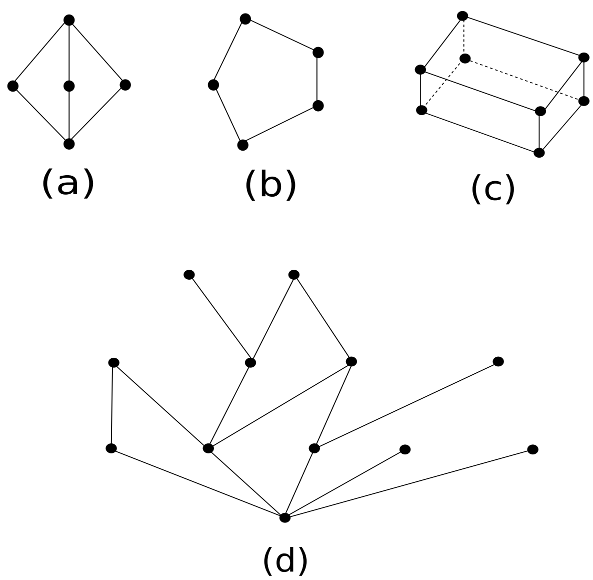


Figure 1: Some examples of Hasse diagrams.

tion, $a_0 \leq a_1$ and $a_1 \leq a_0$. Thus, $a_0 = a_1$ by P_2. A greatest element of X is defined "dually", and its uniqueness is proved similarly. The least and greatest elements of P, when they exist, are denoted by $\perp$ and $\top$ respectively.

Let us elaborate on the use of the adverb "dually" in the previous paragraph. Let $\mathbf{P} = (P, \leq)$ be a poset and let $\leq^d$ denote the relation on P defined by

$$x \leq^d y \text{ iff } y \leq x.$$

It is easy to see, using $P_1 - P_3$, that $\mathbf{P}^d = (P, \leq^d)$ is a poset, called the *dual* of P. The name is suggestive of the fact that the dual of $\mathbf{P}^d$ is $\mathbf{P}$. An important consequence of this is the "*duality principle for posets*", which will be formulated below, after a preliminary definition.

Given a "statement" Φ about posets, we can obtain the *dual* statement Φ^d by replacing each of the relations $\leq$ and $\geq$ by the other whenever they occur in Φ. For example, the dual of the statement "Every subset of a poset has at most one least element" is the statement "Every subset of a poset has at most one greatest element". The duality principle can now be formulated as follows.

DUALITY PRINCIPLE FOR POSETS. If a statement Φ is true in all posets, then its dual Φ^d is also true in all posets.

This is a simple consequence of the fact that if Φ holds in P iff Φ^d holds in P^d, which is also a poset. For example, since we already proved that every subset of a poset has at most one least element, we can conclude that every subset of a poset has at most one greatest element. In general, we shall occasionally need to prove a statement consisting of two parts, the second part being the dual statement of the first. In such a case the duality principle cuts our work in half; we simply need to prove the first part and then remark that the second follows by "duality".

The concept of "least" and "greatest" should be distinguished from the concepts of "minimal" and "maximal" elements. A *minimal* element of a subset X of a poset P is an element $a \in X$ such that there is no $x \in X$ with $x \leq a$;

a *maximal* element is defined dually. Clearly, a least (greatest) element must be minimal (maximal), but the converse need not be true (see fig. 1.1d).

By a *lower* bound of a subset X of a poset P, we mean an element $a \in P$ such that $a \leq x$ for all $x \in X$. When such elements exist, we say that X is *bounded* below in P. The *upper bounds* of X are defined dually. A poset in which every finite poset has a lower bound is said to be *down directed*. An *up directed* poset is defined dually.

By a *join* (or *least upper bound*) of a subset X of a poset P, we mean an element $a \in P$ such that a is an upper bound of X, and $a \leq b$ whenever b is an upper bound of X. A *meet* (or *greatest lower bound*) is defined dually. The join and meet of X, if they exist, are unique and will be denoted by ${}^P\bigvee X$ (or ${}^P\bigwedge X$) respectively. If no confusion is likely to arise, we shall drop the superscript P. If X is a two element set, say $X = \{x, y\}$, then we shall write $x \vee^P y$ for ${}^P\bigvee X$, and $x \wedge^P y$ for ${}^P\bigwedge X$, or simply $x \vee y$ and $x \wedge y$.

The following statement, known as Zorn's lemma, is equivalent to the axiom of choice and will be assumed to hold throughout.

Zorn's Lemma: A nonempty poset in which every chain has an upper bound has a maximal element.

2 Chain Conditions

A poset P is said to satisfy the *minimal (maximal) condition* if every non-empty subset of P has a minimal (maximal) element. A chain satisfying the minimal condition is said to be *well ordered.* A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a poset P is called a *descending (ascending) chain* if $x_n > x_{n+1}$ ($x_n < x_{n+1}$) for all $n \in \mathbb{N}$. P is said to satisfy the *descending (ascending) chain condition* if every descending (ascending) chain in P is finite.

Proposition 2.1 *A poset satisfies the minimal condition iff it satisfies the descending chain condition.*

Proof. If there is a descending chain $x_0 > x_1 > \cdots$ in P, then $X = \{x_1, x_2, \ldots\}$ does not have a maximal element. Inversely, suppose that there is a non-empty subset X which has no maximal element. Pick any $x_0 \in X$. Since x_0 is not a minimal element of X, we can pick $x_1 \in X$ such that $x_1 < x_0$. Similarly, we can pick $x_2 \in X$ such that $x_1 > x_2$, and continuing this process we can produce an infinite descending chain $x_0 > x_1 > x_2 > \cdots$. $\square$

Proposition 2.2 *A poset satisfies the maximal condition iff it satisfies the ascending chain condition.*

Proof. This follows from Proposition 2.1 by duality. $\square$

Proposition 2.3 *A poset P satisfies both the maximal and minimal conditions iff every chain in P is finite.*

Proof. Exercise. $\square$

The descending chain condition is generalized by the atomicity condition. If a poset P has a least element 0, then the elements that cover 0 are called *atoms*. *dual atoms* (or *coatoms*) are defined dually. P is said to be *atomic* if it has a least element and the set $(a] = \{x \in P | x \leq a\}$ contains an atom for each $a \geq 0$. P is said to be *weakly atomic* if, for all $a, b \in P$ with $a < b$, there exists $x, y \in P$ with $a \leq x \prec y \leq b$. Finally, P is called *strongly atomic* if each interval $[a, b] = \{x \in P | a \leq x \leq b\}$ has an element that covers a, whenever $a < b$. For examples of posets with atomicity conditions, see the exercises. An important generalization of the ascending chain condition will be discussed in a later section.

3 Lattices and Complete Lattices

A poset in which every pair of elements has a join is called a *join-semilattice*. By induction it follows that in a join-semilattice every non-empty finite subset has a join. A *meet-semilattice* is defined dually. A poset in which every pair of elements has a join and a meet is called a *lattice*. A poset in which every subset has a join and a meet is called a *complete lattice*. In particular, a complete lattice L always has a greatest element ($= \bigwedge \emptyset = \bigvee L$), and a least element ($= \bigvee \emptyset = \bigwedge L$).

Lattices occur in abundance. For example, any chain is a lattice (though not necessarily complete; example?). For any set X, the set of all subsets of X partially ordered by set inclusion is a complete lattice, the join of any collection is the set union, and the meet is the set intersection. For future use, this lattice will be denoted by $\mathcal{P}(X)$ and will be referred to as "the lattice of all subsets of X".

More generally, a subset I of P is said to be an *order ideal* of P if whenever $x \in P$, $y \in I$, and $x \leq y$, then $x \in I$. Note that $\emptyset$ is an ideal and so is, for each $a \in P$, the *principal* order ideal $(a] = \{x \in P \mid x \leq a\}$. An *order filter* is defined dually. The set of all order ideals of P ordered by set inclusion is a complete lattice; the join of any collection of order ideals is the set union, and the meet is the set intersection. This lattice will be denoted by $\mathcal{O}(P)$ and will be referred to as "the lattice of order ideals of the poset P". We note that an abstract set P may be regarded as a totally unordered set; in this case $\mathcal{O}(P)$ reduces to $\mathcal{P}(P)$.

It should be noted that the defining property of a lattice is self-dual (i.e.. it coincides with it's dual property, see p. 6), as is the defining property of a complete lattice. It follows that the dual of a lattice is again a lattice, and the dual of a complete lattice is complete. Furthermore, joins of L coincide with meets in L^d, and meets in L coincide with joins in L^d.

The following criterion is useful in establishing that a given poset is a complete lattice.

Proposition 3.1 *For every poset P, the following are equivalent:*

(i) P is a complete lattice;

(ii) every subset of P has a meet;

(iii) every subset of P has a join.

Proof.

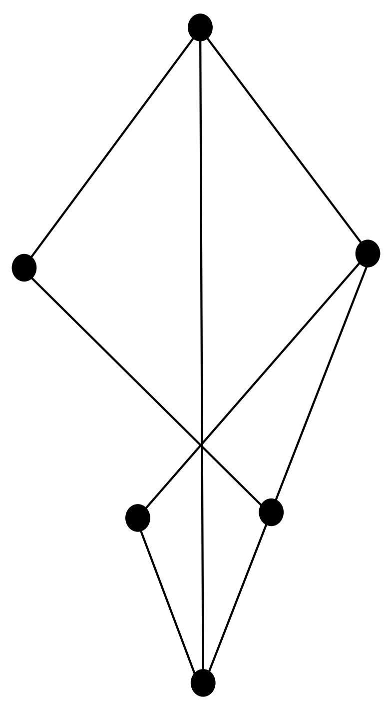
We shall establish the equivalence of (i) and (iii). The equivalence of (i) and (ii) will then follow by duality.

(i) $\Rightarrow$ (iii) By definition of a complete lattice

(iii) $\Rightarrow$ (i) Let $X \subseteq P$, and let Y be the set of all upper bounds of X in P. I.e., $Y = \{y \in P \mid y \geq x \; \forall x \in X\}$. By assumption, $a = \bigwedge Y$ exists. Now every element of X is a lower bound of Y, and so $x \leq a$ for every $x \in X$. Furthermore, if b is an upper bound of X, then $b \in Y$, and hence $a \leq b$. Therefore, $a = \bigvee X$. Thus, every subset of P has a join, so that P is a complete lattice. $\square$

EXERCISES (Sections 1-3)

1. Show that any intersection of partial orderings on a set is a partial ordering
2. Let $T(n)$ denote the number of non-isomorphic posets of n elements. Show that $T(2) = 2, T(3) = 5, T(4) = 16$.
3. Give an example to show that a bijective order homomorphism is not always an order isomorphism.
4. Simplify the following diagram:



5. Show that every partial ordering can be refined to a total ordering. More specifically, if $(P, \leq)$ is a poset, then there is a partial ordering $\trianglelefteq$ on P such that $(P, \trianglelefteq)$ is a chain and $\leq$ is a subset of $\trianglelefteq$ (i.e., for all $x, y \in P$, $x \leq y$ implies $x \trianglelefteq y$). Hint: Consider the set $\mathcal{P}$ of all partial orderings on P refining $\leq$. Note that $\mathcal{P}$ is a poset with respect to set inclusion, and observe that Zorn's lemma can be used to yield a maximal element $\trianglelefteq$ of $\mathcal{P}$. Next show that $(P, \trianglelefteq)$ is a chain.
6. Prove Proposition 2.3

7. Let P be a poset in which every chain has at most m elements and any antichain at most n elements. Show that P has at most mn elements.

8. Show that a poset satisfying the descending chain condition is strongly atomic.

9. Show by means of an example that an atomic poset need not be weakly atomic.

10. Let P be the poset (with respect to the set inclusion) of all vector subspaces of a vector space V. Show that P satisfies the *descending* chain condition iff V is finite dimensional.

11. Consider the set $\mathbb{N}$ of all non-negative integers partially ordered by divisibility ($x \leq y$ iff x divides y). Show that $\mathbb{N}$ is a complete lattice, and describe finite and arbitrary joins and meets.

12. Show that a lattice with least element satisfying the maximal condition is complete.

13. Extend the previous exercise by showing that if L is a lattice with least element in which every well-ordered chain has a join, then L is complete.

14. Prove the following *fixed point theorem*: If L is a complete lattice, then every order endomorphism ϕ of L has a fixed point, i.e. $\phi(a) = a$ for some $a \in L$. (Hint: Let $A = \{x \in L \mid x \leq \phi(x)\}$ and let $a = \bigvee A$; show that $\phi(a) = a$.) Further, show that the fixed points of L form a complete lattice with respect to the ordering induced by L. (Hint: Let F be the set of fixed points; if $X \subseteq F$, show that ${}^F\bigwedge X ={}^L \bigvee Y$, where $Y = \{y \in L \mid y \leq \phi(y) \text{ and } y \leq x,\ \forall x \in X\}$.)

15. A poset Q is said to be an *extension* of a poset P if $P \subseteq Q$, and the partial order of P is the restriction to P of the partial order of Q. If in addition each element $q \in Q$ is the join of a subset $M \subseteq P$, then Q is said to be a *join-extension* of P, and P a *join-dense* subset of Q. If the complete lattice Q is a join-extension of P, then we say that Q is the *join-completion* of P. Suppose that Q is a join-extension of P.

(i) Show that for each $x \in Q$,

$$x =^{Q} \bigvee P \cap (x]_Q.$$

Here, $(x]_Q = \{y | y \in Q, y \leq x\}$.

(ii) Let $\dot{P} = \{(p] | p \in P\}$. Show that $\mathcal{O}(P)$ is a join-completion of $\dot{P}$.

(iii) Show that $\phi : P \to \mathcal{O}(P)$ defined by

$$\phi(p) = (p]$$

for all $p \in P$ is an order embedding with image $\dot{P}$. Deduce that any poset can be order embedded in a complete lattice.

(iv) Show that for each $M \subseteq P$, if ${}^{P}\bigwedge M$ exists, then ${}^{Q}\bigwedge M$ exists, and ${}^{P}\bigwedge M =^{Q} \bigwedge M$.

16. Let P be a poset. Show that there is no order epimorphism from P to $\mathcal{O}(P)$. Deduce as a special case the well known fact that if X is a set, there is no bijection from X to $\mathcal{P}(X)$.

4 Algebras

If A and B are sets, A^B will denote the set of all maps for B to A. If n is some natural number, then we write A^n for $A^{\{0,1,\ldots,n-1\}}$ and call the elements of A^n *ordered n-tuples*. Note that $A^0 = A^\emptyset = \{\emptyset\}$, and we can clearly indentify "naturally" A^1 with A. The ordered n-tuple $a : n = \{0, 1, \ldots, n-1\} \longrightarrow A$ $(n > 0)$ with $a(k) = a_k$ $(0 \leq k \leq n-1)$ will be denoted by $(a_0, a_1, \ldots, a_{n-1})$.

A *type* is a family $\nu = (\nu_i)_{i \in I}$ of natural numbers indexed by some set I.

An *algebra* of type $\nu = (\nu_i)_{i \in I}$ is an ordered pair

$$\mathbf{A} = (A, f)$$

consisting of a *non-empty* set A, called the *universe* of $\mathbf{A}$, and a family $f = (f_i)_{i \in I}$ of maps

$$f_i : A^{\nu_i} \to A$$

called the *fundamental (basic)* operations of $\mathbf{A}$.

The fundamental operation $f_i : A^{\nu_i} \to A$ is usually called a ν_i*-ary* operation, or an operation of *arity* ν_i. Operations of arity zero are also called *nullary operations* or *constants*. The latter term is suggestive of the fact that if f_i has arity zero, then its domain is $A^0 = \{\emptyset\}$, and hence f_i is determined by the unique element $f_i(\emptyset)$ of its range. The usual practice is to identify f_i with $f_i(\emptyset)$.

We use the terms "*unary operation*" for a "1-ary operation", "*binary operation*" for a "2-ary operation", "*ternary operation*" for a "3-ary operation", etc. It is also customary to write the symbol denoting a binary operation in the middle. For example, we write afb for $f(a, b)$, $(afb)fc$ for $f(f(a, b), c)$, etc.

We list a few examples of algebraic structures.

(i) *Groupoids*. A *groupoid* is an algebra $\mathbf{A} = (A, \cdot)$ with a single binary operation. A groupoid satisfying the commutative law

$$x \cdot y = y \cdot x$$

for all $x, y \in A$ is said to be *commutative*.

(ii) *Semigroups.* A *semigroup* is a groupoid $\mathbf{A} = (A, \cdot)$ satisfying the associative law

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for all $x, y, z \in A$.

(iii) *Monoids.* A *monoid* is an algebra $\mathbf{A} = (A, \cdot, 1)$ with a binary operation $\cdot$ and one nullary operation 1 such that $(A, \cdot)$ is a semigroup and

$$x \cdot 1 = 1 \cdot x = x$$

for all $x \in A$. The nullary operation 1 is called an *identity element* of $\mathbf{A}$.

(iv) *Groups.* A *group* is a monoid $\mathbf{A} = (A, \cdot, 1)$ in which the equations $x \cdot a = 1$ and $a \cdot y = 1$ have solutions x, y for any choice of a. As it is well known, there is a unique element $a^{-1} \in A$ such that $x = y = a^{-1}$. Very often it is more convenient to think of a group as an algebra $\mathbf{A} = (A, \cdot, ^{-1}, 1)$ with three operations, one binary $(\cdot)$, one unary $(^{-1})$, and one nullary (1), such that $(A, \cdot, 1)$ is a monoid, and

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

hold for all $a \in A$.

(v) *Semilattices.* In Section 3, a meet-semilattice was defined as a poset in which every pair of elements has a meet, and a join-semilattice was defined dually. Here we show that semilattices can be viewed as commutative idempotent semigroups. That is, groupoids satisfying the identities

S_1: $x \cdot x = x$ (idempotent law)

S_2: $x \cdot y = y \cdot x$ (commutative law)

S_3: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associative law)

Theorem 4.1 *Suppose that the poset $\mathbf{P} = (P, \leq)$ is a meet-semilattice. Set $x \cdot y = \bigwedge \{x, y\}$. Then the algebra $\boldsymbol{P}^\alpha = (P, \cdot)$ is an idempotent commutative semigroup. Conversely, if $\mathbf{S} = (S, \cdot)$ is a commutative idempotent semigroup, and if we define*

$$x \leq y \Leftrightarrow x \cdot y = x$$

then $\mathbf{S}^\beta = (S, \leq)$ is a meet-semilattice in which the meet of any two elements x and y is $x \cdot y$. Furthermore, we always have $\mathbf{P}^{\alpha\beta} = \mathbf{P}$ and $\mathbf{S}^{\beta\alpha} = \mathbf{S}$.

Proof. Exercise. $\square$

Dually, if $\mathbf{P} = (P, \leq)$ is a join-semilattice, then we can obtain a commutative idempotent semigroup $\mathbf{P}^\alpha = (P, \cdot)$ by letting $x \cdot y = \bigvee\{x, y\}$. Conversely, if $\mathbf{S} = (S, \cdot)$ is such a semigroup, and we define $\leq$ by

$$x \leq y \Leftrightarrow x \cdot y = y$$

then $\mathbf{S}^\beta = (S, \leq)$ becomes a join-semilattice.

Thus from the algebraic point of view there is no distinction between join and meet-semilattices. It is only when we introduce the partial ordering on an idempotent commutative semigroup that we have the choice of defining a meet-semilattice or a join-semilattice.

In what follows, the operation of an idempotent commutative semigroup will be denoted by $\wedge$ or $\vee$, depending on whether we view the semigroup as a meet-semilattice or a join-semilattice, respectively.

(vi) *Lattices.* In Section 3 a lattice was defined as a poset $(P, \leq)$ which is both a meet-semilattice and a join-semilattice with respect to $\leq$.

Let $\mathbf{P} = (P, \leq)$ be a lattice. Define now the meet-semilattice $(P, \wedge)$ and the join-semilattice $(P, \vee)$ by

$$x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y. \tag{4.1}$$

In view of Theorem 4.1, the following identities hold in $\mathbf{P}$:

$L_{1\wedge} : x \wedge x = x$ $\quad$ $L_{1\vee} : x \vee x = x$
$L_{2\wedge} : x \wedge y = y \wedge x$ $\quad$ $L_{2\vee} : x \vee y = y \vee x$
$L_{3\wedge} : x \wedge (y \wedge z) = (x \wedge y) \wedge z$ $\quad$ $L_{3\vee} : x \vee (y \vee z) = (x \vee y) \vee z$

Furthermore, since $x \wedge y \leq x \leq x \vee y$, conditions (4.1) imply the *absorption laws*:

$L_{4\wedge} : x \wedge (x \vee y) = x \quad L_{4\vee} : x \vee (x \wedge y) = x$

The next result shows that lattices can be viewed as algebras with two binary operations satisfying $L_1 - L_4$.

Theorem 4.2 *If the poset* $\mathbf{P} = (P, \leq)$ *is a lattice and we define* $x \wedge y = \bigwedge\{x, y\}$, $x \vee y = \bigvee\{x, y\}$, *then the algebra* $\mathbf{P}^\alpha = (P, \wedge, \vee)$ *satisfies* L_1 – L_4. *Conversely, if* $\mathbf{L} = (L, \wedge, \vee)$ *is an algebra satisfying* L_1 – L_4 *and we define*

$$x \leq y \Leftrightarrow x \wedge y = x$$

then $\mathbf{L}^\beta = (L, \leq)$ *is a lattice in which the meet and join of two elements* x *and* y *are, respectively,* $x \wedge y$ *and* $x \vee y$. *Furthermore, we always have* $\mathbf{P}^{\alpha\beta} = P$ *and* $\mathbf{L}^{\beta\alpha} = L$.

Proof. Exercise. $\square$

In view of the previous theorem, a lattice $\mathbf{P}$ can be identified with the algebra $\mathbf{P}^\alpha$, and thus treat lattices as algebras whenever it is convenient to do so.

The concept of duality discussed in Section 1 takes the following form for lattices viewed as algebras:

> If a statement Φ holds in every lattice, then the statement obtained from Φ by interchanging the operations $\wedge$ and $\vee$ also holds in every lattice.

To prove this we just need to observe that if $\mathbf{L} = (L, \wedge, \vee)$ is a lattice, then its dual is the lattice $\mathbf{L}^d = (L, \vee, \wedge)$.

In line with our practice about posets, we shall often identify an algebra $\mathbf{A}$ with its universe A, whenever no confusion is likely to arise.

EXERCISES (Section 4)

1. Prove Theorem 4.1
2. Prove Theorem 4.2 (Hint: Use Theorem 4.1)
3. Prove that $L_4 \Rightarrow L_1$
4. Prove that L_2, L_3, and L_4 are independent (that is, no one of them can be deduced from the other five).
5. How can one define modules over a fixed ring as algebras?

5 Homomorphisms and Subalgebras

Let $\mathbf{A} = (A, f)$ and $\mathbf{B} = (B, g)$ be algebras of the same type $\nu = (\nu_i)_{i \in I}$. A map $\phi : A \to B$ is called a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$, denoted by $\phi : \mathbf{A} \to \mathbf{B}$, provided that

$$\phi(f_i(a_0, \ldots, a_{\nu_i - 1})) = g_i(\phi(a_0), \ldots, \phi(a_{\nu_i - 1})),$$

for each $i \in I$ and each $(a_0, \ldots, a_{\nu_i - 1}) \in A^{\nu_i}$.

An injective (one-to-one) homomorphism is called a *monomorphism*, a surjective (onto) homomorphism is called an *epimorphism* and a bijective (injective and surjective) homomorphism is called an *isomorphism.* We use $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ to denote the set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$. If there is an epimorphism from $\mathbf{A}$ to $\mathbf{B}$, we say that $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$. If there is an isomorphism ϕ from $\mathbf{A}$ to $\mathbf{B}$, then we write $\phi : \mathbf{A} \cong \mathbf{B}$ or simply $\mathbf{A} \cong \mathbf{B}$, and say $\mathbf{A}$ is *isomorphic* to $\mathbf{B}$. Finally, a homomorphism (an isomorphism) from $\mathbf{A}$ to $\mathbf{A}$ is called an *endomorphism* (*automorphism*). the set of all endomorphisms (automorphisms) of $\mathbf{A}$ will be denoted by $\mathrm{End}(\mathbf{A})$ ($\mathrm{Aut}(\mathbf{A})$).

Proposition 5.1 *If* $\phi \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$ *and* $\psi \in \mathrm{Hom}(\mathbf{B}, \mathbf{C})$ *, then* $\psi \circ \phi \in \mathrm{Hom}(\mathbf{A}, \mathbf{C})$.

Proposition 5.2 $\mathrm{id}_A \in \mathrm{Aut}(\mathbf{A})$*, and if* $\phi : \mathbf{A} \cong \mathbf{B}$*, then* $\phi^{-1} : \mathbf{B} \cong \mathbf{A}$.

Proposition 5.3 $\mathrm{End}(\mathbf{A})$ *is a monoid, and* $\mathrm{Aut}(\mathbf{A})$ *is its group of invertible elements.*

We have seen that every lattice can be viewed either as a poset or as a certain type of algebra with two binary operations. Thus the notion of a homomorphism between lattices will depend on the point of view we take. If we view lattices as posets, then we have the notion of an *order-homomorphism* discussed in Section 1. If we view lattices as algebras, then we are led to the notion of a lattice homomorphism. Thus, a lattice homomorphism from the lattice $\mathbf{L}_1 = (L_1, \wedge, \vee)$ to the lattice $\mathbf{L}_2 = (L_2, \wedge, \vee)$ is a map $\phi : L_1 \to Ł_2$ such that

$$\phi(x \wedge y) = \phi(x) \wedge \phi(y) \tag{5.1}$$

and

$$\phi(x \vee y) = \phi(x) \vee \phi(y) \tag{5.2}$$

for each pair of elements $x, y \in Ł$. The notions *lattice monomorphism (embedding), lattice epimorphism, lattice isomorphism*, etc. are specializations in the setting of lattices (viewed as algebras) of the general notions defined at the beginning of this section.

We have an analogous situation with semilattices. A *semilattice homomorphism* from the semilattice $\mathbf{S_1} = (S_1, \cdot)$ to the semilattice $\mathbf{S_2} = (S_2, \cdot)$ is a map $\phi : S_1 \to S_2$ such that

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y),$$

for each pair of elements $x, y \in \ \S_1$. Depending on whether we view a semilattice as a meet-semilattice or as a join-semilattice, we speak of *meet-homomorphisms* or *join-homomorphisms*, respectively. Further, since a lattice $\mathbf{L} = (L, \wedge, \vee)$ is a semilattice both with respect to $\wedge$ and $\vee$, we speak of meet-homomorphisms or join-homomorphisms of lattices. Thus a meet homomorphism between lattices satisfies (5.1), and a join homomorphism satisfies (5.2).

Note that meet-homomorphisms, join-homomorphisms, and lattice homomorphisms are all order homomorphisms. However, the converse does not hold, nor is there any connection between meet and join-homomorphisms.

Let $\mathbf{A} = (A, f)$ be an algebra of type $\nu = (\nu_i)_{i \in I}$. A subset B of A is said to be a *subuniverse* of $\mathbf{A}$, if it is closed under the operations of A. In other words, $f_i(b_0, \ldots, b_{nu_i - 1}) \in B$, for each $i \in I$ and each $(b_0, \ldots, b_{nu_i - 1}) \in B^{\nu_i}$.

Note that every subuniverse of $\mathbf{A}$ contains all the constants (nullary operations) of $\mathbf{A}$, and hence the empty set is a subuniverse of $\mathbf{A}$ iff $\mathbf{A}$ has no constants.

Each subuniverse $B \neq \emptyset$ of $\mathbf{A}$ gives rise to an algebra $\mathbf{B} = (B, g)$, where for each $i \in I,\ g_i$ is the restriction off_i to B^{ν_i}.

Subalgebras of concrete classes of algebras, such as semigroups, groups, rings, lattices etc., will be referred to as subsemigroups, subgroups, subrings, sublattices etc.

Let us point out that in applying the definition of a subalgebra to concrete cases, some caution is necessary. For example, considering a group as a system $\mathbf{A} = (A, \cdot, 1)$ (see example (iv), p. PAGE GOES HERE), the subalgebras of $\mathbf{A}$ are all submonoids of $\mathbf{A}$. If we want to obtain only the subgroups as subalgebras, we have to consider inversion as one of the fundamental operations.

The set of all subuniverses of an algebra $\mathbf{A}$ will be denoted by $\mathrm{Sub}(\mathbf{A})$.

Note that $\mathrm{Sub}(\mathbf{A})$ is closed under arbitrary intersections, and hence, in view of proposition 3.1, $\mathrm{Sub}(\mathbf{A})$ is a complete lattice under set inclusion. Note that if $\mathcal{D} \subseteq \mathrm{Sub}(\mathbf{A})$, then

$$\bigvee \mathcal{D} = \bigcap \{B \mid \bigcup \mathcal{D} \subseteq B \in \mathrm{Sub}(\mathbf{A})\}.$$

More generally, if $X \subseteq A$, we let

$$Sg_{\mathbf{A}}(X) = \bigcap \{B \mid X \subseteq B \in \mathrm{Sub}(\mathbf{A})\}.$$

Note that $Sg_{\mathbf{A}}(X)$ is the smallest subuniverse of $\mathbf{A}$ containing X. The subalgebra of $\mathbf{A}$ over $Sg_{\mathbf{A}}(X)$ is called the *subalgebra of* $\mathbf{A}$ *generated by* X, and will be denoted by $\langle X \rangle_{\mathbf{A}}$. We also say that X is a *set of generators of* $\langle X \rangle_{\mathbf{A}}$. It s clear that $\mathbf{C}$ is a subalgebra of $\mathbf{B}$, and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, then $\mathbf{C}$ is a subalgebra of $\mathbf{A}$. Therefore when we deal with a single algebra $\mathbf{A}$, we shall write $Sg(X)$ for $Sg_{\mathbf{A}}(X)$ and $\langle X \rangle$ for $\langle X \rangle_{\mathbf{A}}$.

EXERCISES (Section 5)

1. Prove Propositions 5.1, 5.2, 5.3.

2. Prove that a map between groups (viewed as algebras of type (2,1,0)) is a group homomorphism if and only if it preserves the binary operation.

3. Prove that join-semillatice and meet-semilattice homomorphisms are order homomorphisms. Is the converse true?

4. Show that a meet-homomorphism between lattices need not be a join-homomorphism.

5. Let $\mathbb{Z}[x]$ denote the additive group of all polynomials in x with coefficients in $\mathbb{Z}$ (integers) and let G be the multiplicative group of all positive rationals. Prove that $\mathbb{Z}[x] \cong G$.

6. Prove that a sublattice generated by two elements has two or four elements.

7. *The Principle of Algebraic Induction.* Consider an algebra $\mathbf{A} = (A, f)$ and a set $X \subseteq A$. Suppose the following:

   (i) *Inductive beginning*: Each $x \in X$ satisfies certain property P.

   (ii) *Inductive steps*: If $a_0, a_1, \ldots, a_{\nu_i - 1} \in Sg(X)$ satisfy P (*inductive hypothesis*), then $f_i(a_0, \ldots, a_{\nu_i - 1})$ satisfies P (*inductive conclusion*).

   Prove (*general conclusion*) that each $x \in Sg(X)$ satisfies P.

   Remark 5.4 Note that (ii) implies in particular that if $\nu_i = 0$, then f_i satisfies P.

8. How does algebraic induction compare with the usual induction on natural numbers?

9. Let $\mathbf{A} = (A, f)$ be an algebra and $X \subseteq A$. We defined $Sg(X)$ as the intersection og all subuniverses of $\mathbf{A}$ that contain X. The statement below gives an elementwise description of $Sg(X)$.

   Show the following:

(i) $Sg(X) = X \cup \bigcup_{i \in I} f_i(Sg(X)^{\nu_i})$. Hint: Use algebraic induction, or show that the RHS is a subalgebra.

(ii) $Sg(X) = \bigcup_{n=1}^{\infty} D^n(X)$ where

$$D^1(X) = D(X) = X \cup \bigcup_{i \in I} f_i X^{\nu_i},$$

and for each $n > 1$,

$$D^{n+1}(X) = D(D^n(X)).$$

10. Let $\mathbf{A} = (A, f)$ be an algebra and $X \subseteq A$. Show that

$$|Sg(X)| \leq |X| + |I| + \aleph_0.$$

11. Show that if $\phi \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$ and $X \subseteq A$, then

$$\phi(Sg_{\mathbf{A}}(X)) = Sg_{\mathbf{B}}(\phi(X)).$$

12. Show that if $\phi, \psi \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$, then $\{x \in \mathbf{A} \mid \phi(x) = \psi(x)\} \in \mathrm{Sub}(\mathbf{A})$.

13. Suppose $\mathbf{A}$ is generated by X. Show that if $\phi, \psi \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$ and $\phi|_X = \psi|_X$, then $\phi = \psi$. Here, $\phi|_X$ means the restriction of ϕ to X.

14. Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of the same type, let $X \subseteq A$ and let $\phi : X \to B$ be an arbitrary map. Show that there is at most one homomorphism $\overline{\phi} \in \mathrm{Hom}(\langle X \rangle_{\mathbf{A}}, B)$ extending ϕ. Show further that the latter exists if and only if for all finite subsets F of X, $\phi_F = \phi|_F$ can be extended to a homomorphism $\overline{\phi}_F \in \mathrm{Hom}(\langle F \rangle_{\mathbf{A}}, B)$.

6 Closure Systems and Closure Operators

In this section we shall consider the important concepts of a “closure system” and a “closure operator”.

Let Let A be a set and let $\emptyset \neq \mathcal{C} \subseteq \mathcal{P}(A)$. $\mathcal{C}$ is said to be a *closure system* (or an *intersection system*) on A if $\mathcal{C}$ is closed under intersections, i.e., $\bigcap \mathcal{D} \in \mathcal{C}$ whenever $\mathcal{D} \subseteq \mathcal{C}$. In particular, taking $\mathcal{D} = \emptyset$, we see that $A = \bigcap \emptyset \in \mathcal{C}$. Since a closure system is closed under arbitrary intersections, it follows by Proposition 3.1 (CHANGE REFERENCE) that it is a complete lattice with respect to set inclusion. Clearly, for any $\mathcal{D} \subseteq \mathcal{C}$,

$$\bigwedge \mathcal{D} = \bigcup \mathcal{D},$$

and

$$\bigvee \mathcal{D} = \bigcap \{X \in \mathcal{C} \mid \bigcup \mathcal{D} \subseteq X\}.$$

We have encountered some examples of closure systems. For example the complete lattice $\mathcal{O}(P)$ of all order ideals of a poset P is a closure system on the set P. For each algebra $\mathbf{A}$, Sub$\mathbf{A}$ is a closure system on A.

The second notion we require (and which will turn out to be equivalent to that of a closure system)is the notion of a closure operator. A *closure operator* on a set A is a mapping $\Gamma : \mathcal{P}(A) \to \mathcal{P}(A)$ satisfying the following conditions for all $X, Y \subseteq A$:

$$\text{if } X \subseteq Y, \text{ then } \Gamma(X) \subseteq \Gamma(Y); \tag{6.1}$$

$$X \subseteq \Gamma(X) \ (\Gamma \text{ is extensive}); \tag{6.2}$$

$$\Gamma\Gamma = \Gamma \ (\Gamma \text{ is idempotent}). \tag{6.3}$$

We have already encountered an important example of a closure operator associated with the closure family Sub$\mathbf{A}$. Namely, the mapping $Sg_{\mathbf{A}} : \mathcal{P}(A) \to \mathcal{P}(A)$ which sends every $X \subseteq A$ to toe subuniverse $Sg_{\mathbf{A}}(X)$ of $\mathbf{A}$ generated by X.

In general, the relationship between closure operators and closure systems is exhibited in the following theorem.

Theorem 6.1 *If $\mathcal{C}$ is a closure system on A, then the mapping $\mathcal{C}^\alpha : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by*

$$\mathcal{C}^\alpha = \bigcap\{Y \in \mathcal{C} \mid Y \supseteq X\} \tag{6.4}$$

is a closure operator on A. Conversely, if Γ is a closure operator on A, then the family

$$\Gamma^\beta = \{X \in A \mid \Gamma(X) = X\} \tag{6.5}$$

is a closure system on A. Furthermore, we have

$$\mathcal{C}^{\alpha\beta} = \mathcal{C} \text{ and } \Gamma^{\beta\alpha} = \Gamma. \tag{6.6}$$

Proof. Suppose $\mathcal{C}$ is a closure family on A and define $\mathcal{C}^\alpha : \mathcal{P}(A) \to \mathcal{P}(A)$ as in 6.4. It is immediate that 6.1 and 6.2 are satisfied, whereas 6.3 follows from the fact that

$$\mathcal{C}^\alpha(X) = X \text{ iff } X \in \mathcal{C}. \tag{6.7}$$

Conversely suppose Γ is a closure operator on A and define Γ^β by 6.5. If $\mathcal{D} \subseteq \Gamma^\beta$ and $X = \bigcap \mathcal{D}$, then $X \supseteq Y$, for each $Y \in \mathcal{D}$. Hence by 6.1, $\Gamma(X) \subseteq \Gamma(Y) = Y$, and so $\Gamma(X) \subseteq \bigcap \mathcal{D} = X$. Together with 6.2 this shows $\Gamma(X) = X$, i.e., $X \in |Gamma^\beta$. We have shown that $\mathcal{C}$ is a closure system, and note that we have done this without using 6.3. We now use condition 6.3 to establish 6.6.

First, let $\mathcal{C}$ be a closure system on A. Then $\mathcal{C}^{\alpha\beta} = \mathcal{C}$ by 6.7. Next let Γ be a closure operator on A, and $X \subseteq A$. In view of 6.3, 6.4 and 6.6, $\Gamma^{\beta\alpha}(X) = \bigcap\{Y \in \Gamma^\beta \mid Y \supseteq X\} = \bigcap\{Y \mid Y \supseteq X,\ \Gamma(Y) = Y\} \supseteq \Gamma(X)$. Thus $\Gamma^{\beta\alpha}(X) \supseteq X$ and note also that

$$\Gamma^{\beta\alpha}(X) = X \text{ iff } \Gamma(X) = X. \tag{6.8}$$

Now $X \subseteq \Gamma(X)$ and so $\Gamma^{\beta\alpha}(X) \subseteq \Gamma^{\beta\alpha}\Gamma(X)$, by 6.1, 6.3 and 6.8. Thus $\Gamma^{\beta\alpha}(X) = \Gamma(X)$, and conditions 6.6 are established. □

An element c of a complete lattice L is said to be *compact* if for every subset X of L with $c \leq \bigvee X$ there exists a finite subset F of X such that $c \leq \bigvee F$.

Proposition 6.2 *For an element c of a complete lattice L the following statements are equivalent:*

1. *c is compact*

2. *For every updirected set $X \subseteq L$, $c \leq \bigvee X$ implies $c \leq x$ for some $x \in X$.*

The set of compact elements of a complete lattice L will be denoted by $c(L)$. Note that $0 \in c(L)$.

Corollary 6.3 *In any complete lattice L, $c(L)$ is a join-subsemilattice of L.*

Consider a closure family $\mathcal{C}$ on a set A, and denote by Γ the closure associated operator. A set $B \in \mathcal{C}$ is said to be *finitely generated* if there is a finite subset X of A such that $B = \Gamma(X)$.

Proposition 6.4 *Each compact element of a closure family is finitely generated.*

The converse is not true in general (see exercise 5). However these two concepts coincide for algebraic closure systems. This concept will be discussed in the next section.

EXERCISES

1. Prove proposition 6.2.

2. Prove corollary 6.3.

3. Prove proposition 6.4.

4. Recall that a subset A of a poset P is called *join-dense* if every element of$P$ is the join of elements of A. Let L be a complete lattice and let A be a join-dense subset of L. Show the following:

   (i) $\mathcal{C} = \{A \cap (x] \mid x \in L\}$ is a closure family on A.

   (ii) L is isomorphic to $\mathcal{C}$.

5. Let L be the lattice obtained by adjoining to the chain of natural numbers a largest element ∞. Let $\mathcal{C} = \{(x] \mid x \in L\}$. Show the following:

   (i) $\mathcal{C}$ is a closure family.

   (ii) For each $x \in L \setminus \{\infty\}$, $(x]$ is a compact element of $\mathcal{C}$ but $(\infty]$ is not compact.

   Infer that a finitely generated member of a closure family need not be compact.

6. The purpose of this exercise is to show that closure operators can be defined on an arbitrary poset.

   Let P be a poset. A mapping $\gamma : P \to P$ is called a **closure operator** if it satisfies the following conditions for all $x, y \in P$:

   If $x \leq y$, then $\gamma(x) \leq \gamma(y)$.

   $x \leq \gamma(x)$.

   $\gamma\gamma = \gamma$.

   Let now $\gamma : P \to P$ be a closure operator. Show that if $X \subseteq {}^{P}\bigwedge X$ exists, then ${}^{\gamma(P)}\bigwedge X$ and ${}^{P}\bigwedge X = {}^{\gamma(P)}\bigwedge X$.

A non-empty subset $C \subseteq P$ is said to be a *closure retract* of P if for all $x \in P$, the set $\{c \in C \mid x \leq c\}$ has a least element.

Show that if C is a closure retract of P, then the mapping $C^\alpha : P \to P$ defined by

$$C^\alpha(x) = min\{c \in C \mid x \leq c\}$$

is a closure operator. Conversely, if $\gamma : P \to P$ is a closure operator, then $\gamma^\beta = \gamma(P)$ is a closure retract of P. Furthermore, $C^{\alpha\beta} = C$ and $\gamma\beta\alpha = \gamma$.

Let now L be a complete lattice. Characterize the closure retracts of L. Are these complete? Are they necessarily closed sublattices of L?

Show further that if $\gamma : P \to P$ is a closure operator and $X \subseteq L$, then

$$\gamma({}^L\bigvee X) = {}^{\gamma(L)}\bigvee \gamma(X).$$

7. The purpose of this exercise is to introduce the important concept of a Galois connection and study some of its properties. Given two posets P and Q, a mapping $\phi : P \to Q$ is called a *dual order homomorphism (isomorphism)* if ϕ is an order homomorphism (isomorphism) from P to the dual Q^d of Q.

   Let P and Q be two posets. A pair of mappings

   $$\phi : P \to Q,\ \psi : Q \to P \tag{6.9}$$

   is said to be a *Galois connection* between P and Q if the following two conditions are satisfied for all $x \in P$ and $y \in Q$:

   $$\phi \text{ and } \psi \text{ are dual order homomorphisms,} \tag{6.10}$$

   $$x \leq \psi\phi(x) \text{ and } y \leq \phi\psi(y). \tag{6.11}$$

   Show that a pair of mappings 6.9 is a Galois connection iff the following condition is satisfied for all $x \in P$ and $y \in Q$:

   $$x \leq \psi(y) \Leftrightarrow y \leq \phi(x). \tag{6.12}$$

Suppose now that the pair 6.9 is a Galois connection. Show (a)-(g) below:

(a) $\psi\phi$ and $\phi\psi$ are closure operators.

(b) $\psi\phi\psi = \psi$ and $\phi\psi\phi = \phi$.

(c) $\phi|_{\psi\phi(P)}$ is a dual order isomorphism whose inverse is $\psi|_{\phi\psi(Q)}$.

(d) If P and Q are complete lattices, then so are $\psi\phi(P)$ and $\phi\psi(Q)$.

(e) Let A and B be two sets and ρ a subset of $A \times B$. Define the mappings

$$\phi : \mathcal{P}(A) \to \mathcal{P}(B),\ \psi : \mathcal{P}(B) \to \mathcal{P}(A) \tag{6.13}$$

in the following fashion. If $X \subseteq A$, set $\phi(X) = \{y \in B \mid (x, y) \in \rho,\ \forall x \in X$, and similarly, if $Y \subseteq B$, set $\psi(Y) = \{x \in A \mid (x, y) \in \rho,\ \forall y \in Y$. Note that $\phi(emptyset) = B$ and $\psi(\emptyset) = A$.

Show that the pair 6.13 is a Galois connection between $\mathcal{P}(A)$ and $\mathcal{P}(B)$. This connection is called *the Galois connection induced by the correspondence* ρ. Most Galois connections encountered in practice arise from a correspondence between sets in the way described above.

(f) Let F be any field, K be any subfield of F and G be the Galois group of F and K. In other words G is the group of all automorphisms of F that leave every element of K fixed (automorphisms of F/K). Let $\mathcal{F}$ denote the lattice of all intermediate fields between F and K and $\mathcal{S}$ denote the lattice of subgroups of G. Then the correspondence

$$\rho = \{(x, \alpha) : \alpha(x) = x\} \subseteq F \times G$$

established a Galois connection

$$\phi : \mathcal{F} \to \mathcal{S},\ \psi : \mathcal{S} \to \mathcal{F}$$

between $\mathcal{F}$ and $\mathcal{S}$. Note that for each $L \in \mathcal{F}$, $\phi(L)$ is the group of all F/L automorphisms, and for each $H \in \mathcal{S}$, $\psi(H)$ is the fixed subfield of F under H.

Galois connections take their name from this example.

(g) Let $\rho \subseteq A \times A$ be a symmetric relation, and let

$$\phi : \mathcal{P}(A) \to \mathcal{P}(A), \ \psi : \mathcal{P}(A) \to \mathcal{P}(A)$$

be the Galois connection induced by ρ. Show that $\phi = \psi$.

8. As an application of Galois connections we shall show that the well known construction of the real numbers from the rationals via Dedekind cuts can be generalized to arbitrary posets.

Let P be a poset and consider the Galois connection

$$Ub : \mathcal{P}(P) \to \mathcal{P}(P), \ Lb : \mathcal{P}(P) \to \mathcal{P}(P)$$

induced by the ordering of P. Thus for each $X \subseteq P$, $Ub(X)$ is the set of all upper bounds of X and $Lb(X)$ is the set of all lower bounds of X. The set $LbUb(X)$ is called a *Dedekind cut* by X. Note that if $a \in P$, then $lbUb(\{a\}) = (a]$. Let $D(P)$ denote the set of all Dedekind cuts determined by the subsets of the poset P. Show the following:

(a) $D(P)$ is a complete lattice with respect to set inclusion.

(b) $D(P)$ is a join and meet-completion of P.

(c) Show that for every poset P there exists a complete lattice L so that P is both join and meet-dense in L. Deduce in particular that if ${}^{P}\bigwedge X$ for $X \subseteq P$, then

$${}^{P}\bigwedge X = {}^{L}\bigwedge X$$

and that the same holds if we replace $\bigwedge$ by $\bigvee$.

L as above is called the *Dedekind-MacNeille completion (or the completion by cuts)* of P.

7 Algebraic Closure Systems and Algebraic Lattices

In this section we introduce the important concepts of an algebraic closure system and an algebraic lattice, and we demonstrate how these two concepts are related. En route, we also define the notion of an ideal of a join-semilattice.

Let A be a set and $\emptyset \neq \mathcal{D} \subseteq \mathcal{P}(A)$. We say that $\mathcal{D}$ is an *updirected system* if the poset $(\mathcal{D}, \subseteq)$ is updirected.

A closure system $\mathcal{C}$ is said to be *algebraic* if $\bigcup \mathcal{D} \in \mathcal{C}$ for every updirected subsystem $\mathcal{D}$ of $\mathcal{C}$.

As the next proposition shows, algebraic closure systems occur in abundance.

Proposition 7.1 *For any algebra* $\mathbf{A} = (A, f)$*, Sub(**A**) is an algebraic closure system.*

The closure operator associated with an algebraic closure system is called an *algebraic closure operator*. Conditions (ii) and (iii) of the next theorem characterize such operators.

Theorem 7.2 *Let A be a set, $\mathcal{C}$ be a closure system on A and Γ the associated closure operator. The following are equivalent:*

(i) $\mathcal{C}$ is an algebraic closure system;

(ii) For each $X \subseteq A$,

$$\Gamma(X) = \bigcup \{\Gamma(F) \mid F \subseteq X,\ F - \mathit{finite}\};$$

(iii) For each updirected system $\mathcal{D} \subseteq \mathcal{P}(A)$,

$$\Gamma(\bigcup \mathcal{D}) = \bigcup \{\Gamma(D) \mid D \in \mathcal{D}\};$$

(iv) Each finitely generated member of $\mathcal{C}$ is compact;

*(v) $\mathcal{C} =$ Sub(**A**), for some algebra $\mathbf{A} = (A, f)$.*

Proof. Recall that for $X \subseteq A$, $X \in \mathcal{C}$ iff $\Gamma(X) = X$.

(i) $\Rightarrow$ (iv) Suppose (i) is satisfied, and let F be a finite subset of A. We need to show that $\Gamma(F)$ is compact in $\mathcal{C}$. For that we shall make use of Proposition 6.2. Let $\mathcal{D}$ be an updirected system contained in $\mathcal{C}$ and suppose that

$$\Gamma(F) \subseteq {}^{\mathcal{C}}\bigvee \mathcal{D}$$

. By virtue of (i), ${}^{\mathcal{C}}\bigvee \mathcal{D} = \bigcup \mathcal{D}$, and so there is $D \in \mathcal{D}$ such that $F \subseteq D$. But then $\Gamma(F) \subseteq \Gamma(D) = D$, and hence $\Gamma(F)$ s compact in $\mathcal{D}$.

(iv) $\Rightarrow$ (iii) Suppose (iv) is satisfied, and let $\mathcal{D}$ be an updirected system on A. Since $\mathcal{D}$ is updirected, it follows that $\{\Gamma(D) \mid D \in \mathcal{D}\}$ is also updirected. Note that for each $D \in \mathcal{D}$, $\Gamma(D) \subseteq \Gamma(\bigcup \mathcal{D})$, and hence

$$\bigcup_{D \in \mathcal{D}} \Gamma(D) \subseteq \Gamma(\bigcup \mathcal{D}).$$

In order to establish the reverse inclusion, note first that $\Gamma(\bigcup \mathcal{D}) \subseteq \Gamma(\bigcup_{D \in \mathcal{D}} \Gamma(D)) = {}^{\mathcal{C}}\bigvee_{D \in \mathcal{D}} \Gamma(D)$. Let now $x \in \Gamma(\bigcup \mathcal{D})$. Then

$$\Gamma(\{x\}) \subseteq {}^{\mathcal{C}}\bigvee_{D \in \mathcal{D}}.$$

Since $\Gamma(\{x\})$ is compact and $\{\Gamma(D) \mid D \in \mathcal{D}$ is updirected, it follows that there exists $D \in \mathcal{D}$ such that $\Gamma(\{x\}) \subseteq \Gamma(D)$. But then

$$x \in \bigcup_{D \in \mathcal{D}} \Gamma(D).$$

This shows that

$$\Gamma(\bigcup \mathcal{D}) \subseteq \{\Gamma(D) \mid D \in \mathcal{D}\}.$$

(iii) $\Rightarrow$ (ii) Suppose (iii) is satisfied, and let $X \subseteq A$. Note that $\{F \mid F \subseteq X,\ F \text{ finite}\}$ is an updirected system, and hence in view of (iii)

$$\Gamma(X) = \bigcup \{\Gamma(F) \mid F \subseteq X,\ F \text{ finite}\}.$$

(ii) $\Rightarrow$ (v) Assume that (ii) is satisfied. Let $f = (f_i)$ be the family of all functions $f_i : A^{\nu_i} \to A$ (indexed by the set I) such that $f_i(C^{\nu_i}) \subseteq C$, for each

$C \in \mathcal{C}$. We claim that if $\mathbf{A} = (A, f)$, then Sub$\mathbf{A} = \mathcal{C}$. It is clear that $\mathcal{C} \subseteq$ Sub$\mathbf{A}$. Let $B \in$ Sub$\mathbf{A}$. We need to show that $B \in \mathcal{C}$ or, what amounts to the same, that $\Gamma(B) \subseteq B$. Note that if $\emptyset \in$ Sub$\mathbf{A}$, then $\emptyset \mathcal{C}$, and thus we may assume that $B \neq \emptyset$. Let $x \in \Gamma(B)$. In view of (ii), there exists $\nu \geq 1$ and elements $b_0, \ldots, b_{\nu-1}$ in B such that $x \in \Gamma(\{b_0, \ldots, b_{\nu-1}\})$.

Define $\phi : A^\nu \to A$: $\phi(x_0, \ldots, x_{\nu-1}) = x$ if $x_0, \ldots, x_{\nu-1} = b_0, \ldots, b_{\nu-1}$ and $\phi(x_0, \ldots, x_{\nu-1}) = x_0$ otherwise. It is immediate that ϕ is one of the fundamental operations of $\mathbf{A}$, and hence $x \in B$, since B is a subuniverse of $\mathbf{A}$. We have shown that $\Gamma(B) \subseteq B$.

(v) $\Rightarrow$ (i) Proposition 7.1. $\square$
The proof of the theorem is now complete.

We define a lattice L to be *algebraic* if L is complete and each of its elements is a join of compact elements.

In view of conditions (ii) and (iv) of Theorem 7.2, every algebraic closure system is an algebraic lattice.

We have already remarked that the ascending chain condition implies compact generation. More specifically, every element of a complete lattice L is compact if and only if L satisfies the ascending chain condition.

We have already seen that the join of two compact elements is compact, but the meet of two compact elements need not be compact. Let us also remark that if K is a closed sublattice of an algebraic lattice L, then K is algebraic.

By an ideal of a join-semilattice S we mean a non-empty subset I of S satisfying the following two conditions for all $x, y \in S$:

(i) $x, y \in I$ implies $x \vee y \in I$;

(ii) $x \leq y \in I$ implies $x \in I$.

Thus the ideals of a join-semilattice S are precisely those non-empty subuniverses of S which are also order ideals.

Note that in case S is a lattice, condition (ii) can be rewritten as:

(ii‘) $y \in I$ implies $x \wedge y \in I$.

The principal order ideals $(a]$, $a \in S$, will simply be called *principal ideals*. The ideal generated by a non-void subset H of a join-semilattice S will be denoted by $(H]$. An ideal I of S is said to be proper if $I \neq S$.

We let $I(S)$ be the set of all ideals of S, and $I_0(S) - I(S) \cup \{\emptyset\}$. We call $I(S)$ the *ideal lattice* of S, and $I_0(S)$ the *augmented ideal lattice* of S.

Theorem 7.3 *Let S be a join-semilattice.*

(i) $I_0(S)$ is an algebraic closure system, and if S has a least element, then so is $I(S)$.

(ii) For each $\emptyset \neq H \subseteq S$,

$$(H] = \{x \in S | x \leq h_0 \vee \cdots \vee h_n,\ \textit{for some } h_0, \ldots, h_n \in H\}$$

.

(iii) The compact elements of $I_0(S)$ are the principal ideals of S and the empty set. If S has a least element, then the compact elements of $I(S)$ are the principal ideals of S.

(iv) The mapping $a \mapsto (a]$ is a join-embedding from S to $I(S)$ and also $I_0(S)$. Further, it is a lattice embedding if L is a lattice.

Note that condition (ii) of theorem 7.3 yields the formula

$$(a] \vee (b] = (a \vee b].$$

where $(a] \vee (b]$ denotes the join of $(a]$ and $(b]$ in $I(S)$.

Furthermore, if S is a lattice, then

$$(a] \wedge (b] = (a \wedge b].$$

Theorem 7.4 *Let L be an algebraic lattice and let C denote the join-semilattice of compact elements of L. Then the mapping $a \mapsto (a] \cap C$ is a lattice isomorphism between L and $I(C)$.*

Proof. **Sketch.** Denote the mapping in the statement of the theorem by ϕ. Consider the mapping $\psi I(C) \to L$ defined by $\psi(I) = \bigvee_{I \in I(C)} I$. Then both ϕ and ψ are order homomorphisms and $\psi\phi = id_L$ and $\phi\psi = id_{I(C)}$. $\Box$

Combining theorems 7.2, 7.3 and 7.4, we obtain the following important result:

Theorem 7.5 *For a lattice L the following statements are equivalent:*

(i) L is an algebraic lattice.

(ii) L is isomorphic to Sub$\mathbf{A}$*, for some algebra $\mathbf{A} = (A, f)$.*

(iii) L is isomorphic to the ideal lattice of some join-semilattice with least element.

In the remainder of the section we study some additional properties of algebraic lattices. As the next theorem shows, compact generation and the atomicity conditions are interrelated.

Theorem 7.6 *Every algebraic lattice is weakly atomic.*

Proof. Suppose that L is an algebraic lattice and let $a < b$ in L. There exists $c \in c(L)$ such that $c \leq b$ and $c \not\leq a$. Hence $a < a \vee c \leq b$. Let now

$$X = \{x \in L | a \leq x < a \vee c\}.$$

Note that $a \in X$, and so X is nonempty. We wish to show that X has a maximal element. To this end, let $\mathcal{C}$ be a chain and suppose that $a \vee c = \bigvee \mathcal{C}$. Then $c \leq \bigvee \mathcal{C}$ and hence $c \leq x_0$ for some $x_0 \in \mathcal{C}$. But then $x_0 \geq a \vee c$, contradicting the fact that $x_0 \in X$. We have shown that every chain in X has an upper bound in X, and hence by Zorn's lemma, X has a maximal element, say m. Then $a \leq m \prec a \vee c \leq b$ and L is weakly atomic. $\Box$

The notion of an algebraic lattice has a useful generalization. Let us define a lattice L to be *upper continuous* if L is complete and, for every updirected subset $D \subseteq L$,

$$a \wedge \bigvee D = \bigvee \{a \wedge d | d \in D\}.$$

Theorem 7.7 *Every algebraic lattice is upper continuous.*

Proof. Suppose L is an algebraic lattice, $a \in L$, and D is an updirected subset of L. Since D is updirected, for every compact element $c \leq a \wedge \bigvee D$, there exists $d \in D$ such that $c \leq d$, and hence $c \leq a \wedge d$. Since $a \wedge \bigvee D$ is the join of compact elements, it follows that

$$a \wedge \bigvee D =\leq \bigvee \{a \wedge d | d \in D\}.$$

The reverse inequality holds in any complete lattice. □

A useful reformulation of theorem 7.7 is the following:

Theorem 7.8 *If L is an algebraic lattice and $\mathcal{F}$ is the family of all finite subsets of some $X \subseteq L$, then for each $a \in L$,*

$$a \wedge \bigvee X = \bigvee \{a \wedge \bigvee F | F \in \mathcal{F}\}.$$

A natural problem that arises in the study of an algebra is that of representing its elements in terms of a "canonical" set of elements under a specific operation of the system. Usually this canonical set is taken to be those elements that cannot be further so represented. An elementary example of this representation is the representation of integers as products of primes. Another fundamental problem is that of representing an algebra as a direct or subdirect product of simpler algebras.

These two problems are related. Indeed, we shall show later that the representations of an algebra $\mathbf{A}$ as a direct or subdirect product correspond to certain meet representations of the least element in the lattice of congruence relations of $\mathbf{A}$.

At this point we wish to show that such a meet representation is available for elements of an algebraic lattice.

Given a lattice L, a *meet-irreducible* element of L is any element $a \neq 1$ for which $a = b \wedge c$ always implies that $a = b$ or $a = c$. A *meet-prime* element of L is any element $a \neq 1$ for which $b \wedge c \leq a$ always implies $b \leq a$ or $c \leq a$. In case L is complete we can define two additional types of elements. An element $a \in L$ is called *completely meet-irreducible* if for every subset S of L, $a = \bigwedge S$ implies $a \in S$; it is called *completely meet-prime* if $\bigwedge S \leq a$ implies

$s \leq a$ for some $s \in S$.

A *join-irreducible, join-prime, completely join-irreducible*, or a *completely join-prime* element is defined dually.

Theorem 7.9 *In an algebraic lattice, every element is a meet of completely meet-irreducible elements.*

Proof. Let L be an algebraic lattice. To establish the theorem, it suffices to show that whenever $a, b \in L$ and $a \not\leq b$, then there is a completely meet-irreducible element q such that $b \leq q$ and $a \not\leq q$. To this end, let $a, b \in L$ such that $a \not\leq b$. There exists $c \in c(L)$ such that $c \leq a$ and $c \not\leq b$. Let $X = \{x \in L | c \not\leq x \text{ and } b \leq x\}$. Note that $b \in X$, and for each $x \in X$, $a \not\leq x$. Now apply Zorn's lemma to get a maximal element $q \in X$. Then of course $b \leq q$ and $a \not\leq q$. It remains to be shown that q is completely meet-irreducible. Let $S \subseteq L$ such that $q = \bigwedge S$. Now $c \not\leq q$ and thus there exists $s \in S$ such that $c \not\leq s$ and $s \geq q$. Ut follows that $s \in X$, and the maximality of q yields that $q = s$. This shows that q is completely meet-irreducible. $\square$

Let us remark that by dualizing the concept of an ideal we get the concept of a *filter* in a meet-semilattice (or a lattice). We write $[a)$ for the *principal filter* generated by a, and if H is a nonempty subset of a meet-semilattice S, we write $[H)$ for the filter of S generated by H. A filter F is called proper if $F \neq S$. We write $F(S)$ for the set of filters of S, and $F_0(S) = F(S) \cup \{\emptyset\}$. Note that for all $a, b \in S$, we have in $F(S)$,

$$[a) \vee [b) = [a \wedge b).$$

Furthermore, if S is a lattice, then

$$[a) \wedge [b) = [a \vee b).$$

EXERCISES

1. Prove proposition 7.1.

2. Prove theorem 7.3.

3. Complete the proof of theorem 7.4.

4. A proper ideal I of a lattice L is called *prime*provided that whenever $a, b \in L$ and $a \wedge b \in I$, then $a \in I$ or $b \in I$. A prime filter is defined dually. Show that an ideal I is prime iff $L \setminus I$ is a filter (in fact a prime filter).

5. Let L be a lattice and let C_2 denote the two-element chain with elements 0 and 1. Show that $I \in I(L)$ is prime iff there is a lattice epimorphism from L to C_2 with $I = \phi^{-1}(\{0\})$.

6. Let $\phi : L \to K$ be a lattice epimorphism. Show the following:

   (i) If $I \in I(L)$, then $\phi(I) \in I(K)$.

   (ii) If $J \in I(K)$, then $\phi^{-1}(J) \in I(L)$.

   (iii) If $J \in I(K)$, then $\phi{-}1(J)$ is prime.

7. Give an alternative (very short) proof of theorem 7.7 by using theorems 7.2 and 7.5.

8. Show that every closed sublattice of an algebraic lattice is itself an algebraic lattice.

9. Find a closure system which is an algebraic lattice but not an algebraic closure system.

10. Let $\mathcal{C}$ be a closure system on a set A, and let Γ be the closure operator associated with it. Show that TFAE:

    (i) Each element in $\mathcal{C}$ is compact.

    (ii) $\mathcal{C}$ is an algebraic closure system and each member of it is finitely generated.

    (iii) For each $M \subseteq A$, $\Gamma(M) = \Gamma(F)$, for some finite $F \subseteq M$.

(iv) For each updirected system $\mathcal{D} \subseteq \mathcal{P}(A)$, $\Gamma(\bigcup \mathcal{D} = \Gamma(D)$, for some $D \in \mathcal{D}$.

(v) For each updirected system $\mathcal{D} \subseteq \mathcal{C}$, $\bigcup \mathcal{D} \in \mathcal{D}$ (ascending chain condition.

11. For a complete lattice L, TFAE:

    (i) $c(L) = L$.

    (ii) L satisfies the ascending chain condition.

    (iii) Every closure system isomorphic to L is an algebraic closure system.

12. Show that if a lattice L satisfies the ascending chain condition, then every element of L is a finite meet of meet-irreducible elements.

13. Show by means of examples that the concepts of a meet-irreducible, meet-prime, completely meet-irreducible, and a completely meet-prime element are all distinct.

14. Let L be a lattice. Show that $I \in I(L)$ is prime iff I is a meet-prime element of $I(L)$.

15. Let S be a join-semilattice with 0 and 1. Show that every proper ideal of S is contained in a maximal ideal of S (i.e., a dual atom of $I(S)$).

16. Formulate and prove the analog of exercise 15 in the setting of algebraic lattice.

8 Distributive Lattices

In any lattice L the following statements are equivalent:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \ \forall x, y, z \in L. \tag{8.1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \ \forall x, y, z \in L. \tag{8.2}$$

Indeed, in view of the duality principle for lattices, we only need to prove the implication 8.1 $\Rightarrow$ 8.2. We have in view of 8.1:

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] \\ &= x \vee [(x \vee y) \wedge z] \\ &= x \vee (x \wedge z) \vee (y \wedge z) \\ &= x \vee (y \wedge z).\end{aligned}$$

A lattice is said to be *distributive* if it satisfies one (and hence both) of the above conditions.

Since the above conditions are dual of each other, the dual of a distributive lattice is also distributive. Examples of distributive lattices include:

1. Every chain is a distributive lattice.
2. Every set-ring is a distributive lattice. Recall that a *set-ring* is simply a sublattice of the lattice of all subsets of a set. As we will see in the next section, every distributive lattice is isomorphic to a set ring.

We say that a lattice L is *modular* if, for all $x, y, z \in L$,

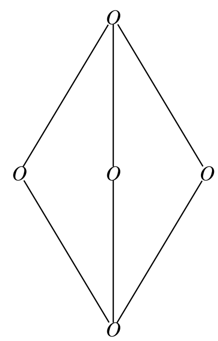
$$x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z. \tag{8.3}$$

Equivalently a lattice L is *modular* if, for all $x, y, z \in L$,

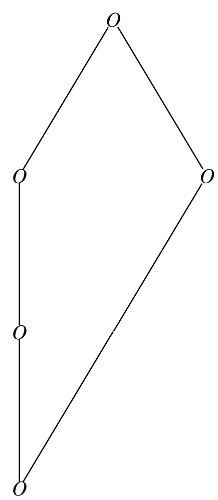
$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z. \tag{8.4}$$

In particular, the dual of a modular lattice is modular. Note that every distributive lattice is modular, but the converse is not true.

Consider the lattices N_5 and M_5 with diagrams given below:



M_5



N_5

The lattice N_5 is not modular. Indeed, $x \geq z$, but

$$x \wedge (y \vee z) = x > z = (x \wedge y) \vee z.$$

The lattice M_5 is easily seen to be modular. Note however that it is not distributive. Indeed,

$$x \wedge (y \vee z) = x > 0 = (x \wedge y) \vee (x \wedge z).$$

N_5 will be referred to as the *pentagon* and M_5 as the *diamond*.

Lemma 8.1 *For a lattice L the following statements are equivalent:*

(i) L is distributive.

(ii) For all $x, y, z \in L$,

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x). \qquad (8.5)$$

Proof. (i) $\Rightarrow$ (ii) Direct computation.

(ii) $\Rightarrow$ (i) If $x \geq y$, then $x \wedge y = y$ and $x \vee y = x$. Substituting into 8.5 we get

$$y \vee (z \wedge x) = (y \vee z) \wedge x.$$

Thus L is modular. Now taking the join of the left-hand side of 8.5 with x we get $x \vee (y \wedge z)$. Further, using modularity, the join of the right-hand side of 8.5 with x is

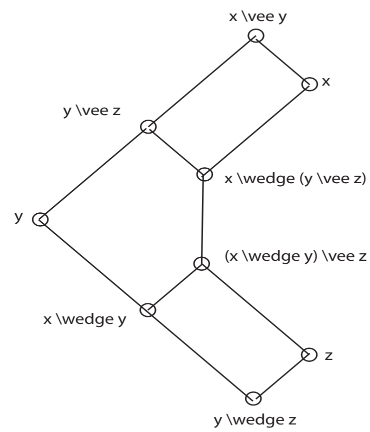
$$\begin{aligned} x \vee (((x \vee y) \wedge (y \vee z)) \wedge (z \vee x)) = \\ (x \vee (x \vee y)) \wedge (x \vee (y \vee z)) \wedge (x \vee z) = \\ (x \vee y) \wedge (x \vee y \vee z) \wedge (x \vee z) = \\ (x \vee y) \wedge (x \vee z). \end{aligned}$$

We have shown that L satisfies 8.2, and hence it is distributive. $\Box$

Theorem 8.2 *A lattice L is modular iff it has no sublattice isomorphic to the pentagon.*

Proof. Clearly, if L is modular, then every sublattice of L is modular, and hence no sublattice of L is isomorphic to the pentagon.

Conversely, suppose that L is non-modular. Then there are elements $x, y, z \in L$ such that $x > z$ and $x \wedge (y \vee z) > (x \wedge y) \vee z$. It follows that L contains a sublattice whose diagram is given in the figure below:



It is easily seen that all the indicated elements are distinct, and hence L contains a sublattice isomorphic to the pentagon. □

Theorem 8.3 *A modular lattice L is distributive iff it has no sublattice isomorphic to the diamond.*

Proof. Clearly, if L is distributive, then every sublattice of L is distributive, and hence no sublattice of L is isomorphic to the diamond.

Suppose that x_1, x_2, x_3 are elements of a modular lattice L. Set

$$\begin{aligned}
u &= (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3), \\
v &= (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3), \\
a_1 &= (x_2 \wedge x_3) \vee (x_1 \wedge (x_2 \vee x_3)), \\
a_2 &= (x_1 \wedge x_3) \vee (x_2 \wedge (x_1 \vee x_3)), \\
a_3 &= (x_1 \wedge x_2) \vee (x_3 \wedge (x_1 \vee x_2)).
\end{aligned}$$

By applying the modular law, we get the following dual expressions for a_1, a_2 and a_3:

$$\begin{aligned}
a_1 &= (x_2 \vee x_3) \wedge (x_1 \vee (x_2 \wedge x_3)), \\
a_2 &= (x_1 \vee x_3) \wedge (x_2 \vee (x_1 \wedge x_3)), \\
a_3 &= (x_1 \vee x_2) \wedge (x_3 \vee (x_1 \wedge x_2)).
\end{aligned}$$

Let us compute $a_1 \vee a_2$. Taking into account that $x_1 \wedge x_3 \leq x_2 \wedge (x_1 \vee x_3)$, $x_2 \wedge x_3 \leq x_2 \wedge (x_1 \vee x_3)$, the modular law yields:

$$\begin{aligned} a_1 \vee a_2 &= (x_2 \wedge (x_1 \vee x_3)) \vee (x_1 \wedge (x_2 \vee x_3)) \\ &= (x_2 \vee x_3) \wedge (x_1 \vee (x_2 \wedge (x_1 \vee x_3))) \\ &= (x_2 \vee x_3) \wedge (x_1 \vee x_2) \wedge (x_1 \vee x_3) \\ &= u. \end{aligned}$$

By symmetry,

$$u = a_1 \vee a_2 = a_2 \vee a_3 = a_3 \vee a_1,$$

and by duality,

$$u = a_1 \wedge a_2 = a_2 \wedge a_3 = a_3 \wedge a_1.$$

Note also that if any two of the elements u, a_1, a_2, a_3, v are equal, then $u = v$. Indeed, clearly this is the case if any two of a_1, a_2, a_3 are equal. Now, suppose $u = a_1$. Then $v = a_1 \wedge a_2 = a_2$ and so $u = a_2 \vee a_3 = a_3$. It follows that $u = a_1 \wedge a_2 = v$. The remaining cases are treated similarly.

Suppose now that L is a modular, non-distributive lattice. In view of lemma 8.1, there exist elements $x_1, x_2, x_3 \in L$ so that if u, a_1, a_2, a_3, v are defined as above, then $u \neq v$. Hence by the preceding discussion, the elements u, a_1, a_2, a_3, v form a sublattice of L isomorphic to the diamond. The proof is now complete. □

Corollary 8.4 *A lattice is distributive iff it has neither the pentagon nor the diamond as a sublattice.*

Corollary 8.5 *A lattice L is distributive iff, for any $x, y, z \in L$, $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ imply $y = z$.*

Proof. Suppose that L is distributive and let $x, y, z \in L$ such that $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$. Then

$$\begin{aligned} y &= (x \vee y) \wedge y \\ &= (x \vee z) \wedge y \\ &= (x \wedge y) \vee (z \wedge y) \\ &= (z \wedge x) \vee (z \wedge y) \\ &= z \wedge (x \vee y) \\ &= z \wedge (x \vee z) \\ &= z. \end{aligned}$$

Conversely, if L is not distributive, then in view of corollary 8.4, the condition in the statement of the corollary is not satisfied. $\square$

The next corollary is established in a similar fashion with the use of theorem 8.2.

Corollary 8.6 *A lattice L is modular iff, for all $x, y, z \in L$, $y \geq z$, $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ imply $y = z$.*

A lattice L is said to be *complemented* if it has a 0 and a 1 and each element $a \in L$ has a *complement*, i.e., an element a' satisfying the conditions

$$a \vee a' = 1, \; a \wedge a' = 0.$$

For example, the pentagon and the diamond are complemented lattices, and the complements in these lattices are not uniquely determined. On the other hand, in view of corollary 8.5, in a distributive lattice with 0 and 1, each element has at most one complement. Furthermore, it is easy to see that if a and b have complements a' and b' in such a lattice, then

$$(a')' = a, \; (a \vee b)' = a' \wedge b', \; (a \wedge b)' = a' \vee b'.$$

A complemented distributive lattice is called a *Boolean lattice*. By a boolean algebra $\mathbf{A} = (A, \wedge, \vee, 0, 1,')$ of type $(2, 2, 0, 0, 1)$ that satisfies a set of identities that define distributive lattices and also

$$1 \wedge x = x, \; x \vee x' = 1,$$

$$0 \vee x = x, \; x \wedge x' = 0.$$

It is clear that if $(A, \wedge, \vee, 0, 1,')$ is a Boolean algebra then $(A, \wedge, \vee)$ is a Boolean lattice, and each Boolean lattice can be made into a Boolean algebra in a unique way.

Some important examples of Boolean lattices are the following: the two-element chain; the complete lattice $\mathcal{P}(X)$ of all subsets of X; a set field. Recall that a set field is a set ring $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $X \setminus F \in \mathcal{F}$ for each $F \in \mathcal{F}$.

We shall show in the next section that each Boolean lattice is isomorphic to a set-field.

Proposition 8.7 *In a Boolean lattice L, we have the following:*

(i) x *wedgey* $= 0$ *iff* $x \leq y'$.

(ii) $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$. *(De Morgan's Laws).*

(iii) Each interval $[a, b]$ *is a Boolean lattice with respect to the induced partial ordering of* L.

(iv) If $Y \subseteq L$, *and* $\bigvee Y$ *exists, then for each* $x \in L$, $\bigvee\{x \wedge y | y \in Y\}$ *exists and* $x \wedge \bigvee Y \bigvee\{x \wedge y | y \in Y\}$.

(v) The dual of (iv).

Proposition 8.8 *Let* L *be a distributive lattice.*

(i) For any two ideals I, J *of* L,

$$I \vee J = \{i \vee j | i \in Y,\ j \in J\}.$$

(ii) $I(L)$ *is a distributive lattice.*

Proof. (i) Let $I, J \in I(L)$. In view of theorem 7.3 (ii), $t \in I \vee J$ iff $t \leq i \vee j$ for some $i \in I$ and $j \in J$. Therefore, $t = (t \wedge i) \vee (t \wedge j), t \wedge i \in I,\ t \wedge j \in J$.

Let $I, J, K \in I(L)$. It is to be shown that

$$I \wedge (J \vee K) \subseteq (I \wedge J) \vee (I \wedge K).$$

Let $t \in I \wedge (J \wedge K)$. In view of (i), $t \in I$ and $t = j \vee k$, for some $j \in J$ and $k \in K$. Hence $t = j \vee k$, and $j \in I \wedge J$, $k \in I \wedge K$. Hence again in view of (i), $t \in (I \wedge J) \vee (I \wedge K)$. $\square$

Lemma 8.9 *Let* I *and* J *be ideals of a distributive lattice* L. *Then if* $I \wedge J$ *and* $I \vee J$ *are principal, then so are* I *and* J.

Proof. Let $I \wedge J = (x]$ and $I \vee J = (y]$. Then $y = i \vee j$ for some $i \in I$ and $j \in J$. Set $c = x \vee i$ and $b = x \vee j$, and note that $c \in I$ and $b \in J$. We claim that $I = (c]$ and $J = (b]$. Indeed, if for instance, $J \neq (b]$, then there exists an element $a\, b$, $a \in J$ such that $\{x, a, b, c, y\}$ is the pentagon.
$\square$

9 The Prime Ideal Theorem

A proper ideal I of a lattice L is called *prime* provided that whenever $a, b \in L$ and $a \wedge b \in I$, then $a \in I$ or $b \in I$. A prime filer is defined dually.

Proposition 9.1 *Let L be an arbitrary lattice, and let I be an ideal of L. The following statements are equivalent:*

(i) I is a prime ideal of L.

(ii) $L \setminus I$ is a filter of L.

(iii) $L \setminus I$ is a prime filter of L.

(iv) I is a meet-prime element of $I(L)$.

If L is distributive, then each of the above is equivalent to:

(v) I is a meet-irreducible element of $I(L)$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Exercises 4 and 14 of section 7.

(i) $\Leftrightarrow$ (v). Proposition 8.8 (ii), and the fact that a meet-irreducible;e element of a distributive lattice is meet-prime. $\square$

An ideal I of L is called *maximal* if it is a maximal element of $I(L) \setminus \{L\}$.

Proposition 9.2 *(i) If a lattice L has a largest element, then every proper ideal is contained in a maximal ideal.*

(ii) Every maximal ideal of a distributive lattice is prime.

The next theorem is one of the most important results in the theory of distributive lattices.

Theorem 9.3 ***(Prime Ideal Theorem)*** *Suppose L is a distributive lattice, I is an ideal of L, F is a filter of L, and $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Let $\mathcal{X}$ be the family of all ideals of L which contain I and are disjoint with F. It is easily verified that if $\mathcal{C}$ is a chain in $\mathcal{X}$, then $\bigcup \mathcal{C} \in \mathcal{X}$, and so by Zorn's lemma, $\mathcal{X}$ has a maximal element P. Then $I \subseteq P$ and $P \cap F = \emptyset$. Suppose that $a, b \in L$ such that $a \wedge b \in P$, but $a, b \notin P$. The maximality of P yields that $(P \vee (a]) \neq \emptyset$ and $(P \vee (b]) \neq \emptyset$. In view of proposition 8.8 (i), there are elements $p, q \in P$ such that $p \vee a \in F$ and $q \vee b \in F$. Then $x = (p \vee a) \wedge (q \vee b) \in F$, since F is a filter. But then

$$x = (p \wedge q) \vee (p \wedge b) \vee (a \wedge q) \vee (a \wedge b) \in F;$$

thus $P \cap F \neq \emptyset$, a contradiction. It follows that P is a prime ideal, and the proof of the theorem is complete. $\Box$

We proceed to discuss some applications of the prime ideal theorem. Note that the corollaries 9.4 - 9.6 can also be obtained with the use of theorem 7.9.

Corollary 9.4 *Let L be a distributive lattice, let I be an ideal of L, and let $a \in L$ such that $a \notin I$. Then there is a prime ideal P such that $I \subseteq P$ and $a \notin P$.*

Proof. Apply theorem 9.3 and $F = [a)$. $\Box$

Corollary 9.5 *In a distributive lattice L, if $a \not\leq b$, then there exists a prime ideal containing B and not a.*

Proof. Apply theorem 9.3 and $F = [a)$, $I = (b]$. $\Box$

Corollary 9.6 *Every ideal I of a distributive lattice is the intersection of all prime ideals containing it.*

Proof. Let $I_1 = \bigcap \{P | P \supseteq I,\ P \text{ is a prime ideal of } L\}$. If $I_1 \neq I$, then there is an element $a \in I_1 \setminus I$. It follows from corollary 9.4 that there is a prime ideal P, with $P \supseteq I$ and $a \notin P$. But then $a \notin P \supseteq I$, a contradiction. $\Box$

Theorem 9.7 *A lattice is distributive iff it is isomorphic to a set-ring.*

Proof. Let L be a distributive lattice and denote by X the set of all prime ideals of L. For each $a \in L$, set

$$\alpha(a) = \{P | a \notin P,\ P \in X\}.$$

It is easy to see that for all $a, b \in L$,

$$\alpha(a \vee b) = \alpha(a) \cup \alpha(b)$$

$$\alpha(a \wedge b) = \alpha(a) \cap \alpha(b),$$

and hence $\alpha(L) = \{\alpha(a) | a \in L\}$ is a set-ring. Further $a \leq b$ implies $\alpha(a) \subseteq \alpha(b)$ and by corollary 9.5, $\alpha(a) \subseteq \alpha(b)$ implies $a \leq b$. Thus the mapping $a \mapsto \alpha(a)$ is an isomorphism between L and $\alpha(L)$. $\square$

Theorem 9.8 *A lattice is Boolean iff it is isomorphic to a set-field.*

Proof. Use the representation of theorem 9.7. Obviously for each a, $\alpha(a') = X \setminus \alpha(a)$, and thus α preserves complements. $\square$

For a distributive lattice L with more than one element, let $P(L)$ denote the poset of all prime ideals of L with respect to set inclusion. The importance of $P(L)$ should be clear from the previous result. A topology on $P(L)$ will be discussed in the next section.

Some interesting properties of L are reflected in $P(L)$. An important result of this type is the following theorem.

Theorem 9.9 *Let L be a distributive lattice with $0 \neq 1$. Then L is a Boolean lattice iff $P(L)$ is an antichain.*

Proof. Suppose L is a Boolean lattice. Let P be a prime ideal of L and let I be an ideal of L properly containing P. Choose $a \in I \setminus P$. Since $a \notin P$ and $a \wedge a' = 0 \in P$, it follows that $a' \in P \subseteq I$. It follows that $1 = a \vee a' \in I$ and hence $I = L$. We have shown that if L is a Boolean lattice, then every prime ideal of L is maximal. In particular, $P(L)$ is an antichain.

Conversely, suppose L is not a Boolean lattice. Then L has a non-complemented element a. Set $F = \{x | a \vee x = 1\}$. Then F is a filter. Take $F_1 = F \vee [a) = \{x | x \geq f \wedge a,\ \text{for some } f \in F\}$. The filter F_1 does not contain 0 since

$0 = f \wedge a,\ 1 = f \vee a$ would mean that f is the complement of a. Thus by the prime ideal theorem there exists a prime ideal P disjoint from F_1. Note that $1 \notin (a] \vee P$, otherwise $1 = a \vee p$ for some $p \in P$, contradicting $P \cap F = \emptyset$. Thus some prime ideal Q contains $(a] \vee P$, and so $P \subseteq Q$. It follows that $P(L)$ is not an antichain.

$\square$

EXERCISES

A complete lattice L is said to satisfy the *join-infinite distributive law* (JID) if for every $x \in L$ and every family $(x_i | i \in I)$ of elements of elements of L,

$$x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \wedge x_i).$$

The dual of (JID) is the *meet-infinite distributive law* (MID).

A complete lattice L is called *completely meet-distributive* if it satisfies the infinite distributive law (CMD)

$$\bigwedge_{i \in I} \bigvee_{j \in J} x_{ij} = \bigvee_{f \in J^I} \bigwedge_{i \in I} x_{if(i)}.$$

for every doubly indexed family $(x_{ij} | i \in I,\ j \in J)$ of elements of L.

A *completely join-distributive* lattice is defined dually.

A closed sublattice of the Boolean lattice of all subsets of a set is known as a *complete set-ring*. Note that a complete set-ring is completely join- and meet-distributive.

1. Show that a distributive algebraic lattice satisfies (JID).
2. Show that a complete chain is completely join- and meet-distributive.
3. Show that a Boolean lattice is atomic iff its greatest element is the join of atoms. Equivalently, if it is dually atomic.
4. Show that for a complete Boolean lattice L TFAE:

   (i) L is completely meet-distributive.

   (ii) L is completely join-distributive.

   (iii) L is atomic.

   (iv) L is dually atomic.

   (v) L is isomorphic to the Boolean lattice of all subsets of a set.

Remark 9.10 It suffices to establish the equivalence if (i), (iii) and (v). For (i) $\Rightarrow$ (iii), note that $1 = \bigwedge_{x \in L}(x \vee x')$, and proceed from there to show that 1 is the join of atoms. Then use exercise 3.

Let L_1 and L_2 be complete lattices. A mapping $\phi : L_1 \to L_2$ is called a *complete homomorphism* if, for every family $(x_i | i \in I$ of elements of L_1,

$$\phi({}^{L_1}\bigvee x_i) = {}^{L_2}\bigvee \phi(x_i),$$

and

$$\phi({}^{L_1}\bigwedge x_i) = {}^{L_2}\bigwedge \phi(x_i).$$

If $\phi : L_1 \to L_2$ is a surjective complete homomorphism, we say that L_2 is a *complete homomorphic image* of L_1.

5. Let L be a complete lattice, and let $\mathcal{O}(L)$ be the complete lattice of all (non-empty) order ideals of L. Define $\phi : \mathcal{O}(L) \to L$ by $\phi(I) = \bigvee I$ for all $I \in \mathcal{O}(L)$. Show that TFAE:

   (i) L is completely meet- and join- distributive.

   (ii) L is completely meet-distributive.

   (iii) ϕ is a complete surjective homomorphism.

   (iv) L is a complete homomorphic image of a complete set-ring.

6. Use exercise 5 to show that a complete lattice is completely meet-distributive iff it is completely join-distributive.

   In view of exercise 6, a completely meet (join)-distributive lattice will be referred to as a completely distributive lattice.

7. Let L be a complete lattice. Show that TFAE:

   (i) L is isomorphic to $\mathcal{O}(P)$ for some poset P.

   (ii) L is isomorphic to a complete set-ring.

   (iii) L is distributive, and both L and its dual are algebraic lattices.

   (iv) L is completely distributive, and the dual of L is an algebraic lattice.

   (v) Every element of L is the join of completely join-prime elements.

(vi) If X denotes the set of all completely join-prime elements of L, then the mapping $L \to \mathcal{O}(X)$ given by $a \mapsto (a] \cap X$ is a lattice isomorphism.

8. Let L be a complemented modular lattice. Show the following:

   (i) If $a \leq x \leq b$ in L and y is a complement of x, then

   $$z = (a \vee y) \wedge b = a \vee (y \wedge b)$$

   is the complement of x in the interval $[a, b]$.

   (ii) L is *relatively complemented*, that is each interval $[a, b]$, $a < b$, in L is a complemented lattice.

   (iii) Let $a \leq x \leq b$ in L, and let z be a complement of x in $[a, b]$. Then there exists a complement y of x in L, so that

   $$z = (a \vee y) \wedge b = a \vee (y \wedge b).$$

   A complemented lattice L is called *uniquely complemented*, if every element of L has a unique complement. Clearly a Boolean lattice is a uniquely complemented lattice and a widely accepted conjecture in the thirties - supported by results such as those in exercises 9 and 10 below - was that the converse was also true. R. P. Dilworth showed however that this is not the case. He in fact showed, using free lattice techniques, that *every* lattice can be embedded into a uniquely complemented lattice. For a proof of this fact see the book by Crawley and Dilworth or that by Gratzer.

9. Show that a uniquely complemented modular lattice is a Boolean lattice. (Hint: Use exercise 8.)

10. Show that a uniquely complemented atomic lattice is an atomic Boolean lattice. (Hint: show first that two elements of the lattice are equal iff they exceed the same atoms.)

11. Show that a uniquely complemented, weakly atomic lattice is an atomic Boolean lattice. Conclude in particular that a uniquely complemented algebraic lattice is an atomic Boolean lattice. (Hint: Show that if $a \prec b$, then $a' \wedge b$ is an atom. Then use exercise 10.)

10 Congruence Relations - Subdirect Products

$E(A)$ will denote the set of all equivalence relations on a set A.

Lemma 10.1 *For any set A, $E(A)$ is an algebraic closure system on A^2, and hence an algebraic lattice. For $\emptyset \neq \mathcal{K} \subseteq E(A)$, $x \bigvee \mathcal{K} y$ iff $x = z_0 \epsilon_0 \cdots z_n \epsilon_n z_{n+1} = y$ for some elements $z_i \in A$ and relations $\epsilon_j \in \mathcal{K}$.*

Let $\mathbf{A} = (A, f)$ be an algebra of type ν. A *congruence relation* on $\mathbf{A}$ is an equivalence relation Θ on A satisfying the following substitution property: for each basic operation $f_i : A^{\nu_i} \to A$, if $a_k \Theta b_k$ for $0 \leq k < \nu_i$, then $f_i(a_0, \ldots, a_{\nu_i - 1}) \Theta f_i(b_0, \ldots, b_{\nu_i - 1})$.

Note that if f_i is nullary, then the substitution property is vacuously satisfied.

The set of all congruence relations on $\mathbf{A}$ will be denoted by $\mathrm{Con}(\mathbf{A})$.

For each mapping $\phi : A \to B$, the *kernel* of ϕ (denoted by $\mathrm{Ker}(\phi)$) is the relation on A defined by

$$(a, a') \in \mathrm{Ker}(\phi) \text{ iff } \phi(a) = \phi(a').$$

It is clear that $\mathrm{Ker}(\phi) \in E(A)$ for each mapping $\phi : A \to B$.

Proposition 10.2 *Let $\mathbf{A} = (A, f)$ be an algebra, B be an arbitrary set, and ϕ be a surjective mapping from A to B. Then there exists at most one algebra $\mathbf{B}$ with universe B so that ϕ becomes an epimorphism from $\mathbf{A}$ to $\mathbf{B}$. This is possible iff $\mathrm{Ker}(\phi) \in \mathrm{Con}(\mathbf{A})$.*

Corollary 10.3 *Given a congruence relation Θ on an algebra $\mathbf{A}$, there exists exactly one way to make the decomposition A/Θ to an algebra so that the canonical mapping from A to A/Θ becomes an epimorphism.*

This algebra is called the *the quotient of $\mathbf{A}$ mod Θ* and denoted by $\mathbf{A}/\Theta$.

Proposition 10.4 *(The general homomorphism theorem) Let $\phi : \mathbf{A} \to \mathbf{B}$ be an epimorphism and $\chi : \mathbf{A} \to \mathbf{C}$ be any homomorphism. There exists at most one homomorphism $\psi : \mathbf{B} \to \mathbf{C}$ such that $\psi\phi = \chi$. This homomorphism exists iff $\mathrm{Ker}(\phi) \subseteq \mathrm{Ker}(\chi)$. Furthermore, ψ is surjective iff χ is surjective, and χ is injective iff $\mathrm{Ker}(\phi) = \mathrm{Ker}(\chi)$.*

Corollary 10.5 *(The special homomorphism theorem) Let* $\phi : \mathbf{A} \to \mathbf{B}$ *be an epimorphism. Then there exists exactly one isomorphism* $\psi : \mathbf{A}/\mathrm{Ker}(\phi) \to \mathbf{B}$ *so that* $\psi\pi = \phi$ *where* $\pi : \mathbf{A} \to \mathbf{A}/\mathrm{Ker}(\phi)$ *is the canonical epimorphism. In short,*

$$\mathbf{B} \cong \mathbf{A}/\mathrm{Ker}(\phi).$$

Proposition 10.6 *For each algebra* $\mathbf{A} = (A, f)$, $\mathrm{Con}(\mathbf{A})$ *is a closed sublattice of* $E(A)$. *In particular,* $\mathrm{Con}(\mathbf{A})$ *is an algebraic closure system on* A^2, *and hence an algebraic lattice.*

For each $H \subseteq A^2$, $\Theta(H)$ will denote the smallest congruence relation on $\mathbf{A}$ containing H. If $H = \{(a, b)\}$, then $\Theta(\{(a, b)\})$ will be denoted by $\Theta(a, b)$ and will be called the *principal congruence relation* generated by (a, b). In view of theorem 7.2, each principal congruence is a compact member of $\mathrm{Con}(\mathbf{A})$, and the compact members of $\mathrm{Con}(\mathbf{A})$ are all finite joins of principal congruence relations.

Note that for each nonempty $H \subseteq A^2$,

$$\Theta(H) = \bigvee\{\Theta(a, b)|(a, b) \in H\}.$$

Proposition 10.7 *Let* $\mathbf{A}$ *be an algebra and* $a \neq b \in A$. *Then there exists* $\Theta \in \mathrm{Con}(\mathbf{A})$ *maximal with respect to not containing* (a, b). *Furthermore,* Θ *is a completely meet-irreducible element of* $\mathrm{Con}(\mathbf{A})$.

Proposition 10.8 *Let* $\phi : \mathbf{A} \to \mathbf{B}$ *be an epimorphism, and let* $\Theta = \mathrm{Ker}(\phi)$.

(i) If $\sigma \in \mathrm{Con}(\mathbf{A})$ *and* $\sigma \supseteq \Theta$, *then* $\phi(\sigma) = \{(\phi(x), \phi(y))|(x, y) \in \sigma\} \in \mathrm{Con}(\mathbf{B})$.

(ii) If $\tau \in \mathrm{Con}(\mathbf{B})$, *then* $\phi^{-1}(\tau) = \{(x, y) \in A^2|(\phi(x), \phi(y)) \in \tau\} \in \mathrm{Con}(\mathbf{A})$ *and* $\phi^{-1}(\tau) \supseteq \Theta$.

(iii) The mapping $\sigma \mapsto \phi(\sigma)$ *is a lattice isomorphism from the interval* $[\Theta)$ *of* $\mathrm{Con}(\mathbf{A})$ *to* $\mathrm{Con}(\mathbf{B})$. *The inverse is the mapping* $\tau \mapsto \phi^{-1}(\tau)$.

Let $\mathbf{A}_i = (A_i, f^{\mathbf{A}_i})$, $i \in I$ be algebras of type $\nu = (\nu_k)$, $k \in K$. The *direct product* of the algebras $\mathbf{A}_i$ is the algebra $\prod_{i\in I} \mathbf{A}_i = (\prod_{i\in I} A_i, f)$, where for each $k \in K$ and $(x_0, \ldots, x_{\nu_i-1}) \in (\prod A_k)^{\nu_k}$,

$$f_k(x_0, \ldots, x_{\nu_i-1})(i) = f_k^{\mathbf{A}_i}(x_0(i), \ldots, x_{\nu_i-1}(i)),$$

for each $i \in I$.

For each $i \in I$, the mapping $\pi_i : \prod A_i \to A_i$ defined by

$$\pi_i(x) = x(i),\ x \in \prod A_i,$$

is an epimorphism from $\prod \mathbf{A}_i$ to $\mathbf{A}_i$. We refer to this epimorphism as the ith *projection* of $\prod \mathbf{A}_i$.

In the case $I = \emptyset$, the direct product is a one-element algebra of type ν, and clearly it has no projections.

By a *subdirect product* of a family $(\mathbf{A}_i | i \in I)$ of algebras we mean an subalgebra $\mathbf{B}$ of $\prod \mathbf{A}_i$ with the property that $\pi_i(B) = A_i$ for each $i \in I$.

An algebra $\mathbf{B}$ is said to be *subdirectly irreducible* if (i) $|B| > 1$, and (ii) if ϕ is an isomorphism from $\mathbf{B}$ onto a subdirect product of $(\mathbf{A}_i | i \in I$, then $\pi_i\phi$ is an isomorphism for some $i \in I$.

Lemma 10.9 *If* $\mathbf{B}$ *is a subdirect product of* $(\mathbf{A}_i | i \in I$ *and if* $\Theta_i = \mathrm{Ker}(\pi_i|_{\mathbf{B}})$, *then in* $\mathrm{Con}(\mathbf{B})$,

$$0 = \bigwedge_{i \in I} \Theta_i.$$

Conversely, if $(\Theta_i | i \in I)$ *is a family of congruence relations of an algebra* $\mathbf{B}$ *such that* $0 = \bigvee i \in I$, *then there is an isomorphism* ϕ *of* $\mathbf{B}$ *onto a subdirect product of* $(\mathbf{B}/\Theta_i | i \in I)$ *so that, for each* $i \in I$, $\pi_i\phi$ *is the canonical mapping from* $\mathbf{B}$ *to* $\mathbf{B}/\Theta_i$.

Proof. We only need to sketch the proof for the second part. Define $\phi : B \to \prod_{i \in I} B/\Theta_i$ by

$$\phi(b)(i) = [b]_{\Theta_i},\ \forall i \in I.$$

Here $[b]_{\Theta_i}$ denotes the Θ_i class of b in B/Θ_i. It is easy to see that the mapping ϕ has the required properties. □

Lemma 10.10 *Let* $\mathbf{B}$ *be an algebra with* $|B| > 1$. *TFAE:*

(i) $\mathbf{B}$ *is subdirectly irreducible.*

(ii) The least element of $\mathrm{Con}(\mathbf{B})$ *is completely meet-irreducible.*

(iii) $\mathrm{Con}(\mathbf{B})$ *is an atomic lattice having a single atom.*

(iv) There exist two distinct elements $a, b \in \mathbf{B}$ *such that* $(a, b) \in \Theta$ *for every non-trivial congruence relation* Θ *on* $\mathbf{B}$.

Theorem 10.11 *Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

Proof. Let $\mathbf{B}$ be an algebra. If $|B| = 1$, then $\mathbf{B}$ is the subdirect product of an empty family of subdirectly irreducible algebras. So suppose $|B| > 1$. Since $\mathrm{Con}(\mathbf{B})$ is an algebraic lattice, by theorem 7.9, there exists a family $(\Theta_i | i \in I)$ of completely meet-irreducible congruence relations on $\mathbf{B}$ such that $0 = \bigwedge \Theta_i$ in $\mathrm{Con}(\mathbf{B})$. In view of lemma **??**, $\mathbf{B}$ is isomorphic to a subdirect product of $(\mathbf{B}_i/\Theta_i | i \in I)$. Furthermore, in view of proposition 10.8 (iii) and lemma 10.10(ii), each $\mathbf{B}/\Theta_i$ is subdirectly irreducible. $\square$

Let $\mathcal{K}$ be a class of algebras of type ν. Define:

$I\mathcal{K}$ = class of all algebras of type ν that are isomorphic to some algebra in $\mathcal{K}$.

$H\mathcal{K}$ = class of all algebras of type ν that are isomorphic to a subalgebra of some algebra in $\mathcal{K}$.

$S\mathcal{K}$ = class of all algebras of type ν that are isomorphic to a direct product of algebras in $\mathcal{K}$.

$\mathcal{K}$ is called *abstract* if $I\mathcal{K} = \mathcal{K}$. Note that $H\mathcal{K}, S\mathcal{K}, P\mathcal{K}$ are abstract classes, no matter what $\mathcal{K}$ might be.

Note also that the operators I, S, P and H are closure operators. That is, if C is one of H, I, S or P, then for any classes $\mathcal{K}$ and $\mathcal{L}$ of algebras of type ν,

(i) $\mathcal{K} \subseteq \mathcal{L}$ implies $C\mathcal{K} \subseteq C\mathcal{L}$.

(ii) $\mathcal{K} \subseteq C\mathcal{K}$.

(iii) $CC\mathcal{K} = C\mathcal{K}$.

Proposition 10.12 *Let $\mathcal{K}$ be any class of algebras of type ν. Then*

(i) $SH\mathcal{K} \subseteq HS\mathcal{K}$.

(ii) $PS\mathcal{K} \subseteq SP\mathcal{K}$.

(iii) $PH\mathcal{K} \subseteq HP\mathcal{K}$.

A class $\mathcal{K}$ of algebras is called a *variety* if $H\mathcal{K} \subseteq \mathcal{K}$, $S\mathcal{K} \subseteq \mathcal{K}$, and $P\mathcal{K} \subseteq \mathcal{K}$.

Corollary 10.13 *For every class $\mathcal{K}$ of algebras of the same type ν, $HSP\mathcal{K}$ is the smallest variety containing $\mathcal{K}$.*

The following result is an immediate consequence of theorem 10.11.

Corollary 10.14 *Let $\mathcal{K}$ be a variety. An algebra $\mathbf{A}$ is a subdirect product of subdirectly irreducible algebras in $\mathcal{K}$ iff $\mathbf{A} \in \mathcal{K}$.*

EXERCISES

1. Let L be an arbitrary lattice and $\Theta \in \mathrm{Con}(L)$. Show that for any $a, b \in L$, $(a, b) \in \Theta$ iff $(a \wedge b, a \vee b) \in \Theta$ iff $(x, y) \in \Theta$, for all $x, y \in [a \wedge b, a \vee b]$. Deduce in particular that every Θ-class is a convex sublattice of L.

2. Let L be an arbitrary lattice.

   (i) Show that if $\Theta, \Phi \in \mathrm{Con}(L)$, the join $\Theta \vee \Phi$ can be described as follows: $(a, b) \in \Theta \vee \Phi$ iff there exist elements $x_0 = a \wedge b, x_1, \ldots, x_n = a \vee b$ in L such that for each $0 < k < n$, $a \wedge b \leq x_{k+1} \leq a \vee b$ and $(x_k, x_{k+1}) \in \Theta$ or $(x_k, x_{k+1}) \in \Phi$.

   (ii) Show that $\mathrm{Con}(L)$ is a distributive lattice.

   (iii) Combine (ii) with exercise 1 (section 9), to conclude that $\mathrm{Con}(L)$ satisfies the join-infinite distributive law.

Given two intervals $[a, b]$ and $[c, d]$ in a lattice L, we say that $[a, b]$ is a *lower transpose* of $[c, d]$ and $[c, d]$ is an *upper transpose* of $[a, b]$ if $b \wedge c = a$ and $b \vee c = d$. We describe the situation symbolically by $[a, b] \nearrow [c, d]$ or $[c, d] \searrow [a, b]$. For simplicity, $[a, b]$ will be called a *transpose* of $[c, d]$ if $[a, b] \nearrow [c, d]$ or $[a, b] \searrow [c, d]$.

Note that $[a_0, b_0] \nearrow [a_1, b_1] \nearrow [a_2, b_2]$ implies $[a_0, b_0] \nearrow [a_2, b_2]$ and the same is true for lower transposes.

Two intervals $[a, b]$ and $[c, d]$ in a lattice L are said to be *projective* if there exist finitely many intervals $[c, d] = [x_0, y_0], [x_1, y_1], \ldots, [x_n, y_n] = [a, b]$ such that for each $k < n$, $[x_k, y_k]$ is a transpose of $[x_{k+1}, y_{k+1}]$.

Note that if $[a, b]$ and $[c, d]$ are projective, then $\Theta(a, b) = \Theta(c, d)$, but the converse is easily seen to be false. This leads to the following generalization of the notion of projectivity.

The interval $[c, d]$ is said to be *weakly projective* into the interval $[a, b]$ if there exist finitely many intervals $[c, d] = [x_0, y_0], [x_1, y_1], \ldots, [x_n, y_n] = [a, b]$ such that for each $k < n$, $[x_k, y_k]$ is a transpose of a subinterval of $[x_{k+1}, y_{k+1}]$.

3. Let L be a lattice, $\Theta \in \mathrm{Con}(L)$, and $a, b, c, d \in L$. Prove that if $(a, b) \in \Theta$ and $[c \wedge d, c \vee d]$ is weakly projective into $[c \wedge d, c \vee d]$, then $(c, d) \in \Theta$.

4. Show that a nonempty binary relation Θ on a lattice L is a congruence relation iff it satisfies the following conditions for all $a, b, c \in L$:

   (i) $(a, b) \in \Theta$ iff $(a \wedge b, a \vee b) \in \Theta$.

   (ii) $a \leq b \leq c$, $(a, b), (b, c) \in \Theta$ imply $(a, c) \in \Theta$.

   (iii) $a \leq b$ and $(a, b) \in \Theta$ imply $(a \vee c, b \vee c), (a \wedge c, b \wedge c) \in \Theta$.

5. Let L be a lattice, $H \subseteq L^2$, and $c, d \in L$. Show that $(c, d) \in \Theta(H)$ iff there exist elements $c \wedge d = x_0 \leq x_1 \leq \cdots \leq x_n = c \vee d$, and $\{(a_k.a_{k+1}) | k < n\} \subseteq H$ such that for each $k < n$, the interval $[x_k, x_{k+1}]$ is weakly projective into $[a_k \wedge a_{k+1}, a_k \vee a_{k+1}]$. Hint: Use exercises 3 and 4.

   Weak projectivity can be simplified in the case when L is a modular lattice.

6. Let L be a modular lattice. Show the following:

   (i) If $[a, b]$ is a transpose of $[c, d]$, then every subinterval of $[a, b]$ is a transpose of a subinterval of $[c, d]$.

   (ii) If $[c, d]$ is weakly projective into $[a, b]$, then $[c, d]$ is and a subinterval of $[a, b]$ are projective.

   Combining exercises 5 and 6 we get:

7. Let L be a modular lattice and $H \subseteq L^2$. Then $(c, d) \in \Theta(H)$ iff there exist elements $c \wedge d = x_0 \leq x_1 \leq \cdots \leq x_n = c \vee d$, and $\{(a_k.a_{k+1}) | k < n\} \subseteq H$ such that for each $k < n$, the interval $[x_k, x_{k+1}]$ is projective to a subinterval of $[a_k \wedge a_{k+1}, a_k \vee a_{k+1}]$.

   Let us call a nontrivial interval $[a, b]$ of a lattice L *primitive* if for each nontrivial interval $[c, d]$ weakly projective into $[a, b]$, there exist elements $a < x_0 < x_1 < \cdots < x_n = b$ such that each interval $[x_k, x_{k+1}]$, $k < n$, is weakly projective into $[a, b]$.

8. Show that a congruence relation Θ in a lattice L is an atom of $\mathrm{Con}(L)$ iff $\Theta = \Theta(a, b)$ for some primitive interval $[a, b]$. Hint: Use exercise 5.

9. Let L be a modular lattice and $a \prec b, c \prec d$ in L. Show the following:

   (i) $\Theta(a,b) = \Theta(c,d)$ iff $[a,b]$ and $[c,d]$ are projective.

   (ii) $\Theta(a,b)$ is an atom of $\mathrm{Con}(L)$.

   The next exercise simplifies projectivity in distributive lattices.

10. For any intervals $[a,b]$, $[c,d]$ in a distributive lattice L, TFAE:

    (i) $[a,b]$ and $[c,d]$ are projective.

    (ii) $[a,b] \nearrow [x,y] \searrow [c,d]$ for some interval $[x,y]$.

    (iii) $[a,b] \searrow [x,y] \nearrow [c,d]$ for some interval $[x,y]$.

    Hint: Show first that $[a_0,b_0] \nearrow [a_1,b_1] \searrow [a_2,b_2]$, then $[a_0,b_0] \searrow [a_0 \wedge a_2, b_0 \wedge b_2] \nearrow [a_2,b_2]$.

11. If $a < b \leq c < d$ in a distributive lattice, prove that

    (i) $[a,b]$ and $[c,d]$ are not projective.

    (ii) $\Theta(a,b) \wedge \Theta(c,d) = 0$.

    Hint: Show that $[a,b]$ and $[c,d]$ cannot have a common transpose. Then apply exercise 10.

12. If L is a weakly atomic modular lattice, show $\mathrm{Con}(L)$ is atomic.

13. Show that a distributive lattice is weakly atomic iff $\mathrm{Con}(L)$ is atomic.

11 A Topological Representation of Distributive Lattices

In this section we show that every distributive lattice L with 0 can be associated with a certain space, called the *representation space* (or *spectral space*) of L. We show that the representation space of L determines L up to an isomorphism, and we characterize the topological spaces that arise as representation spaces of distributive lattices with 0.

Until further L *will denote a distributive lattice with 0*. The poset of prime ideals of L will be denoted by $\mathcal{S}(L)$.

The complete lattice of open subsets of a topological space X will be denoted by
$mathcalO_X$, and that of closed subsets of X by $\mathcal{C}_X$. Further, we write $\mathcal{A}(X)$ for the poset of compact open subsets of X, and we denote the topological closure of a subset $Y \subseteq X$ by $\overline{Y}$.

Let $\Theta \subseteq L \times \mathcal{S}(L)$ be defined by

$$(a, P) \in \Theta \Leftrightarrow a \in P.$$

Θ induces a Galois connection

$$h : \mathcal{P}(L) \to \mathcal{P}(\mathcal{S}(L)),\ k : \mathcal{P}(\mathcal{S}(L)) \to \mathcal{P}(L)$$

between lattices $\mathcal{P}(L)$ and $\mathcal{P}(\mathcal{S}(L))$ of all subsets of L and $\mathcal{S}(L)$ respectively (see exercise 6, section 8).

Note that for all $X \subseteq L$, $Y \subseteq \mathcal{S}(L)$,

$$h(X) = \{P \in \mathcal{S}(L) | P \supseteq X\}, \tag{11.1}$$

and

$$k(Y) = \bigcap \{P | P \in Y\}. \tag{11.2}$$

$h(X)$ is usually referred to as the *hull* of X, and $k(Y)$ as the *kernel* of Y.

For each $X \subseteq L$, let

$$\sigma(X) = \mathcal{S}(L) \setminus h(X) = \{P \in \mathcal{S}(L) | P \not\supseteq X\}. \tag{11.3}$$

The next lemma is an immediate consequence of corollary 9.6.

Lemma 11.1 *The following statements hold for the ideal $(X]$ generated by a subset X of L.*

(i) $h(X) = h((X])$.

(ii) $kh(X) = (X]$.

(iii) $kh(X) = X$ *iff* $X \in I(L)$.

(iv) $\sigma(X) = \sigma((X])$.

Lemma 11.2 *Let $I, J \in I(L)$, I_t $(t \in T)$ subsets of L, and $a, b \in L$.*

(i) $h(\bigcup I_t) = \bigcap h(I_t)$.

(ii) If each I_t is an ideal of L, then $h(\bigvee I_t) = \bigcap(I_t)$.

(iii) $h(I \cap J) = h(I) \cup h(J)$.

(iv) $h(a \wedge b) = h(a) \cup h(b)$.

(v) $h(a \vee b) = h(a) \wedge h(b)$.

(vi) $\sigma(\bigcup I_t) = \bigcup \sigma(I_t)$.

(vii) If each I_t is an ideal of L, then $\sigma(\bigvee I_t) = \bigcup \sigma(I_t)$.

(viii) $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$.

(ix) $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b)$.

(x) $\sigma(a \vee b) = \sigma(a) \cup \sigma(b)$.

We introduce a topology on $\mathcal{S}(L)$ by choosing all sets of the form $\sigma(a)$, $a \in A$, as a subbase for the topology. In view of conditions (ix) and (x) of the preceding lemma, these sets form actually a base for the topology.

$\mathcal{S}(L)$ endowed with this topology is called the *representation space* of L. Note incidentally that $|L| = 1$ iff $\mathcal{S}(L) = \emptyset$.

Theorem 11.3 *(i)* $\mathcal{C}_{\mathcal{S}(L)} = \{h(I) | I \in I(L)\}$.

(ii) The mapping $I \mapsto h(I)$ is a dual lattice isomorphism from $I(L)$ to $\mathcal{C}_{\mathcal{S}(L)}$, with inverse $Y \mapsto k(Y)$.

(iii) $\mathcal{O}_{\mathcal{S}(L)} = \{\sigma(I) | I \in I(L)\}$.

(iv) The mapping $I \mapsto \sigma(I)$ is a lattice isomorphism from $I(L)$ to $\mathcal{O}_{\mathcal{S}(L)}$ with inverse

$$Y \mapsto k(\mathcal{S}(L) \setminus Y).$$

(v) $\mathcal{A}_{\mathcal{S}(L)} = \{\sigma(a) | a \in L\}$, and the mapping $a \mapsto \sigma(a)$ is a lattice isomorphism from L to $\mathcal{A}_{\mathcal{S}(L)}$. In particular, $\mathcal{S}(L)$ determines L up to isomorphism.

Proof.(iii) In view of lemma 11.2, each $\sigma(I)$, $I\imath I(L)$, is open and sets of this type form the smallest collection of open sets containing $\{\sigma(a)|a \in L\}$. It follows that $\mathcal{O}_{\mathcal{S}(L)} = \{\sigma(I)|I \in I(L)\}$.

(i) It is an immediate consequence of (iii)

(ii) In view of (i) and lemma 11.1 (iii), $I \mapsto h(I)$ is bijective and its inverse is $Y \mapsto k(Y)$. Since each of these mappings is a dual order homomorphism, the result follows.

(iv) Immediate consequence of (ii) and (iii).

(v) In view of corollary **??**, $I(L)$ is an algebraic lattice and its compact elements are the principal ideals of L. Hence by (iv), $\mathcal{O}_{\mathcal{S}(L)}$ is an algebraic lattice and its compact elements are the sets $\sigma(a) = \sigma((a])$, $a \in L$. That is $\mathcal{A}_{\mathcal{S}(L)} = \{\sigma(a)|a \in L\}$. $\square$

Our next objective is to characterize the representation spaces of distributive lattices with 0. We start with a lemma:

Lemma 11.4 *(i) For each $Y \subseteq \mathcal{S}(L)$,*

$$\overline{Y} = hk(Y).$$

Recall that $\overline{Y}$ denotes the topological closure of Y.

(ii) For each $P \in \mathcal{S}(L)$,

$$\overline{\{P\}} = h(P).$$

(iii) $\mathcal{S}(L)$ is a T_0-space.

Proof. (i) Let $Y \subseteq \mathcal{S}(L)$. In view of theorem 11.3 (i), $hk(Y)$ is closed, since $k(Y) \in I(L)$. Suppose Z is a closed subset of $\mathcal{S}(L)$ such that $Y \subseteq Z$. Then $hk(Y) subseteq hk(Z) = Z$, by theorem 11.3 (ii).

(ii) Note that $k(\{P\}) = P$, and use (i).

(iii) If $P, Q \in \mathcal{S}(L)$ and $P \neq Q$, then in view of (ii), $\overline{\{P\}} \neq \overline{\{S\}}$. Thus $\mathcal{S}(L)$ is a T_0 space. □

In view of the preceding lemma, the topological closure of each subset Y of $\mathcal{S}(L)$ is the hull of the kernel of Y. For this reason the topology is usually called the *hull-kernel topology* on $\mathcal{S}(L)$. Obviously we could have defined the topology on $\mathcal{S}(L)$ by using the Kuratowski operator hk.

In view of the isomorphism theorem 11.3, $\mathcal{U} \in \mathcal{O}_{\mathcal{S}(L)}$ is meet-irreducible in $\mathcal{O}_{\mathcal{S}(L)}$ iff $\mathcal{U} = \sigma(P)$ for some $P \in \mathcal{S}(L)$, and $C \in \mathcal{C}_{\mathcal{S}(L)}$ is join-irreducible in $\mathcal{C}_{\mathcal{S}(L)}$ iff $c = h(P)$ for some $P \in \mathcal{S}(L)$.

Combining these remarks with lemma 11.4 we obtain:

Corollary 11.5 *(i) $C \in \mathcal{C}_{\mathcal{S}(L)}$ is join-irreducible in $\mathcal{C}_{\mathcal{S}(L)}$ iff there exists (a necessarily unique) $P \in \mathcal{S}(L)$ such that $C = \overline{\{O\}}$. Furthermore, if this is the case, then*

$$P = k(\mathcal{S}(L)\ \mathcal{U}) = \bigcap\{P | P \in \mathcal{S}(L) \setminus \mathcal{U}\}.$$

A topological space X is called a *Stone space* if it satisfies the following conditions.

(S_1) X is a T_0 space.

(S_2) The family $\mathcal{A}(X)$ of compact open subsets of X is a set-ring and a base for X.

(S_3) If C is a join-irreducible in $\mathcal{C}_X$, $C = \overline{\{x\}}$, for some $x \in X$.

It can be shown without much difficulty that a topological space X is a Stone space iff it satisfies (S_1), (S_2) and

(s_3') If $(\mathcal{U}_i)$ and $(\mathcal{V}_j)$ $(i \in I,\ j \in J)$ are nonempty families of compact open sets and $\bigcap \mathcal{U}_i \subseteq \bigcup \mathcal{V}_j$, then there exist finite non-empty subsets $I' \subseteq I$ and $J' \subseteq J$ such that

$$\bigcap_{i \in I'} \mathcal{U}_i \subseteq \bigcup_{j \in J'} \mathcal{V}_j.$$

By lemma 11.4 and corollary **??**, $\mathcal{S}(L)$ is a Stone space. The next theorem shows that the Stone spaces are precisely the representation spaces of distributive lattices with 0.

Theorem 11.6 *(i) For each distributive lattice L with 0, $a \mapsto \sigma(a)$ is an isomorphism from L to $\mathcal{A}(\mathcal{S}(L))$.*

(ii) Foe each Stone space X,

$$x \mapsto \rho(x) = \{\mathcal{U} \in \mathcal{A}(X) | x \notin mathcalU\}.$$

is a homeomorphism from X to $\mathcal{S}(\mathcal{A}(X))$.
There is thus, up to isomorphisms and homeomorphisms, a bijective correspondence between distributive lattices with 0 and Stone spaces.

Proof. (i) Theorem 11.3 (v).

(ii) Let X be a Stone space. We may clearly assume that $X \neq \emptyset$.

(a) $\emptyset \neq \rho(x) \neq \mathcal{A}(X)$, for all $x \in X$. Indeed, $\emptyset \in \rho(x)$, for each $x \in X$, and hence $\rho(x) \neq \emptyset$. Further, since $\mathcal{A}(X)$ is a base for X by (S_2), for each $x \in X$ there is $\mathcal{U} \in \mathcal{A}(X)$ such that $x \in \mathcal{U}$. Hence $\mathcal{U} \notin \rho(x)$ and $\rho(x) \neq \mathcal{A}(X)$.

(b) $\rho(x)$ is a prime ideal of $\mathcal{A}(X)$, for all $x \in X$. Indeed, (a) establishes that $\rho(x)$ is a nonempty proper subset of $\mathcal{A}(X)$. The rest is a simple verification. In view of (b), ρ is indeed a mapping from X to $\mathcal{S}(\mathcal{A}(X))$.

(c) ρ is injective. This is clear by (S_1). Indeed, if $x \neq y$, then there exists $\mathcal{U} \in \mathcal{A}(X)$ containing only one of X and y. Hence $\rho(x) \neq \rho(y)$.

(d) ρ is surjective. Let P be a prime ideal of $\mathcal{A}(X)$. We first claim that

$$\mathcal{U} \in P \text{ iff} \mathcal{U} \subseteq \bigcup P. \tag{11.4}$$

Clearly if $\mathcal{U} \in P$ then $\mathcal{U} \subseteq \bigcup P$. Suppose now that $\mathcal{U} \subseteq \bigcup P = \bigcup\{\mathcal{V} | \mathcal{V} \in P\}$. Since $\mathcal{U}$ is compact, there exist $\mathcal{V}_1, \ldots, \mathcal{V}_m \in P$ such that $\mathcal{U} \subseteq \bigcup \mathcal{V}_i \in P$. Thus $\mathcal{U} \subseteq P$. Using 11.4 we can easily infer that $\bigcup P$ is meet-irreducible in $\mathcal{O}_X$, and hence $X \setminus \bigcup P$ is join-irreducible in $\mathcal{C}_X$. By (S_3), there is $x \in X$ such that

$$X \setminus \bigcup P = \overline{\{x\}}. \tag{11.5}$$

Now in view of 11.5, $x \notin \mathcal{U}$, for each $\mathcal{U} \in P$. On the other hand, if $\mathcal{U} \in \mathcal{A}(X) \setminus P$, then $\mathcal{U} \cap (X \setminus \bigcup P) = \mathcal{U} \cap \overline{\{x\}} \neq \emptyset$. It follows that $x \in \mathcal{U}$. It is now clear by the definition of the mapping ρ that $\rho(x) = P$. We have shown that ρ is surjective.

(e) ρ and ρ^{-1} are continuous. Let $\emptyset \neq \mathcal{U} \in \mathcal{A}(X)$. Then $x \in \rho^{-1}(\sigma(\mathcal{U}))$ iff $\rho(x) \in \sigma(\mathcal{U})$ iff $\mathcal{U} \notin \rho(x)$ iff $x \in \mathcal{U}$. Thus $\rho^{-1}(\sigma(\mathcal{U})) = \mathcal{U}$ and $\sigma(\mathcal{U}) = \rho(\mathcal{U})$. As every basic open set in $\mathcal{S}(\mathcal{F}(L))$ is of the form $\sigma(\mathcal{U}), \mathcal{U} \in \mathcal{S}(S)$, it follows that both ρ and ρ^{-1} are continuous. The proof of the theorem is now complete. $\Box$

Many important properties of distributive lattices manifest themselves clearly in their representation spaces.

Proposition 11.7 *Let L be a distributive lattice with0.*

(i) L has a greatest element iff $\mathcal{S}(L)$ is compact.

(ii) L is a complete lattice iff the interior of the intersection of a family of compact open sets in $\mathcal{S}(L)$ is a compact open set.

12 Boolean Duality

A topological space X is called a *Boolean space* if it is a compact T_2 space in which the clopen sets form a base of open sets.

In this section we shall show that the category of Boolean algebras and Boolean homomorphisms is "dually equivalent" to the category of Boolean spaces and continuous maps.

We start with a lemma.

Lemma 12.1 *For a topological space X TFAE:*

(i) X is a Boolean space.

(ii) X is the representation space of some Boolean lattice.

Proof. (i) $\Rightarrow$ (ii). Suppose X us a Boolean space. Recall that a subset of a compact T_2-space is compact iff it is closed. Hence in such a space, a subset is compact open iff it is clopen. It follows that $\mathcal{A}(X)$ is the Boolean lattice of clopen subsets of X. Hence in view of theorem 11.6, the proof of the implication will be completed iff we show that X is a Stone space. Evidently X satisfies (S_1) and in view of the preceding remarks it also satisfies (S_2). We proceed to show that is satisfies (S_3). To this end, let C be join-irreducible in $\mathcal{C}_X$. By definition, $C \neq \emptyset$. Let D be a closed subset of X properly contained in C. There is a clopen subset $\mathcal{U}$ such that $D \subseteq \mathcal{U}$ and $C \nsubseteq \mathcal{U}$. Now $C \subseteq \mathcal{U} \cup (X \setminus \mathcal{U}) = X$. Since X is join-irreducible and $C \nsubseteq \mathcal{U}$, it follows that $C \subseteq X \setminus \mathcal{U}$¿ But then $D \subseteq \mathcal{U} \cap (X \setminus \mathcal{U}) = \emptyset$. It is now clear that if $x \in C$, then $C = \overline{\{x\}}$, and hence (S_3) is satisfied.

(ii) $\Rightarrow$ (i). Straightforward by theorem 11.6 and by proposition **??** (i). $\square$

Lemma 12.2 *Let A, B be Boolean lattices and $\phi : A \to B$ be a Boolean homomorphism. Then $\mathcal{S}(\phi) : \mathcal{S}(B) \to \mathcal{S}(A)$ defined by*

$$\mathcal{S}(\phi)(P) = \phi^{-1}(P)$$

(P a prime ideal of B) is continuous. Conversely, if $\psi : \mathcal{S}(B) \to \mathcal{S}(A)$ is continuous, then there is a unique Boolean homomorphism $\phi : A \to B$ such that $\mathcal{S}(\phi) = \psi$.

Proof. Let $\mathcal{U}$ be a clopen subset of $\mathcal{S}(A)$. There is $a \in A$ such that $\mathcal{U} = \sigma(a)$. Now,

$$\begin{aligned}(\mathcal{S}(\phi))^{-1}(\mathcal{U}) &= \{P \in \mathcal{S}(B) | \mathcal{S}(\phi)(P) \in \mathcal{U}\} \\ &= \{P \in \mathcal{S}(B) | \phi^{-1}(P) \in \mathcal{O}(a)\} \\ &= \{P \in \mathcal{S}(B) | a \notin \phi^{-1}(P)\} \\ &= \{P \in \mathcal{S}(B) | \phi(a) \notin P\} \\ &= \sigma(\phi(a)) \in \mathcal{A}(\mathcal{S}(A))\end{aligned}$$

Thus $\mathcal{S}(\phi)$ is continuous.

Suppose now that $\psi : \mathcal{S}(B) \to \mathcal{S}(A)$ is a continuous mapping. Note that if $\mathcal{U} \in \mathcal{A}(\mathcal{S}(A))$, then $\phi^{-1}(\mathcal{U}) \in \mathcal{A}(\mathcal{S}(B))$. Thus if $\mathcal{U} = \sigma(a)$, $a \in A$, then $\phi^{-1}(\mathcal{U}) = \phi : A \to B$, $a \mapsto$ (**NOTE: SOMETHING IS MISSING HERE**) is a Boolean homomorphism and $\psi = \mathcal{S}(\phi)$. □

Lemma 12.3 *Let X, Y be Boolean spaces and let $\phi : X \to Y$ be a continuous mapping. Then $\mathcal{A}(\phi) : \mathcal{A}(Y) \to \mathcal{A}(X)$ defined by $\mathcal{U} \mapsto \phi^{-1}(\mathcal{U})$ is a Boolean homomorphism. Conversely, if $\psi : \mathcal{A}(Y) \to \mathcal{A}(X)$ is a Boolean homomorphism, then there exists a unique continuous mapping $\phi : X \to Y$ such that $\mathcal{A}(\phi) = \psi$.*

The category of Boolean algebras and Boolean homomorphisms will be denoted by $\mathcal{BA}$. The objects of $\mathcal{BA}$ are all Boolean algebras and the morphisms are the Boolean homomorphisms. The category of Boolean spaces and continuous mappings will be denoted by $\mathcal{BS}$.

Proposition 12.4 *(i) $\mathcal{S} : \mathcal{BA} \to \mathcal{BS}$ is a contravariant functor. That is,*

- *(a) $\mathcal{S}(id_A) = id_{\mathcal{S}(A)}$, for each Boolean algebra A.*
- *(b) If $\phi : A \to B$ and $\psi : B \to C$ are Boolean homomorphisms, then $\mathcal{S}(\psi\phi) = \mathcal{S}(\phi)\mathcal{S}(\psi)$.*

(ii) $\mathcal{A} : \mathcal{BS} \to \mathcal{B}(A)$ is a covariant functor.

(iii) $\mathcal{AS} : \mathcal{BS} \to \mathcal{BS}$ is a covariant functor.

Let now A be a Boolean algebra. In view of theorem 11.3, the mapping

$$\sigma_A : A \to \mathcal{A}(\mathcal{S}(A)),$$

defined by

$$\sigma_A(a) = \sigma(a) = \{P \in \mathcal{S}(A) | a \notin P\}$$

is a Boolean homomorphism.

Again in view of theorem 11.6, if X is a Boolean space, then the mapping

$$\rho_X : X \to \mathcal{S}(\mathcal{A}(X))$$

defined by

$$\rho_A(x) = \rho(x) = \{\mathcal{U} \in \mathcal{A}(X) | x \notin \mathcal{U}\}$$

is a homomorphism.

We denote by $I_{\mathcal{BA}}$ the identity functor of $\mathcal{BA}$ and $I_{\mathcal{BS}}$ the identity functor of $\mathcal{BS}$.

Theorem 12.5 *The categories $\mathcal{BS}$ and $\mathcal{BA}$ are dually equivalent. More specifically,*

(i) $\mathcal{S} : \mathcal{BA} \to \mathcal{BS}$ is a contravariant functor.

(ii) $\mathcal{A} : \mathcal{BS} \to \mathcal{BA}$ is a contravariant functor.

(iii) $(\sigma_A)_{A \in \mathcal{BA}} : I_{\mathcal{BA}} \to \mathcal{AS}$ is a natural equivalence (isomorphism). That is, for all $A \in \mathcal{BA}$, $\sigma_A : A \to \mathcal{AS}(A)$ is an isomorphism, and for all $A, B \in \mathcal{BA}$ and every Boolean homomorphism $\phi : A \to B$, the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[r, "\sigma_A"] \arrow[d, "\phi"'] & \mathcal{A}(\mathcal{S}(A)) \arrow[d, "\mathcal{A}(\mathcal{S}(\phi))"] \\
B \arrow[r, "\sigma_B"] & \mathcal{A}(\mathcal{S}(B))
\end{tikzcd}$$

(iv) $(P_X)_{X \in \mathcal{BS}} : I_{\mathcal{BS}} \to \mathcal{SS}$ is a natural equivalence.

13 Free Algebras

Let $\mathcal{K}$ be a class of algebras of the same similarity type ν, and let X be an arbitrary set. We say that algebra $\mathbf{A} \in \mathcal{K}$ is a *$\mathcal{K}$-free algebra over X*, if

(i) $Sg(X) = A$.

(ii) Foe every algebra $\mathbf{B} \in \mathcal{K}$ and every mapping $\phi : X \to B$, there exists a homomorphism $\overline{\phi} : \mathbf{A} \to \mathbf{B}$ extending ϕ.

In the preceding definition, the set X is called a *free generating set* and is said to *freely generate* $\mathbf{A}$.

We remark that, in view of exercise 7 on page (**INSERT PAGE HERE**), the homomorphism $\overline{\phi}$ is uniquely determined. It is also easy to see that a $\mathcal{K}$-free algebra is determined up to isomorphism, by the cardinality of any free generating set (proof?). Note also that if $\emptyset \neq Y \subseteq X$, X is a free generating set for $\mathbf{A}$ and $[Y]_{\mathbf{A}} \in \mathcal{K}$, then Y is a free generating set for $[Y]_{\mathbf{A}}$.

For each cardinal m, we shall choose an isomorphic copy of the $\mathcal{K}$-free algebra with m free generators and denote it by $\mathbf{F}_{\mathcal{K}}(m)$,

The notation $\mathbf{F}_{\mathcal{K}}(X)$ will also be employed whenever the free generating set is specified.

The main objective of this section is to establish the existence of free algebras in every variety of algebras. We shall also pursue the important goal of intrinsically describe free algebras.

We proceed by first considering the variety $\mathcal{S}_1$ of all semigroups. The study of free semigroups will illustrate the general principles and provide the foundation for the general theory.

Lemma 13.1 *If X is a subset of a semigroup $\mathbf{A} = (A, f)$, then*

$$Sg(X) = \{\prod_{k=1}^{n} x_k \mid x_k \in X\}.$$

Let $\mathbf{A} = (A, f)$ be a semigroup and $X \subseteq A$. We say that $\mathbf{A}$ and X satisfy property (FS) provided that each element $a \in A$ is uniquely represented in the form $a = \prod_{k=1}^{n}$ with $n \geq 1$ and $x_k \in X$.

Lemma 13.2 *For each $X \neq \emptyset$, there exists a semigroup $\boldsymbol{A}$ such that $A \supseteq X$ and $\mathbf{A}, X$ satisfy condition (FS).*

Proposition 13.3 *Let X be a subset of a semigroup $\boldsymbol{A}$ such that $Sg(X) = A$. TFAE:*

(i) $\mathbf{A}, X$ satisfy condition (FS).

(ii) $\mathbf{A}$ is the free semigroup over X.

Corollary 13.4 *For each cardinal $m > 0$, $F_{\mathcal{S}}(m)$ exists.*

Let $\mathcal{K}_\nu$ be the class of all algebras of a given similarity type $\nu = (\nu_i)$. As an intermediate step in accomplishing the main objective of this section, we shall establish the existence and give an intrinsic description of $\mathcal{K}$-free algebras.

An algebra $\mathbf{A} \in \mathcal{K}$ is said to be a *Peano algebra* on a set X provided it satisfies the following postulates for all $i, j \in I$, $a, b \in A^{\nu_i}$:

(P1) $f_j(a) \notin X$

(P2) $f_i(a) = f_j(b)$ implies $i = j$ and $a = b$

(P3) $Sg(X) = A$.

Note that (P1) states that the images of operations are disjoint to the generating set, whereas (P2) states that the images of operations are pairwise disjoint and each operation is injective. This gives the decomposition of

$$A = X \bigcup_{i \in I} \mathrm{Im}(f_i)$$

of A into disjoint classes. Examples are:

(i) $(\mathbb{N}_1^+)$ is a Peano algebra of type (1) on $\{0\}$.

(ii) $(\mathbb{N}_1^+, 0)$ is a Peano algebra of type (1,0) on $\emptyset$.

Let $\mathbf{A} = F_{\mathcal{S}}(X)$ be the free semigroup over a nonempty set X. For each $x \in X$, let l_x denote left multiplication by x. Then

$$\Theta = (A, l_x x)$$

is a Peano algebra of type $(1x)x \in X$ on X.

Theorem 13.5 *For each nonempty set X and each type ν, there is a Peano algebra of type ν on X. If $X = \emptyset$, a Peano algebra exists iff the type has nullary operations.*

Proof. (Sketch) Let $\nu = (\nu_i)_{i\in I}$. We may assume that $I \cap X = \emptyset$. Let $\mathbf{S} = F_{\mathcal{S}}(I \cup X)$ be the free semigroup on $I \cup X$. For each $i \in I$, let g_i be the ν_i-ary operation on S defined by

$$g_i(a_0, \ldots, a_{\nu_i-1}) = \begin{cases} ia_0 \ldots a_{\nu_i-1}, & \text{if } \nu_i > 0 \\ i, & \text{if } \nu_i = 0 \end{cases}$$

The algebra $\mathbf{B} = (S, (g_i|i \in I))$ has type ν. We claim that the subalgebras $\mathbf{A} = (A, (f_i)_{i\in I})$, of $\mathbf{B}$ generated by X satisfies (P1)-(P3). The verification of (P1) and (P3) is immediate. The verification of (P2) is an immediate consequence of the following lemma. □

Lemma 13.6 *For all $a, b \in A$ and $x, y \in S$, $ax = by \Rightarrow a = b$ and $x = y$.*

Proof. Hint: Use algebraic induction on $a \in A$. □

Proposition 13.7 *Let $\mathbf{A} \in \mathcal{K}_\nu$ and let X be a subset of A. TFAE:*

(i) $\boldsymbol{A}$ is a Peano algebra of type ν on X.

(ii) $\boldsymbol{A}$ is the $\mathcal{K}_\nu$-free subalgebra over X.

Theorem 13.8 *For each type ν and each cardinal $m > 0$, $\mathbf{F}(m)$ exists. Moreover, $\mathbf{F}_{\mathcal{K}_\nu}(0)$ exists iff ν has nullary operations.*

Let $\mathcal{K}$ be a variety of algebras of type ν which contains a nontrivial member. Let X be a set such that $m = |X| \geq 1$. Define the congruence $\Theta_{\mathcal{K}}(X)$ on $\mathbf{P}_\nu(X)$ (the Peano algebra on X) by

$$\Theta_{\mathcal{K}}(X) = \bigcap\{\Theta \in \mathrm{Con}(\mathbf{P}_\nu(X)) : \mathbf{P}_\nu(X)/\Theta \in \mathcal{K}\}.$$

Theorem 13.9

$$\mathbf{F}_{\mathcal{K}}(X) \cong \mathbf{P}_\nu(X)/\Theta_{\mathcal{K}}(X).$$

Proof. Hints:

(a) Show that $\mathbf{P}_\nu(X)/\Theta_{\mathcal{K}}(X) \in \mathcal{K}$.

(b) Let $Y = \{[x]|x \in X\}$. Show that $|Y| = |X|$.

(c) Show that Y $\mathcal{K}$-freely generates $\mathbf{P}_\nu(X)/\Theta_{\mathcal{K}}(X)$.

□

EXERCISES

$\mathcal{D}$: the variety of distributive lattices.

$\mathcal{B}$: the variety of Boolean algebras.

Notations: $X \Subset Y$: X is a finite nonempty subset of Y. If X is a subset of a Boolean algebra then we write X^- for the set $\{\overline{x}|x \in X\}$.

1. Let $(T_i)|1 \leq i \leq n)$ and $(S_j|1 \leq j \leq m)$ be families of finite nonempty subsets of a distributive lattice L. Then

   (i)
$$\{\bigvee_{i=1}^{n}(\wedge T_i)\} \wedge \{\bigvee_{j=1}^{m}(\wedge S_j)\} = \bigvee_{(i,j),\ 1\leq i\leq n,\ 1\leq j\leq m}[\bigwedge(T_i \cup S_j)].$$

   (ii)
$$\{\bigwedge_{i=1}^{n}(\vee T_i)\} \vee \{\bigwedge_{j=1}^{m}(\vee S_j)\} = \bigwedge_{(i,j),\ 1\leq i\leq n,\ 1\leq j\leq m}[\bigvee(T_i \cup S_j)].$$

   (iii)
$$\bigvee_{i=1}^{n}(\bigwedge T_i)\} = \bigwedge\{\bigvee_{i=1}^{n} f(i)|f \in X_{i=1}^{n} T_i\}.$$

   (iv)
$$\bigwedge_{i=1}^{n}(\bigvee T_i)\} = \bigvee\{\bigwedge_{i=1}^{n} f(i)|f \in X_{i=1}^{n} T_i\}.$$

2. If $L \in \mathcal{D}$ and $\emptyset \neq X \leq L$, then

$$\begin{aligned} Sg(X) &= \{\bigvee_{i=1}^{n}(\bigwedge T_i) : T_i \Subset X, n \geq 1\} \\ &= \{\bigwedge_{j=1}^{m}(\bigvee S_j) : S_j \Subset X, m \geq 1\} \end{aligned}$$

3. If $\mathbf{A} \in \mathcal{B}$ and $\emptyset \neq X \subseteq A$, then

$$Sg(X) = \{\bigvee_{i=1}^{n}(\bigwedge T_i) : T_i \Subset X \cup X^-, n \geq 1\}$$
$$= \{\bigwedge_{j=1}^{m}(\bigvee S_j) : S_j \Subset X \cup X^-, m \geq 1\}$$

4. Let $\mathbf{M}, \mathbf{L} \in \mathcal{D}$ and let X be a nonempty subset of L such that $L = Sg(X)$. A mapping $\phi : X \to M$ can be extended to a homomorphism $\overline{\phi} : \mathbf{L} \to \mathbf{M}$ iff the following condition is satisfied:

$$\wedge T_1 \leq \vee T_2 \Rightarrow \wedge\phi(T_1) \leq \vee\phi(T_2)$$

whenever $T_1, T_2 \Subset X$.

5. Let $\mathbf{A}, \mathbf{B} \in \mathcal{B}$ and let X be a nonempty subset of A such that $A = Sg(X)$. A mapping $\phi : X \to B$ can be extended to a homomorphism $\overline{\phi} : \mathbf{A} \to \mathbf{B}$ iff the following condition is satisfied:

$$(\wedge T_1) \wedge (\wedge T_2^-) = 0 \Rightarrow (\wedge\phi(T_1)) \wedge (\wedge\phi(T_2))^- = 0$$

whenever $T_1 \cup T_2 \Subset X$.

6. For a lattice $L \in \mathcal{D}$ and a nonempty generating subset X of L, TFAE:

   (i) X $\mathcal{D}$-freely generates L, that is, $L \cong \mathbf{F}_{\mathcal{D}}(X)$.

   (ii) Whenever $T_1 \cup T_2 \Subset X$ and $\wedge T_1 \leq \vee T_2$, then $T_1 \cap T_2 \neq \emptyset$.

7. For a Boolean algebra $\mathbf{A}$ and a nonempty generating subset X of A, TFAE:

   (i) X $\mathcal{B}$-freely generates $\mathbf{A}$, that is, $\mathbf{A} = \mathbf{F}_{\mathcal{B}}(X)$.

   (ii) Whenever $T_1 \cup T_2 \Subset X$ and $(\cap T_1)(\cap T_2^-)$, then $T_1 \cap T_2 \neq \emptyset$.

   For each cardinal m, the existence of $\mathbf{F}_{\mathcal{B}}(m)$ and $\mathbf{F}_{\mathcal{D}}(m)$ is assured by the results of this section 13. An elementary direct construction is descried in the next exercise.

8. Let X be a set of cardinality $m > 0$. For each $x \in X$, let $A_x = \{T \subseteq X | x \in T\}$ and let $Y = \{A_x | x \in X\}$.

(i) $\mathbf{F}_{\mathcal{D}}(m)$ is isomorphic to the sublattice of $\mathcal{P}(X)$ generated by Y and Y is a $\mathcal{D}$-free set of generators for it.

(ii) $\mathbf{F}_{\mathcal{B}}(m)$ is isomorphic to the Boolean algebra of $\mathcal{P}(X)$ generated by Y and Y is a $\mathcal{B}$-free set of generators for it.

9. If $\mathbf{B}$ is a subalgebra of a Peano algebra $\mathbf{A}$, then there exists a unique subset Y of B such that $\mathbf{B}$ is a Peano algebra on Y.

10. Describe the free objects in the variety of abelian groups.

11. Let G be a group and let X be a generating set for G. TFAE:

    (i) X freely generates G.

    (ii) Each $w \in G$, $w \neq 1$, is uniquely representable in the form
    $$w = \prod_{i=1}^{n} x_i^{\epsilon_i}$$
    where $n \geq 1$, $x_1 \in X$, $\epsilon_i = \pm 1$, and if $x_i = x_{i+1}$, then $\epsilon_i = \epsilon_{i+1}$.

    (iii) Each $w \in G$, $w \neq 1$, is uniquely representable in the form
    $$w = \prod_{i=1}^{n} x_i^{k_i}$$
    where $n \geq 1$, $x_1 \in X$, $x_i \neq x_{i+1}$, and $k_i \in \mathbb{Z} \setminus \{0\}$.

12. Let $\mathbb{R}[t]$ denote the polynomial ring in one indeterminate over the reals. The subgroup of the general linear group $GL_2(\mathbb{R}[t])$ generated by the transvections
$$T_{12}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$
and
$$T_{21}(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$
is freely generated by them.

The next exercise presents an alternative construction for the free objects in the variety $\mathcal{D}$ of distributive lattices.

13. Let X be a nonempty set. Let (L_i) be the collection of all totally ordered sets with underlying set X. Let $L = \prod L_i$. For each $x \in X$, let $\rangle x \langle$ be the element of L such that $\rangle x \langle(i) = x$ for all $i \in I$. Set $Y = \{\rangle x \langle | x \in X\}$. Show:

   (i) $|X| = |Y|$.

   (ii) The sublattice F of L generated by Y is $\mathcal{D}$-freely generated by Y. That is, $\mathbf{F}_{\mathcal{D}}(X) \cong \mathbf{F}$.

14 The HSP Theorem

The objective of this section is to establish Birkhoff's celebrated result which asserts that a class of similar algebras is a variety iff it is an equational class.

An *identity (equation)* of type ν is an expression of the form

$$p \approx q,$$

where $p, q \in \mathbf{P}_\nu(\omega)$. Here $\mathbf{P}_\nu(\omega)$ denotes a fixed copy of the isomorphic Peano algebras of type ν on a countably infinite set. This set of all identities of type ν will be denoted by Id_ν.

An algebra $\mathbf{A} \in \mathcal{K}_\nu$ is said to *satisfy* the identity $p \approx q$ (or that the identity is *true* in $\mathbf{A}$) if $\phi(p) = \phi(q)$ for every homomorphism $\phi : \mathbf{P}_\nu(\omega) \to \mathbf{A}$. This relation is written as

$$\mathbf{A} \vDash p \approx q.$$

A class $\mathcal{K} \subseteq \mathcal{K}_\nu$ is said to satisfy the identity $p \approx q$, written

$$\mathcal{K} \vDash p \approx q,$$

if each member of $\mathcal{K}$ satisfies $p \approx q$. If Σ is a set of identities of type ν, we say that $\mathcal{K}$ satisfies Σ, written

$$\mathcal{K} \vDash \Sigma,$$

if $\mathcal{K}$ satisfies every identity in Σ.

For a class $\mathcal{K} \subseteq \mathcal{K}_\nu$, we write $\mathcal{V}(\mathcal{K})$ for the smallest variety containing $\mathcal{K}$. In view of section 12, $\mathcal{V}(\mathcal{K}) = HSP(\mathcal{K})$.

Lemma 14.1 *For any class $\mathcal{K} \subseteq \mathcal{K}_\nu$, all the classes $\mathcal{K}, P\mathcal{K}, S\mathcal{K}, H\mathcal{K}$ and $\mathcal{V}(\mathcal{K})$ satisfy the same identities.*

Let $\mathcal{K}$ be a class of algebras of type ν and $\Sigma \in Id_\nu$. We define

$$Th(\mathcal{K}) = \{p \approx q | \mathcal{K} \vDash p \approx q\}$$

and

$$Mod(\Sigma) = \{\mathbf{A} | \mathbf{A} \vDash \Sigma\}.$$

$\mathcal{K}$ is called an *equational class* if $\mathcal{K} = Mod(\Gamma)$ for some $\Gamma \subseteq Id_\nu$. Σ is called an *equational theory* if $\Sigma = Th(\mathcal{L})$ for some class $\mathcal{L} \subseteq \mathcal{K}_\nu$. In this case we call Σ the equational theory of $\mathcal{L}$.

The next result is a direct consequence of lemma 14.1.

Corollary 14.2 *Let $\mathcal{K} \subseteq \mathcal{K}_\nu$ and $\Sigma \subseteq Id_\nu$.*

(i) $Mod(\Sigma)$ is a variety.

(ii) $\mathcal{V}(\mathcal{K}) \subseteq Mod(Th(\mathcal{K}))$.

We shall establish that every variety is an equational class by showing that the inclusion of the preceding result is actually an equality. The proof of this fact depends on the following result:

Lemma 14.3 *Let $\mathcal{K} \subseteq \mathcal{K}_\nu$ be a variety containing a non-trivial algebra and let $p \approx q \in Id_\nu$. TFAE:*

(i) $p \approx q \in Th(\mathcal{K})$.

(ii) $(p, q) \in \Theta_\mathcal{K}(\omega)$.

(iii) $\mathbf{F}_\mathcal{K}(\mathcal{K}) \models p \approx q$.

(iv) $\overline{p} = \overline{q}$, where $p \mapsto \overline{p}$ is the canonical projection of $\mathbf{P}_\nu(\omega)$ onto $\mathbf{F}_\mathcal{K}(\omega) = \mathbf{P}_\nu(\omega)/\Theta_\mathcal{K}(\omega)$.

(v) $\mathbf{F}_\mathcal{K}(X) \models p \approx q$ for every infinite set X.

Theorem 14.4 *For any class $\mathcal{K} \subseteq \mathcal{K}_\nu$, $\mathcal{V}(\mathcal{K}) = Mod(Th(\mathcal{K}))$. Thus, $\mathcal{K}$ is a variety iff $\mathcal{K}$ is an equational class.*

Proof. (Sketch) In view of corollary 14.2, we need to show that $Mod(Th(\mathcal{K}) \subseteq \mathcal{V}(\mathcal{K})$. If $\mathcal{K}$ consists of trivial algebras, then $\mathbf{A} \models Th(\mathcal{K})$ iff $|A| = 1$. Thus, in this case $Mod(Th(\mathcal{K})) = \mathcal{V}(\mathcal{K})$. Suppose that $\mathcal{K}$ contains a non-trivial member so that $\mathbf{F}_{\mathcal{V}(\mathcal{K})}(m)$ exists for all $m > 1$. Let $\mathbf{A} \in Mod(Th(\mathcal{K}))$ and let X be a set such that $|X| \geq |A|$. Use lemma lemma 14.3 to show that $\mathbf{A}$ is an isomorphic image of $\mathbf{F}_\mathcal{K}(X)$. $\square$

Corollary 14.5 *There are at most $2^{I+\aleph_0}$ varieties of type $\nu = (\nu_i)_{i \in I}$.*

Corollary 14.6 *If $\mathcal{K}$ is a variety having a non-trivial member and $|X| \geq \aleph_0$, then $\mathcal{K} = \mathcal{V}(\mathbf{F}_\mathcal{K}(X))$.*

15 Free Products of Algebras

The concept of a free product is fundamental to the study of an algebraic system. Intuitively, a free product takes a family of algebras from a given class and combines it in the “loosest” or “freest” way possible. By this it is meant that any other algebra generated by the given family must be a homomorphic image of the free product.

Free products have been widely used in areas such as group theory and lattice theory to provide general methods of construction, establish various embedding theorems, and also produce pathological algebras.

Let $\mathcal{K}$ be a class of algebras of the same similarity type and let $(\mathbf{A}_i)$ be a family of members of $\mathcal{K}$. The $\mathcal{K}$-*free product* of this family is an algebra $\mathbf{A} \in \mathcal{K}$, denoted by ${}^{\mathcal{K}}\bigsqcup \mathbf{A}_i$, together with a family of injective homomorphisms $(\alpha_i : \mathbf{A}_i \to \mathbf{A})$ such that

(i) $Sg(\bigcup \alpha_i(A_i)) = A$.

(ii) If $\mathbf{B} \in \mathcal{K}$ and $(\beta_i : \mathbf{A}_i \to \mathbf{B})$ is a family of homomorphisms, then there exists a (necessarily unique) homomorphism $\gamma : \mathbf{A} \to \mathbf{B}$ satisfying $\beta_i = \gamma\alpha_i$ for all i.

We shall find it convenient to identify each factor $\mathbf{A}_i$ with its image $\alpha_i(\mathbf{A}_i)$ in ${}^{\mathcal{K}}\bigsqcup \mathbf{A}_i$ and view each α_i as the inclusion map.

It is easy to see that if the free product of a family exists, then it is unique up to isomorphism. The interpretation of the concept of a free product of a universal-mapping problem is due to Sikorski (1952). Free products of groups had been previously studied extensively in the realm of combinatorial group theory.

We address first the question of existence of free products.

Theorem 15.1 *Let $\mathcal{K}$ be a variety of algebras and let $(\mathbf{A}_i)$ be a family in $\mathcal{K}$. The free product ${}^{\mathcal{K}}\bigsqcup \mathbf{A}_i$ exists provided the following property is satisfied: (EP) There is an algebra $\mathbf{B} \in \mathcal{K}$ and a family of injective homomorphisms $(\phi_i : \mathbf{A}_i \to \mathbf{B})$.*

The preceding result applies, in particular, to varieties in which every algebra has a one-element subalgebra. In this case, one can let $\mathbf{B}$ be the direct product of the algebras $\mathbf{A}_i$ and let $\phi_i : \mathbf{A}_i \to \mathbf{B}$ be the canonical embedding for each i.

EXERCISES

1. Give an explicit construction for the free products in the variety of abelian groups.

2. Do the same for the class of all modules over a fixed ring R.

3. Show that the homomorphism γ in the definition of a free product is unique.

4. Assume that $\mathbf{F}_{\mathcal{K}}(1)$ and $\mathbf{F}_{\mathcal{K}}(m)$ exist for $m \geq 1$. Prove that $\mathbf{F}_{\mathcal{K}}(m)$ is the $\mathcal{K}$-free product of m copies of $\mathbf{F}_{\mathcal{K}}(1)$.

5. Let L be a distributive lattice and let $(L_i | i \in I)$ be a family of sublattices of L such that $\cup L_i$ generates L.

   (i) $L \cong^{\mathcal{D}} \bigsqcup \mathbf{L}_i$.

   (ii) If $J, K \Subset I$, $a_j \in L_j$ for all $j \in J$ and $b_k \in L_k$ for all $k \in K$, the relation
   $$\bigwedge_{j \in J} a_j \leq \bigvee_{k \in K} b_k$$
   implies the existence of $i \in J \cap K$ with $a_i \leq b_i$.