Continued Fractions

Author: *Aleksa Vučković, second grade student, Mathematical Grammar School, Belgrade*
Supervisor: *Danica Zečević*
Center for talented youth "Mihajlo Pupin", Dimitrija Tucovića 2, Pančevo

Introduction

Simple continued fraction is expression with form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}, \qquad (*)$$

where a_1, a_2,... are natural numbers and a_0 whole number. We often write expression $(*)$ as $[a_0, a_1, a_2, ..., a_n]$ for compactness. In this record, we do not necessarily imply that a_i are integers. We call expression $[a_0, a_1, ..., a_k]$ k-th **covergence** for $[a_0, a_1, ..., a_n]$, and expression $a'_k = [a_k, a_{k+1}, ..., a_n]$ k-th complete quotient $(n \geq k)$. These two terms will be very important for us to prove the properties of continued fractions and their convergences. Continued fractions have existed for hundreds of years. Every final record of the continued fraction corresponds to a rational number, while the infinite corresponds to irrational. Continued fractions are related to Euclid's algorithm, have their application in calendar, writing down the number *pi* and the number e in the form of fractions, and are used to find the solution of the Pell's equation$(x^2 - dy^2 = 1)$.

Objective

The aim of this research paper is to prove the properties of continued fractions, to represent real numbers in the form of continued fractions, to find the best approximation of an irrational number to rational ones, or to approximate a fraction with large values of denominators and numerators to smaller numerical values in order for easier calculations and practical application with minimal errors.

Materials and methods

Methods of research are collecting literature, as well as their interpretation and resolution. The correlation between theoretical and practical considerations was sought application, the process of reaching the solution is presented in detail. The combined method of research was used and literature with a diverse application of continued fractions was studied in this scientific work. In addition, the development of a rational number into a continued fraction is shown graphically, for easier understanding.

Results

The subject of this paper are continued fractions, convergence of continued fractions, as well as their properties and applications. Below you will have the opportunity to see some of the many theorems and their consequences. A selection of the most important things on this topic has been made in the author's opinion which the reader will encounter in the forthcoming text.

Continued fractions are not suitable for basic calculus operations. However, it is not difficult to perform operations $x \to \frac{1}{x}$ and $x \to -x$ with them, by using equalities:

$$[a_0, a_1, ..., a_n] \cdot [0, a_0, a_1, ..., a_n] = 0 \qquad \text{for } a_0 \geq 1, \text{ and}$$
$$[a_0, a_1, ..., a_n] + [-1 - a_0, 1, a_1 - 1, a_2, ..., a_n] \qquad \text{for } a_1 > 1.$$

Continued fraction $[a_0, a_1, ..., a_n]$ is equal to $\frac{p_n}{q_n}$, where arrays (p_n) and (q_n) meet the following requirements

$$\begin{matrix} p_{-1} = 1, & p_0 = a_0, & p_k = a_k p_{k-1} + p_{k-2} \\ q_{-1} = 1, & q_0 = 1, & q_k = a_k q_{k-1} + q_{k-2} \end{matrix} \qquad \text{for } 2 \leq k \leq n.$$

Apart from the above operations and the recursive connection between the convergence of continued fractions, it is important to mention:
Theorem, respectively, the next 2 identities

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \quad \text{and}$$
$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$$

on which the following consequences are based:

(1) Convergents $\frac{p_n}{q_n}$ of simple continued fraction are irreducible: $\gcd(p_n, q_n) = 1$.

(2) $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} \dots < \frac{p_n}{q_n} < \dots \frac{p_3}{q_3} < \frac{p_1}{q_1}$.

And at last, the identity that connects continued fraction and "reverse" counterpart
If it is $[a_0, a_1, ..., a_n] = \frac{p_n}{q_n}$, then $[a_n, a_{n-1}, ..., a_0] = \frac{p_n}{p_{n-1}}$.

Conclusion

Although these fractions are no longer subject in school books, their use is still unusually large. Continued fractions appear in many different branches of mathematics: the theory of Diophantine approximations, algebraic number theory, coding theory, toric geometry, dynamical systems, ergodic theory, topology, etc. One of the mathematical explanations of this phenomenon is based on an interesting structure of the set of real numbers endowed with two operations: addition and inversion. This structure appeared for the first time in the Euclidean algorithm, which was known several thousand years ago. That is the reason why continued fractions can be encountered far away from number theory.

Literature

[1] D. Djukić *Verižni razlomci*, Belgrade, 2010/11. `https://imomath.com/srb/dodatne/veriznirazlomci_ddj.pdf`
[2] S. Krushchev *Orthogonal Polynomials and Continued Fractions*, Cambridge University Press, 2008.
[3] W. Bosma, C. Kraaikamp *Continued Fractions* `https://www.math.ru.nl/~bosma/Students/CF.pdf`